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Extrapolation from $A_\infty^{\rho,\infty}$, vector-valued inequalities and applications in the Schrödinger settings

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Abstract. In this paper, we generalize the A_∞ extrapolation theorem (Cruz-Uribe–Martell–Pérez, Extrapolation from A_∞ weights and applications, *J. Funct. Anal.* **213** (2004), 412–439) and the A_p extrapolation theorem of Rubio de Francia to Schrödinger settings. In addition, we also establish weighted vector-valued inequalities for Schrödinger-type maximal operators by using weights belonging to $A_p^{\rho,\infty}$ which includes A_p. As applications, we establish weighted vector-valued inequalities for some Schrödinger-type operators.

1. Introduction

In this paper, we consider the Schrödinger differential operator

$$L=-\Delta+V(x) \quad \text{on } \mathbb{R}^n,\ n\geq 3,$$

where $V(x)$ is a nonnegative potential satisfying a certain reverse Hölder inequality.

A nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q for $1<q\leq\infty$ if there exists $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)\,dy\right)^{1/q}\leq C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)\,dy\right)$$

holds for every $x\in\mathbb{R}^n$ and $0<r<\infty$, where $B(x,r)$ denotes the ball centered at x with radius r. In particular, if V is a nonnegative polynomial, then $V\in B_\infty$. Throughout this paper, we always assume that $0\not\equiv V\in B_{n/2}$.

The study of the Schrödinger operator $L=-\Delta+V$ has recently attracted much attention; see [3], [4], [12], [11], [16], [23], [28] and [29]. In particular, it should

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be pointed out that Shen [23] proved that Schrödinger-type operators, such as $\nabla(-\Delta+V)^{-1}\nabla$, $\nabla(-\Delta+V)^{-1/2}$, $(-\Delta+V)^{-1/2}\nabla$ with $V\in B_n$, and $(-\Delta+V)^{i\gamma}$ with $\gamma\in\mathbb{R}$ and $V\in B_{n/2}$, are standard Calderón–Zygmund operators.

Recently, Bongioanni–Harboure–Salinas [3] proved $L^p(\mathbb{R}^n)$, $1<p<\infty$, boundedness for commutators of Riesz transforms associated with Schrödinger operator with $\mathrm{BMO}_\infty(\rho)$ functions, which include the BMO functions, and they [4] established the weighted boundedness for Riesz transforms, fractional integrals and Littlewood–Paley functions associated with Schrödinger operators with weights in the $A_p^{\rho,\infty}$ class, which includes the Muckenhoupt weights. Very recently, the author ([25] and [26]) established weighted norm inequalities for some Schrödinger-type operators, which include commutators of Riesz transforms, fractional integrals and Littlewood–Paley functions associated with Schrödinger operators.

On the other hand, extrapolation of weights plays an important role in harmonic analysis. In particular, Rubio de Francia [22] proved the A_p extrapolation theorem: If the operator T is bounded on $L^{p_0}(\omega)$ for some p_0, $1<p_0<\infty$, and every $\omega\in A_{p_0}$, then for every p, $1<p<\infty$, T is bounded on $L^p(\omega)$, $\omega\in A_p$ (see also [9] and [14]). Recently, Cruz-Uribe–Martell–Pérez in [5] extended this theorem from A_p weights to A_∞ weights, to pairs of operators, and to the range $0<p<\infty$ in the context of Muckenhoupt bases; see also [6], [7], [8], [10], [17] and [18].

In this paper, we generalize the A_∞ extrapolation theorem in [5] and the A_p extrapolation theorem of Rubio de Francia to Schrödinger settings and give some applications.

The paper is organized as follows. In Section 2, we give factorization of $A_p^{\rho,\infty}$, and establish weighted vector-valued inequalities for Schrödinger-type maximal operators, these results play a crucial role in this paper. In Section 3, we obtain extrapolation theorems from $A_\infty^{\rho,\infty}$ and $A_p^{\rho,\infty}$. Finally, we establish weighted vector-valued inequalities for some Schrödinger-type operators in Section 4.

Throughout this paper, we let C denote constants that are independent of the main parameters involved but whose value may differ from line to line. By $A\sim B$, we mean that there exists a constant $C>1$ such that $1/C\le A/B\le C$.

2. Factorization and vector-valued inequalities

In this section, we give the factorization of $A_p^{\rho,\infty}$ and weighted vector-valued inequalities for Schrödinger-type maximal operators.

We first recall some notation. Given $B=B(x,r)$ and $\lambda>0$, we will write λB for the λ-dilate ball, which is the ball with the same center x and with radius λr. Similarly, $Q(x,r)$ denotes the cube centered at x with the sidelength r (here and below only cubes with sides parallel to the coordinate axes are considered), and

$\lambda Q(x,r)=Q(x,\lambda r)$. Let $f=\{f_k\}_{k=1}^\infty$ be a sequence of locally integrable functions on $\mathbb{R}^n$, $|f(x)|_r=(\sum_{k=1}^\infty |f_k(x)|^r)^{1/r}$, and $|Tf(x)|_r=(\sum_{k=1}^\infty |Tf_k(x)|^r)^{1/r}$.

The function $m_V(x)$ is defined by

$$\rho(x)=\frac{1}{m_V(x)}=\sup_{r>0}\biggl\{r:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)\,dy\le 1\biggr\}.$$

Obviously, $0<m_V(x)<\infty$ if $V\neq 0$. In particular, $m_V(x)=1$ if $V=1$, and $m_V(x)\sim(1+|x|)$ if $V=|x|^2$.

Lemma 2.1. ([23]) *There exists $l_0>0$ and $C_0>1$ such that*

$$\frac{1}{C_0}(1+|x-y|m_V(x))^{-l_0}\le\frac{m_V(x)}{m_V(y)}\le C_0(1+|x-y|m_V(x))^{l_0/(l_0+1)}.$$

In particular, $m_V(x)\sim m_V(y)$ if $|x-y|<C/m_V(x)$.

In this paper, we write $\Psi_\theta(B)=(1+r/\rho(x_0))^\theta$, where $\theta>0$, and x_0 and r denotes the center and radius of B respectively.

A weight will always mean a nonnegative function which is locally integrable. As in [4], we say that a weight ω belongs to the class $A_p^{\rho,\theta}$ for $1<p<\infty$, if there is a constant C such that for all balls B

$$\biggl(\frac{1}{\Psi_\theta(B)|B|}\int_B\omega(y)\,dy\biggr)\biggl(\frac{1}{\Psi_\theta(B)|B|}\int_B\omega^{-1/(p-1)}(y)\,dy\biggr)^{p-1}\le C.$$

We also say that a nonnegative function ω satisfies the $A_1^{\rho,\theta}$ condition if there exists a constant C such that

$$M_{V,\theta}(\omega)(x)\le C\omega(x)\quad\text{for a.e. } x\in\mathbb{R}^n,$$

where

$$M_{V,\theta}f(x)=\sup_{x\in B}\frac{1}{\Psi_\theta(B)|B|}\int_B|f(y)|\,dy.$$

When $V=0$, we denote $M_0f(x)$ by $Mf(x)$ (the standard Hardy–Littlewood maximal function). It is easy to see that $|f(x)|\le M_{V,\theta}f(x)\le Mf(x)$ for a.e. $x\in\mathbb{R}^n$ and any $\theta\ge 0$.

Since $\Psi_\theta(B)\ge 1$ if $\theta\ge 0$, we then have $A_p\subset A_p^{\rho,\theta}$ for $1\le p<\infty$, where A_p denotes the classical Muckenhoupt weights; see [15] and [20]. We will see that $A_p\Subset A_p^{\rho,\theta}$ for $1\le p<\infty$ in some cases. In fact, letting $\theta>0$ and $0\le\gamma\le\theta$, it is easy to check that $\omega(x)=(1+|x|)^{-(n+\gamma)}\notin A_\infty$ and $\omega(x)\,dx$ is not a doubling measure, but $\omega(x)=(1+|x|)^{-(n+\gamma)}\in A_1^{\rho,\theta}$ provided that $V=1$ and $\Psi_\theta(B(x_0,r))=(1+r)^\theta$.

We remark that balls can be replaced by cubes in the definitions of $A_p^{\rho,\theta}$ and $M_{V,\theta}$, since $\Psi_\theta(B)\le\Psi_\theta(2B)\le 2^n\theta\Psi_\theta(B)$.

Next we give the weighted boundedness for $M_{V,\theta}$.

Lemma 2.2. ([27]) *Let* $1<p<\infty$, $p'=p/(p-1)$ *and assume that* $\omega\in A_p^{\rho,\theta}$. *There exists a constant* $C>0$ *such that*

$$\|M_{V,p'\theta}f\|_{L^p(\omega)}\le C\|f\|_{L^p(\omega)}.$$

Similar to the classical Muckenhoupt weights (see [15], [19] and [24]), we give some properties for the weight class $A_p^{\rho,\theta}$ for $p\ge1$.

Proposition 2.3. *Let* $A_p^{\rho,\infty}:=\bigcup_{\theta\ge0}A_p^{\rho,\theta}$ *for* $p\ge1$. *Then the following are true*:

(i) *If* $1\le p_1<p_2<\infty$, *then* $A_{p_1}^{\rho,\theta}\subset A_{p_2}^{\rho,\theta}$;
(ii) $\omega\in A_p^{\rho,\theta}$ *if and only if* $\omega^{-1/(p-1)}\in A_{p'}^{\rho,\theta}$, *where* $1/p+1/p'=1$;
(iii) *If* $\omega\in A_p^{\rho,\infty}$, $1<p<\infty$, *then there exists* $\varepsilon>0$ *such that* $\omega\in A_{p-\varepsilon}^{\rho,\infty}$;
(iv) *Let* $f\in L_{\rm loc}(\mathbb{R}^n)$, $0<\delta<1$, *then* $(M_{V,\theta}f)^\delta\in A_1^{\rho,\theta}$;
(v) *Let* $1<p<\infty$, *then* $\omega\in A_p^{\rho,\infty}$ *if and only if* $\omega=\omega_1\omega_2^{1-p}$, *where* $\omega_1,\omega_2\in A_1^{\rho,\infty}$.

Proof. (i) and (ii) are obvious by the definition of $A_p^{\rho,\theta}$. (iii) is proved in [4]. In fact, from Lemma 5 in [4], we know that if $\omega\in A_p^{\rho,\theta}$, then $\omega\in A_{p_0}^{\rho,\theta_0}$, where $p_0=1+(p-1)/(1+\delta)<p$ with $\delta>0$ (δ is a constant depending only on the $A_p^{\rho,{\rm loc}}$ constant of ω, see [4]) and

$$\theta_0=\frac{\theta p+\eta(p-1)}{p_0}\quad\text{with }\eta=\theta p+(\theta+n)\frac{pl_0}{l_0+1}+(l_0+1)\frac{n\delta}{1+\delta}.$$

We now prove (iv). It will suffice to show that there exists a constant C such that for every f, every cube Q and almost every $x\in Q$,

$$\frac{1}{\Psi_\theta(Q)|Q|}\int_Q M_{V,\theta}f(y)^\delta\,dy\le CM_{V,\theta}f(x)^\delta.$$

Fix Q and decompose f as $f=f_1+f_2$, where $f_1=f\chi_{2Q}$ and $f_2=f-f_1$. Then $M_{V,\theta}f(x)\le M_{V,\theta}f_1(x)+M_{V,\theta}f_2(x)$, and so for $0\le\delta<1$,

$$M_{V,\theta}f(x)^\delta\le M_{V,\theta}f_1(x)^\delta+M_{V,\theta}f_2(x)^\delta.$$

Since $M_{V,\theta}$ is weak-$(1,1)$, by Kolmogorov's inequality (see [21])

$$\begin{aligned}\frac{1}{\Psi_\theta(Q)|Q|}\int_Q(M_{V,\theta}f_1)^\delta(y)\,dy&\le\frac{C}{\Psi_\theta(Q)|Q|}|Q|^{1-\delta}\|f_1\|_1^\delta\\&\le C\bigg(\frac{1}{\Psi_\theta(Q)|Q|}\int_{2Q}|f(y)|\,dy\bigg)^\delta\end{aligned}$$

$$\leq C\left(\frac{1}{\Psi_\theta(2Q)|2Q|}\int_{2Q}|f(y)|\,dy\right)^\delta$$
$$\leq CM_{V,\theta}f(x)^\delta.$$

To estimate $M_{V,\theta}f_2$, note that letting Q' be a cube such that $x\in Q'$, we have that if $Q'\cap(\mathbb{R}^n\setminus 2Q)\neq\varnothing$, then $Q\subset 4nQ'$. Hence, for any $z\in Q$,

$$\frac{1}{\Psi_\theta(Q')|Q'|}\int_{Q'}|f_2(y)|\,dy\leq\frac{C}{\Psi_\theta(4nQ')|4nQ'|}\int_{4nQ'}|f_2(y)|\,dy\leq CM_{V,\theta}(z).$$

So $M_{V,\theta}(y)\leq CM_{V,\theta}(x)$ for any $y\in Q$. Thus

$$\frac{1}{\Psi_\theta(Q')|Q'|}\int_{Q'}M_{V,\theta}f_2(y)^\delta\,dy\leq CM_{V,\theta}f(x)^\delta.$$

It remains to prove (v). We first assume that $\omega_1\in A_1^{\rho,\theta_1}$ and $\omega_2\in A_1^{\rho,\theta_2}$. Since

$$\left(\frac{1}{\Psi_{\theta_1}(Q)|Q|}\int_Q\omega_1(y)\,dy\right)\left(\inf_Q\omega_1(y)\right)^{-1}\leq C_1,$$
$$\left(\frac{1}{\Psi_{\theta_2}(Q)|Q|}\int_Q\omega_2(y)\,dy\right)\left(\inf_Q\omega_2(y)\right)^{-1}\leq C_2,$$

moreover

$$\frac{1}{\Psi_\theta(Q)|Q|}\int_Q\omega(y)\,dy=\frac{1}{\Psi_\theta(Q)|Q|}\int_Q\omega_1(y)\omega_2^{1-p}(y)\,dy$$
$$\leq\left(\frac{1}{\Psi_\theta(Q)|Q|}\int_Q\omega_1(y)\,dy\right)\left(\inf_Q\omega_2(y)\right)^{1-p},$$

$$\left(\frac{1}{\Psi_\theta(Q)|Q|}\int_Q\omega^{-1/(p-1)}(y)\,dy\right)^{p-1}=\left(\frac{1}{\Psi_\theta(Q)|Q|}\int_Q\omega_1^{-1/(p-1)}(y)\omega_2(y)\,dy\right)^{p-1}$$
$$\leq\left(\frac{1}{\Psi_\theta(Q)|Q|}\int_Q\omega_2(y)\,dy\right)^{p-1}\left(\inf_Q\omega_1(y)\right)^{-1}.$$

From these inequalities and choosing $\theta=\max\{\theta_1,\theta_2\}$, we get that

$$\left(\frac{1}{\Psi_\theta(Q)|Q|}\int_Q\omega(y)\,dy\right)\left(\frac{1}{\Psi_\theta(Q)|Q|}\int_Q\omega^{-1/(p-1)}(y)\,dy\right)^{p-1}\leq C_1C_2^{p-1}.$$

To prove the converse, we consider first $p\geq 2$, let $\omega\in A_p^{\rho,\theta}$, and define T by

$$Tf=[\omega^{-1/p}M_{V,p\theta}(f^{p/p'}\omega^{1/p})]^{p'/p}+\omega^{1/p}M_{V,p\theta}(f\omega^{-1/p}).$$

Because $\omega^{-p'/p}\in A_{p'}^{\rho,\theta}$, then T is bounded on L^p by Lemma 2.2, that is,

$$\|Tf\|_{L^p}\le A\|f\|_{L^p},$$

for some $A>0$. Also, since $p\ge 2$, we have $p/p'\ge 1$, and Minkowski's inequality gives $T(f_1+f_2)\le Tf_1+Tf_2$. Fix now a nonnegative f with $\|f\|_{L^p}=1$ and write

$$\eta=\sum_{k=1}^{\infty}(2A)^{-k}T^k(f),$$

where $T^k(f)=T(T^{k-1}(f))$. Then $\|\eta\|_{L^p}\le 1$. Furthermore, since T is positivity-preserving and subadditive, we have the pointwise inequality

$$T\eta\le\sum_{k=1}^{\infty}(2A)^{-k}T^{k+1}(f)=\sum_{k=2}^{\infty}(2A)^{1-k}T^k(f)\le 2A\eta.$$

Thus, if $\omega_1=\omega^{1/p}\eta^{p/p'}$, then

$$M_{V,p\theta}(\omega_1)\le T(\eta)^{p/p'}\omega^{1/p}\le(2A\eta)^{p/p'}\omega^{1/p}=(2A)^{p/p'}\omega_1$$

and $\omega\in A_1^{\rho,p\theta}$. Similarly, if $\omega_2=\omega^{-1/p}\eta$, then $M_{V,p\theta}(\omega_1)\le 2A\omega_2$, so $\omega_2\in A_1^{\rho,p\theta}$. Moreover,

$$\omega=\omega_1\omega_2^{1-p}=\omega^{1/p}\eta^{p/p'}(\omega^{-1/p}\eta)^{1-p},$$

since $p/p'=p-1$, finishing the proof for $p\ge 2$.

The case $p\le 2$ is similar. In fact, let $\omega\in A_p^{\rho,\theta}$, then $\omega^{-p'/p}\in A_{p'}^{\rho,\theta}$, and define T by

$$Tf=[\omega^{1/p}M_{V,p'\theta}(f^{p'/p}\omega^{-1/p})]^{p/p'}+\omega^{-1/p}M_{V,p'\theta}(f\omega^{1/p}).$$

Then T is bounded on L^p by Lemma 2.2, that is,

$$\|Tf\|_{L^{p'}}\le B\|f\|_{L^{p'}},$$

for some $A>0$. Also, since $p\le 2$, we have $p'/p\ge 1$, and Minkowski's inequality gives $T(f_1+f_2)\le Tf_1+Tf_2$. Fix now a nonnegative f with $\|f\|_{L^{p'}}=1$ and write

$$\eta=\sum_{k=1}^{\infty}(2B)^{-k}T^k(f),$$

where $T^k(f)=T(T^{k-1}(f))$. Then $\|\eta\|_{L^{p'}}\le 1$. Furthermore, since T is positivity-preserving and subadditive, we have the pointwise inequality

$$T\eta\le\sum_{k=1}^{\infty}(2B)^{-k}T^{k+1}(f)=\sum_{k=2}^{\infty}(2B)^{1-k}T^k(f)\le 2B\eta.$$

Thus, if $\omega_1 = \omega^{-1/p}\eta^{p'/p}$, then

$$M_{V,p\theta}(\omega_1) \leq T(\eta)^{p'/p}\omega^{-1/p} \leq (2B\eta)^{p'/p}\omega^{1/p} = (2B)^{p'/p}\omega_1$$

and $\omega \in A_1^{\rho,p'\theta}$. Similarly, if $\omega_2 = \omega^{1/p}\eta$, then $M_{V,p'\theta}(\omega_1) \leq 2B\omega_2$, so $\omega_2 \in A_1^{\rho,p'\theta}$. Moreover,

$$\omega = \omega_2\omega_1^{1-p} = \omega^{1/p}\eta(\omega^{-1/p}\eta^{p'/p})^{1-p},$$

since $p/p' = p-1$, finishing the proof for $p \leq 2$. The proof is complete. □

We remark that the referee has pointed out that in fact (v) of Proposition 2.3 can also be obtained by a direct argument in [17]. We leave this as an exercise for interested readers.

C. Fefferman and E. Stein [13] obtained vector-valued inequalities for Hardy–Littlewood maximal operators. Later, K. Andersen and R. John [1] generalized the Fefferman–Stein vector-valued inequalities to the A_p weight case. We next give some weighted vector-valued inequalities for maximal operators $M_{V,\eta}$ by using the new weights above. The following interpolation results will be used. Let $\mathcal{S}$ denote the linear space of sequences $f = \{f_k\}_{k=1}^{\infty}$ of the form: $f_k(x)$ is a simple function on $\mathbb{R}^n$ and $f_k(x) \equiv 0$ for all sufficient large k. $\mathcal{S}$ is dense in $L^p_\omega(l^r)$, $1 \leq p, r < \infty$; see [2].

Lemma 2.4. ([1]) *Let $\omega \geq 0$ be locally integrable on $\mathbb{R}^n$, $1 < r < \infty$, $1 \leq p_i \leq q_i < \infty$ and suppose T is a sublinear operator defined on $\mathcal{S}$ satisfying*

$$\omega(\{x \in \mathbb{R}^n : |Tf(x)|_r > \alpha\}) \leq \frac{M_i^{q_i}}{\alpha^{q_i}}\left(\int_{\mathbb{R}^n} |f(x)|_r^{p_i}\omega(x)\,dx\right)^{q_i/p_i}$$

for $i = 0, 1$ and $f \in \mathcal{S}$. Then T extends uniquely to a sublinear operator on $L^p_\omega(l^r)$ and there is a constant M_θ such that

$$\left(\int_{\mathbb{R}^n} |Tf(x)|_r^q\omega(x)\,dx\right)^{1/q} \leq M_\theta\left(\int_{\mathbb{R}^n} |f(x)|_r^p\omega(x)\,dx\right)^{1/p},$$

where $(1/p, 1/q) = (1-\theta)(1/p_0, 1/q_0) + \theta(1/p_1, 1/q_1)$, $0 < \theta < 1$.

Lemma 2.5. ([1]) *Let $\omega \geq 0$ be locally integrable on $\mathbb{R}^n$, $1 < r_i, s_i < \infty$, $1 \leq p_i, q_i < \infty$ and suppose T is a sublinear operator defined on $\mathcal{S}$ satisfying*

$$\left(\int_{\mathbb{R}^n} |Tf(x)|_{s_i}^{q_i}\omega(x)\,dx\right)^{1/q_i} \leq M_i\left(\int_{\mathbb{R}^n} |f(x)|_{r_i}^{p_i}\omega(x)\,dx\right)^{1/p_i}$$

for $i=0,1$ *and* $f\in\mathcal{S}$. *Then* T *extends uniquely to a sublinear operator on* $L^p_\omega(l^r)$ *such that*

$$\left(\int_{\mathbb{R}^n}|Tf(x)|_r^q\omega(x)\,dx\right)^{1/q}\le M_0^{1-\theta}M_1^\theta\left(\int_{\mathbb{R}^n}|f(x)|_r^p\omega(x)\,dx\right)^{1/p},$$

where $(1/p,1/q,1/s,1/r)=(1-\theta)(1/p_0,1/q_0,1/s_0,1/r_0)+\theta(1/p_1,1/q_1,1/s_1,1/r_1)$, $0<\theta<1$.

We define the dyadic maximal operator $M^\Delta_{V,\theta}f(x)$ by

$$M^\Delta_{V,\theta}f(x):=\sup_{x\in Q(\text{dyadic cube})}\frac{1}{\psi_\theta(Q)|Q|}\int_Q|f(x)|\,dx,$$

where $\psi_\theta(Q)=(1+r/\max_{\overline{Q}}\rho(x))^\theta$, r is the side-length of Q, $\overline{Q}$ is the closure of Q and $\theta>0$.

Lemma 2.6. *Let* f *be a locally integrable function on* $\mathbb{R}^n$, $\lambda>0$, *and* $\Omega_\lambda=\{x\in\mathbb{R}^n:M^\Delta_{V,\theta}f(x)>\lambda\}$. *Then* Ω_λ *may be written as a disjoint union of dyadic cubes* $\{Q_j\}_{j=1}^\infty$ *with*

(i) $\lambda<(\psi_\theta(Q_j)|Q_j|)^{-1}\int_{Q_j}|f(x)|\,dx$;

(ii) $(\psi_\theta(Q_j)|Q_j|)^{-1}\int_{Q_j}|f(x)|\,dx\le(4n)^\theta 2^n\lambda$;

for each cube Q_j. *This has the immediate consequences*:

(iii) $|f(x)|\le\lambda$ *for a.e.* $x\in\mathbb{R}^n\setminus\bigcup_{j=1}^\infty Q_j$;

(iv) $|\Omega_\lambda|\le\lambda^{-1}\int_{\mathbb{R}^n}|f(x)|\,dx$.

The proof follows from the same argument as of Lemma 1 on p. 150 of [24].

Theorem 2.7. *Let* $1<r<\infty$ *and* $\theta>0$.

(a) *If* $1\le p<\infty$, $\omega\in A_p^{\rho,\theta}$ *and* $\eta=p_0\theta_0$, *where* $p_0=4(l_0+1)^5(p+\frac{1}{2}(r+1)')$ *and* $\theta_0=p((3\theta+n)p+(l_0+1)n)$, *there is a constant* C_{r,p,θ,l_0,C_0} *such that*

$$\omega(\{x\in\mathbb{R}^n:|M_{V,\eta}f(x)|_r>\alpha\}|\le\frac{C}{\alpha^p}\int_{\mathbb{R}^n}|f(x)|_r^p\omega(x)\,dx. \tag{2.1}$$

(b) *If* $1<p<\infty$, $\omega\in A_p^{\rho,\theta}$ *and* η *is as above, there is a constant* C_{r,p,θ,l_0,C_0} *such that*

$$\int_{\mathbb{R}^n}|M_{V,\eta}f(x)|_r^p\omega(x)\,dx\le\frac{C}{\alpha^p}\int_{\mathbb{R}^n}|f(x)|_r^p\omega(x)\,dx. \tag{2.2}$$

Proof. Observe first that (2.2) for the case $r=p$ is an easy consequence of Lemma 2.2 since $\eta>r'\theta$ and

$$\begin{aligned}(2.3)\qquad \int_{\mathbb{R}^n}|M_{V,\eta}f(x)|_r^r\omega(x)\,dx &= \sum_{k=1}^\infty\int_{\mathbb{R}^n}|M_{V,\eta}f_k(x)|^r\omega(x)\,dx\\ &\le C\sum_{k=1}^\infty\int_{\mathbb{R}^n}|f_k(x)|^r\omega(x)\,dx\\ &= C\sum_{k=1}^\infty\int_{\mathbb{R}^n}|f_k(x)|_r^r\omega(x)\,dx.\end{aligned}$$

Now suppose $r>p$, $\omega\in A_p^{\rho,\theta}$ and $\alpha>0$. As usual, we can assume that $f\in C_0^\infty$. Let $\theta_1=\theta(l_0+1)$. From Lemma 2.6, we obtain a sequence of nonoverlapping cubes $\{Q_j\}_{j=1}^\infty$ such that

$$(2.4)\qquad |f(x)|_r\le\alpha,\quad x\notin\Omega=\bigcup_{j=1}^\infty Q_j,$$

and

$$(2.5)\qquad \alpha<\frac{1}{\psi_{\theta_1}(Q_j)|Q_j|}\int_{Q_j}|f(x)|_r\,dx\le 2^n(4n)^{\theta_1}\alpha,\quad j=1,2,\dots.$$

Let $f=f'+f''$, where $f'=\{f_k'\}_{k=1}^\infty$, $f_k'(x)=f_k(x)\chi_{\mathbb{R}^n\setminus\Omega}(x)$. Then

$$|M_{V,\eta}f(x)|_r\le|M_{V,\eta}f'(x)|_r+|M_{V,\eta}f''(x)|_r.$$

From this, (2.1) will follow if we show that

$$(2.6)\qquad \omega(\{x\in\mathbb{R}^n:|M_{V,\eta}f'(x)|_r>\alpha\})\le\frac{C}{\alpha^p}\int_{\mathbb{R}^n}|f(x)|_r^p\omega(x)\,dx$$

and

$$(2.7)\qquad \omega(\{x\in\mathbb{R}^n:|M_{V,\eta}f''(x)|_r>\alpha\})\le\frac{C}{\alpha^p}\int_{\mathbb{R}^n}|f(x)|_r^p\omega(x)\,dx.$$

Since $\omega\in A_r^{\rho,\theta}$ by (i) of Proposition 2.3, from (2.3) and (2.4), we then have

$$\begin{aligned}\omega(\{x\in\mathbb{R}^n:|M_{V,\eta}f'(x)|_r>\alpha\}) &\le\frac{C}{\alpha^r}\int_{\mathbb{R}^n}|f(x)|_r^r\omega(x)\,dx\\ &\le\frac{C}{\alpha^p}\int_{\mathbb{R}^n}|f(x)|_r^p\omega(x)\,dx.\end{aligned}$$

Thus, (2.6) is proved. To prove (2.7), define $\bar{f}=\{\bar{f}_k\}_{k=1}^{\infty}$ by

$$\bar{f}_k(x)=\frac{1}{\psi_{\theta_1}(Q_j)|Q_j|}\int_{Q_j}|f_k(y)|\,dy,\quad \text{if } x\in Q_j,\ j=1,2,...,$$

and zero, otherwise. Let $\widetilde{Q}_j=2nQ_j$. We now claim that for any $x\in\widetilde{\Omega}=\bigcup_{j=1}^{\infty}\widetilde{Q}_j$,

$$M_{V,\eta}f_k''(x)\le CM_{V,\bar{\eta}}\bar{f}_k(x)\quad \text{for all } k,$$

where $\bar{\eta}=\eta/2(l_0+1)^2$.

In fact, for all $x\notin\widetilde{\Omega}$, and any cube $Q\ni x$, if $Q_j\cap Q\neq\varnothing$, then $Q_j\subset\widetilde{Q}=4nQ$, and hence

$$\begin{aligned}\frac{1}{\Psi_\eta(Q)|Q|}\int_Q|f_k''(x)|\,dx&=\frac{1}{\Psi_\eta(Q)|Q|}\sum_{j=1}^{\infty}\int_{Q_j\cap Q}|f_k(x)|\,dx\\&\le\frac{1}{\Psi_\eta(Q)|Q|}\sum_{Q_j\subset\widetilde{Q}}\int_{Q_j}|f_k(x)|\,dx\\&\le\frac{1}{\Psi_\eta(Q)|Q|}\sum_{Q_j\subset\widetilde{Q}}\psi_{\theta_1}(Q_j)\int_{Q_j}\bar{f}_k(x)\,dx\\&\le C\frac{\Psi_{\theta_2}(\widetilde{Q})}{\Psi_\eta(Q)|Q|}\int_{\widetilde{Q}}\bar{f}_k(x)\,dx\\&\le CM_{V,\bar{\eta}}\bar{f}_k(x),\end{aligned}$$

where $\theta_2=\theta_1(l_0+1)=\theta(l_0+1)^2$.

By the claim above, it is easy to see that (3.8) will follow if we show that

$$\omega(\widetilde{\Omega})\le\frac{C}{\alpha^p}\int_{\mathbb{R}^n}|f(x)|_r^p\omega(x)\,dx \tag{2.8}$$

and that

$$\omega(\{x\in\mathbb{R}^n:|M_{V,\bar{\eta}}\bar{f}(x)|_r>\alpha\})\le\frac{C}{\alpha^p}\int_{\mathbb{R}^n}|f(x)|_r^p\omega(x)\,dx. \tag{2.9}$$

If $p>1$, by (2.5), we then have

$$\begin{aligned}\omega(\widetilde{Q}_j)&=\int_{\widetilde{Q}_j}\omega(x)\,dx\\&\le\frac{1}{\alpha^p(\psi_{\theta_1}(Q)|Q|)^p}\left(\int_{Q_j}|f(x)|_r\,dx\right)^p\int_{\widetilde{Q}_j}\omega(x)\,dx\end{aligned} \tag{2.10}$$

$$\leq \frac{1}{\alpha^p}\left(\int_{Q_j}|f(x)|_r^p\omega(x)\,dx\right)\left(\frac{1}{(\Psi_\theta(Q)|Q|)}\int_{Q_j}\omega^{-1/(p-1)}(x)\,dx\right)^{p-1}$$
$$\times\left(\frac{1}{(\Psi_\theta(Q)|Q|)}\int_{\tilde{Q}_j}\omega(x)\,dx\right)$$
$$\leq \frac{1}{\alpha^p}\int_{Q_j}|f(x)|_r^p\omega(x)\,dx,$$

since $\omega\in A_p^{\rho,\theta}$.

A similar argument shows that (2.10) holds also if $p=1$. Hence, (2.8) follows from (2.10) upon summing over j. Note that $|\bar f(x)|_r\leq 2^n(4n)^{\theta_1}\alpha$, and since $|\bar f(x)|_r$ is supported in Ω, using Lemma 2.2, we obtain

$$\omega(\{x\in\mathbb{R}^n:|M_{V,\bar\eta}\bar f(x)|_r>\alpha\})\leq C\alpha^{-r}\int_{\mathbb{R}^n}|\bar f(x)|_r^r\omega(x)\,dx\leq C\int_\Omega\omega(x)\,dx$$

which together with (2.10) yields (2.9) as required. This complete the proof of (2.1) in the case $r\geq p$. If $r>p>1$, by (iii) of Proposition 2.3, we know that for $\omega\in A_p^{\rho,\theta}$, there exist constants p_1, p_2 and θ_3 (depending only on ω) such that $(r+1)/2<p_1<p<p_2<r$ and $\theta_3\leq\theta_0$ so that (2.1) holds with $\omega\in A_{p_1}^{\theta_3}$ and $\omega\in A_{p_2}^{\theta}$ respectively. Obviously, $\bar\eta>2p_1'\theta_3$, and so Lemmas 2.2 and 2.4 yields (2.2) for $r>p>1$.

Suppose now that $p>r$ and $\omega\in A_p^{\rho,\theta}$. By (iii) of Proposition 2.3, there exist constants $\theta_4\leq\theta_0$ and $1<r_0<p$ such that $\omega\in A_q^{\rho,\theta_4}, q\geq p/r_0$. In particular, (i) of Proposition 2.3 yields $\omega(x)>0$ a.e. and $\omega(x)^{1-q'}\in A_{q'}^{\rho,\theta_4}$, so that by Lemma 2.2, for any nonnegative function $\|\varphi\|_{L_\omega^{q'}}\leq 1$, we then have

$$\int_{\mathbb{R}^n}|M_{V,\eta_1}(\varphi\omega)(x)|^{q'}\omega(x)^{1-q'}\,dx\leq C_q\int_{\mathbb{R}^n}|\varphi(x)|^{q'}\omega(x)\,dx=C_q,$$

where $\eta_1=\bar\eta/(l_0+1)^3>q\theta_4$, and hence

$$\int_{\mathbb{R}^n}|M_{V,\bar\eta}f(x)|_r^r\varphi(x)\omega(x)\,dx\leq C\int_{\mathbb{R}^n}|f(x)|_r^r\frac{M_{V,\eta_1}(\varphi\omega)(x)}{\omega^{1/q}(x)}\omega^{1/q}(x)\,dx \tag{2.11}$$
$$\leq C\left(\int_{\mathbb{R}^n}|f(x)|_r^{rq}\omega(x)\,dx\right)^{1/q}.$$

In the first inequality of (2.11), we used the fact that for any nonnegative measurable functions f and g, and $q>1$, we have

$$\int_{\mathbb{R}^n}(M_{V,\bar\eta}f)^q g\,dx\leq C\int_{\mathbb{R}^n}f^q(M_{V,\eta_1}g)\,dx. \tag{2.12}$$

Taking the supremum in (2.11) over such φ then yields (2.2) for $1<r\le r_0$ upon taking $q=p/r$, and this together with the case $p=r$ provided in (2.3) yields (3.3) for $r_0<r<p$ by an application of Lemma 2.4. Thus, the proof of (a) and (b) is complete.

It remains to prove (2.12), let $\eta_2=\eta_1(l_0+1)=\bar{\eta}/(l_0+1)^2$, we shall begin by proving

$$\int_{\mathbb{R}^n}(M^{\Delta}_{V,\eta_2}f)^q g\,dx\le C\int_{\mathbb{R}^n} f^q(M_{V,\eta_1}g)\,dx. \tag{2.13}$$

We do this as follows: Hold g fixed, and look at the mapping $T\colon f\to M^{\Delta}_{V,\eta_2}f$. Then (2.13) says that T is bounded from $L^q(\mathbb{R}^n, M_{V,\eta_1}g(x)\,dx)$ to $L^q(\mathbb{R}^n, g(x)\,dx)$. Clearly, T is bounded from $L^\infty(\mathbb{R}^n, M_{V,\eta_1}g(x)\,dx)$ to $L^\infty(\mathbb{R}^n, g(x)\,dx)$. If we can show that T is of weak-$(1,1)$ type, then (2.13) holds by the Marcinkiewicz interpolation theorem.

Lemma 2.6 shows that $\{x\in\mathbb{R}^n : M^{\Delta}_{V,\eta_2}f(x)>\lambda\}=\bigcup_{j=1}^\infty Q_j$, where the Q_j are pairwise disjoint cubes satisfying the condition

$$\lambda\le\frac{1}{\psi_{\eta_2}(Q_j)|Q_j|}\int_{Q_j} f(x)\,dx\le 2^n(4n)^{\eta_2}\lambda.$$

Then

$$\begin{aligned}\int_{Q_j} g(y)\,dy &\le \int_{Q_j} g(y)\,dy\frac{1}{\lambda\psi_{\eta_2}(Q_j)|Q_j|}\int_{Q_j} f(x)\,dx\\ &\le\frac{C}{\lambda}\int_{Q_j} f(x)\left(\frac{1}{\Psi_{\eta_1}(Q_j)|Q_j|}\int_{Q_j} g(y)\,dy\right)dx\\ &\le\frac{C}{\lambda}\int_{Q_j} f(x)M_{V,\eta_1}g(x)\,dx.\end{aligned}$$

Summing over j, we obtain that

$$\int_{\{x\in\mathbb{R}^n:(M^{\Delta}_{V,\eta_2}f)(x)>\lambda\}} g(y)\,dy\le C\int_{\mathbb{R}^n} f(x)M_{V,\eta_1}g(x)\,dx.$$

Thus, (2.13) holds. To complete the proof of (2.12), we first define

$$M'_{V,\eta_3}f(x)=\sup_{r>0}\frac{1}{(1+r/\rho(x))^{\eta_3}|Q|}\int_{Q(x,r)}|f(y)|\,dy.$$

Obviously, $(4n)^{\bar{\eta}}C_0M'_{V,\eta_3}f(x)\ge M_{V,\bar{\eta}}f(x)$, where $\eta_3=\bar{\eta}/(l_0+1)=\eta_2(l_0+1)$.

Hence, to end the proof, it will suffice to show that

$$\{x\in\mathbb{R}^n : M'_{V,\eta_3}f(x)>c_0\lambda\}\subset\bigcup_{j=1}^\infty 2Q_j, \tag{2.14}$$

where $c_0=C_0^2 4^{l_0+1+n}(4n)^{\bar{\eta}}$.

Fix $x\notin\bigcup_{j=1}^\infty 2Q_j$ and let Q be any cube centered at x. Let r denote the side length of Q, and choose $k\in\mathbb{Z}$ such that $2^{k-1}\le r<2^k$. Then Q intersects m $(\le 2^n)$ dyadic cubes with sidelength 2^k; call them $R_1=R_1(x_1,2^k), R_2=R_2(x_2,2^k), ..., R_m=R_m(x_m,2^k)$. None of these cubes is contained in any of the Q_j's, for otherwise we would have $x\in\bigcup_{j=1}^\infty 2Q_j$. Hence

$$\begin{aligned}\frac{1}{(1+r/\rho(x))^{\eta_3}|Q|}\int_{Q(x,r)}|f(y)|\,dy &= \frac{1}{(1+r/\rho(x))^{\eta_3}|Q|}\sum_{i=1}^m\int_{Q\cap R_i}|f(y)|\,dy\\ &\le \sum_{i=1}^m \frac{C_0 4^{l_0+1}2^{kn}}{(1+2^k/\max_Q\rho(x))^{\eta_2}|Q|\,|R_i|}\int_{R_i}|f(y)|\,dy\\ &\le 2^n 4^{l_0+1}C_0 m\lambda\\ &\le 4^{l_0+1+n}C_0\lambda.\end{aligned}$$

Thus, (2.14) holds, so (2.12) is proved. □

We remark that the referee has pointed out that in fact Theorem 2.7 can be also obtained by a similar argument found in [7]. This is left as an exercise for the interested readers.

3. Extrapolation theorems

In this section, $\mathcal{F}$ will denote a family of ordered pairs of nonnegative, measurable functions (f,g). If we say that for p, $0<p<\infty$, and $\omega\in A_\infty^{\rho,\infty}=\bigcup_{p=1}^\infty A_p^{\rho,\infty}$,

$$\int_{\mathbb{R}^n} f(x)^p\omega(x)\,dx\le C\int_{\mathbb{R}^n} g(x)^p\omega(x),\quad (f,g)\in\mathcal{F},$$

we mean that this inequality holds for any $(f,g)\in\mathcal{F}$ such that the left-hand side is finite, and that the constant C depends only on p and the $A_\infty^{\rho,\infty}$ constant of ω. We will make similar abbreviated statements involving Lorentz spaces. For vector-valued inequalities we will consider sequences $\{(f_j,g_j)\}_{j=1}^\infty$, where each pair (f_j,g_j) is contained in $\mathcal{F}$.

In addition, we will use following classes: given a pair of operators (T,S), let $\mathcal{F}(T,S)$ denote the family of pairs of functions $(|Tf|,|Sf|)$, where f lies in the common domain of T and S, and the left-hand side of the corresponding inequality is finite. To achieve this, the function f may be restricted in some other way, e.g. $f\in C_0^\infty$. In this case we may indicate this by writing $\mathcal{F}(|Tf|,|Sf|:f\in C_0^\infty)$.

We can now state our main results of this paper.

Theorem 3.1. *Given a family $\mathcal{F}$, suppose that for some p_0, $0<p_0<\infty$, and for every weight $\omega\in A_\infty^{\rho,\infty}$,*

$$\int_{\mathbb{R}^n} f(x)^{p_0}\omega(x)\,dx \le C\int_{\mathbb{R}^n} g(x)^{p_0}\omega(x), \quad (f,g)\in\mathcal{F}. \tag{3.1}$$

Then the following are true:

- *For all $0<p<\infty$ and $\omega\in A_\infty^{\rho,\infty}$,*

$$\int_{\mathbb{R}^n} f(x)^{p}\omega(x)\,dx \le C\int_{\mathbb{R}^n} g(x)^{p}\omega(x)\,dx, \quad (f,g)\in\mathcal{F}; \tag{3.2}$$

- *For all $0<p<\infty$, $0<s\le\infty$ and $\omega\in A_\infty^{\rho,\infty}$,*

$$\|f\|_{L^{p,s}(\omega)} \le C\|g\|_{L^{p,s}(\omega)}, \quad (f,g)\in\mathcal{F}; \tag{3.3}$$

- *For all $0<p,q<\infty$ and $\omega\in A_\infty^{\rho,\infty}$,*

$$\left\|\left(\sum_{j=1}^{\infty} f_j^q\right)^{1/q}\right\|_{L^p(\omega)} \le C\left\|\left(\sum_{j=1}^{\infty} g_j^q\right)^{1/q}\right\|_{L^p(\omega)}, \quad \{(f_j,g_j)\}_{j=1}^{\infty}\subset\mathcal{F}; \tag{3.4}$$

- *For all $0<p,q<\infty$, $0<s\le\infty$, and $\omega\in A_\infty^{\rho,\infty}$,*

$$\left\|\left(\sum_{j=1}^{\infty} f_j^q\right)^{1/q}\right\|_{L^{p,s}(\omega)} \le C\left\|\left(\sum_{j=1}^{\infty} g_j^q\right)^{1/q}\right\|_{L^{p,s}(\omega)}, \quad \{(f_j,g_j)\}_{j=1}^{\infty}\subset\mathcal{F}. \tag{3.5}$$

Our second main result shows that we can also extrapolate from an initial Lorentz space inequality.

Theorem 3.2. *Given a family $\mathcal{F}$, suppose that for some p_0, $0<p_0<\infty$, and for every weight $\omega\in A_\infty^{\rho,\infty}$,*

$$\|f\|_{L^{p_0,\infty}(\omega)} \le C\|g\|_{L^{p_0,\infty}(\omega)}, \quad (f,g)\in\mathcal{F}. \tag{3.6}$$

Then, for all $0<p<\infty$ and $\omega\in A_\infty^{\rho,\infty}$,

$$\|f\|_{L^{p,\infty}(\omega)} \le C\|g\|_{L^{p,\infty}(\omega)}, \quad (f,g)\in\mathcal{F}. \tag{3.7}$$

Our third main result is a generalization of the A_p extrapolation theorem of Rubio de Francia.

Theorem 3.3. *Fix $\gamma\ge1$ and r, $\gamma<r<\infty$. If T is a bounded operator on $L^r(\omega)$ for any $\omega\in A_{r/\gamma}^{\rho,\infty}$, with operator norm depending only the $A_{r/\gamma}$ constant of ω, then T is bounded on $L^p(\omega)$, $\gamma<p<\infty$, for any $\omega\in A_{p/\gamma}^{\rho,\infty}$.*

As a consequence of Theorem 3.3, we have the following result.

Corollary 3.4. *Fix $\gamma\geq 1$. Let $\gamma<p,q<\infty$ and T satisfy the conditions in Theorem 3.3. Then for any $\omega\in A_{p/\gamma}^{\rho,\infty}$ such that*

$$\bigg\|\bigg(\sum_{j=1}^{\infty}|Tf_j|^q\bigg)^{1/q}\bigg\|_{L^p(\omega)}\leq C\bigg\|\bigg(\sum_{j=1}^{\infty}|f_j|^q\bigg)^{1/q}\bigg\|_{L^p(\omega)}.$$

We shall adapt an argument in [5] for proving Theorems 3.1 and 3.2, and prove Theorem 3.3 by using an argument in [9]. We first give the proof of Theorem 3.1.

3.1. Proof of inequality (3.2)

Step 1. We first show that hypothesis (3.1) is equivalent to the family of weighted inequalities with $A_1^{\rho,\infty}$ weights.

Proposition 3.5. *Hypothesis (3.1) of Theorem 3.1 is equivalent to the fact that for all $0<q<p_0$, $\omega\in A_1^{\rho,\infty}$, and $(f,g)\in\mathcal{F}$,*

$$\int_{\mathbb{R}^n} f(x)^q\omega(x)\,dx\leq C\int_{\mathbb{R}^n} g(x)^q\omega(x)\,dx. \tag{3.8}$$

Proof. We will prove that (3.1) implies (3.8). If (3.2) is proved, then the converse is proved. Fix $(f,g)\in\mathcal{F}$. Without loss of generality, we can assume that $g\in L^q(\omega)$ and $\|f\|_{L^q(\omega)}>0$. Let $s=p_0/q$. Since $\omega\in A_1^{\rho,\infty}$, there is a $\theta>0$ such that $\omega\in A_1^{\rho,\theta}\subset A_{s'}^{\rho,\theta}$, and $M_{V,s\theta}$ is bounded on $L^{s'}(\omega)$ by Lemma 2.2, that is,

$$\|M_{V,s\theta}h\|_{L^{s'}(\omega)}\leq A\|h\|_{L^{s'}(\omega)},$$

for some $A>0$. For $h\in L^{s'}(\omega)$, $h\geq 0$, we apply the algorithm of Rubio de Francia to define

$$\mathcal{R}h(x)=\sum_{k=0}^{\infty}\frac{M_{V,s\theta}^k h(x)}{(2A)^k},$$

where $M_{V,s\theta}^k$ is the operator $M_{V,s\theta}$ iterated k times if $k\geq 1$, and for $k=0$ is just the identity. From the definition of $\mathcal{R}$, it easy to see that

(a) $h(x)\leq\mathcal{R}h(x)$;

(b) $\|\mathcal{R}h\|_{L^{s'}(\omega)}\leq 2\|h\|_{L^{s'}(\omega)}$;

(c) $M_{V,s\theta}(\mathcal{R}h)(x)\leq 2A\mathcal{R}h(x)$, so $\mathcal{R}h(x)\in A_1^{\rho,s\theta}$ with constant independent of h.

Since $f,g\in L^{s'}(\omega)$ with positive norms, from (b), we then have

$$H(x)=\mathcal{R}\bigg(\Big(\frac{f}{\|f\|_{L^{s'}(\omega)}}\Big)^{q/s'}\Big(\frac{g}{\|g\|_{L^{s'}(\omega)}}\Big)^{q/s'}\bigg)(x)\in L^{s'}(\omega).$$

By (a),

$$\Big(\frac{f}{\|f\|_{L^{s'}(\omega)}}\Big)^{q/s'}\leq H(x)\quad\text{and}\quad\Big(\frac{g}{\|g\|_{L^{s'}(\omega)}}\Big)^{q/s'}\leq H(x). \tag{3.9}$$

So $H(x)>0$ whenever $f(x)>0$. Further, H is finite a.e. on the set where $\omega>0$ because $h\in L^{s'}(\omega)$. Hence,

$$\int_{\mathbb{R}^n} f(x)^q\omega(x)\,dx \leq \Big(\int_{\mathbb{R}^n} f(x)^{p_0}H(x)^{-s}\omega(x)\,dx\Big)^{1/s}\Big(\int_{\mathbb{R}^n} H(x)^{s'}\omega(x)\,dx\Big)^{1/s'}$$
$$=: \mathrm{I}\cdot\mathrm{II}.$$

Obviously, $\mathrm{II}\leq 4$ by (b).

To estimate I, since $\omega\in A_1^{\rho,\theta}\subset A_1^{\rho,s\theta}$, and $H\in A_1^{\rho,s\theta}$ by (c), we have $wH^{-s}=wH^{1-(1+s)}\in A_{1+s}^{\rho,s\theta}\subset A_\infty^{\rho,\infty}$ by (v) of Proposition 2.3. On the other hand, by (3.9), we get that

$$\int_{\mathbb{R}^n} f(x)^{p_0}H(x)^{-s}\omega(x)\,dx\leq\|f\|_{L^s(\omega)}^{qs/s'}\int_{\mathbb{R}^n} f(x)^{p_0-qs/s'}\omega(x)\,dx=\|f\|_{L^s(\omega)}^{qs}<\infty.$$

So, we can use (3.1); by (3.9), we get that

$$I\leq\Big(\int_{\mathbb{R}^n} g(x)^{p_0}H(x)^{-s}\omega(x)\,dx\Big)^{1/s}\leq C\int_{\mathbb{R}^n} g(x)^p\omega(x)\,dx.$$

By I and II, we obtain the desired result.

Step 2. We now show that for all $0<p<\infty$ and for every $\omega\in A_\infty^{\rho,\infty}$, (3.2) holds. Fix $0<p<\infty$ and $\omega\in A_\infty^{\rho,\infty}$. Assume that $(f,g)\in\mathcal{F}$ with $f,g\in L^p(\omega)$. By (i) of Proposition 2.3, we know that $A_{p_1}^{\rho,\theta}\subset A_{p_2}^{\rho,\theta}$ if $1\leq p_1\leq p_2$, and thus there exist $\theta>0$ and $0<q<\min\{p,p_0\}$ such that $\omega\in A_{p/q}^{\rho,\theta}$. Let $r=p/q>1$. Since $\omega\in A_r^{\rho,\theta}$, we get that $\omega^{1-r'}\in A_{r'}^{\rho,\theta}$ by (ii) of Proposition 2.3. Given $h\in L^{r'}(\omega^{1-r'})$, $h\geq 0$, we use the algorithm of Rubio de Francia to define

$$\mathcal{R}h(x)=\sum_{k=0}^{\infty}\frac{M_{V,r\theta}^k h(x)}{(2B)^k},$$

where B is the operator norm of $M_{V,r\theta}$ on $L^{r'}(\omega^{1-r'})$; this is finite since $\omega^{1-r'}\in A_{r'}^{\rho,\theta}$. Then

(a) $h(x) \le \mathcal{R}h(x)$;
(b) $\|\mathcal{R}h\|_{L^{r'}(\omega^{1-r'})} \le 2\|h\|_{L^{r'}(\omega^{1-r'})}$;
(c) $M_{V,sr}(\mathcal{R}h)(x) \le 2B\mathcal{R}h(x)$, so $\mathcal{R}h(x) \in A_1^{\rho,r\theta}$ with constant independent of h.

By duality

$$\|f\|_{L^p(\omega)}^q = \|f^q\|_{L^r(\omega)} = \sup_{\|h\|_{L^{r'}(\omega)} \le 1} \int_{\mathbb{R}^n} f(x)^q h(x)\omega(x)\, dx.$$

Fix such a function $h \ge 0$. Then $h\omega \in L^{r'}(\omega^{1-r'})$ and $\|h\omega\|_{L^{r'}(\omega^{1-r'})} = \|h\|_{L^{r'}(\omega)} = 1$. By (c), $\mathcal{R}(h\omega) \in A_1^{\rho,r\theta}$. By (a) and (3.1), we then have

$$\int_{\mathbb{R}^n} f(x)^q h(x)\omega(x)\, dx \le \int_{\mathbb{R}^n} f(x)^q \mathcal{R}(h\omega)(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^q \mathcal{R}(h\omega)(x)\, dx,$$

provided that the middle term is finite.

The same argument also holds for g instead of f. Hence,

$$\int_{\mathbb{R}^n} f(x)^q h(x)\omega(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^q \mathcal{R}(h\omega)(x)\, dx \le C\|g\|_{L^p(\omega)}^q.$$

From this, we obtain the desired result. □

3.2. Proof of inequality (3.3)

We need two lemmas. We first give a result about the operator M_ω defined by

$$M_\omega(f)(x) = \sup_{x \in B} \frac{1}{\omega(5B)} \int_B |f(x)|\omega(x)\, dx.$$

Lemma 3.6. *Let* $1 \le p < \infty$. *If* $\omega \in A_\infty^{\rho,\infty}$, *then*

$$\omega(\{x \in \mathbb{R}^n : M_\omega f(x) > \lambda\}) \le C\left(\frac{\|f\|_{L^p(\omega)}}{\lambda}\right)^p \quad \text{for all } \lambda > 0 \text{ and } f \in L^p(\omega).$$

In particular, for $1 < p \le \infty$,

$$\|M_\omega f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Proof. We set $x \in E_\lambda = \{x \in \mathbb{R}^n : M_\omega f(x) > \lambda\}$ with any $\lambda > 0$. Then, there exists a ball $B_x \ni x$ such that

$$\frac{1}{\omega(5B_x)} \int_{B_x} |f(y)|\omega(y)\, dy > \lambda. \tag{3.10}$$

Thus, $\{B_x\}_{x\in E_\lambda}$ covers E_λ. By Vitali's lemma, there exists a collection of disjoint cubes $\{B_{x_j}\}_{j=1}^\infty$ such that $\bigcup_{j=1}^\infty B_{x_j}\subset E_\lambda\subset\bigcup_{j=1}^\infty 5B_{x_j}$ and

$$\omega(E_\lambda)\le\sum_{j=1}^\infty\omega(5B_{x_j}). \tag{3.11}$$

From (3.10) and by Hölder's inequality, we have

$$\lambda<\frac{1}{\omega(5B_x)^{1/p}}\left(\int_{B_x}|f(y)|^p\omega(y)\,dy\right)^{1/p}.$$

From this and by (3.11), we get that

$$\begin{aligned}\omega(E_\lambda)x&\le\sum_{j=1}^\infty\omega(5B_{xj})\le\frac{C}{\lambda^p}\sum_{j=1}^\infty\int_{B_{x_j}}|f(y)|^p\omega(y)\,dy\\&=\frac{C}{\lambda^p}\int_{\bigcup_{j=1}^\infty B_{x_j}}|f(y)|^p\omega(y)\,dy\le\frac{C}{\lambda^p}\int_{\mathbb{R}^n}|f(y)|^p\omega(y)\,dy.\end{aligned}$$

Thus, Lemma 3.6 is proved. □

Given two weights u and v, we say that $u\in A_1(v)$ if for every x, $M_vu(x)\le Cu(x)$.

Lemma 3.7. *If $\omega_1\in A_p^{\rho,\theta}$, $1\le p\le\infty$, and $\omega_2\in A_1(\omega_1)$, then $\omega_1\omega_2\in A_p^{\rho,\theta p}$.*

Proof. If $\omega_2\in A_1(\omega_1)$, then for any ball B,

$$\begin{aligned}\frac{1}{(\Psi_\theta(B))^{p^2}|B|}\int_B\omega_1(x)\omega_2(x)\,dx&=\frac{\omega_1(5B)}{\Psi_\theta(B)^{p^2}|B|}\frac{1}{\omega_1(5B)}\int_B\omega_2(x)\omega_1(x)\,dx\\&\le C\frac{\omega_1(5B)}{\Psi_\theta(B)^{p^2}|B|}\operatorname*{ess\,inf}_B\omega_2\\&\le C\frac{\omega_1(B)}{\Psi_\theta(B)^{p}|B|}\operatorname*{ess\,inf}_B\omega_2,\end{aligned}$$

where in the last inequality we used the fact that (see [25])

$$\omega_1(5B)\le C\Psi_\theta(B)^p\omega_1(B).$$

On the other hand,

$$\left(\frac{1}{|B|}\int_B(\omega_1(x)\omega_2(x))^{-1/(p-1)}\,dx\right)^{p-1} \le \left(\frac{1}{|B|}\int_B \omega_1(x)^{-1/(p-1)}\,dx\right)^{p-1}\left(\operatorname*{ess\,inf}_B \omega_2\right)^{-1}.$$

From these two inequalities, we get the desired result. □

Proof of (3.3). Fix p, s, $\omega\in A_\infty^{\rho,\infty}$ and $(f,g)\in\mathcal{F}$ with $f,g\in L^{p,s}(\omega)$. Fix $0<q<\min\{p,s\}$ and set $r=p/q>1$ and $\tilde r=s/q>1$. (If $s=\infty$, take $0<q<p$ and $\tilde r=\infty$.) Then

$$\|f\|^q_{L^{p,s}(\omega)}=\|f^q\|_{L^{r,\tilde r}(\omega)}=\sup_h\int_{\mathbb{R}^n}f(x)^q h(x)\omega(x)\,dx,$$

where the supremum is taken over all $h\in L^{r',\tilde r}(\omega)$ with $h\ge 0$ and $\|h\|_{L^{r',\tilde r'}}=1$. Fix such a function h. Using the algorithm of Rubio de Francia to define

$$\mathcal{R}_\omega h(x)=\sum_{k=0}^{\infty}\frac{M_\omega^k h(x)}{(2A_\omega)^k},$$

where A_ω is the operator norm of M_ω on $L^{r',\tilde r}(\omega)$ endowed with a norm equivalent to $\|\cdot\|_{L^{r',\tilde r}(\omega)}$. Since M_ω is bounded on $L^p(\omega)$ by Lemma 3.6, and by Marcinkiewicz interpolation in the scale of Lorentz space, it is bounded on $L^{r',\tilde r}(\omega)$. Then,

(a) $h(x)\le\mathcal{R}_\omega h(x)$;

(b) $\|\mathcal{R}_\omega h\|_{L^{r',\tilde r}(\omega^{1-r'})}\le C\|h\|_{L^{r',\tilde r}(\omega^{1-r'})}=C$;

(c) $M_{V,s\theta}(\mathcal{R}h)(x)\le 2A_\omega\mathcal{R}h(x)$, so $\mathcal{R}_\omega h(x)\in A_1(\omega)$ with constant independent of h.

By Lemma 3.7, $\omega\mathcal{R}_\omega h\in A_\infty^{\rho,\infty}$. As above, (3.2) holds with exponent q and the $A_\infty^{\rho,\infty}$ weight $\omega\mathcal{R}_\omega h$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^n}f(x)^q h(x)\omega(x)\,dx &\le \int_{\mathbb{R}^n}f(x)^q\mathcal{R}_\omega h(x)\omega(x)\,dx\le C\int_{\mathbb{R}^n}g(x)^q\mathcal{R}_\omega h(x)\omega(x)\,dx\\ &\le C\|g^q\|_{L^{r,\tilde r}(\omega)}\|\mathcal{R}_\omega h\|_{L^{r',\tilde r'}(\omega)}\le C\|g\|^q_{L^{r,\tilde r}(\omega)},\end{aligned}$$

since

$$\int_{\mathbb{R}^n}f(x)^q\mathcal{R}_\omega h(x)\omega(x)\,dx\le\|f^q\|_{L^{r,\tilde r}(\omega)}\|\mathcal{R}_\omega h\|_{L^{r',\tilde r'}(\omega)}\le C\|f\|^q_{L^{r,\tilde r}(\omega)}<\infty.$$

Thus, the desired inequality is obtained. □

3.3. Proof of inequalities (3.4) and (3.5)

Fix $0<q<\infty$. It suffices to prove the vector-valued inequalities only for finite sums by the monotone convergence theorem. Fix $N\geq 1$ and define

$$f_q(x)=\left(\sum_{j=1}^N f_j(x)^q\right)^{1/q} \quad \text{and} \quad g_q(x)=\left(\sum_{j=1}^N g_j(x)^q\right)^{1/q},$$

where $\{(f_j,g_j)\}_{j=1}^N\subset\mathcal{F}$. Now form a new family $\mathcal{F}_q$ consisting of the pairs (f_q,g_q). Then, for every $\omega\in A_\infty^{\rho,\infty}$ and $(f_q,g_q)\in\mathcal{F}_q$, by (3.2) we get

$$\|f_q\|_{L^q(\omega)}^q=\sum_{j=1}^N\int_{\mathbb{R}^n} f_j(x)^q\omega(x)\,dx\leq C\sum_{j=1}^N\int_{\mathbb{R}^n} g_j(x)^q\omega(x)\,dx=C\|g_q\|_{L^q(\omega)}^q,$$

which implies that the hypotheses of Theorem 3.1 are fulfilled by $\mathcal{F}_q$ with $p_0=q$. Hence, by (3.2) and (3.3), for all $0<p<\infty$, $0<s\leq\infty$, $\omega\in A_\infty^{\rho,\infty}$ and $(f_q,g_q)\in\mathcal{F}_q$, $\|f_q\|_{L^p(\omega)}\leq C\|g_q\|_{L^p(\omega)}$ and $\|f_q\|_{L^{p,s}(\omega)}\leq C\|g_q\|_{L^{p,s}(\omega)}$. □

3.4. Proof of Theorem 3.2

This is similar to the proof of Theorem 3.1, adapting the same argument of Theorem 2.2 in [5], we omit the details here.

3.5. Proof of Theorem 3.3

We first need the following lemma, which is different from Lemma 2.2.

Lemma 3.8. *Let $1\leq p<\infty$ and suppose that $\omega\in A_p^{\rho,\theta}$. If $p<p_1<\infty$, then*

$$\int_{\mathbb{R}^n}|M_{V,\theta}f(x)|^{p_1}\omega(x)\,dx\leq C\int_{\mathbb{R}^n}|f(x)|^{p_1}\omega(x)\,dx.$$

Proof. In fact,

$$\begin{aligned}
&\frac{1}{\Psi_\theta(B)|B|}\int_B|f(y)|\,dy\\
&=\frac{1}{\Psi_\theta(B)|B|}\int_B|f(y)|\omega^{1/p}(y)\omega^{-1/p}(y)\,dy\\
&\leq\left(\frac{1}{\Psi_\theta(B)|B|}\int_B|f(y)|^p\omega(y)\,dy\right)^{1/p}\left(\frac{1}{\Psi_\theta(B)|B|}\int_B\omega^{-1/(p-1)}(y)\,dy\right)^{(p-1)/p}
\end{aligned}$$

$$\le C\biggl(\frac{1}{\Psi_\theta(B)|B|}\int_B |f(y)|^p\omega(y)\,dy\biggr)^{1/p}\biggl(\frac{1}{\Psi_\theta(5B)|B|}\int_{5B}\omega^{-1/(p-1)}(y)\,dy\biggr)^{(p-1)/p}$$
$$\le C\biggl(\frac{1}{\Psi_\theta(B)|B|}\int_B |f(y)|^p\omega(y)\,dy\biggr)^{1/p}\biggl(\frac{1}{\Psi_\theta(B)|B|}\int_{5B}\omega(y)\,dy\biggr)^{-1/p}$$
$$\le C\biggl(\frac{1}{\omega(5B)}\int_B |f|^p\omega(y)\,dy\biggr)^{1/p}.$$

Therefore,

$$M_{V,\theta}f(x)\le CM_\omega(|f|^p)(x)^{1/p},\quad x\in\mathbb{R}^n.$$

From this and using Lemma 3.6, we can deduce Lemma 3.8. □

Proof of Theorem 3.3. We only consider the case $\gamma=1$, the case $\gamma>1$ is similar. We first show that if $1<q<r$ and $\omega\in A_1^{\rho,\infty}$ then T is bounded on $L^q(\omega)$. Without loss of generality, we assume that $\omega\in A_1^{\rho,\eta}$ for some $\eta>0$. By (iv) of Proposition 2.3 the function $M_{V,\eta}^{(r-q)/(r-1)}$ is in $A_1^{\rho,\eta}$, and $\omega(M_{V,\eta}f)^{q-r}\in A_r^{\rho,\eta}$ by (iv) of Proposition 2.3. Hence,

$$\begin{aligned}\int_{\mathbb{R}^n}|Tf|^q\omega &= \int_{\mathbb{R}^n}|Tf|^q(M_{V,\eta}f)^{-(q-r)q/r}(M_{V,\eta}f)^{(q-r)q/r}\omega\,dx\\ &\le\biggl(\int_{\mathbb{R}^n}|Tf|^r\omega(M_{V,\eta}f)^{q-r}\,dx\biggr)^{q/r}\biggl(\int_{\mathbb{R}^n}(M_{V,\eta}f)^q\omega\,dx\biggr)^{(r-q)/r}\\ &\le\biggl(\int_{\mathbb{R}^n}|f|^r\omega(M_{V,\eta}f)^{q-r}\,dx\biggr)^{q/r}\biggl(\int_{\mathbb{R}^n}|f|^q\omega\,dx\biggr)^{(r-q)/r}\\ &\le C\int_{\mathbb{R}^n}|f|^q\omega\,dx,\end{aligned}$$

where the second inequality holds by our hypothesis on T and by Lemma 3.8 (since $\omega\in A_1^{\rho,\eta}$), and the third inequality holds since $|f(x)|\le M_{V,\eta}f(x)$ a.e. for any $\eta\ge0$, so $M_{V,\eta}f(x)^{q-r}\le|f(x)|^{q-r}$ a.e.

Given any $1<p<\infty$ and $\omega\in A_p^{\rho,\theta}$, by (iii) of Proposition 2.3 there exists $q>1$ and $\theta_1\ge\theta$ such that $\omega\in A_{p/q}^{\rho,\theta_1}$. Hence we only need to prove that T is bounded on $L^p(\omega)$ if $\omega\in A_{p/q}^{\rho,\theta_1}$.

Fix $\omega\in A_{p/q}^{\rho,\theta_1}$. Then by duality there exists $u\in L^{(p/q)'}(\omega)$ with norm 1 such that

$$\biggl(\int_{\mathbb{R}^n}|Tf|^p\omega\,dx\biggr)^{q/p}=\int_{\mathbb{R}^n}|Tf|^q\omega u\,dx.$$

For any $s>1$, $\omega u \le M_{V,\eta}((\omega u)^s)^{1/s}$ for any $\eta>0$ and $M_{V,\eta}((\omega u)^s)^{1/s} \in A_1^{\rho,\eta}$. Hence, by the first part of the proof,

$$\begin{aligned}\int_{\mathbb{R}^n} |Tf|^q \omega u\,dx &\le \int_{\mathbb{R}^n} |Tf|^q M_{V,\eta}((\omega u)^s)^{1/s}\,dx \\ &\le C\int_{\mathbb{R}^n} |f|^q M_{V,\eta}((\omega u)^s)^{1/s}\,dx \\ &= C\int_{\mathbb{R}^n} |f|^q \omega^{q/p} M_{V,\eta}((\omega u)^s)^{1/s}\omega^{-q/p}\,dx \\ &\le C\bigg(\int_{\mathbb{R}^n} |f|^p \omega\,dx\bigg)^{q/p} \\ &\quad\times\bigg(\int_{\mathbb{R}^n} M_{V,\eta}((\omega u)^s)^{(p/q)'/s}\omega^{1-(p/q)'}\,dx\bigg)^{1/(p/q)'}.\end{aligned}$$

Since $\omega \in A_{p/q}^{\rho,\theta_1}$, we have $\omega^{1-(p/q)'} \in A_{(p/q)'}^{\rho,\theta_1}$ by (ii) of Proposition 2.3. Therefore, if we take s sufficiently close to 1, then there exists θ_s such that $\omega^{1-(p/q)'} \in A_{(p/q)'/s}^{\rho,\theta_s}$ by (iii) of Proposition 2.3. If we choose $\eta=((p/q)'/s)'\theta_s$, then by Lemma 2.2 the second integral is dominated by

$$C\int_{\mathbb{R}^n} (\omega u)^{(p/q)'}\omega^{1-(p/q)'}\,dx = C.$$

The proof is complete. □

We remark that an interesting problem posed by the referee is how to extend Theorem 3.3 to the context of rearrangement-invariant Banach function spaces, as considered in [8].

4. Some applications

Let T be a Schrödinger-type operator. From Theorem 3.1 in [25] we know that for all $0<p<\infty$ and $\omega \in A_\infty^{\rho,\infty}$, for any $\eta>0$, there exists a constant C depending only on η, p, q, C_0, l_0 and the $A_\infty^{\rho,\infty}$ constant of ω such that

$$\|Tf\|_{L^p(\omega)} \le C\|M_{V,\eta}f\|_{L^p(\omega)}.$$

By applying Theorem 3.1 to the family $\mathcal{F}_\eta(|Tf|, M_{V,\eta}f : f \in C_0^\infty)$, we obtain that

- for all $0<p,q<\infty$ and $\omega \in A_\infty^{\rho,\infty}$,

(4.1)
$$\bigg\|\bigg(\sum_{j=1}^\infty |Tf_j|^q\bigg)^{1/q}\bigg\|_{L^p(\omega)} \le C\bigg\|\bigg(\sum_{j=1}^\infty (M_{V,\eta}f_j)^q\bigg)^{1/q}\bigg\|_{L^p(\omega)}, \quad \{(f_j,g_j)\}_{j=1}^\infty \subset \mathcal{F}_\eta;$$

- for all $0<p,q<\infty$, $0<s\leq\infty$ and $\omega\in A_\infty^{\rho,\infty}$,

(4.2)
$$\left\|\left(\sum_{j=1}^{\infty}|Tf_j|^q\right)^{1/q}\right\|_{L^{p,s}(\omega)}\leq C\left\|\left(\sum_{j=1}^{\infty}(M_{V,\eta}f_j)^q\right)^{1/q}\right\|_{L^{p,s}(\omega)},\quad \{(f_j,g_j)\}_{j=1}^{\infty}\subset\mathcal{F}_\eta.$$

If we combine this with Theorem 2.7, we have the following inequalities:

- If $1<q<\infty$, then for every $\omega\in A_1^{\rho,\infty}$, there exists a constant C depending only on η, q, C_0, l_0 and the $A_1^{\rho,\infty}$ constant of ω such that

$$\left\|\left(\sum_{j=1}^{\infty}|Tf_j|^q\right)^{1/q}\right\|_{L^{1,\infty}(\omega)}\leq C\left\|\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{1/q}\right\|_{L^{1}(\omega)};\tag{4.3}$$

- If $1<q<\infty$ and $1<p<\infty$, then for every $\omega\in A_p^{\rho,\infty}$, there exists a constant C depending only on η, p, q, C_0, l_0 and the $A_p^{\rho,\infty}$ constant of ω such that

$$\left\|\left(\sum_{j=1}^{\infty}|Tf_j|^q\right)^{1/q}\right\|_{L^{p}(\omega)}\leq C\left\|\left(\sum_{j=1}^{\infty}f_j^q\right)^{1/q}\right\|_{L^{p}(\omega)}.\tag{4.4}$$

Let T be a Schrödinger-type operator as above. From Theorem 3.1 in [25] we have that for all $0<p<\infty$ and $\omega\in A_\infty$, for any $\eta>0$, there exists a constant C depending only on η, p, q, C_0, l_0 and the $A_\infty^{\rho,\infty}$ constant of ω such that

$$\|[b,T]f\|_{L^p(\omega)}\leq C\|b\|_{\mathrm{BMO}_\infty(\rho)}\|M_{V,\eta}(M_{V,\eta}f)\|_{L^p(\omega)}.$$

By applying Theorem 3.1 to the family $\mathcal{F}_\eta(|[b,T]f|,M_{V,\eta}(M_{V,\eta}f):f\in C_0^\infty)$, we obtain that

- for all $0<p,q<\infty$, $\omega\in A_\infty^{\rho,\infty}$ and $\{(f_j,g_j)\}_{j=1}^{\infty}\subset\mathcal{F}_\eta$,

(4.5)
$$\left\|\left(\sum_{j=1}^{\infty}|[b,T]f_j|^q\right)^{1/q}\right\|_{L^{p}(\omega)}\leq C\|b\|_{\mathrm{BMO}_\infty(\rho)}\left\|\left(\sum_{j=1}^{\infty}(M_{V,\eta}(M_{V,\eta}f_j))^q\right)^{1/q}\right\|_{L^{p}(\omega)};$$

- for all $0<p,q<\infty$, $0<s\leq\infty$, $\omega\in A_\infty^{\rho,\infty}$ and $\{(f_j,g_j)\}_{j=1}^{\infty}\subset\mathcal{F}_\eta$,

(4.6)
$$\left\|\left(\sum_{j=1}^{\infty}|[b,T]f_j|^q\right)^{1/q}\right\|_{L^{p,s}(\omega)}\leq C\|b\|_{\mathrm{BMO}_\infty(\rho)}\left\|\left(\sum_{j=1}^{\infty}(M_{V,\eta}(M_{V,\eta}f_j))^q\right)^{1/q}\right\|_{L^{p,s}(\omega)},$$

where the new space $\mathrm{BMO}_\theta(\rho)$ introduced in [3] is defined by

$$\|f\|_{\mathrm{BMO}_\theta(\rho)}=\sup_{B\subset\mathbb{R}^n}\frac{1}{\Psi_\theta(B)|B|}\int_B|f(x)-f_B|\,dx<\infty,$$

where $f_B = \frac{1}{|B|}\int_B f(y)\,dy$, $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$, $B = B(x_0, r)$ and $\theta > 0$. We also let $\mathrm{BMO}_\infty(\rho) = \bigcup_{\theta>0} \mathrm{BMO}_\theta(\rho)$.

If we combine this with Theorem 2.7, we have the following inequality: If $1<q<\infty$ and $1<p<\infty$, then for every $\omega \in A_p^{\rho,\infty}$, there exists a constant C depending only on η, p, q, C_0, l_0 and the $A_p^{\rho,\infty}$ constant of ω such that

$$\left\| \left(\sum_{j=1}^{\infty} |[b,T]f_j|^q \right)^{1/q} \right\|_{L^p(\omega)} \le C\|b\|_{\mathrm{BMO}_\infty(\rho)} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{1/q} \right\|_{L^p(\omega)}. \tag{4.7}$$

We remark that the inequalities (4.1)–(4.7) are all new.

Next we consider another class $V \in B_q$, with $q \ge \frac{1}{2}n$ for Riesz transforms associated with Schrödinger operators. Let $T_1 = (-\Delta+V)^{-1}V$, $T_2 = (-\Delta+V)^{-1/2}V^{1/2}$ and $T_3 = (-\Delta+V)^{-1/2}\nabla$. By using Theorem 3.3 in [26] and Corollary 3.4, we have the following result.

Theorem 4.1. *Suppose $V \in B_q$ and $q \ge \frac{1}{2}n$. Then*

(i) *if $q' < p, r < \infty$ and $\omega \in A_{p/q'}^{\rho,\infty}$,*

$$\big\| |T_1 f|_r \big\|_{L^p(\omega)} \le C \big\| |f|_r \big\|_{L^p(\omega)};$$

(ii) *if $(2q)' < p, r < \infty$ and $\omega \in A_{p/(2q)'}^{\rho,\infty}$,*

$$\big\| |T_2 f|_r \big\|_{L^p(\omega)} \le C \big\| |f|_r \big\|_{L^p(\omega)};$$

(iii) *if $p_0' < p, r < \infty$ and $\omega \in A_{p/p_0'}^{\rho,\infty}$, where $1/p_0 = 1/q - 1/n$ and $\frac{1}{2}n \le q < n$,*

$$\big\| |T_3 f|_r \big\|_{L^p(\omega)} \le C \big\| |f|_r \big\|_{L^p(\omega)}.$$

Let $T_1^* = V(-\Delta+V)^{-1}$, $T_2^* = V^{1/2}(-\Delta+V)^{-1/2}$ and $T_3^* = \nabla(-\Delta+V)^{-1/2}$. By duality we can easily get the following result.

Corollary 4.2. *Suppose $V \in B_q$ and $q \ge \frac{1}{2}n$. Then*

(i) *if $1 < p, r < q$ and $\omega^{-1/(p-1)} \in A_{p'/q'}^{\rho,\infty}$,*

$$\big\| |T_1^* f|_r \big\|_{L^p(\omega)} \le C \big\| |f|_r \big\|_{L^p(\omega)};$$

(ii) *if $1 < p, r < 2q$ and $\omega^{-1/(p-1)} \in A_{p'/(2q)'}^{\rho,\infty}$,*

$$\big\| |T_2^* f|_r \big\|_{L^p(\omega)} \le C \big\| |f|_r \big\|_{L^p(\omega)};$$

(iii) *if* $1<p,r<p_0$ *and* $\omega^{-1/(p-1)}\in A^{\rho,\infty}_{p'/p_0'}$, *where* $1/p_0=1/q-1/n$ *and* $\frac{1}{2}n\le q<n$,

$$\big\||T_3^*f|_r\big\|_{L^p(\omega)}\le C\big\||f|_r\big\|_{L^p(\omega)}.$$

Let T_1, T_2 and T_3 be as above. By using Theorem 4.5 in [26] and Corollary 3.4, we have the following result.

Theorem 4.3. *Suppose* $V\in B_q$ *and* $q\ge\frac{1}{2}n$. *Let* $b\in\mathrm{BMO}_\infty(\rho)$. *Then*
(i) *if* $q'<p,r<\infty$ *and* $\omega\in A^{\rho,\infty}_{p/q'}$,

$$\big\||[b,T_1]f|_r\big\|_{L^p(\omega)}\le C\|b\|_{\mathrm{BMO}_\infty(\rho)}\big\||f|_r\big\|_{L^p(\omega)};$$

(ii) *if* $(2q)'<p,r<\infty$ *and* $\omega\in A^{\rho,\infty}_{p/(2q)'}$,

$$\big\||[b,T_2]f|_r\big\|_{L^p(\omega)}\le C\|b\|_{\mathrm{BMO}_\infty(\rho)}\big\||f|_r\big\|_{L^p(\omega)};$$

(iii) *if* $p_0'<p,r<\infty$ *and* $\omega\in A^{\rho,\infty}_{p/p_0'}$, *where* $1/p_0=1/q-1/n$ *and* $\frac{1}{2}n\le q<n$,

$$\big\||[b,T_3]f|_r\big\|_{L^p(\omega)}\le C\|b\|_{\mathrm{BMO}_\infty(\rho)}\big\||f|_r\big\|_{L^p(\omega)}.$$

Let T_1^*, T_2^* and T_3^* be as above. By duality we can easily get the following result.

Corollary 4.4. *Suppose* $V\in B_q$ *and* $q\ge\frac{1}{2}n$. *Let* $b\in\mathrm{BMO}_\infty(\rho)$. *Then*
(i) *if* $1<p,r<q$ *and* $\omega^{-1/(p-1)}\in A^{\rho,\infty}_{p'/q'}$,

$$\big\||[b,T_1^*]f|_r\big\|_{L^p(\omega)}\le C\|b\|_{\mathrm{BMO}_\infty(\rho)}\big\||f|_r\big\|_{L^p(\omega)};$$

(ii) *if* $1<p,r<2q$ *and* $\omega^{-1/(p-1)}\in A^{\rho,\infty}_{p'/(2q)'}$,

$$\big\||[b,T_2^*]f|_r\big\|_{L^p(\omega)}\le C\|b\|_{\mathrm{BMO}_\infty(\rho)}\big\||f|_r\big\|_{L^p(\omega)};$$

(iii) *if* $1<p,r<p_0$ *and* $\omega^{-1/(p-1)}\in A^{\rho,\infty}_{p'/p_0'}$, *where* $1/p_0=1/q-1/n$ *and* $\frac{1}{2}n\le q<n$,

$$\big\||[b,T_3^*]f|_r\big\|_{L^p(\omega)}\le C\|b\|_{\mathrm{BMO}_\infty(\rho)}\big\||f|_r\big\|_{L^p(\omega)}.$$

Finally, we consider the Littlewood–Paley g-function related to Schrödinger operators defined by

$$g(f)(x)=\left(\int_0^\infty\left|\frac{d}{dt}e^{-tL}(f)(x)\right|^2t\,dt\right)^{1/2},$$

and the commutator g_b of g with $b \in \mathrm{BMO}(\rho)$ defined by

$$g_b(f)(x) = \left(\int_0^\infty \left| \frac{d}{dt} e^{-tL}((b(x)-b(\cdot))f)(x) \right|^2 t\,dt \right)^{1/2}.$$

The maximal operator of the diffusion semi-group is defined by

$$T^* f(x) = \sup_{t>0} |e^{-tL} f(x)| = \sup_{t>0} \left| \int_{\mathbb{R}^n} k_t(x,y) f(y)\,dy \right|,$$

and its commutator

$$T_b^* f(x) = \sup_{t>0} \left| \int_{\mathbb{R}^n} k_t(x,y)(b(x)-b(y)) f(y)\,dy \right|,$$

where k_t is the kernel of the operator e^{-tL}, $t>0$.

By combining Theorems 1 and 2 in [4], Theorems 1.1 and 3.1 in [26] and Corollary 3.4 together, we obtain the following result.

Theorem 4.5. *Let $b \in \mathrm{BMO}_\infty(\rho)$ and T, T_b^*, g and g_b be as above.*
(i) If $1<p, r<\infty$ and $\omega \in A_p^{\rho,\infty}$, then there exists a constant C such that

$$\big\| |g(f)|_r \big\|_{L^p(\omega)} + \big\| |T^* f|_r \big\|_{L^p(\omega)} \le C \big\| |f|_r \big\|_{L^p(\omega)}.$$

(ii) If $1<p, r<\infty$ and $\omega \in A_p^{\rho,\infty}$, then there exists a constant C such that

$$\big\| |g_b(f)|_r \big\|_{L^p(\omega)} + \big\| |T_b^* f|_r \big\|_{L^p(\omega)} \le C \|b\|_{BMO_\infty(\rho)} \big\| |f|_r \big\|_{L^p(\omega)}.$$

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