Tangential touch between the free and the fixed boundary in a semilinear free boundary problem in two dimensions

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Abstract. We study minimizers of the functional

$$\int_{B_1^+} (|\nabla u|^2 + 2(\lambda^+ (u^+)^p + \lambda^- (u^-)^p))\, dx,$$

where $B_1^+ = \{x \in B_1 : x_1 > 0\}$, $u = 0$ on $\{x \in B_1 : x_1 = 0\}$, $\lambda^{\pm}$ are two positive constants and $0 < p < 1$. In two dimensions, we prove that the free boundary is a uniform C^1 graph up to the flat part of the fixed boundary and also that two-phase points cannot occur on this part of the fixed boundary. Here, the free boundary refers to the union of the boundaries of the sets $\{x : \pm u(x) > 0\}$.

1. Introduction

1.1. Problem setting

Let $u^{\pm} = \max(\pm u, 0)$, $\Pi = \{x \in \mathbb{R}^n : x_1 = 0\}$, B_r be the open ball in $\mathbb{R}^n$ centered at 0 with radius $r > 0$, and $B_r^+ = \{x \in B_r : x_1 > 0\}$, and consider minimizers of the functional

$$(1) \qquad E(u) = \int_{B_1^+} (|\nabla u|^2 + 2\lambda^+ (u^+)^p + 2\lambda^- (u^-)^p)\, dx,$$

over

$$\mathcal{K} = \{u \in W^{1,2}(B_1^+) : u = 0 \text{ on } B_1 \cap \Pi \text{ and } u = f \text{ on } \partial B_1^+ \setminus \Pi\}.$$

Here

$$f \in W^{1,2}(B_1) \cap L^{\infty}(B_1), \quad \lambda^{\pm} > 0, \quad \text{and} \quad 0 < p < 1.$$

By classical methods in calculus of variations it is straightforward to prove the existence of a minimizer. The corresponding Euler–Lagrange formulation of (1)

reads

$$(2) \quad \begin{cases} \Delta u = p(\lambda^+ (u^+)^{p-1} \chi_{\{u>0\}} - \lambda^- (u^-)^{p-1} \chi_{\{u<0\}}) & \text{in } \{x \in B_1^+ : u(x) \neq 0\}, \\ u = f & \text{on } \partial B_1^+ \setminus \Pi, \\ u = 0 & \text{on } B_1 \cap \Pi. \end{cases}$$

Due to the singularity of the Euler–Lagrange equation, it is not clear that any minimizer satisfies the equation everywhere. Moreover, since the energy is not convex, there might be more than one minimizer with given boundary data.

We use the notation $\Omega^+ = \{x : u(x) > 0\}$, $\Omega^- = \{x : u(x) < 0\}$, $\Gamma^\pm = \partial \Omega^\pm$ and $\Gamma = \Gamma^+ \cup \Gamma^-$, and refer to Γ as the *free boundary* which is not known a priori, i.e., it is a part of the solution of the problem.

The main result of this paper concerns the behavior of the free boundary close to the fixed boundary Π, in two dimensions. In order to state our main theorem, we define the class of solutions within which we will work.

Definition 1.1. Let M and R be two positive constants. We define $P_R(M)$ to be the class of minimizers u of (1) in B_R^+ such that $0 \in \Gamma \cap \Pi$ and

$$\|u\|_{L^\infty(B_R^+)} \leq M.$$

Remark 1.2. If u is a minimizer of (1) in B_1^+ such that

$$\|u\|_{L^\infty(B_1^+)} \leq M,$$

and $x_0 \in \Gamma \cap \Pi$ but $0 \notin \Gamma \cap \Pi$, one can by translating and rescaling u obtain a function in $P_1(M')$ for another constant M'.

Theorem 1.3. *Let $u \in P_1(M)$ in dimension two. Then, in a neighborhood of the origin u does not change sign. Moreover, the free boundary is a C^1 graph with a modulus of continuity depending only on M, $\lambda^\pm$ and p.*

Remark 1.4. In the theorem above, the modulus of continuity refers to a function $\sigma \colon [0, 1) \to [0, \infty)$ such that

$$\sigma(0) = \lim_{r \to 0} \sigma(r) = 0.$$

1.2. Known result

The one-phase case of the problem, i.e., the case when u does not change signs, has been well studied before. Phillips has proved in [14] that minimizers are locally in $C^{1,\beta-1}$ for $\beta=2/(2-p)$.

Furthermore, Phillips (cf. [13]) and Alt and Phillips (cf. [2]) showed that the free boundary is fully regular in dimension two. For the two-phase case, when u is allowed to change signs, it was proved in [8] that u is locally $C^{1,\beta-1}$. Moreover, the second author and Petrosyan proved in [11], the C^1 regularity of the free boundary in dimension two. However, none of these results say anything about the behavior near the fixed boundary, they are all interior results.

For the particular case of the problem when $p=0$, Alt, Caffarelli and Friedman introduced in [1] a monotonicity formula and showed the optimal Lipschitz regularity of minimizers and the C^1 regularity of the free boundary in dimension two. In the case $p=1$, (2) reduces to the two-phase obstacle problem which was introduced by Weiss in [19]. For this problem, Uraltseva [17] and Shahgholian [15] proved the optimal $C^{1,1}$ regularity. Furthermore, in [16], Shahgholian, Uraltseva and Weiss proved the C^1 regularity of the free boundary close to so-called *branching points* (see Section 2). The mentioned results are all interior regularity results. But for the cases $p=0$ and $p=1$ there are also some results concerning the behavior of the free boundary near the fixed boundary. See for instance [3] and [10] where it is proved for $p=1$ and $p=0$, respectively, that the free boundary approaches the fixed one in a tangential fashion.

1.3. Organization of the paper

The paper is organized as follows:

– In Section 2, we introduce the notion of blow-ups and also the different notions of free boundary points.

– In Section 3, we prove $C^{1,\alpha}$-estimates up to the fixed boundary.

– In Section 4, we state and prove some technicalities that are important for the rest of the paper, such as growth estimates, non-degeneracy, classification of global minimizers and Weiss's monotonicity formula.

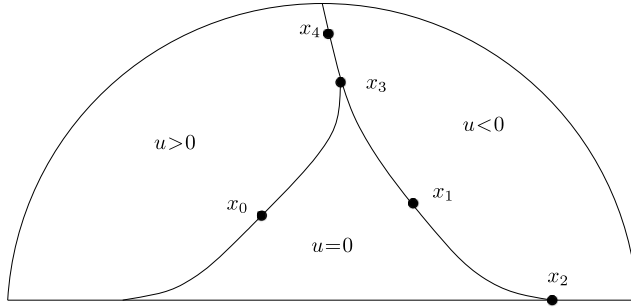– In Section 5, we prove the main result.

Figure 1. This figure illustrates the different types of free boundary points. The point x_0 is a positive one-phase free boundary point, x_1 is a negative one-phase point, x_2 is a negative one-phase point touching the fixed boundary, x_3 is a branching point and x_4 is a two-phase point which might or might not be a branching point.

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2. Free boundary points and the notion of blow-ups

Suppose that u is a minimizer of (1) and $x_0 \in \Gamma$. Then we divide the free boundary points into the following parts (see Figure 1):

1. We say that x_0 is a *positive (negative) one-phase free boundary point* if there exist a neighborhood of x_0 such that u is non-negative (non-positive) in it. In other words, $x_0 \in \Gamma^+ \backslash \Gamma^-$ ($x_0 \in \Gamma^- \backslash \Gamma^+$).

2. We say that x_0 is a *two-phase free boundary point* if $x_0 \in \Gamma^+ \cap \Gamma^-$. Moreover, if $|\nabla u(x_0)|=0$ then x_0 is said to be a *branching point*.

A useful notion when studying properties of free boundary problems is the so-called blow-ups.

Definition 2.1. For a given minimizer u of (1) and $x_0 \in \Gamma$ (a one phase or branching point) we define the rescaled functions

$$u_{x_0,r}(x) = \frac{u(x_0+rx)}{r^\beta}, \quad \beta = \frac{2}{2-p} \text{ and } r > 0.$$

In the case $x_0=0$ we use the notation $u_r=u_{0,r}$. If we can find a sequence u_{x_0,r_j}, $r_j \to 0$, such that

$$u_{x_0,r_j} \longrightarrow u_0 \quad \text{in } C^1_{\text{loc}}(\{x \in \mathbb{R}^n : x_1 > 0\}) \text{ (or } C^1_{\text{loc}}(\mathbb{R}^n)),$$

we say that u_0 is a *blow-up of u at x_0*. It is easy to see that u_0 is a global minimizer of (1), i.e., a minimizer of (1) in D for all $D \subset \{x \in \mathbb{R}^n : x_1 > 0\}$ (or sometimes in $\mathbb{R}^n$), and with a certain growth condition (see below).

We also define the following class of global minimizers.

Definition 2.2. Let M be a positive constant. We define $P_\infty(M)$ to be the *class of local minimizers u of* (1), i.e., minimizers of (1) in D for all $D \subset \{x \in \mathbb{R}^n : x_1 > 0\}$, defined in $\{x \in \mathbb{R}^n : x_1 > 0\}$ such that $0 \in \Gamma \cap \Pi$ and

$$\|u\|_{L^\infty(B_R^+)} \leq MR^\beta \quad \text{for all } R > 0.$$

3. Regularity

In this section we will prove that any minimizer is $C^{1,\alpha}$ up to the fixed boundary. It is possible that parts of the results in this section can be found in the literature, however we have not been able to find any good reference for that. For instance, in [8] the interior C^1-regularity is proved for minimizers of functionals of the type (1), but nowhere can any statement about the regularity up to the fixed boundary be found, even though the technique properly used, probably would imply the same regularity up to the boundary in this case.

Lemma 3.1. (Estimates in L^∞) *Let u be a minimizer of* (1)*. Then $u \in L^\infty(B_1^+)$ and we have the estimate*

$$\|u\|_{L^\infty(B_1^+)} \leq C(p, \|f\|_{L^\infty}).$$

Proof. Any minimizer of (1) is a solution of (2) when $\{x : u(x) \neq 0\}$. Let

$$v(x) = \max(u(x), 1).$$

Then $\Delta v \geq -pC$ for some positive constant C. By the maximum principle

$$\sup v \leq \max(1, \sup f) + pC.$$

Similar arguments for

$$v(x) = \max(-u(x), 1)$$

show that u is bounded from below and we get

$$\|u\|_{L^\infty} \leq C(pC + \|f\|_{L^\infty}). \quad \square$$

3.1. Hölder regularity

We can now prove that minimizers are Hölder continuous for all exponents less than one. Throughout the rest of the paper, the harmonic replacement of a function u in an open set D, will refer to the function v satisfying

$$\begin{cases} \Delta v = 0 & \text{in } D, \\ u = v & \text{on } \partial D. \end{cases}$$

In what follows we let $B_r^+(x) = \{x \in B_r(x) : x_1 > 0\}$, where $B_r(x)$ denotes the ball with center x and radius r.

Proposition 3.2. (Hölder regularity) *Let u be a minimizer of* (1). *Then for each $\gamma < 1$ there is a constant $C = C(\gamma, \lambda^{\pm}, p, \|u\|_{L^{\infty}(B_1^+)})$ such that*

$$\|u\|_{C^{0,\gamma}(\overline{B_{1/2}^+})} \leq C.$$

Proof. Take $x_0 \in B_{1/2}^+$ and let $0 < r < \frac{1}{2}$. The idea is to prove that for all $\gamma \in (0,1)$ there is a constant C_γ independent of r and x_0 such that

(3)
$$\int_{B_r^+(x_0)} |\nabla u|^2 \, dx \leq C_\gamma r^{n-\gamma}.$$

By Morrey's embedding this will imply the desired result, see Theorem 7.19 in [9]. With v as the harmonic replacement of u in $B_r^+(x_0)$ we have, due to the Dirichlet principle,

$$\int_{B_r^+(x_0)} |\nabla v|^2 \, dx \leq \int_{B_r^+(x_0)} |\nabla u|^2 \, dx \leq E(u).$$

Since v is harmonic and $u = v$ on $\partial B_r^+(x_0)$,

$$\int_{B_r^+(x_0)} |\nabla v - \nabla u|^2 \, dx = \int_{B_r^+(x_0)} (|\nabla v|^2 - |\nabla u|^2) \, dx.$$

Putting these to together and using Lemma 3.1 we can conclude that

$$\int_{B_r^+(x_0)} |\nabla v - \nabla u|^2 \, dx \leq \int_{B_r^+(x_0)} 2(\lambda_1 (u^+)^p + \lambda_2 (u^-)^p) \, dx \leq C(p, \lambda^{\pm}, \|u\|_{L^{\infty}(B_1^+)}) r^n.$$

If $r < R < \frac{1}{2}$ and v is the harmonic replacement in $B_R^+(x_0)$, the estimate above implies via Young's inequality that

$$\int_{B_r^+(x_0)} |\nabla u|^2 \, dx \leq 2 \int_{B_r^+(x_0)} |\nabla u - \nabla v|^2 \, dx + 2 \int_{B_r^+(x_0)} |\nabla v|^2 \, dx$$

$$\leq C r^n + 2C \left(\frac{r}{R}\right)^n \int_{B_R^+(x_0)} |\nabla v|^2 \, dx$$

$$\leq C r^n + 2C \left(\frac{r}{R}\right)^n \int_{B_R^+(x_0)} |\nabla u|^2 \, dx,$$

where we have again used that v minimizes the Dirichlet energy and the estimate

$$\int_{B_r^+(x_0)} |\nabla v|^2 \, dx \leq C \left(\frac{r}{R}\right)^n \int_{B_R^+(x_0)} |\nabla v|^2 \, dx,$$

which follows from interior gradient estimates for harmonic functions, upon reflecting v in an odd manner across Π. Taking $r = \sigma^{j+1}$ and $R = \sigma^j$, where σ is small enough and $j \in \mathbb{N}$, then this turns into

$$\int_{B_{\sigma^{j+1}}^+(x_0)} |\nabla u|^2 \, dx \leq C \sigma^{(j+1)n} + C \sigma^n \int_{B_{\sigma^j}^+(x_0)} |\nabla u|^2 \, dx.$$

Now it is clear that if (3) holds for $r = \sigma^j$ for some γ and C_γ, then the estimate above implies that

$$\int_{B_{\sigma^{j+1}}^+(x_0)} |\nabla u|^2 \, dx \leq C \sigma^{(j+1)n} + C C_\gamma \sigma^n \sigma^{j(n-\gamma)} \leq C_\gamma \sigma^{(j+1)(n-\gamma)} \left(\frac{C}{C_\gamma} + C \sigma^\gamma\right).$$

If we choose C_γ large enough and σ small enough then

$$\int_{B_{\sigma^{j+1}}^+(x_0)} |\nabla u|^2 \, dx \leq C_\gamma \sigma^{(j+1)(n-\gamma)}.$$

Iterating this, yields (3). $\quad\square$

3.2. $C^{1,\alpha}$-estimates up to the fixed boundary

Now we turn our attention to the $C^{1,\alpha}$-regularity. The idea is to use the method in [12]. We are going to employ the following result, which is a special case of Theorem I.2 in [6].

Proposition 3.3. *Let $u \in H^1(B_1^+)$. Assume there exist C and α such that for each $x_0 \in B_{1/2}^+$ there is a vector $A(x_0)$ with the property*

$$(4) \qquad \int_{B_r(x_0) \cap B_1^+} |\nabla u(x) - A(x_0)|^2 \, dx \leq C r^{n+2\alpha} \quad \text{for every } r < \tfrac{1}{2}.$$

Then $u \in C^{1,\alpha}(\overline{B_{1/2}^+})$ *and we have the estimate*

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \leq C_0(C).$$

The only non-standard in the proposition above is that we get $C^{1,\alpha}$-estimates up to the fixed boundary. Below we present a technical result concerning harmonic functions. First we just make the following remark.

Remark 3.4. Let $x_0 \in B_{1/2}^+$. Then for any $r < \frac{1}{2}$, we have the following estimates for any harmonic function u in $B_r^+(x_0)$, where either u vanishes on $B_1 \cap \Pi$ or $B_r^+(x_0) = B_r(x_0)$, i.e., $B_r(x_0)$ does not intersect Π,

$$(5) \qquad \sup_{B_{r/2}^+(x_0)} |D^2 u(x)| \leq \frac{C}{r^{n/2+1}} \left(\int_{B_r^+(x_0)} |\nabla u|^2 \, dx \right)^{1/2}$$

and

$$(6) \qquad |\nabla u(x_0)| \leq \frac{C}{r^{n/2}} \left(\int_{B_r^+(x_0)} |\nabla u|^2 \, dx \right)^{1/2}.$$

Moreover, for $\alpha \in [0, 1)$ we get

$$(7) \quad \|u\|_{C^{1,\alpha}(B_{r/2}^+(x_0))} \leq C r^{1-\alpha} \|\Delta u\|_{L^\infty(B_r^+(x_0))} + C r^{-n/2-\alpha} \left(\int_{B_r^+(x_0)} |\nabla u|^2 \, dx \right)^{1/2}.$$

All the above estimates follow from standard interior estimates for the Poisson equation if $B_r^+(x_0) = B_r(x_0)$. To obtain these estimates in the case $B_r^+(x_0) \neq B_r(x_0)$, assume $r = 1$ and simply reflect u oddly across Π. Then we can apply usual interior estimates in $B_{r/2}^+(x_0) \cup (B_{r/2}^+(x_0))^{\text{reflected}}$. The estimate (5) will now follow from rescaling the estimate

$$\sup_{B_{1/2}^+(x_0)} |D^2 u(x)| \leq C \left(\int_{B_1^+(x_0)} u^2 \, dx \right)^{1/2} \leq C \left(\int_{B_1^+(x_0)} |\nabla u|^2 \, dx \right)^{1/2},$$

where the first estimate comes from interior C^2-estimates for harmonic functions (see Theorem 7 on p. 29 in [7]). Similarly, (6) follows from rescaling the gradient estimate for harmonic functions

$$|\nabla u(x_0)| \leq C \left(\int_{B_r^+(x_0)} u^2 \, dx \right)^{1/2} \leq C \left(\int_{B_r^+(x_0)} |\nabla u|^2 \, dx \right)^{1/2}.$$

Finally, (7) is a consequence of interior $C^{1,\alpha}$-estimates for the Poisson equation (cf. Theorem 4.15 on p. 68 in [9])

$$\|u\|_{C^{1,\alpha}(B^+_{1/2}(x_0))} \leq C\|\Delta u\|_{L^\infty(B^+_1(x_0))} + C\left(\int_{B^+_1(x_0)} u^2\,dx\right)^{1/2}$$

$$\leq C\|\Delta u\|_{L^\infty(B^+_1(x_0))} + C\left(\int_{B^+_1(x_0)} |\nabla u|^2\,dx\right)^{1/2}.$$

Lemma 3.5. *Let* $x_0 \in B^+_{1/2}$ *and let* v *be harmonic in* $B^+_r(x_0)$. *Assume also either that* v *vanishes on* $B_1 \cap \Pi$ *or that* $B^+_r(x_0) = B_r(x_0)$. *Then for* $\sigma < 1$ *we have*

$$\int_{B^+_{\sigma r}(x_0)} |\nabla v(x) - \nabla v(x_0)|^2\,dx \leq C\sigma^{n+2}\int_{B^+_r(x_0)} |\nabla v(x)|^2\,dx.$$

Proof. From estimate (5) we have

$$\sup_{B^+_{\sigma r}(x_0)} |D^2 v(x)| \leq \frac{C}{r^{n/2+1}}\left(\int_{B^+_r(x_0)} |\nabla v|^2\,dx\right)^{1/2},$$

from which it follows that for $x \in B^+_{\sigma r}(x_0)$,

$$|\nabla v(x) - \nabla v(x_0)|^2 \leq \frac{C\sigma r}{r^{n+2}}\int_{B^+_r(x_0)} |\nabla v|^2\,dx.$$

If we integrate this over $B^+_{\sigma r}(x_0)$ we obtain

$$\int_{B^+_{\sigma r}(x_0)} |\nabla v(x) - \nabla v(x_0)|^2\,dx \leq C\sigma^{n+2}\int_{B^+_r(x_0)} |\nabla v|^2\,dx. \quad \square$$

Now we are ready to prove the desired estimate.

Proposition 3.6. ($C^{1,\alpha}$-*estimates*) *Let* u *be a minimizer of* (1). *Then there are constants* $\alpha = \alpha(\lambda^\pm, p, \|u\|_{L^\infty(B^+_1)})$ *and* $C = C(\lambda^\pm, p, \|u\|_{L^\infty(B^+_1)})$ *such that*

$$\|u\|_{C^{1,\alpha}(\overline{B^+_{1/2}})} \leq C.$$

Proof. We will find appropriate constants α and C such that (4) holds for all $r < \frac{1}{2}$. Then the result will follow from Proposition 3.3.

The way we will do this is by proving that for some small α and σ and for all $x_0 \in B^+_{1/2}$ we can find a sequence A_j such that

$$(8) \qquad \int_{B^+_{\sigma^j}(x_0)} |\nabla u - A_j|^2\,dx \leq C_1\sigma^{j(n+2\alpha)}$$

and

$$|A_j - A_{j-1}| \leq C_2 \sigma^{j\alpha} \tag{9}$$

for all j, as long as we have

$$\inf_{B^+_{\sigma^j}(x_0)} |u| \leq \sigma^j. \tag{10}$$

Intuitively this will imply the desired inequality since if (10) holds for all j then we can pass to the limit in (8) and we are done, if not, (10) must fail for some j, but then u does not vanish in the corresponding ball so that the equation for u is non-singular there, and we can use estimates for the Poisson equation with bounded inhomogeneity.

For the sake of clarity we split the proof into three different steps.

Step 1. (8) *holds as long as* (10) *holds.* The proof is by induction. Clearly, this is true for $j=1$ and some A_1 (which we might choose to only point in the x_1-direction) if we pick C_1 large enough. So assume this is true for $j=k$ and then we prove that it holds also for $j=k+1$. Take v to be the harmonic replacement of u in $B^+_{\sigma^k}(x_0)$. Then $v - A_k \cdot x$ is the replacement of $u - A_k \cdot x$. Hence, by the Dirichlet principle,

$$\int_{B^+_{\sigma^k}(x_0)} |\nabla v - A_k|^2 \, dx \leq \int_{B^+_{\sigma^k}(x_0)} |\nabla u - A_k|^2 \, dx =: I_1.$$

Now we need to treat two cases differently. In the case $\sigma^k \geq (x_0)_1$, define $A_{k+1} = \partial_1 v(x_0)$ and remark also that we can then assume that A_k only points in the x_1-direction; indeed if $\sigma^k \geq (x_0)_1$, then also $\sigma^{k-1} \geq (x_0)_1$. In the other case, when $\sigma^k < (x_0)_1$ let $A_{k+1} = \nabla v(x_0)$. We see now that either $v - A_k \cdot x = 0$ on Π or $B^+_{\sigma^k}(x_0) = B_{\sigma^k}(x_0)$. Hence, Lemma 3.5 applied with $r = \sigma^k$ implies

$$\int_{B^+_{\sigma^{k+1}}(x_0)} |\nabla v - A_{k+1}|^2 \, dx \leq C\sigma^{n+2} \int_{B^+_{\sigma^k}(x_0)} |\nabla v - A_k|^2 \, dx \leq C\sigma^{n+2} I_1.$$

Since u is a minimizer of (1), we have

$$\int_{B^+_{\sigma^k}(x_0)} |\nabla v|^2 \, dx \leq \int_{B^+_{\sigma^k}(x_0)} |\nabla u|^2 \, dx \leq \int_{B^+_{\sigma^k}(x_0)} (|\nabla u|^2 + \lambda_1 (u^+)^p + \lambda_2 (u^-)^p) \, dx.$$

Using that (10) is assumed to hold up to $j=k$, the Hölder regularity of u implies that

$$I_2 := \int_{B^+_{\sigma^k}(x_0)} |\nabla u - \nabla v|^2 \, dx \leq \int_{B^+_{\sigma^k}(x_0)} (\lambda_1 (u^+)^p + \lambda_2 (u^-)^p) \, dx$$

$$\leq C \max(\lambda_1, \lambda_2) \sigma^{kn} \sup_{B^+_{\sigma^k}(x_0)} |u|^p \leq C \max(\lambda_1, \lambda_2) \sigma^{k(n+\beta p)}.$$

Now pick β so that $\beta p > 2\alpha$. By Young's inequality

$$
\int_{B^+_{\sigma^{k+1}}(x_0)} |\nabla u - A_{k+1}|^2 \, dx \leq 2 \int_{B^+_{\sigma^{k+1}}(x_0)} |\nabla v - A_{k+1}|^2 \, dx
$$

$$
+ 2 \int_{B^+_{\sigma^{k+1}}(x_0)} |\nabla u - \nabla v|^2 \, dx
$$

$$
\leq 2C\sigma^{n+2} I_1 + 2C\sigma^{k(n+\beta p)}
$$

$$
\leq 2C_1 C \sigma^{n+2} \sigma^{k(n+2\alpha)} + 2C\sigma^{k(n+\beta p)}
$$

$$
\leq C_1 \sigma^{(k+1)(n+2\alpha)} \left(C\sigma^{2-2\alpha} + 2\frac{C}{C_1} \sigma^{\beta p k - n - 2\alpha(k+1)} \right)
$$

$$
\leq C_1 \sigma^{(k+1)(n+2\alpha)} (1 + 2C\sigma^{-n-2\alpha})
$$

$$
\leq C_1 \sigma^{(k+1)(n+2\alpha)},
$$

if C_1 is chosen to be large enough and σ is small enough. This proves that (8) holds for $j = k+1$.

Step 2. (9) *holds as long as* (10) *holds.* We remark that $A_{k+1} - A_k$ is the gradient of $v - A_k \cdot x$ at x_0, where v is as in Step 1. Moreover, either $v - A_k \cdot x = 0$ on Π or $B^+_{\sigma^k}(x_0) = B_{\sigma^k}(x_0)$. Therefore, by the C^1-estimates in (6) we get that

$$
|A_{k+1} - A_k| \leq C\sigma^{-kn/2} \left(\int_{B^+_{\sigma^k}(x_0)} |\nabla v - A_k|^2 \, dx \right)^{1/2} \leq C\sqrt{C_1} \sigma^{\alpha k},
$$

from (8) for $j = k$, which holds due to Step 1. Hence, if C_2 is large enough, then $|A_{k+1} - A_k| \leq C_2 \sigma^{\alpha(k+1)}$.

Step 3. *Conclusion.* First of all, in the case when (10) holds for all j then from (9) we have that

$$
|A_j - A_k| \leq \sum_{i=j}^{k-1} |A_i - A_{i+1}| \leq C' \sigma^{\alpha j}.
$$

Hence, the sequence A_j converges to a limit $A(x_0)$. This together with (8) implies (4) immediately.

If (10) holds for $j < k$ but fails for $j = k$ then

$$
\inf_{B^+_{\sigma^k}(x_0)} |u| > \sigma^k,
$$

so that from (2) we have

$$|\Delta u| \le C(p, \lambda^\pm)\sigma^{k(p-1)} \quad \text{in } B^+_{\sigma^k}(x_0).$$

Furthermore, either $u - A_k \cdot x = 0$ on Π or $B^+_{\sigma^k}(x_0) = B_{\sigma^k}(x_0)$. Hence, $u - A_k \cdot x$ has $C^{1,\alpha}$-estimates in $B^+_{\sigma^k/2}(x_0)$. In particular, from (7) we have

$$|\nabla u(x_0) - A_k| \le C(p, \lambda^\pm)\sigma^{kp} + C\sigma^{-kn/2}\left(\int_{B^+_{\sigma^k}(x_0)} |\nabla u - A_k|^2 \, dx\right)^{1/2}$$

$$\le C(p, \lambda^\pm)\sigma^{kp} + C\sqrt{C_1}\sigma^{k\alpha} \le \sigma^{k\alpha}(C(p, \lambda^\pm) + C\sqrt{C_1}) \le C\sigma^{k\alpha},$$

if $p \ge \alpha$, and also from (7) it follows that for $r \le \sigma^k$,

$$r^{-\alpha}\operatorname{osc}_{B^+_{r/2}(x_0)} |\nabla u - A_k| \le C(p, \lambda^\pm)r^{(p-\alpha)} + Cr^{-n/2-\alpha}\left(\int_{B^+_r(x_0)} |\nabla u - A_k|^2 \, dx\right)^{1/2}$$

$$\le C(p, \lambda^\pm)r^{(p-\alpha)} + C\sqrt{C_1}r^{(\alpha-\alpha)} \le C,$$

if again $p \ge \alpha$. With $A(x_0) = \nabla u(x_0)$ and $\sigma \le \frac{1}{2}$, integrating the last two estimates over $B^+_r(x_0)$ yields for any $r \le \sigma^{k+1}$,

$$\int_{B^+_r(x_0)} |\nabla u - A(x_0)|^2 \, dx \le Cr^{n+2\alpha}.$$

For $r = \sigma^j$ for $j \le k$ we have from Young's inequality and (8) that

$$\int_{B^+_r(x_0)} |\nabla u - A(x_0)|^2 \, dx \le 2\int_{B^+_r(x_0)} |\nabla u - A_j|^2 \, dx + 2\int_{B^+_r(x_0)} |A(x_0) - A_j|^2 \, dx$$

$$\le 2C_1\sigma^{n+j2\alpha} + 2\sigma^n|A(x_0) - A_j|^2.$$

From (9) for $j \le k$ it follows that

$$|A(x_0) - A_j| \le |A(x_0) - A_k + A_k - A_{k-1} + ... + A_{j+1} - A_j| \le C\sigma^{j\alpha}.$$

This yields the estimate, still with $r = \sigma^j$, for $j \le k$,

$$\int_{B^+_r(x_0)} |\nabla u - A(x_0)|^2 \, dx \le 2C_1\sigma^{n+j2\alpha} + 2C\sigma^{n+j2\alpha}.$$

Thus, we obtain the desired inequality for all r. $\quad\square$

4. Technical tools

Here we present some technical lemmas which we will use later to prove our main result.

4.1. Optimal growth

In the proof of the proposition below, we will use techniques similar to those in for instance [3] and [4] to prove that u will have the optimal growth of order $\beta = 2/(2-p)$ at branching points.

Proposition 4.1. (Optimal growth) *Suppose that* $u \in P_1(M)$, $x_0 \in \Gamma \cap \Pi$ *and* $|\nabla u(x_0)| = 0$. *Then there exists* $C = C(\lambda^{\pm}, p, M)$ *such that with* $\beta = 2/(2-p)$,

$$\sup_{B_r^+(x_0)} |u| \leq C r^\beta \quad \text{for all } 0 < r < \tfrac{1}{2}.$$

Proof. The proof is by contradiction. Without loss of generality, assume $x_0 = 0$ and define

$$S_r(u) = \sup_{B_r^+} u \quad \text{for } 0 < r < \tfrac{1}{2}.$$

We will show that either

(11) $$S_r \leq C r^\beta$$

for a constant C, or there exists a $k \in \mathbb{N}$ with $2^k r \leq 1$ such that

(12) $$S_r \leq 2^{-k\beta} S_{2^k r}.$$

Indeed, if this holds true, then take j to be the first integer such that (11) fails for $r = 2^{-j-1}$ but holds true for $r = 1, 2^{-1}, ..., 2^{-j}$. Then (12) holds for $r = 2^{-j-1}$. Hence

$$S_{2^{-j-1}} \leq 2^{-k\beta} S_{2^{k-j-1}} \leq C 2^{-\beta(j+1)},$$

since (11) holds for $r = 2^{k-j-1} \leq 1$. Thus, the result follows by iteration.

Now back to the proof of (11) or (12). Suppose both these assertions fail. Then one can find sequences $r_j \to 0$ and $u_j \in P_1(M)$ such that with $S_j := S_{r_j}$ one has

$$S_j > C_j r_j^\beta,$$

where $C_j \to \infty$ and

$$S_j > 2^{-k\beta} S_{r_j 2^k} \quad \text{for all } k \in \mathbb{N} \text{ and } 2^k r_j \leq 1.$$

Define

$$w_j(x) = \frac{u_j(r_j x)}{S_j}.$$

Then

(a) $\sup_{B_1^+}|w_j(x)|=1$;

(b) $\sup_{B_{2^k}^+}|w_j(x)|\leq 2^{k\beta}$;

(c) $w_j(0)=|\nabla w_j(0)|=0$;

(d) $w_j=0$ on $B_{1/r_j}\cap\Pi$;

(e) w_j is a minimizer of

$$\int_{B_{2^j}^+}\left(\frac{|\nabla v|^2}{2}+T_j(\lambda^+(v^+)^p+2\lambda^-(v^-)^p)\right),$$

where $T_j=r_j^{-2}/S_j^{2-p}\to 0$ as $j\to\infty$.

By using Proposition 3.6, we can find a subsequence of w_j which converges to a limiting function w_0 in $C^1(\overline{B_R^+})$ for all $R>0$. Due to (a)–(e), w_0 satisfies

(1) $\sup_{B_1^+}|w_0(x)|=1$;

(2) $\sup_{B_{2^k}^+}|w_0(x)|\leq 2^{k\beta}$ for all k;

(3) $w_0(0)=|\nabla w_0(0)|=0$;

(4) $w_0=0$ on Π;

(5) $\Delta w_0=0$ in $(\mathbb{R}^n)^+$.

We reflect the function w_0 in an odd manner with respect to Π to get a harmonic function in the whole $\mathbb{R}^n$. By interior estimates for harmonic functions and (2), we have

$$\sup_{B_{2^k}}|D^2 w_0(z)|\leq\frac{C}{2^{k(n+2)}}\|w_0\|_{L^1(B_{2^k})}\leq C2^{k(\beta-2)}\quad\text{for every }k\geq 1.$$

Since $\beta<2$, letting $k\to\infty$ implies that $D^2 w_0=0$ and consequently w_0 is a linear function. Then (3) implies $w_0=0$, contradicting (1). □

4.2. Non-degeneracy

The next lemma shows that blow-ups cannot vanish identically. This property is usually referred to as non-degeneracy and to prove it, we use the idea in [11] which in turn is an adaptation of a similar proof given in [5].

Lemma 4.2. (Non-degeneracy) *Suppose that u is a minimizer of* (1) *and $x_0\in\Gamma^+\cap\Pi$. Then for some constant $c^+=c^+(\lambda^+,p)$,*

$$(13)\qquad\sup_{\partial B_r^+(x_0)\cap\Omega^+}u\geq c^+r^\beta,\quad 0<r<\tfrac{1}{2}.$$

Similarly if $x_0\in\Gamma^-\cap\Pi$, then there exists a constant $c^-=c^-(\lambda^-,p)$ such that

$$(14)\qquad\inf_{\partial B_r^+(x_0)\cap\Omega^-}u\leq -c^-r^\beta,\quad 0<r<\tfrac{1}{2}.$$

Proof. We prove only (13). The inequality (14) can derived analogously. Suppose that $y \in \Omega^+$, $B_r^+(y) \subset B_1^+$ and u is a minimizer of (1). Define the function

$$w(x) = |u(x)|^{2/\beta} - c|x-y|^2,$$

where c is a constant which we will determine later. By a simple computation we find that

$$\Delta w = \frac{2p\lambda^+}{\beta} + \frac{2}{\beta}\left(\frac{2}{\beta}-1\right)\frac{|\nabla u|^2}{|u|^p} - 4c \quad \text{in } \Omega^+ \cap B_r^+(y).$$

If we choose $c = p\lambda^+/2\beta$ then $\Delta w \geq 0$ in $B_r^+(y) \cap \Omega^+$ and by the maximum principle, the maximum of w occurs on $\partial(B_r^+(y) \cap \Omega^+)$. We know that

$$\begin{cases} w(y) \geq 0, \\ \Delta w \geq 0 & \text{in } B_r^+(y) \cap \Omega^+, \\ w \leq 0 & \text{on } \partial\Omega^+, \\ w \leq 0 & \text{on } B_r^+(y) \cap \Pi, \end{cases}$$

and consequently w attains its maximum on $\partial B_r^+(y)$ and

$$\sup_{\partial B_r^+(y) \cap \Omega^+} w > 0.$$

In other words,

(15) $$\sup_{\partial B_r^+(y) \cap \Omega^+} u^{2/\beta} > cr^2.$$

Now let $x_0 \in \Gamma^+ \cap \Pi$. Then one can find a sequence y_j in Ω^+ such that $y_j \to x_0$. By considering (15) for y_j and passing to the limit, one obtains

$$\sup_{\partial B_r^+(x_0) \cap \Omega^+} u^{2/\beta} \geq cr^2,$$

or equivalently,

$$\sup_{\partial B_r^+(x_0) \cap \Omega^+} u \geq c^+ r^\beta. \quad \square$$

One important consequence of Lemma 4.2 is that the free boundary is stable in the sense that limits of free boundary points are always free boundary points. In particular, it implies that if u_j is a sequence of minimizers converging to u_0 and $x_j \in \Gamma^\pm(u_j)$ with $x_j \to x_0$, then $x_0 \in \Gamma^\pm(u_0)$.

4.3. Monotonicity formula

The next lemma is a crucial monotonicity formula due to Weiss, proved in [20]. See Theorem 3.1 in [18], where the monotonicity formula was introduced in the interior setting.

Lemma 4.3. (Weiss's monotonicity formula) *Suppose that $u \in P_R(M)$ and $G(u) = 2\lambda^+(u^+)^p + 2\lambda^-(u^-)^p$. Let*

$$W(r, x_0, u) = r^{-2\beta} \int_{B_r^+(x_0)} (|\nabla u|^2 + 2G(u)) \, dx - \frac{\beta}{r^{1+2\beta}} \int_{\partial B_r^+(x_0)} u^2(x) \, ds$$

for $r > 0$. Then W is monotonically increasing with respect to r if $r < d(\partial B_R^+, x_0)$. Moreover, W is constant if and only if u is a homogeneous function of degree β.

4.4. Global minimizers

The next theorem classifies the homogeneous global minimizers of (1) in two dimensions. This result is basically a result from [11]. From this we can then classify all global minimizers. From now on we will be working only in two dimensions.

Theorem 4.4. *Let $u \in P_\infty(M)$ be homogeneous and assume the dimension to be two. Then for some suitable constants $c^\pm$ one of the following holds*:
  (1) $u(x) = c^+(x_1^+)^\beta$ *for one phase non-negative points*;
  (2) $u(x) = -c^-(x_1^-)^\beta$ *for one phase non-positive points.*

Proof. Let $0 \in \Gamma^+ \cap \Pi$. Assume first that u is a homogeneous global minimizer of (1). From the homogeneity assumption, we conclude that any connected component of Ω^+ is a cone. Lemma 4.2 in [11] asserts that it has opening $\gamma \in (\pi/\beta, \pi)$ for $\beta = 2/(2-p)$. Since $\beta \in (1, 2)$, there can only be one component. Applying the second part of Lemma 4.2, we obtain $\gamma = \pi$, which up to rotations corresponds to $u(x) = c^+(x_1^+)^\beta$. Since u must vanish on Π, no other rotation except the identity is possible. The case $0 \in \Gamma^- \cap \Pi$ can be handled similarly. $\square$

The theorem above implies in particular that there can be no two-phase points touching the fixed boundary.

Corollary 4.5. *Suppose $u \in P_1(M)$. Then the origin is a one-phase point.*

Proof. If there were to be a two-phase branching point touching Π, then we could, by Proposition 4.1 and the C^1-estimates, perform a blow-up at the origin.

Due to Lemma 4.2, the blow-up will have both phases non-empty, which by Theorem 4.4 above is not possible. Now, if there is a two-phase point in Π where the gradient does not vanish, then the gradient must be perpendicular to Π, which would imply that it is a one-phase point, a contradiction. $\quad\square$

Lemma 4.6. *Suppose $u \geq 0$ is a minimizer of* (1) *in* $\{x \in \mathbb{R}^n : x_1 > -A\}$ *for some constant $A > 0$ and that*

$$0 \in \Gamma \cap \Pi \quad and \quad \sup_{B_r} |u| \leq C r^\beta$$

for $r > 0$ and some $C > 0$. Then u is one of the alternatives in Theorem 4.4.

Proof. We prove that u is homogeneous of degree β. Then $u \in P_\infty(C)$ for some C and the result follows from Theorem 4.4.

Since u grows at most like r^β at infinity,

$$u_r(x) = \frac{u(rx)}{r^\beta}$$

is bounded as $r \to \infty$. Using Proposition 4.1, the C^1-estimates and Lemma 4.2, we can extract a subsequence $u_j = u_{r_j}$, with $r_j \to \infty$ so that $u_j \to u_\infty$, where u_∞ is a minimizer of (1) in $\{x \in \mathbb{R}^n : x_1 > 0\}$, $u_\infty = 0$ on $\{x \in \mathbb{R}^n : x_1 = 0\}$, $0 \in \Gamma(u_\infty)$ and

$$W(u_\infty, s) = \lim_{r \to \infty} W(u_r, s) = \lim_{r \to \infty} W(u, rs) = \lim_{r \to \infty} W(u, r).$$

Then Lemma 4.3 implies that u_∞ is homogeneous of degree β and $u \in P_\infty(C)$. From Theorem 4.4, we have $u_\infty = c^+(x_1^+)^\beta$.

We have also that u_r is uniformly bounded when r is small enough. Hence, by Proposition 4.1, the C^1-estimates and Lemma 4.2, we can extract a subsequence $u_{r_j} \to u_0$ for some subsequence $r_j \to 0$ such that u_0 is a minimizer of (1) in $\mathbb{R}^n$, $0 \in \Gamma(u_\infty)$ and

$$W(u_0, s) = \lim_{r \to 0} W(u_r, s) = \lim_{r \to 0} W(u, rs) = \lim_{r \to 0} W(u, r),$$

which is a constant since W is monotone. Hence, $W(u_0, s)$ is constant and then by Lemma 4.3, u_0 must be homogeneous of degree β. Since $u \geq 0$, Theorem 4.1 in [11] implies that $u_0 = u_\infty$.

Using Lemma 4.3 again, it follows that

$$W(u_0, 1) \leq W(u, r) \leq W(u_\infty, 1) = W(u_0, 1),$$

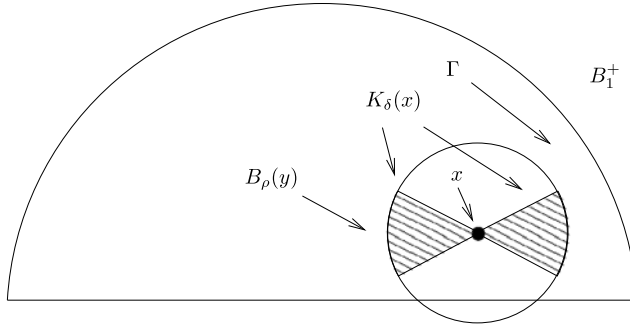so that $W(u, r)$ is constant and u must be homogeneous of degree β. $\quad\square$

Figure 2. Γ is inside $K_\delta(x)$ when x is close to Π.

5. Proof of the main theorem

In this section we prove our main theorem. In the proposition that follows we prove that near Π, the free boundary will have a normal very close to e_1 (see Figure 2), still in two dimensions. By Corollary 4.5, any free boundary point touching Π must be a one-phase point, and hence we can work under the assumption that u has a sign near the origin. In what follows, we will use the notation

$$K_\delta(z) = \big\{ x : |x_1 - z_1| < \delta |x_2 - z_2| \big\}.$$

Proposition 5.1. *Let $u \in P_1(M)$. For any $\delta > 0$ there are $\varepsilon = \varepsilon(\lambda^\pm, p, M, \delta)$ and $\rho = \rho(\lambda^\pm, p, M, \delta)$ so that $x \in \Gamma$ and $x_1 < \varepsilon$ imply that*

$$\Gamma \cap B_\rho^+(x) \subset K_\delta(x) \cap B_\rho^+(x).$$

Proof. We argue by contradiction and we treat only the case when $u \geq 0$ near the origin. If the assertion is not true then for some $\delta > 0$ there are sequences $u_j \in P_1(M)$, $\varepsilon_j \to 0$, $x^j \in \Gamma(u_j)$ and

$$y^j \in \Gamma(u_j) \cap K_\delta^c(x^j).$$

Let $r_j = |x^j - y^j|$. We split the proof into two different cases, depending on whether y^j is very close to x^j or not.

Case 1. x_1^j / r_j bounded. By choosing a subsequence we can assume that $x^j / r_j \to A < \infty$. Let

$$v_j(x) = \frac{u_j(r_j x + x^j)}{r_j^\beta}.$$

Then v_j satisfies the following conditions:

(1) From the optimal growth,

$$\sup_{B_R} |v_j| \le CR^\beta \quad \text{when } Rr_j < 1;$$

(2) v_j is a minimizer of (1) in $\{x \in B_{1/r_j} : x_1 > -x^j/r_j\}$;
(3) $v_j = 0$ on $\{x \in B_{1/r_j} : x_1 = -x^j/r_j\}$;
(4) $0 \in \Gamma(v_j)$;
(5) $z_j = (x^j - y^j)/r_j \in \partial B_1 \cap K_\delta^c \cap \Gamma(v_j)$;
(6) $v_j \ge 0$ in B_R when R is small enough or j is large enough.

Therefore, invoking Lemma 4.2 and using the C^1-estimates for minimizers, we can assume that $v_j \to v_0$ locally uniformly and $z_j \to z_0$ such that:

(1) $\sup_{B_R} |v_0| \le CR^\beta$ for all $R > 0$;
(2) v_0 is a minimizer of (1) in $\{x \in \mathbb{R}^n : x_1 > -A\}$;
(3) $v_0 = 0$ on $\{x \in \mathbb{R}^n : x_1 = -A\}$;
(4) $0 \in \Gamma(v_0)$;
(5) $z_0 \in \partial B_1 \cap K_\delta^c \cap \Gamma(v_0)$;
(6) $v_0 \ge 0$.

Lemma 4.6 implies that $v_0 = c^+(x_1^+)^\beta$. This contradicts (5).

Case 2. $x_1^j/r_j \to \infty$. Define in this case

$$v_j(x) = \frac{u_j(x_1^j x + x^j)}{(x_1^j)^\beta}.$$

Then the following holds:

(1) From the optimal growth,

$$\sup_{B_R} |v_j| \le CR^\beta \quad \text{when } Rx_1^j < 1;$$

(2) v_j is a minimizer of (1) in $\{x \in B_{1/x_1^j} : x_1 > -1\}$;
(3) $v_j = 0$ on $\{x \in B_{1/x_1^j} : x_1 = -1\}$;
(4) $0 \in \Gamma(v_j)$;
(5) $z_j = (x^j - y^j)/x_1^j \in \partial B_1 \cap K_\delta^c \cap \Gamma(v_j)$;
(6) $v_j \ge 0$ in B_R when R is small enough or j is large enough.

From the assumptions on x_1^j and r_j it is clear that $z_j \to 0$. Moreover, from Theorem 8.2 in [2], $\Gamma(v_j)$ is a uniform (in j) C^1-graph near the origin. Hence, (5) implies that $\Gamma(v_j)$ has asymptotically a tangent lying in K_δ^c. Therefore, we can assume that $v_j \to v_0$ locally uniformly where v_0 satisfies:

(1) $\sup_{B_R} |v_0| \le CR^\beta$ for all $R > 0$;
(2) v_0 is a minimizer of (1) in $\{x \in \mathbb{R}^n : x_1 > -1\}$;
(3) $v_0 = 0$ on $\{x \in \mathbb{R}^n : x_1 = -1\}$;

   (4) $0 \in \Gamma(v_0)$;

   (5) $\Gamma(v_0)$ has a tangent in K_δ^c at the origin;

   (6) $v_0 \geq 0$.

From Lemma 4.6 we have $v_0 = c^+(x_1^+)^\beta$. This is in contradiction with (5). $\quad\square$

Now the situation is as follows. Away from Π, Theorem 8.2 in [2] applies, so there the free boundary is a C^1-graph. Moreover, from Proposition 5.1, we know that the normal of the free boundary approaches e_1 as we approach Π. This is enough to assure that the free boundary is a uniform C^1-graph up to Π. We spell out the details below.

Proof of Theorem 1.3. Since any free boundary point in Π must be a one-phase point, we can assume that $0 \in \Gamma^+ \cap \Pi$. Denote by ν_x the normal of Γ at a point x. We need to prove that ν_x is uniformly continuous. From Theorem 8.2 in [2] it follows that Γ is a C^1-graph away from Π. In particular, around any point $x \in \Gamma$, ν is continuous with a modulus of continuity $\sigma(\,\cdot\,/x_1)$, where σ is some modulus of continuity. Moreover, by Proposition 5.1, we know that for any $\tau > 0$, there is a $\delta_\tau = \delta_\tau(\lambda^\pm, M, p)$ such that $x_1 < \delta_\tau$ implies that $\|\nu_x - e_1\| < \tau/2$.

Take two points $x, y \in \Gamma$. Now we split the proof into three cases:

Case 1. $x_1, y_1 < \delta_\tau/2$. Then obviously $\|\nu_x - \nu_y\| \leq \tau$.

Case 2. $x_1 < \delta_\tau/2$ *and* $y_1 > \delta_\tau/2$. Then $|x - y| < \delta_\tau/2$ implies $\|\nu_x - \nu_y\| \leq \tau$.

Case 3. $x_1, y_1 > \delta_\tau/2$. From the arguments above,

$$\|\nu_x - \nu_y\| \leq \sigma\left(\frac{2|x-y|}{\delta_\tau}\right),$$

which implies that $\|\nu_x - \nu_y\| \leq \tau$, if $|x - y|$ is small enough.

Combining all the three cases above, we can conclude the following estimate of the modulus of continuity for ν_x: For any $\tau > 0$, there is a δ_τ such that

$$\|\nu_x - \nu_y\| \leq \max\left(\tau, \sigma\left(\frac{2|x-y|}{\delta_\tau}\right)\right).$$

Hence, ν_x is uniformly continuous. $\quad\square$

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