Ark. Mat., **52** (2014), 149–173
DOI: 10.1007/s11512-013-0180-5
© 2013 by Institut Mittag-Leffler. All rights reserved

Banach analytic sets and a non-linear version of the Levi extension theorem

Sergey Ivashkovich

Abstract. We prove a certain non-linear version of the Levi extension theorem for meromorphic functions. This means that the meromorphic function in question is supposed to be extendable along a sequence of complex curves, which are arbitrary, not necessarily straight lines. Moreover, these curves are not supposed to belong to any finite-dimensional analytic family. The conclusion of our theorem is that nevertheless the function in question meromorphically extends along an (infinite-dimensional) analytic family of complex curves and its domain of existence is a pinched domain filled in by this analytic family.

1. Introduction

1.1. Statement of the main result

By (λ, z) we denote the standard coordinates in $\mathbb{C}^2$. For $\varepsilon>0$ consider the following ring domain

$$R_{1+\varepsilon}=\{(\lambda,z)\in\mathbb{C}^2:1-\varepsilon<|\lambda|<1+\varepsilon \text{ and } |z|<1\}=A_{1-\varepsilon,1+\varepsilon}\times\Delta, \tag{1}$$

i.e., $R_{1+\varepsilon}$ is the product of the annulus $A_{1-\varepsilon,1+\varepsilon}:=\{z\in\mathbb{C}:1-\varepsilon<|\lambda|<1+\varepsilon\}$ and the unit disk Δ. Let a sequence of holomorphic functions $\{\phi_k\colon \Delta_{1+\varepsilon}\to\Delta\}_{k=1}^{\infty}$ be given such that ϕ_k converge uniformly on $\Delta_{1+\varepsilon}$ to some $\phi_0\colon \Delta_{1+\varepsilon}\to\Delta$. We say that such a sequence is a *test sequence* if $(\phi_k-\phi_0)|_{\partial\Delta}$ does not vanish for $k\gg0$ and

$$\operatorname*{Var\,Arg}_{\partial\Delta}(\phi_k-\phi_0) \quad \text{stays bounded when } k\to\infty. \tag{2}$$

Denote by C_k the graph of ϕ_k in $\Delta_{1+\varepsilon}\times\Delta$, and by C_0 the graph of ϕ_0.

This research is supported in part by the grant ANR-10-BLAN-0118.

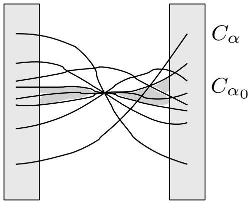

Figure 1. The *brighter dashed zone* on this picture represents the ring domain $R_{1+\varepsilon}$ and the *curves* are the graphs C_α. Around C_{α_0}, the graph of $\phi_0=\phi_{\alpha_0}$, the analytic family $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ fills in an *another* (*darker*) *dashed zone*, a pinched domain $\mathcal{P}$. On this picture there is exactly one pinch, *the point at which most of the graphs intersect.*

Theorem 1.1. *Let f be a meromorphic function on $R_{1+\varepsilon}$ and $\{\phi_k\}_{k=1}^{\infty}$ be a test sequence such that for every k the restriction $f|_{C_k\cap R_{1+\varepsilon}}$ is well defined and extends to a meromorphic function on the curve C_k, and that the number of poles counted with multiplicities of these extensions is uniformly bounded. Then there exists an analytic family of holomorphic graphs $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ parameterized by a Banach ball $\mathcal{A}$ of infinite dimension such that:*

(i) *$f|_{C_\alpha\cap R_{1+\varepsilon}}$ extends to a meromorphic function on C_α for every $\alpha\in\mathcal{A}$ and the number of poles counted with multiplicities of these extensions is uniformly bounded;*

(ii) *f meromorphically extends as a function of two variables (λ,z) to the pinched domain $\mathcal{P}:=\mathrm{Int}(\bigcup_{\alpha\in\mathcal{A}}C_\alpha)$ swept by C_α.*

Here every C_α is a graph of some holomorphic function ϕ_α. The notion of a *pinched domain*, though intuitively clear, see Figure 1, is discussed in details at the beginning of Section 2.

Definition 1.2. The graphs $\{C_k\}_{k=1}^{\infty}$ of our functions $\{\phi_k\}_{k=1}^{\infty}$ are in *general position* if for every point $\lambda_0\in\Delta$ there exists a subsequence $\{\phi_{k_p}\}_{p=1}^{\infty}$ such that the zeroes of $\phi_{k_p}-\phi_0$ do not accumulate to λ_0.

Theorem 1.1 implies the following *non-linear* Levi-type extension theorem.

Corollary 1.3. *If under the conditions of Theorem 1.1 the curves $\{C_k\}_{k=1}^{\infty}$ are in general position then f extends to a meromorphic function in the bidisk $\Delta_{1+\varepsilon}\times\Delta$.*

Remark 1.4. Let us explain the condition of a general position. Take the sequence $C_k=\{z:z=\lambda/k\}$ in $\mathbb{C}^2$. Then the function $f(\lambda,z)=e^{z/\lambda}$ is holomorphic in $R:=\mathbb{C}^*\times\mathbb{C}$ and extends holomorphically along every curve C_k. But it is not holomorphic (even not meromorphic) in $\mathbb{C}^2$. It is also holomorphic when restricted to any curve $C=\{z:z=\phi(\lambda)\}$ provided $\phi(0)=0$. Therefore the subspace H_0 of $\phi\in\mathrm{Hol}(\Delta_{1+\varepsilon},\Delta)$ such that f extends along the corresponding curve is of codimension one. In fact this is the general case: the Banach ball $\mathcal{A}$ in Theorem 1.1 appears as a neighborhood of the limit point α_0 in the subspace of finite codimension of a well-chosen Banach space of holomorphic functions.

Remark 1.5. To explain the condition (2) we impose on our test sequences we construct in Section 5 for the following *non-test* sequence $\{\phi_k(\lambda)=\left(\frac{2}{3}\lambda\right)^k\}_{k=1}^{\infty}$ a holomorphic function f in $\mathbb{C}^*\times\mathbb{C}$ which holomorphically extends along every graph C_k but which is not extendable meromorphically along any one-parameter analytic family $\{\phi_\alpha\}_{\alpha\in\mathcal{A}}$, see Example 5.1 there.

1.2. Meromorphic mappings

The assumption that f is a function in Theorem 1.1 is not really important. We prove also a non-linear version of an extension theorem for meromorphic mappings with values in general complex spaces putting it into a form suitable for applications. Let us call a family $\{\phi_t\in\mathrm{Hol}(\Delta_{1+\varepsilon},\Delta):t\in T\}$ a *test family* if there exists $N\in\mathbb{N}$ such that for every pair $s\neq t\in T$ there exists a radius $1-\varepsilon/2<r<1+\varepsilon/2$ such that $(\phi_s-\phi_t)|_{\partial\Delta_r}$ does not vanish and has winding number $\leq N$. As usual by C_t we denote the graph of ϕ_t.

Corollary 1.6. *Let X be a reduced disk-convex complex space and $f\colon R_{1+\varepsilon}\to X$ be a meromorphic mapping. Suppose that there exists an uncountable test family of holomorphic functions $\{\phi_t\in\mathrm{Hol}(\Delta_{1+\varepsilon},\Delta):t\in T\}$ such that $f|_{C_t\cap R_{1+\varepsilon}}$ holomorphically extends to C_t for every $t\in T$. Then f extends to a meromorphic mapping from a pinched domain $\mathcal{P}$ to X.*

Moreover, there exists, like in Theorem 1.1, an infinite-dimensional family of graphs C_α parameterized by a Banach ball $\mathcal{A}$ such that $f|_{C_\alpha\cap R_{1+\varepsilon}}$ holomorphically extends to C_α for all $\alpha\in\mathcal{A}$. The condition that $f|_{C_t\cap R_{1+\varepsilon}}$ is assumed to extend *holomorphically* should not be confusing because meromorphic functions on curves are precisely holomorphic mappings to the Riemann sphere $\mathbb{P}^1$. That is, the meromorphic functions case is the case $X=\mathbb{P}^1$ in this corollary.

1.3. Structure of the paper and notes

Theorem 1.1 is proved in Section 2. The set $\mathcal{A}$ such that $f|_{C_\alpha\cap R_{1+\varepsilon}}$ meromorphically extends to C_α for $\alpha\in\mathcal{A}$ is always a Banach analytic subset of a neighborhood of ϕ_0 in the Banach space $\mathrm{Hol}(\Delta_{1+\varepsilon},\Delta)$. In particular the sequence $\phi_k(\lambda)=\left(\frac{2}{3}\lambda\right)^k$ of Example 5.1 is a Banach analytic set, namely the zero set of an appropriate singular integral operator, see Section 3 for more details. The (known) problem however is that a *every* metrizable compact set can be endowed with a structure of a Banach analytic set (in an appropriate complex Banach space), see [Mz]. For the case of a converging sequence of points see Remark 3.9 for a very simple example. Therefore in the infinite-dimensional case from the fact that our Banach analytic set contains a non-isolated point we cannot deduce that it contains an analytic disk. Our major task here is to overcome this difficulty.

In the case when $C_k=\Delta_{1+\varepsilon}\times\{z_k\}$ with $z_k\to 0$, i.e., when C_k are horizontal disks, this result is exactly the theorem of E. Levi, see [L] (the case of holomorphic extension is due to Hartogs, see [H]). It should be said that Levi's theorem is usually stated in the form as our Corollary 1.6: *if f as above extends along an uncountable family of horizontal disks $C_t=\Delta_{1+\varepsilon}\times\{t\}$, then f meromorphically extends to Δ^2.* But the proof goes as follows: one notes that then there exists a sequence $\{t_k\}_{k=1}^{\infty}$ (in fact an uncountable subfamily) such that extensions along the C_{t_k} have uniformly bounded number of poles, and then the statement similar to our Theorem 1.1 is proved.

If the number of poles of the extensions $f|_{C_k}$ is not uniformly bounded then the conclusion of Theorem 1.1 fails to be true even in the case of horizontal disks. This is shown by Example 5.3.

In the case when $\{C_t\}_{t\in T}$ are non-horizontal straight disks, i.e., intersections of lines with Δ^2, Corollary 1.3 is due to Dinh, see [D], Corollaire 1. The proof in [D] uses results on the complex Plateau problem in projective space (after an appropriate Segre imbedding) and is essentially equivalent to the solution of this problem. From the point of view of this paper this is a special case when $\{C_k\}_{k=1}^{\infty}$ ad hoc belong to a finite-dimensional analytic family: in the Levi case the family is one-dimensional, in the case of Dinh two-dimensional. In Section 3, after recalling the necessary facts about singular integral transforms, we give a very short proof of a non-linear extension theorem, see Theorem 3.5, in the case when $\{C_k\}_{k=1}^{\infty}$ are ad hoc included in an *arbitrary* finite-dimensional family. In the straight case, i.e., when $\{C_t\}_{t\in T}$ are non-horizontal straight disks, the result of Corollary 1.6 for Kähler X was proved in [Sk] following the approach of [D].

It is important to outline that we do not suppose a priori that $\{C_k\}_{k=1}^{\infty}$ are included into any finite-dimensional family of complex curves (e.g. any family of algebraic curves of uniformly bounded degree) and, in fact, it is the main point of this paper to develop techniques for producing analytic disks C_α in families.

Corollary 1.6 is proved in Section 4, where also a general position assumption is discussed. Examples 5.1 and 5.3 are treated in Section 5.

Acknowledgement. At the end I would like to give my thanks to the referee of this paper for the valuable remarks and suggestions.

2. Extension to pinched domains

2.1. Analytic families and pinched domains

By an *analytic family of holomorphic mappings from* Δ *to* Δ we understand the quadruple $(\mathcal{X},\pi,\mathcal{A},\Phi)$ where:

– $\mathcal{X}$ is a complex manifold, which is either finite-dimensional or a Banach manifold;

– a holomorphic submersion $\pi\colon \mathcal{X}\to\mathcal{A}$, where $\mathcal{A}$ is a positive-dimensional complex (Banach) manifold such that for every $\alpha\in\mathcal{A}$ the preimage $\mathcal{X}_\alpha:=\pi^{-1}(\alpha)$ is a disk;

– a holomorphic map $\Phi\colon \mathcal{X}\to\mathbb{C}^2$ of generic rank 2 such that for every $\alpha\in\mathcal{A}$ the image $\Phi(\mathcal{X}_\alpha)=C_\alpha$ is a graph of a holomorphic function $\phi_\alpha\colon \Delta\to\Delta$.

A family $(\mathcal{X},\pi,\mathcal{A},\Phi)$ we shall often also call an *analytic family of complex disks in* Δ^2. In our applications $\mathcal{A}$ will always be a neighborhood of some α_0 and without loss of generally we may assume for convenience that $\phi_{\alpha_0}\equiv 0$, i.e., that $C_{\alpha_0}=\Delta\times\{0\}$. In this local case, after shrinking $\mathcal{X}$ and Δ^2 if necessary, we can suppose that $\mathcal{X}=\Delta\times\mathcal{A}$, and we shall regard in this case Φ as a natural universal map

$$\Phi\colon (\lambda,\alpha)\longmapsto(\lambda,\phi_\alpha(\lambda)) \tag{3}$$

from $\Delta\times\mathcal{A}$ to Δ^2, writing $\Phi(\lambda,\alpha)=(\lambda,\phi(\lambda,\alpha))$ when convenient, meaning $\phi(\lambda,\alpha)=\phi_\alpha(\lambda)$. We shall often consider the case when $\mathcal{A}$ is a one-dimensional disk, in that case we say that our family is a *complex one-parameter* analytic family. In this case taking as $\mathcal{A}$ a sufficiently small neighborhood of α_0 and perturbing $\partial\Delta$ in the λ-variable slightly we can suppose without loss of generality that ϕ_α does not vanish on $\partial\Delta$ if $\alpha\neq\alpha_0$. In particular the winding number of $\phi_\alpha|_{\partial\Delta}$ is constant for $\alpha\in\mathcal{A}\setminus\{\alpha_0\}$, see Proposition 3.8.

Denote the image $\Phi(\mathcal{X})$ by $\overline{\mathcal{P}}_{\mathcal{X},\Phi}$, where $(\mathcal{X},\pi,\Delta,\Phi)$ is some complex one-parameter analytic family of complex disks in Δ^2. A point λ_0 such that $\phi(\lambda_0,\alpha)\equiv 0$ as a function of α we call *a pinch of* $\overline{\mathcal{P}}_{\mathcal{X},\Phi}$ and say that $\overline{\mathcal{P}}_{\mathcal{X},\Phi}$ has a pinch at λ_0. Let us describe the shape of $\overline{\mathcal{P}}_{\mathcal{X},\Phi}$ near a pinch λ_0. Since $\phi(\lambda_0,\alpha)\equiv 0$ we can divide it by $(\lambda-\lambda_0)^{l_0}$ with some (taken to be maximal) $l_0\geq 1$. That is, in a neighborhood of $(\lambda_0,\alpha_0)\in\Delta\times\mathcal{A}$ we can write

$$\phi(\lambda,\alpha)=(\lambda-\lambda_0)^{l_0}\phi_1(\lambda,\alpha), \tag{4}$$

where $\phi_1(\lambda_0,\alpha)\not\equiv 0$. Set

$$\Phi_1\colon (\lambda,\alpha)\longmapsto(\lambda,\phi_1(\lambda,\alpha)). \tag{5}$$

The image of Φ_1 contains a bidisk $\Delta_r^2(\lambda_0,0)$ of some radius $r>0$ centered at $(\lambda_0,0)$. Therefore

$$\overline{\mathcal{P}}_{\mathcal{X},\Phi}\supset\{z\in\Delta_r^2(\lambda_0,0):|z|<c|\lambda-\lambda_0|^{l_0}\} \tag{6}$$

with some constant $c>0$.

Definition 2.1. By a *pinched domain* we shall understand an open neighborhood $\mathcal{P}$ of $\bar{\Delta}\setminus\Lambda$, where Λ is a finite set of points in Δ, such that in a neighborhood of every $\lambda_0\in\Lambda$ the domain $\mathcal{P}$ contains

$$\{z\in\Delta_r^2(\lambda_0,0):|z|<c|\lambda-\lambda_0|^{l_0}\}\setminus\{(\lambda_0,0)\}. \tag{7}$$

We shall call l_0 the *order of the pinch* λ_0.

After shrinking Δ (in the λ-variable) if necessary, we can suppose that the set $\overline{\mathcal{P}}_{\mathcal{X},\Phi}$ which corresponds to the complex one-parameter analytic family $(\mathcal{X},\pi,\mathcal{A},\Phi)$ has only finite number of pinches, say at $\lambda_1,...,\lambda_N$ of orders $l_1,...,l_N$ respectively, and therefore $\mathcal{P}_{\mathcal{X},\Phi}:=\overline{\mathcal{P}}_{\mathcal{X},\Phi}\setminus\{\lambda_1,...,\lambda_N\}$ is a pinched domain. Note that $\overline{\mathcal{P}}_{\mathcal{X},\Phi}$ obviously contains every curve in a neighborhood $\mathcal{B}$ of $\phi_0\equiv 0$ of the subspace

$$\{\phi\in\mathrm{Hol}(\Delta,\Delta):\mathrm{ord}_0(\phi,\lambda_j)\geq l_j\}\subset\mathrm{Hol}(\Delta,\Delta), \tag{8}$$

which is of finite codimension.

Remark 2.2. (a) Therefore, let us make the following assumption: our pinched domains will always be supposed to have only finitely many pinches and moreover, these pinches *do not belong* to the corresponding pinched domain *by definition.*

(b) The Hilbert manifold structure on $\mathcal{B}$ (if needed) can be ensured by considering instead of $\mathrm{Hol}(\Delta,\Delta)$ the Hilbert space $H_+^{1,2}(\mathbb{S}^1)$ of Sobolev functions on the circle, which holomorphically extends to Δ, for example. This will be done later in Section 3. At that point it will be sufficient for us to note that extension along one-parameter analytic families is equivalent to that along infinite-dimensional ones, and both imply the extension to pinched domains. More precisely, the following is true:

Proposition 2.3. *Let* $(\mathcal{X},\pi,\mathcal{A},\Phi)$ *be a complex one-parameter analytic family of complex disks in* Δ^2 *and let* $\mathcal{P}_{\mathcal{X},\Phi}$ *be the corresponding pinched domain. Suppose that a holomorphic function* f *on* $R_{1+\varepsilon}$ *meromorphically extends along every* C_α, $\alpha\in\mathcal{A}$. *Let* $\mathcal{B}$ *be the* infinite-dimensional *analytic family of complex disks in* Δ^2 *constructed as in* (8). *Then*

(i) f *meromorphically extends to* $\mathcal{P}_{\mathcal{X},\Phi}$ *as a function of two variables*;

(ii) *for every* $\beta\in\mathcal{B}$ *the restriction* $f|_{C_\beta\cap R_{1+\varepsilon}}$ *extends to a meromorphic function on* C_β *and the number of poles of these extensions are uniformly bounded.*

Proof. Writing $\mathcal{X}=\Delta\times\mathcal{A}$ with $\mathcal{A}=\Delta$ and $\alpha_0=0$, and taking the preimage $W:=\Phi^{-1}(R_{1+\varepsilon})$ in $\Delta\times\mathcal{A}\equiv\Delta^2$ we find ourselves in the following situation:

(i) W contains a ring domain (denote it by W as well), and $g:=f\circ\Phi$ is meromorphic (or holomorphic, after shrinking) on W.

(ii) For every $\alpha\in\mathcal{A}$ the restriction $g|_{(\Delta\times\{\alpha\})\cap W}$ meromorphically extends to $\Delta\times\{\alpha\}$.

The classical theorem of Levi, see [L] and [Si1], implies now that g meromorphically extends to $\mathcal{X}=\Delta\times\mathcal{A}$, and this gives us the extension of f to $\mathcal{P}_{\mathcal{X},\Phi}$.

For the proof of the extendability of $f|_{C_\beta\cap R_{1+\varepsilon}}$ to C_β for every $\beta\in\mathcal{B}$ close enough to zero let us first of all remark that $\Phi^{-1}(C_\beta)$ is contained in a relatively compact part of $\mathcal{X}$. Indeed, take a pinch λ_0 and suppose without loss of generality that $\lambda_0=0$. Write

$$\phi(\lambda,\alpha)=\lambda^{l_0}\phi_1(\lambda,\alpha) \tag{9}$$

as in (4). Since $\phi_1(0,\alpha)\not\equiv 0$ we can use the Weierstrass preparation theorem and represent

$$\phi_1(\lambda,\alpha)=u(\alpha^k+g_1(\lambda)\alpha^{k-1}+\ldots+g_k(\lambda)) \tag{10}$$

with $u(0,0)\neq 0$ and $g_1(0)=\ldots=g_k(0)=0$. Take the corresponding $\phi_\beta\in\mathcal{B}$ with graph C_β and write it as $\phi_\beta(\lambda)=c_0\lambda^{l_0}\tilde{\phi}(\lambda)$. Consider the equation $\phi(\lambda,\alpha)=\phi_\beta(\lambda)$, i.e.,

$$\lambda^{l_0}(\alpha^k+g_1(\lambda)\alpha^{k-1}+\ldots+g_k(\lambda))=u^{-1}c_0\lambda^{l_0}\tilde{\phi}(\lambda), \tag{11}$$

or, equivalently

$$\alpha^k+g_1(\lambda)\alpha^{k-1}+\ldots+g_k(\lambda)=u^{-1}c_0\tilde{\phi}(\lambda). \tag{12}$$

For $\lambda\sim 0$ all solutions $\alpha_1(\lambda),\ldots,\alpha_k(\lambda)$ of (12) are close to zero, provided c_0 is small enough. This proves our assertion that $\Phi^{-1}(C_\beta)\Subset\mathcal{X}$ and implies that $f|_{C_\beta\cap R_{1+\varepsilon}}$ meromorphically extends to C_β.

The orders of the poles of the meromorphic function g of two variables (λ,α) is bounded on every relatively compact part of $\mathcal{X}=\Delta\times\mathcal{A}$ and therefore the orders of the poles of our extensions are also bounded. □

Remark 2.4. Note that f meromorphically extends to the pinched domain $\mathcal{P}_{\mathcal{B}}$ swept by the family $\mathcal{B}$ as well, simply because it is the same domain (up to shrinking).

2.2. Proof of Theorem 1.1

We start with the proof of (ii). Without loss of generality we may assume that $\phi_0 \equiv 0$. Indeed, the condition $\phi_0 \equiv 0$ is not a restriction neither here, nor anywhere else in this paper, because it can be always achieved by the coordinate change

$$\begin{cases} \lambda \longmapsto \lambda, \\ z \longmapsto z - \phi_0(\lambda). \end{cases} \tag{13}$$

Furthermore, when considering the extension of a meromorphic function f from a ring domain $R_{1+\varepsilon}$ to the bidisk $\Delta_{1+\varepsilon} \times \Delta$ one can suppose that f is holomorphic on $R_{1+\varepsilon}$ (after shrinking $R_{1+\varepsilon}$ if necessary and after multiplying by some power of z), and moreover, after decomposing $f = f^+ + f^-$, where f^+ is holomorphic in $\Delta_{1+\varepsilon} \times \Delta$ and f^- in $(\mathbb{P}^1 \setminus \bar{\Delta}) \times \Delta$, one can subtract f^+ from f and suppose that $f^+ \equiv 0$. This means that we can suppose that f has the Taylor decomposition

$$f(\lambda, z) = \sum_{n=0}^{\infty} A_n(\lambda) z^n \tag{14}$$

in $R_{1+\varepsilon}$ with

$$A_n(\lambda) = \sum_{l=-\infty}^{-1} a_{n,l} \lambda^l. \tag{15}$$

As a result, in this proof we may suppose that $f = f^-$ and f^- is holomorphic in $A_{1-\varepsilon,1+\varepsilon} \times \Delta_{1+2\varepsilon}$. Therefore for $|\lambda|$ near 1 the Taylor expansion of f is

$$f(\lambda, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f(\lambda, 0)}{\partial z^n} z^n = \sum_{n=0}^{\infty} A_n(\lambda) z^n, \tag{16}$$

and we have the estimates

$$|A_n(\lambda)| = \frac{1}{n!} \left| \frac{\partial^n f(\lambda, 0)}{\partial z^n} \right| \leq \frac{C}{(1+\varepsilon)^n}, \tag{17}$$

for some constant C, all $k \in \mathbb{N}$ and all $\lambda \in \mathbb{S}^1 := \partial\Delta$. Under the assumptions of the theorem we see that meromorphic extensions $f_k(\lambda)$ of $f(\lambda, \phi_k(\lambda))$ have a uniformly bounded number of poles counted with multiplicities. As well as the number of zeroes of ϕ_k are uniformly bounded too. Up to taking a subsequence we can suppose that

(a) the number of poles of the f_k's, counted with multiplicities, is constant, say M, and these poles converge to the finite set $b_1, ..., b_M \in \Delta_{1-\varepsilon}$ with corresponding multiplicities, i.e., some of the $b_1, ..., b_M$ may coincide;

(b) the number of zeroes of ϕ_k, counted with multiplicities, is also constant, say N and these zeroes converge to a finite set with corresponding multiplicities. We shall denote it by $a_1, ..., a_N$, meaning that some of them can coincide.

Step 1. For every k take a Blaschke product P_k having zeroes exactly at the poles of f_k with corresponding multiplicities and extract from $\{P_k\}_{k=1}^{\infty}$ a converging subsequence with the limit

$$P_0(\lambda) = \prod_{j=1}^{M} \frac{\lambda - b_j}{1 - \bar{b}_j \lambda}. \tag{18}$$

The holomorphic functions $g_k := P_k f_k$ have uniformly bounded modulus on Δ and converge to some g_0, with modulus bounded by C (a constant from (17)). Therefore f_k converge on compact subsets of $\Delta \setminus \{b_1, ..., b_M\}$ to a meromorphic function, which is nothing but A_0, and which satisfies the estimate

$$|A_0(\lambda)| \leq \frac{CC_1}{|\lambda - b_1| ... |\lambda - b_M|}, \tag{19}$$

where $C_1 = \max\{\prod_{j=1}^{M} |1 - \bar{b}_j \lambda| : |\lambda| \leq 1\}$.

Step 2. Consider the function

$$f_1(\lambda, z) := \frac{f(\lambda, z) - A_0(\lambda)}{z}, \tag{20}$$

and the functions

$$f_{1,k}(\lambda) := f_1(\lambda, \phi_k(\lambda)) = \frac{f(\lambda, \phi_k(\lambda)) - A_0(\lambda)}{\phi_k(\lambda)}. \tag{21}$$

These functions are well defined and meromorphic on $\Delta_{1+\varepsilon}$, the equality in (21) makes sense on $A_{1-\varepsilon,1+\varepsilon}$. After taking a subsequence we see that the poles of $f_{1,k}$, which are different from the zeroes of ϕ_k, converge to the same points $b_1, ..., b_N$. So the multiplicities do not increase. Let again P_k be the Blaschke product having zeroes at the poles of $f_{1,k}$ with corresponding multiplicities. Take a subsequence P_k uniformly converging to a corresponding Blaschke product P_0 and holomorphic functions $g_k := P_k f_{1,k}$ uniformly converging to some holomorphic function g_0. In $A_{1-\varepsilon,1+\varepsilon}$ it is straightforward to see that $g_0 = P_0 A_1$. This proves that A_1 (if not identically zero), has at most $N+M$ poles counting with multiplicities and these poles are located at $a_1,\dots,a_N,b_1,\dots,b_M$.

Moreover, for $|\lambda|=1$ from (17) we have the estimate

$$|P_0(\lambda)A_1(\lambda)|\le\frac{C}{1+\varepsilon}, \tag{22}$$

which implies that

$$|A_1(\lambda)|\le\frac{1}{|\lambda-a_1|\dots|\lambda-a_N|\,|\lambda-b_1|\,|\lambda-b_N|}\frac{CC_1C_2}{1+\varepsilon} \tag{23}$$

for $\lambda\in\Delta\setminus\{a_1,\dots,a_N,b_1,\dots,b_M\}$. Here $C_1=\max\{\prod_{j=1}^N|1-\bar a_j\lambda|:|\lambda|\le1\}$. Let from now on $C'=CC_1C_2$.

Step 3. Suppose that we have proved that A_n extends to a meromorphic function in Δ with the estimate

$$|A_n(\lambda)|\le\frac{1}{\prod_{j=1}^N|\lambda-a_j|^n\prod_{j=1}^M|\lambda-b_j|}\frac{C'}{(1+\varepsilon)^n} \tag{24}$$

for $\lambda\in\Delta\setminus\{a_1,\dots,a_N,b_1,\dots,b_M\}$. Note that inequality (24) means, in particular, that $A_0,\dots,A_n$ have no other poles than $a_1,\dots,b_N$ with corresponding multiplicities. Apply considerations as above to

$$f_{n+1}(\lambda,z)=\frac{1}{z^{n+1}}\left(f(\lambda,z)-\sum_{j=0}^nA_j(\lambda)z^j\right),$$

i.e., consider

$$f_{n+1,k}(\lambda)=\frac{1}{\phi_k^{n+1}}\left(f(\lambda,\phi_k)-\sum_{j=0}^nA_j(\lambda)\phi_k^j\right)$$

and repeat the same process with Blaschke products. Note only that the products $A_j(\lambda)\phi_k^j$ have no poles at the zeroes of ϕ_k. On the boundary $\{\lambda:|\lambda|=1\}$ the functions $|f_{n+1,k}(\lambda)|$ are bounded by $C/(1+\varepsilon)^{n+1}$ due to Cauchy inequalities and therefore we get the conclusion that A_{n+1} meromorphically extends to Δ with the estimate

$$|A_{n+1}(\lambda)|\le\frac{1}{\prod_{j=1}^N|\lambda-a_j|^{n+1}\prod_{j=1}^M|\lambda-b_j|}\frac{C'}{(1+\varepsilon)^{n+1}}. \tag{25}$$

Estimate (25) implies that (16) converges in the domain

$$\{(\lambda,z)\in\Delta^2:|z|<c|\lambda-a_{j_1}|^{l_1}\dots|\lambda-a_{N_1}|^{l_{N_1}}\}\setminus\bigcup_{j=1}^M\{(\lambda,z):\lambda=b_j\}, \tag{26}$$

for an appropriately chosen $c>0$. Here N_1 is the number of *different* a_j's, which are denoted as $a_{j_1}, ..., a_{N_1}$ having the corresponding multiplicities $l_1, ..., l_{N_1}$. In particular we mean here that b_k are different from a_{j_1} for all k and j_1. The estimate (25) implies that the extension of $f\prod_{j=1}^{M}(\lambda-b_j)$ to (26) is locally bounded near every vertical disk $\{(\lambda,z):\lambda=b_{k_1}\}$ and therefore extends across it by Riemann's extension theorem. We conclude that f extends as a meromorphic function to the pinched domain

$$\mathcal{P}=\{(\lambda,z)\in\Delta^2 : |z|<c|\lambda-a_{j_1}|^{l_1}...|\lambda-a_{N_1}|^{l_{N_1}}\}, \tag{27}$$

and this proves part (ii) of Theorem 1.1.

(i) Take now any holomorphic function ϕ in $\Delta_{1+\varepsilon}$ of the form

$$\phi(\lambda)=(\lambda-a_{j_1})^{l_1}...(\lambda-a_{N_1})^{l_{N_1}}\psi$$

with ψ small enough in order that the graph C_ϕ is contained in $\mathcal{P}$ (more precisely so that $C_\phi\cap((\Delta\setminus\{a_{j_1},...,a_{N_1}\})\times\Delta)\subset\mathcal{P}$). To prove the part (i) of our theorem we need to prove the following statement.

Step 4. *$f(\lambda,\phi(\lambda))$ meromorphically extends from $A_{1-\varepsilon,1+\varepsilon}$ to $\Delta_{1+\varepsilon}$.* Indeed, $f(\lambda,\phi(\lambda))$ is meromorphic on $\Delta\setminus\{a_{j_1},...,a_{N_1}\}$. At the same time from the estimate (24) we see that the terms in the series

$$f(\lambda,\phi(\lambda))=\sum_{n=0}^{\infty}A_n(\lambda)\phi^n(\lambda) \tag{28}$$

are, in fact, holomorphic in a neighborhood of every a_j and converge normally there, provided $\|\psi\|_\infty$ was taken small enough. Uniform boundedness of the number of poles follows now from Proposition 2.3. Part (i) is proved.

Remark 2.5. In order to prove Corollary 1.3 note that pinches that appeared in the proof of Theorem 1.1 are limits of zeroes of ϕ_k. General position assumption means that for every $\lambda_0\in\Delta$ we can take a subsequence such that the resulting pinched domain will not have a pinch in λ_0. The rest follows.

3. Extension along finite-dimensional families

3.1. Properties of the singular integral transform

By $L^{1,2}(\mathbb{S}^1)$ we denote the Sobolev space of complex-valued functions on the unit circle having their first derivative in L^2. This is a complex Hilbert space with the scalar product $(h,g)=\int_0^{2\pi}[h(e^{i\theta})\bar{g}(e^{i\theta})+h'(e^{i\theta})\bar{g}'(e^{i\theta})]\,d\theta$. Recall that by the Sobolev imbedding theorem $L^{1,2}(\mathbb{S}^1)\subset\mathcal{C}^{1/2}(\mathbb{S}^1)$, where $\mathcal{C}^{1/2}(\mathbb{S}^1)$ is the space of Hölder $\frac{1}{2}$-continuous functions on $\mathbb{S}^1$.

For the convenience of the reader we recall a few well known facts about the Hilbert transform in $L^{1,2}(\mathbb{S}^1)$.

Lemma 3.1. *A function $\phi\in L^{1,2}(\mathbb{S}^1)$ extends holomorphically to Δ if and only if the following condition is satisfied*:

$$P(\phi)(\tau):=-\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\phi(t)-\phi(\tau)}{t-\tau}\,dt\equiv 0. \tag{29}$$

Proof. The fact that ϕ extends holomorphically to Δ can be obviously expressed as

$$\lim_{\substack{z\to\tau\\ z\in\Delta}}\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\phi(t)}{t-z}\,dt=\phi(\tau) \tag{30}$$

for all $\tau\in\mathbb{S}^1$. Write then

$$\lim_{\substack{z\to\tau\\ z\in\Delta}}\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\phi(t)}{t-z}\,dt=\lim_{\substack{z\to\tau\\ z\in\Delta}}\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\phi(t)-\phi(\tau)}{t-z}\,dt+\lim_{\substack{z\to\tau\\ z\in\Delta}}\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\phi(\tau)}{t-z}\,dt \tag{31}$$
$$=-P(\phi)(\tau)+\phi(\tau).$$

From (30) and (31) we immediately get (29). □

Denote by $\mathbb{S}^1_\varepsilon(\tau)$ the circle $\mathbb{S}^1$ without the ε-neighborhood of τ. Consider the following singular integral operator (the Hilbert transform)

$$S(\phi)(\tau):=\text{p.v.}\frac{1}{\pi i}\int_{\mathbb{S}^1}\frac{\phi(t)}{t-\tau}\,dt:=\lim_{\varepsilon\to 0}\frac{1}{\pi i}\int_{\mathbb{S}^1_\varepsilon(\tau)}\frac{\phi(t)}{t-\tau}\,dt. \tag{32}$$

In the sequel we shall write simply

$$S(\phi)(\tau):=\frac{1}{\pi i}\int_{\mathbb{S}^1}\frac{\phi(t)}{t-\tau}\,dt, \tag{33}$$

i.e., the integral in the right-hand side will be always understood in the sense of the principal value.

Lemma 3.2. *The following relation between the operators S and P holds*:

$$S=-2P+\text{Id}. \tag{34}$$

Therefore a function $\phi\in L^{1,2}(\mathbb{S}^1)$ holomorphically extends to the unit disk if an only if

$$S(\phi)(\tau)\equiv\phi(\tau). \tag{35}$$

Proof. Write

$$\frac{1}{2}S(\phi)(\tau)=\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\phi(t)}{t-\tau}\,dt=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\mathbb{S}^1_\varepsilon(\tau)}\frac{\phi(t)}{t-\tau}\,dt$$
$$=\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\phi(t)-\phi(\tau)}{t-\tau}+\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\mathbb{S}^1_\varepsilon(\tau)}\frac{\phi(\tau)}{t-\tau}\,dt=-P(\phi)(\tau)+\frac{1}{2}\phi(\tau).$$

Therefore one has

$$S(\phi)=-2P(\phi)+\phi, \tag{36}$$

which is (34), and which implies (35). □

Denote by $H^{1,2}_+(\mathbb{S}^1)$ the subspace of $L^{1,2}(\mathbb{S}^1)$ which consists of functions holomorphically extendable to the unit disk Δ. By $H^{1,2}_-(\mathbb{S}^1)$ denote the subspace of functions holomorphically extendable to the complement of the unit disk in the Riemann sphere $\mathbb{P}^1$ and with zero at infinity. Observe the following orthogonal decomposition

$$L^{1,2}(\mathbb{S}^1)=H^{1,2}_+(\mathbb{S}^1)\oplus H^{1,2}_-(\mathbb{S}^1). \tag{37}$$

We finish this review with the following lemma.

Lemma 3.3. (i) *P and S are bounded linear operators on $L^{1,2}(\mathbb{S}^1)$ and*

$$S^2=\mathrm{Id}. \tag{38}$$

(ii) *Moreover, on the space $H^{1,2}_+(\mathbb{S}^1)$ the operator S acts as identity and on the space $H^{1,2}_-(\mathbb{S}^1)$ as $-\mathrm{Id}$.*

(iii) *Consequently P is an orthogonal projector onto $H^{1,2}_-(\mathbb{S}^1)$.*

For the proof of (38) we refer to [MP], pp. 46, 50 and 69. In fact, since $S=-2P+\mathrm{Id}$ and because $\operatorname{Ker} P=H^{1,2}_+(\mathbb{S}^1)$, we see that $S=\mathrm{Id}$ on $H^{1,2}_+(\mathbb{S}^1)$. From (38) and (34) we also see that $P=\mathrm{Id}$ on $H^{1,2}_-(\mathbb{S}^1)$, i.e., P projects $L^{1,2}(\mathbb{S}^1)$ onto $H^{1,2}_-(\mathbb{S}^1)$ parallel to $H^{1,2}_+(\mathbb{S}^1)$.

This lemma clearly implies the following corollary.

Corollary 3.4. *A function $\phi\in L^{1,2}(\mathbb{S}^1)$ extends to a meromorphic function in Δ with not more than N poles if and only if $P(\phi)$ is rational, is zero at infinity and has not more than N poles.*

Indeed, decompose $\phi = \phi^+ + \phi^-$ according to (37). ϕ is meromorphic with at most N poles, all in Δ, if and only if ϕ^- is such, which means that ϕ^- should be rational with at most N poles. But since, according to (iii) of Lemma 3.3, one has $P(\phi) = \phi^-$, the last is equivalent to the fact that $P(\phi)$ is rational with at most N poles.

3.2. The case of finite-dimensional families

To clarify the finite versus infinite-dimensional issues in this paper let us give a simple proof of Theorem 1.1 in the special case when ϕ_k belong to an analytic family $\{\phi_\alpha\}_{\alpha\in\mathcal{A}}$ parameterized by a finite-dimensional complex manifold $\mathcal{A}$. More precisely, as in Section 2, we are given a complex manifold $\mathcal{X}$, a holomorphic submersion $\pi\colon \mathcal{X}\to\mathcal{A}$ such that for every $\alpha\in\mathcal{A}$ the preimage $\mathcal{X}_\alpha := \pi^{-1}(\alpha)$ is a disk. We are also given a holomorphic map $\Phi\colon \mathcal{X}\to\mathbb{C}^2$ such that for every $\alpha\in\mathcal{A}$ the image $\Phi(\mathcal{X}_\alpha) = C_\alpha$ is a graph of a holomorphic function $\phi_\alpha\colon \Delta_{1+\varepsilon}\to\Delta$. We shall regard $\mathcal{A}$ as a (locally closed) complex submanifold of $H^{1,2}_+(\mathbb{S}^1)$. And, finally, by saying that $\{\phi_k\}_{k=1}^\infty$ belong to $\{\phi_\alpha\}_{\alpha\in\mathcal{A}}$ we mean that there exist $\alpha_k\in\mathcal{A}$, $\alpha_k\to\alpha_0\in\mathcal{A}$, such that $\phi_k = \phi_{\alpha_k}$ for $k\geq 0$.

After shrinking, if necessary, we can suppose that our function f is holomorphic on $R_{1+\varepsilon} = A_{1-\varepsilon,1+\varepsilon}\times\Delta_{1+\varepsilon}$. Consider an analytic mapping $F\colon L^{1,2}(\mathbb{S}^1)\to L^{1,2}(\mathbb{S}^1)$ such that

$$F\colon \phi(\lambda)\longmapsto f(\lambda,\phi(\lambda)), \tag{39}$$

and consider also the following integral operator $\mathcal{F}\colon H^{1,2}_+(\mathbb{S}^1)\to H^{1,2}_-(\mathbb{S}^1)$,

$$\mathcal{F}(\phi)(\lambda) = -\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{f(\zeta,\phi(\zeta)) - f(\lambda,\phi(\lambda))}{\zeta-\lambda}\,d\zeta = P(F(\phi)). \tag{40}$$

According to Lemma 3.1, $f(\lambda,\phi(\lambda))$ extends to a holomorphic function in $\Delta_{1+\varepsilon}$ if and only if $\mathcal{F}(\phi) = 0$, and according to Corollary 3.4 it extends meromorphically to $\Delta_{1+\varepsilon}$ with at most N poles in $\Delta_{1-\varepsilon}$ if an only if $\mathcal{F}(\phi)$ is a boundary value of a rational function with at most N poles all in $\Delta_{1-\varepsilon}$.

Theorem 3.5. *Let f be meromorphic on $R_{1+\varepsilon}$ and $\{\phi_k\colon \Delta_{1+\varepsilon}\to\Delta\}_{k=1}^\infty$ be a sequence of holomorphic functions converging to some $\phi_0\colon \Delta_{1+\varepsilon}\to\Delta$, $\phi_k\not\equiv\phi_0$ for all k. Suppose that*

(a) *$\{\phi_k\}_{k=1}^\infty$ belong to a finite-dimensional analytic family $\{\phi_\alpha\}_{\alpha\in\mathcal{A}}$, i.e., $\phi_k = \phi_{\alpha_k}$ for some $\alpha_k\in\mathcal{A}$ and $\alpha_k\to\alpha_0$ in $\mathcal{A}$ with $\phi_0 = \phi_{\alpha_0}$;*

(b) *for every k the restriction $f|_{C_k\cap R_{1+\varepsilon}}$ is well defined and extends to a meromorphic function on the curve C_k;*

(c) *the number of poles counted with multiplicities of these extensions is uniformly bounded.*

Then there exists a complex disk $\Delta\subset\mathcal{A}$ *containing* α_0 *such that for every* $\alpha\in\Delta$ *the restriction* $f|_{C_\alpha\cap R_{1+\varepsilon}}$ *meromorphically extends to* C_α*, and the number of poles of these extensions counted with multiplicities is uniformly bounded.*

Proof. (i) Consider the holomorphic case first. Restrict $\mathcal{F}$ to $\mathcal{A}$ to obtain a holomorphic map $\mathcal{F}_{\mathcal{A}}\colon \mathcal{A}\to H^{1,2}_{-}(\mathbb{S}^1)$. The restriction $f|_{C_\alpha\cap R_{1+\varepsilon}}$ holomorphically extends to C_α if and only if $\mathcal{F}(\alpha)=0$. Therefore we are interested in the zero set $\mathcal{A}^0$ of $\mathcal{F}_{\mathcal{A}}$. But the zero set $\mathcal{A}^0$ of a holomorphic mapping from a finite-dimensional manifold is a *finite-dimensional* analytic set. Since this set contains a converging sequence $\{\alpha_k\}_{k=1}^{\infty}$ it has positive dimension. Denote (with the same letter) by $\mathcal{A}^0$ a positive-dimensional irreducible component of our zero set which contains an infinite number of the ϕ_{α_k}. Suppose, up to replacing α_k by a subsequence, that all α_k are in $\mathcal{A}^0$. Let $\mathcal{X}^0$ be the corresponding universal family, i.e., the restriction of $\pi\colon \mathcal{X}\to\mathcal{A}$ to $\mathcal{A}^0$, and $\Phi^0\colon \mathcal{X}^0\to\Delta_{1+\varepsilon}\times\Delta$ be the corresponding evaluation map. Φ^0 should be of generic rank two, otherwise ϕ_k would be constant. Therefore $\mathcal{A}^0$ contains a complex disk through α_0 with properties as required.

(ii) The meromorphic extension in this case is also quite simple. Without loss of generality we suppose that all extensions $f_{\alpha_k}(\lambda)$ have at most N poles counting with multiplicities. Since $f_{\alpha_k}(\lambda)=f(\lambda,\phi_{\alpha_k}(\lambda))$ for $\lambda\in A_{1-\varepsilon,1+\varepsilon}$, all poles of these extensions are contained in $\bar{\Delta}_{1-\varepsilon}$. Denote by $R^N(1-\varepsilon)$ the subset of $H^{1,2}_{-}(\mathbb{S}^1)$ which consists of rational functions, which are holomorphic on $\mathbb{P}^1\setminus\bar{\Delta}$, zero at infinity and having not more than N poles, all contained in $\bar{\Delta}_{1-\varepsilon}$. $R^N(1-\varepsilon)$ can be explicitly described as the set

(41)
$$R^N(1-\varepsilon)=\left\{\sum_j (z-a_j)^{-m_j}\sum_{k=0}^{m_j-1} c_{jk}(z-a_j)^k : c_{jk}\in\mathbb{C}, a_j\in\bar{\Delta}_{1-\varepsilon} \text{ and } \sum_j m_j=N\right\}.$$

Let us note that $\mathcal{F}_{\mathcal{A}}(\phi_{\alpha_k})\in R^N(1-\varepsilon)$ for all k and that the set $\mathcal{A}^N$ of those $\alpha\in\mathcal{A}$ for which $f(\lambda,\phi_\alpha(\lambda))$ is meromorphically extendable to Δ with not more than N poles, all in $\bar{\Delta}_{1-\varepsilon}$, is in fact $\mathcal{F}_{\mathcal{A}}^{-1}(R^N(1-\varepsilon))$.

Set $g_0=\mathcal{F}_{\mathcal{A}}(\phi_{\alpha_0})$. From (41) we see that $R^N(1-\varepsilon)$ is a finite-dimensional subspace of $H^{1,2}_{-}(\mathbb{S}^1)$. Therefore we can take an orthogonal complement $H\subset H^{1,2}_{-}(\mathbb{S}^1)$ to it at g_0 in such a way that $H^{1,2}_{-}(\mathbb{S}^1)=R^N(1-\varepsilon)\times H$ locally in a neighborhood of g_0. Denote by Ψ the composition of $\mathcal{F}_{\mathcal{A}}$ with the projection onto H. Now $\mathcal{A}^N$ is the zero set of Ψ and therefore we are done as in the case (i). $\square$

Let us make a few remarks concerning the finite-dimensional case of the last theorem.

Remark 3.6. (a) Horizontal disks belong to a one-dimensional family, non-horizontal straight disks to a two-dimensional family. Therefore Theorem 3.5 generalizes the Hartogs–Levi theorem and the result of Dinh.

(b) We do not claim in Theorem 3.5 that the disks Δ contains α_k for $k \gg 1$ and this is certainly not true in general. What is true is that the set $\mathcal{A}^0$ (or $\mathcal{A}^N$), of all $\alpha \in \mathcal{A}$ such that f extends along C_α, is an analytic set of positive dimension, and so contains an analytic disk with center at α_0.

Remark 3.7. At the same time Theorem 3.5 is a particular case of Theorem 1.1 because of the following observation.

Proposition 3.8. *Let $(\mathcal{X}, \pi, \mathcal{A}, \Phi)$ be a finite-dimensional analytic family of holomorphic maps $\Delta_{1+\varepsilon} \to \Delta$ and let $\alpha_0 \in \mathcal{A}$ be a point. Then there exist a neighborhood $V \ni \alpha_0$, a complex hypersurface A in V and a radius $r \sim 1$ such that for $\alpha \in V \setminus A$ the restriction $(\phi_\alpha - \phi_0)|_{\partial\Delta_r}$ does not vanish and therefore* $\operatorname{Var}\operatorname{Arg}_{\partial\Delta}(\phi_\alpha - \phi_{\alpha_0})$ *is constant on $V \setminus A$. If, in particular, $(\mathcal{X}, \pi, \mathcal{A}, \Phi)$ is a one-parameter family then $A = \{\alpha_0\}$.*

Proof. After shrinking we can suppose that $\mathcal{X} = \Delta_{1+\varepsilon} \times \Delta^n$, $\alpha_0 = 0$ and $\phi_0 \equiv 0$. Since $\Phi\colon \mathcal{X} \to \Delta_{1+\varepsilon} \times \Delta$ acts as $(\lambda, \alpha) \mapsto (\lambda, \phi(\lambda, \alpha))$ we can consider the zero divisor $\mathcal{Z} = \phi^{-1}(0)$ of ϕ. $\mathcal{Z}$ is not empty, because $\phi(\lambda, 0) \equiv 0$, and is proper, because Φ is of generic rank two. Denote by $\mathcal{Z}_1$ the union of all irreducible components of $\mathcal{Z}$ which contain $\Delta_{1+\varepsilon} \times \{0\}$. Set $A := \mathcal{Z}_1 \cap \{0\}$ and note that A is a hypersurface in Δ^n.

Let $\mathcal{Z}_0$ be the union of all irreducible components of $\mathcal{Z}$ which do not contain $\Delta_{1+\varepsilon} \times \{0\}$. The intersection $\mathcal{Z}_0 \cap \Delta_{1+\varepsilon} \times \{0\}$ is a discrete set. Therefore we can find $r \sim 1$ such that $\mathcal{Z}_0 \cap \partial\Delta_r = \varnothing$. Now it is clear that for a sufficiently small neighborhood $V \ni 0$ in Δ^n we have that $\phi(\,\cdot\,, \alpha)$ does not vanish on $\partial\Delta_r$ provided $\alpha \in V \setminus A$. Then $\operatorname{Var}\operatorname{Arg}_{\partial\Delta}(\phi_\alpha)$ is clearly constant. In the one-parameter case A is discrete but contains α_0. □

Note that in general a test sequence does not belong to any finite-dimensional family. Take for example $\phi_k(\lambda) = \lambda^2/k + e^{-k}\lambda^k$. Therefore Theorem 1.1 properly contains Theorem 3.5.

Remark 3.9. (a) If C_k are intersections of $\Delta_{1+\varepsilon} \times \Delta$ with algebraic curves of bounded degree, then they are included in a finite-dimensional analytic (even algebraic in this case) family.

(b) If $\phi_k(\partial\Delta)\subset M$, where M is totally real in $\partial\Delta\times\bar{\Delta}$, and have bounded Maslov index then they are included in a finite-dimensional analytic family.

(c) If we do not suppose *ad hoc* that ϕ_k belong to some finite-dimensional analytic family of holomorphic functions then the argument above is clearly not sufficient. The following example is very instructive. Consider a holomorphic map $\mathcal{F}\colon l^2\to l^2\oplus l^2$ defined as

$$\mathcal{F}\colon \{z_k\}_{k=1}^{\infty}\to\{\{z_k(z_k-1/k)\}\oplus\{z_kz_j\}_{j>k}\}. \tag{42}$$

The zero set of $\mathcal{F}$ is a sequence $\{Z_k=(0,...,0,1/k,0,...)\}_{k=1}^{\infty}\subset l^2$ together with the origin. These Z_k's might well be our ϕ_k's and therefore we cannot conclude the existence of families in the zero set of our $\mathcal{F}$ from (40) at this stage.

(d) Example 5.1 has precisely the feature as above with $\mathcal{F}$ being the integral operator (40).

4. Mappings to complex spaces

In this section we shall prove Corollary 1.6. The proof consists in making a reduction to the holomorphic function case of Theorem 1.1. This reduction will follow the lines of arguments developed in [I1], [I2], [I3] and [I4]. For the convenience of the reader we shall briefly recall the key statements from these papers which are relevant to our present task.

4.1. Continuous families of analytic disks

An *analytic disk* in a complex space X is a holomorphic map $h\colon\Delta\to X$ continuous up to the boundary. Recall that a complex space X is called *disk-convex* if for every compact $K\Subset X$ there exists another compact $\widehat{K}$ such that for every analytic disk $h\colon\bar{\Delta}\to X$ with $h(\partial\Delta)\subset K$ one has $h(\bar{\Delta})\subset\widehat{K}$. $\widehat{K}$ is called the *disk envelope* of K. All compact Stein 1-convex complex spaces are disk-convex.

Given a meromorphic mapping $f\colon R_{1+\varepsilon}\to X$, where $R_{1+\varepsilon}=A_{1-\varepsilon,1+\varepsilon}\times\Delta$, we can suppose without loss of generality that f is holomorphic on $R_{1+\varepsilon}$ and that $f(R_{1+\varepsilon})$ is contained in some compact K. We suppose that our space X is reduced and that it is equipped with some Hermitian metric form ω. Denote by $\nu=\nu(\widehat{K})$ the minima of the areas of rational curves in the disk envelope $\widehat{K}$ of K. Note that ν is achievable by some rational curve and therefore $\nu>0$. We are given an *uncountable* family of disks $\{C_t{:}t\in T\}$ which are the graphs of holomorphic functions $\phi_t\colon\Delta_{1+\varepsilon}\to\Delta$. Note that the condition on our family of disks to be a *test* family (see the introduction) implies, in particular, that they are all distinct. In the sequel when writing $f(C_t)$ we mean more precisely $f|_{C_t}(C_t)$, i.e., the restriction of f to C_t. This is an analytic disk in X and since $f|_{C_t}(\partial C_t)\subset K$ we see that for every $t\in T$ one has $f(C_t)\subset\widehat{K}$. For every natural k set

$$T_k=\Big\{t\in T:\operatorname{area}(\Gamma_{f|_{C_t}})\le k\frac{\nu}{2}\Big\}, \tag{43}$$

where $\Gamma_{f|_{C_t}}=:\Gamma_t$ is the graph of $f|_{C_t}$ in $\Delta^2_{1+\varepsilon}\times X$ and the area is taken with respect to the standard Euclidean form $\omega_e=dd^c(|\lambda|^2+|z|^2)$ on $\mathbb{C}^2$ and ω on X. For some k the set $T_k\setminus T_{k-1}$ is uncountable, so denote this set by T again.

It will be convenient in the sequel to consider our parameter space T as a subset of the space of 1-cycles in $\Delta^2_{1+\varepsilon}\times X$. Let us say a few words about this issue. For general facts about cycle spaces we refer to [B], for more details concerning our special situation to §1 of [I4]. Recall that a 1-cycle in a complex space Y is a formal sum $Z=\sum_j n_jZ_j$, where $\{Z_j\}_j$ is a locally finite sequence of irreducible analytic subsets of Y of pure dimension one. The space of analytic 1-cycles in Y will be denoted as $\mathcal{C}^{\mathrm{loc}}_1(Y)$. It carries a natural topology, i.e., the topology of currents.

From now on $Y=\Delta^2_{1+\varepsilon}\times X$. Denote by $\mathcal{C}_T$ the subset of $\mathcal{C}^{\mathrm{loc}}_1(Y)$ which consists of graphs Γ_t, i.e., $\mathcal{C}_T=\{\Gamma_t:t\in T\}$. We see $\mathcal{C}_T$ as a topological subspace of $\mathcal{C}$ and in the sequel we shall identify T with $\mathcal{C}_T$. Indeed, note that $t\mapsto\Gamma_t$ is injective, because already $t\mapsto C_t$ is injective. Since T was supposed to be uncountable, so is also $\{\Gamma_t:t\in T\}=\mathcal{C}_T$.

Denote by $\overline{\mathcal{C}}_T$ the closure of $\mathcal{C}_T$ in our space of 1-cycles $\mathcal{C}^{\mathrm{loc}}_1(Y)$ on $\Delta^2_{1+\varepsilon}\times X$. Cycles Z in $\overline{\mathcal{C}}_T$ are characterized by following two properties:

(i) Z has an irreducible component Γ which is a graph of the extension of the restriction $f|_{C\cap R_{1+\varepsilon}}$, where C is a graph of some holomorphic function ϕ: $\Delta_{1+\varepsilon}\to\bar{\Delta}$;

(ii) the other irreducible components for Z (if any) are a finite number of rational curves projecting to points in $\Delta\times\Delta_{1+\varepsilon}$.

This directly follows from the theorem of Bishop, because the areas of the graphs Γ_t are uniformly bounded, and from Lemma 7 in [I1], which says that a limit of a sequence of disks is a disk plus a finite number of rational curves. More precisely in (i) we mean that C is a graph of some holomorphic ϕ: $\Delta_{1+\varepsilon}\to\bar{\Delta}$ and $f|_{C\cap R_{1+\varepsilon}}$ holomorphically extends to C with $\Gamma_{f|_C}=\Gamma$, see Lemma 1.3 from [I4] for more details. Note that by the choice we made we have that

$$(k-1)\frac{\nu}{2}\le\operatorname{area}(Z)\le k\frac{\nu}{2} \tag{44}$$

for all $Z\in\overline{\mathcal{C}}_T$. Indeed (44) is satisfied for $Z=\Gamma_t$ and therefore for their limits.

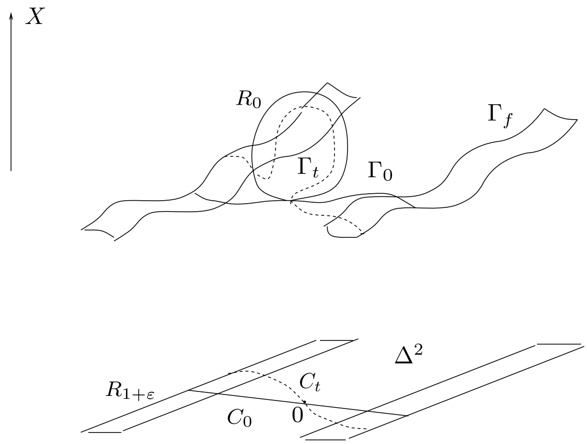

Figure 2. When C_t (the *punctured curve downstairs*) approaches C_0 (the *bold line*) the graph $\Gamma_t=Z_t$ (the *punctured curve upstairs*) of $f|_{C_t}$ stays irreducible and approaches Z_0 (the *bold curve upstairs*). Z_0 is reducible, its irreducible component Γ_0 is a graph over C_0, and its second irreducible component R_0 is a rational curve, which is contained in $\{0\}\times X$. $\Gamma_0=\Gamma_0'\cup R_0$ is an element of $\overline{\mathcal{C}}_T\setminus\mathcal{C}_T=\mathcal{R}_T$ and Γ_f is the graph of f over $R_{1+\varepsilon}$.

Remark 4.1. Let us mention that we write Z both for a 1-cycle as an analytic subset of $Y=\Delta^2_{1+\varepsilon}\times X$ and for the corresponding *point* in the cycle space $\mathcal{C}_1^{\mathrm{loc}}(Y)$.

Denote by $\mathcal{R}_T$ the subset of reducible cycles in $\overline{\mathcal{C}}_T$. This is a closed subset of $\overline{\mathcal{C}}_T$. Indeed, if Z_n is a converging sequence from $\mathcal{R}_T$ then every Z_n has at least one irreducible component, say R_n, which is a rational curve. Therefore the limit $Z:=\lim_{n\to\infty} Z_n$ contains a limit $R:=\lim_{n\to\infty} R_n$ (up to taking a subsequence, if necessary). This R can only be a union of rational curves, i.e., Z is reducible. The difference $\overline{\mathcal{C}}_T\setminus R_{\overline{T}}$ is uncountable since it contains $\mathcal{C}_T$.

From here we get easily that there exists a point $Z_0\in\overline{\mathcal{C}}_T\setminus\mathcal{R}_T$ having a fundamental system of neighborhoods $\{U_n\}_{n=1}^{\infty}$ in $\mathcal{C}_1^{\mathrm{loc}}(Y)$ such that $U_n\cap\overline{\mathcal{C}}_T\subset\overline{\mathcal{C}}_T\setminus\mathcal{R}_T$ for all n and such that all these intersections are uncountable and relatively compact. The last is again by the theorem of Bishop.

Note now that for every $Z_1, Z_2\in\overline{\mathcal{C}}_T\cap U_1$ we have

$$|\mathrm{area}(Z_1)-\mathrm{area}(Z_2)|\leq\frac{\nu}{2}. \tag{45}$$

This readily follows from (44). The first step in the proof of Lemma 2.4.1 from [I3] states that a family of cycles satisfying (45) is continuous in the cycle-space topology. Note that since Z_1 and Z_2 are irreducible we have $Z_1=\Gamma_{f|_{C_1}}$ and $Z_2=\Gamma_{f|_{C_2}}$ for some disks $C_i=\{z:z=\phi(\lambda)\}$, and (45) means that $f|_{C_1}$ is close to $f|_{C_2}$.

4.2. Proof of Corollary 1.6

For every radius r close to 1 consider the subfamily T_r of T such that for $t\in T_r$ the function $\phi_t-\phi_{t_0}$ does not vanish on $\partial\Delta_r$. Take a sequence $r_m\nearrow 1-\varepsilon/2$. Suppose that T_{r_m} is at most countable for every r_m. Then for all $t\in T\setminus(T_{r_1}\cup\ldots\cup T_{r_m})$, which is an uncountable set, the function $\phi_t-\phi_{t_0}$ vanishes on all $\partial\Delta_{r_j}$, $j=1,\ldots,m$. For $m>N$ we get a contradiction (N here bounds the winding numbers of our test family, see the discussion before Corollary 1.6 in the introduction). Therefore for some radius $r\sim 1-\varepsilon/2$ (we can suppose that $r=1$ after all) we find an uncountable subfamily $T'\subset T$ such that for every $t\in T'$ the function $\phi_t-\phi_{t_0}$ does not vanish on $\partial\Delta$. Take this T' as T and make the reductions of the previous subsection for this T.

Now note that Z_0 is irreducible, i.e., is an analytic disk, simply because Z_0 was taken from $\overline{\mathcal{C}}_T\setminus\mathcal{R}_T$. According to (i), Z_0 is a graph of the extension of $f|_{C_0\cap R_{1+\varepsilon}}$ to C_0 for some curve $C_0=\{z:z=\phi_0(\lambda)\}$. Take a Stein neighborhood W of the disk $Z_0=\Gamma_{f|_{C_0}}$, see [Si2], and note that by continuity of the family $\{Z:Z\in\overline{\mathcal{C}}_T\cap U_1\}$ we have that $Z\subset W$ for all $Z\in U\cap(\overline{\mathcal{C}}_T\setminus\mathcal{R}_T)$ for some neighborhood $U\subset U_1$ of Z_0 in the space of cycles. Every such Z is the graph of the extension to some $C=\{z:z=\phi(\lambda)\}$ of the restriction $f|_{C\cap R_{1+\varepsilon}}$. Via an imbedding of W to an appropriate $\mathbb{C}^n$, our f is an n-tuple of holomorphic functions which holomorphically extend to every corresponding C. We are in position to apply the (holomorphic functions case of) Theorem 1.1 and get a holomorphic extension of f to an appropriate pinched domain. This finishes the proof.

4.3. General position and further assumptions

In practice one looks for extending f to a bidisk Δ^2. As we have seen this depends first of all on whether a test sequence/family is in general position, which can be expressed in several different ways. One of them was given in the introduction. Another one was given in [D] and used also in [Sk]. It sounds as follows: a family (or, a sequence) $\{C_t\}_t$ is said to be in general position if for any distinct t_1, t_2 and t_3 one has

$$C_{t_1}\cap C_{t_2}\cap C_{t_3}=\varnothing, \tag{46}$$

i.e., if no three of our curves pass through one point. When C_t ad hoc belong to a finite-dimensional analytic family this notion is equivalent to ours, simply because the set of α such that f extends along C_α is an analytic set and a fortiori forms a pinched domain, to which all but a finite number of C_t should belong. In general these notions seem to be different. However let us note that for an uncountable family condition (46) implies ours. Indeed, given $\lambda_0 \in \Delta$ if for every bidisk $\Delta^2((\lambda_0,0),1/n)$ the set of $t \in T$ such that $C_t \cap \Delta^2((\lambda_0,0),1/n) = \varnothing$ is at most countable then T would be at most countable, unless almost all C_t pass through $(\lambda_0,0)$. Since this is forbidden by (46) we see that there exists an uncountable $T' \subset T$ such that $C_t \cap \Delta^2((\lambda_0,0),1/n) = \varnothing$ for $t \in T'$. Taking a convergent sequence from T' we have that the zeroes of this sequence do not accumulate to λ_0. Applying Theorem 1.1 we extend f to a pinched domain which has no pinch at λ_0. Repeating this argument a finite number of times we extend f to a neighborhood of $\Delta \times \{0\}$.

One can try to define the general position condition so that it ensures the "non-pinching". Again if ϕ_k a priori belong to a finite-dimensional family this condition will be equivalent to both conditions above. Indeed, after all we know that the set of ϕ's such that f extends along its graph is an analytic set in a finite-dimensional parameter space. Therefore all ϕ_k except *finitely many* fit into a positive-dimensional families, i.e., all (except finitely many) pass (or not) through some fixed number of points. When ϕ_k do not belong to a finite-dimensional family (but is a test sequence) the situation is unclear. It may happen that the Banach analytic family $\{\phi_\alpha\}_{\alpha \in \mathcal{A}}$ of those ϕ_α along which f extend does not contain any ϕ_k, and therefore it is not clear how to "read off" the "non-pinching" property of the family $\{\phi_\alpha\}_{\alpha \in \mathcal{A}}$ from the behavior of ϕ_k.

For the last point suppose now that our sequence/family is in general position, as in the introduction, and therefore f extends to a neighborhood of $\Delta \times \{0\}$ (or to a neighborhood of the graph C_{ϕ_0}, but this is the same). The extendability of f further to the whole of Δ^2 depends now on the image space X. More precisely it depends on whether a Hartogs-type extension theorem is valid for meromorphic mappings with values in this particular X. If X is projective or, more generally Kähler, then this is true and was proved in [I2]. For more general X this is not always the case, see [I4] for examples and further statements on this subject.

5. Examples

5.1. Construction of the example

Example 5.1. Let the function f be defined by the following series

$$\sum_{n=1}^{\infty} 3^{-4n^3} \prod_{j=1}^{n} \left[z - \left(\frac{2}{3}\lambda \right)^j \right] \lambda^{-n^2} z^n. \tag{47}$$

Then f is holomorphic in the ring domain $R := \mathbb{C}^* \times \mathbb{C}$, holomorphically extends along every $C_k := \left\{ z : z = \left(\frac{2}{3}\lambda \right)^k \right\}$, but there does not exist an analytic family $\{\phi_\alpha\}_{\alpha \in \mathcal{A}}$

parameterized by a disk $\mathcal{A}\ni 0$, with $\phi_0\equiv 0$, such that $f|_{C_\alpha\cap(\mathbb{C}^*\times\mathbb{C})}$ meromorphically extends to C_α for all $\alpha\in\mathcal{A}$.

First of all the terms of this series are holomorphic and converge normally to a holomorphic function in the ring domain $R=\mathbb{C}^*\times\mathbb{C}$. Indeed, fix any $0<\varepsilon<\frac{1}{3}$, then for $\varepsilon<|\lambda|<1/\varepsilon$ and $|z|<\frac{1}{3}\varepsilon$ one has

$$\prod_{j=1}^{n}\left|z-\left(\frac{2}{3}\lambda\right)^{j}\right|\leq\prod_{j=1}^{n}\left(\frac{1}{\varepsilon}\right)^{j}=\left(\frac{1}{\varepsilon}\right)^{n(n+1)/2},$$

and therefore

$$\sum_{n=1}^{\infty}3^{-4n^3}\prod_{j=1}^{n}\left|z-\left(\frac{2}{3}\lambda\right)^{j}\right|\frac{|z|^n}{|\lambda|^{n^2}}\leq\sum_{n=1}^{\infty}3^{-4n^3-n}\left(\frac{1}{\varepsilon}\right)^{n(n+1)/2}\left(\frac{1}{\varepsilon}\right)^{n^2+n}$$
$$\leq\sum_{n=1}^{\infty}3^{-4n^3-n}\left(\frac{1}{\varepsilon}\right)^{3(n^2+n)/2},$$

i.e., the series (47) normally converges on compact sets in R to a holomorphic function, which will still be denoted as $f(\lambda,z)$. Note that for

$$z=\phi_l(\lambda)=\left(\tfrac{2}{3}\right)^{l}\lambda^{l},\quad l\geq 2,\tag{48}$$

the sum in (47) is finite and is equal to

$$\sum_{n=1}^{l-1}3^{-4n^3}\prod_{j=1}^{n}\left[z-\left(\frac{2}{3}\lambda\right)^{j}\right]\frac{z^n}{\lambda^{n^2}}=\sum_{n=1}^{l-1}3^{-4n^3}\prod_{j=1}^{n}\left[\left(\frac{2}{3}\lambda\right)^{l}-\left(\frac{2}{3}\lambda\right)^{j}\right]\left(\frac{2}{3}\right)^{nl}\lambda^{n(l-n)},$$

with all terms being polynomials, because $l>n$ there.

Proposition 5.2. *There does not exist a complex one-parameter analytic family $\{\phi_\alpha\}_{\alpha\in\Delta}$ of holomorphic functions in Δ_2 with values in $\bar{\Delta}$ with $\phi_0\equiv 0$ and such that for every $\alpha\in\Delta$ the restriction $f(\lambda,\phi_\alpha(\lambda))$ extends from Δ_2^* to a meromorphic function in Δ_2.*

Proof. Suppose such a family exists and let $\mathcal{P}$ be a corresponding pinched domain. All pinches of $\mathcal{P}$ except at zero can be removed using graphs C_k and Theorem 1.1. For this it is sufficient to observe that on a small disk Δ_δ around such a pinch ϕ_k never vanishes and therefore our sequence is a test sequence on Δ_δ. After that by Proposition 2.3 one can take as our one-parameter family

$$\phi_\alpha(\lambda)=\alpha\lambda^{n_0-1}\tag{49}$$

with some $n_0 \geq 1$. From (49) we see that for λ close to zero the image of $\phi_\alpha(\lambda)$ as a function of α will contain a disk of radius $\sim c|\lambda|^{n_0}$. Therefore for every $\lambda \in \mathbb{R}^+$ close to zero there exists $\alpha \in \Delta_{1/2}$ such that $\phi_\alpha(\lambda) \in \mathbb{R}^+$ and $\phi_\alpha(\lambda) \geq c\lambda^{n_0}$ for some constant $c>0$.

Take some $n_1 > n_0$ such that $\left(\frac{2}{3}\right)^{n_1} < c/2$. First of all represent our function as

$$f(\lambda, z) = f_1(\lambda, z) + \prod_{j=1}^{n_1} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] f_2(\lambda, z), \tag{50}$$

where

$$f_1(\lambda, z) = \sum_{n=1}^{n_1} 3^{-4n^3} \prod_{j=1}^{n} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] \lambda^{-n^2} z^n$$

and

$$f_2(\lambda, z) = \prod_{j=1}^{n_1} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] \sum_{n=n_1+1}^{\infty} 3^{-4n^3} \prod_{j=n_1+1}^{n} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] \lambda^{-n^2} z^n.$$

Since f_1 is a rational function its restriction $f_1(\lambda, \phi_\alpha(\lambda))$ will be meromorphic in Δ_2. Therefore if $f(\lambda, \phi_\alpha(\lambda))$ would be meromorphic in Δ_2 we would conclude that $f_2(\lambda, \phi_\alpha(\lambda))$ is meromorphic in Δ_2 to, unless

$$\prod_{j=1}^{n_1} \left[\phi_\alpha(\lambda) - \left(\frac{2}{3}\lambda\right)^j \right]$$

is identically zero. The latter is possible only if ϕ_α is one of ϕ_l in (48). This is not the case and actually by Proposition 3.8 any complex one-parameter family cannot contain a converging sequence with infinitely growing winding numbers. Therefore we have that ϕ_α is not one of ϕ_l for all non-zero α small enough. Hence $f_2(\lambda, \phi_\alpha(\lambda))$ should be meromorphic in Δ_2 with pole only at zero if we suppose that $f(\lambda, \phi_\alpha(\lambda))$ is such a function. This implies that $f_2(\lambda, \phi_\alpha(\lambda))$ should be meromorphic in $\Delta_2 \times \Delta$ as a function of the two variables (λ, α). But the series

$$f_2(\lambda, \phi_\alpha(\lambda)) = \sum_{n=n_1+1}^{\infty} 3^{-4n^3} \prod_{j=n_1+1}^{n} \left[\phi_\alpha(\lambda) - \left(\frac{2}{3}\lambda\right)^j \right] \lambda^{-n^2} \phi_\alpha(\lambda)^n \tag{51}$$

representing $f_2(\lambda, \phi_\alpha(\lambda))$ at the point $(\lambda, \phi_\alpha(\lambda)) \in \mathbb{R}^+ \times \mathbb{R}^+$ can be estimated as follows. Since

$$\prod_{j=n_1+1}^{n} \left[\phi_\alpha(\lambda) - \left(\frac{2}{3}\lambda\right)^j \right] \geq \lambda^{n_0(n-n_1)} \prod_{j=n_1+1}^{n} \left[c - \frac{c}{2}\lambda^{j-n_0} \right] \geq \lambda^{n_0(n-n_1)} \left(\frac{c}{2}\right)^{n-n_1},$$

we get that

$$\sum_{n=n_1+1}^{\infty} 3^{-4n^3} \prod_{j=n_1+1}^{n} \left[\phi_\alpha(\lambda) - \left(\frac{2}{3}\lambda\right)^j\right] \lambda^{-n^2} \phi_\alpha(\lambda)^n$$
$$\geq \sum_{n=n_1+1}^{\infty} 3^{-4n^3} \lambda^{n_0(n-n_1)-n^2} \left(\frac{c}{2}\right)^n c^n \lambda^{n_0 n}$$
$$= \sum_{n=n_1+1}^{\infty} 2^{-n} 3^{-4n^3} \lambda^{n_0(2n-n_1)-n^2} c^{2n}. \tag{52}$$

The right-hand side in (52) grows faster than any polynomial of $1/\lambda$ as $\lambda \to 0$, $\lambda \in \mathbb{R}^+$. Therefore $f_2(\lambda, \phi_\alpha(\lambda))$ has an essential singularity at $\{(\lambda, \alpha) : \lambda = 0\}$, which is a contradiction. □

5.2. One more example

The following example can be found in [Si1], p. 16.

Example 5.3. Let $\{z_k\}_{k=0}^{\infty}$ be a sequence converging to zero, $z_k \neq 0$. Let $P_l(z)$ be a polynomial of degree $l+1$ such that $P_l(z_0) = \dots = P_l(z_l) = 0$ and $P_l(0) \neq 0$ with $\|P_l\|_{L^\infty(\Delta)} = 1/l!$. Set

$$f(\lambda, z) = \sum_{l=1}^{\infty} P_l(z) \lambda^{-l}. \tag{53}$$

The function f is holomorphic in $\mathbb{C}^* \times \mathbb{C}$ and $\{0\} \times \mathbb{C}$ is its essential singularity. For every z_k the restriction $f|_{C_k} := f(\cdot, z_k)$, where $C_k := \Delta \times \{z_k\}$, is rational, having a pole of order k at zero. Moreover the disks C_k form a test sequence and are in general position. Therefore the conclusions of Theorem 1.1 and Corollary 1.3 fail when the orders of the poles of the restrictions are not uniformly bounded.

References

[B] Barlet, D., Espace analytique reduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie, in *Fonctions de Plusieurs Variables Complexes II, Séminaire François Norguet, Janvier 1974–Juin 1975*, Lecture Notes in Math. **482**, pp. 1–157, Springer, Berlin–Heidelberg, 1975.

[D] Dinh, T.-C., Problème du bord dans l'espace projectif complexe, *Ann. Inst. Fourier* (*Grenoble*) **48** (1998), 1483–1512.

[H] Hartogs, F., Zur Theorie der analytischen Funktionen mehrerer unabhängiger Verändenlichen insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten, *Math. Ann.* **62** (1906), 1–88.

[I1] Ivashkovich, S., The Hartogs phenomenon for holomorphically convex Kähler manifolds, *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), 866–873, 879 (Russian). English transl.: *Math. USSR-Izv.* **29** (1986), 225–232.

[I2] Ivashkovich, S., The Hartogs-type extension theorem for meromorphic mappings into compact Kähler manifolds, *Invent. Math.* **109** (1992), 47–54.

[I3] Ivashkovich, S., An example concerning separate analyticity and extension properties of meromorphic mappings, *Amer. J. Math.* **121** (1999), 97–130.

[I4] Ivashkovich, S., Extension properties of meromorphic mappings with values in non-Kähler complex manifolds, *Ann. of Math.* **160** (2004), 795–837.

[L] Levi, E., Studii sui punti singolari essenziali delle funzioni analitiche di due o più variabili complesse, *Ann. Mat. Pura Appl.* **17** (1910), 61–87.

[Mz] Mazet, P., *Analytic Sets in Locally Convex Spaces*, North-Holland, Amsterdam, 1984.

[MP] Mikhlin, S. and Prössdorf, S., *Singular Integral Operators*, Springer, Berlin, 1986.

[Sk] Sarkis, F., Problème de Plateau complexe dans les variétés kählériennes, *Bull. Soc. Math. France* **130** (2002), 169–209.

[Si1] Siu, Y.-T., *Extension of Analytic Objects*, Dekker, New York, 1974.

[Si2] Siu, Y.-T., Every Stein subvariety admits a Stein neighborhood, *Invent. Math.* **38** (1976), 89–100.

Sergey Ivashkovich
UFR de Mathématiques
Université de Lille-1
FR-59655 Villeneuve d'Ascq
France
ivachkov@math.univ-lille1.fr
and
IAPMM Nat. Acad. Sci. Ukraine
Naukova 3b
79601 Lviv
Ukraine

Received April 6, 2012
in revised form November 28, 2012
published online March 29, 2013