Ark. Mat., **52** (2014), 329–354
DOI: 10.1007/s11512-012-0177-5

Global integral gradient bounds for quasilinear equations below or near the natural exponent

Nguyen Cong Phuc

Abstract. We obtain sharp integral potential bounds for gradients of solutions to a wide class of quasilinear elliptic equations with measure data. Our estimates are global over bounded domains that satisfy a mild exterior capacitary density condition. They are obtained in Lorentz spaces whose degrees of integrability lie below or near the natural exponent of the operator involved. As a consequence, nonlinear Calderón–Zygmund type estimates below the natural exponent are also obtained for $\mathcal{A}$-superharmonic functions in the whole space $\mathbb{R}^n$. This answers a question raised in our earlier work (On Calderón–Zygmund theory for p- and $\mathcal{A}$-superharmonic functions, to appear in *Calc. Var. Partial Differential Equations*, DOI 10.1007/s00526-011-0478-8) and thus greatly improves the result there.

1. Introduction

The main goal of this paper is to obtain maximal global regularity for gradients of weak solutions to nonhomogeneous quasilinear equations with measure data of the form

$$\begin{cases} -\operatorname{div} \mathcal{A}(x,\nabla u)=\mu & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for a given finite measure μ on a bounded domain $\Omega\subset\mathbb{R}^n$, $n\geq 2$.

In (1.1) the nonlinearity $\mathcal{A}\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a Carathédory vector-valued function, i.e., $\mathcal{A}(x,\xi)$ is measurable in x for every ξ and continuous in ξ for a.e. x. We assume that $\mathcal{A}$ satisfies the following growth and monotonicity conditions: for some $1<p\leq n$ there holds

$$|\mathcal{A}(x,\xi)| \leq \beta|\xi|^{p-1}, \tag{1.2}$$

Supported in part by NSF grant DMS-0901083.

$$(1.3)\qquad \langle \mathcal{A}(x,\xi)-\mathcal{A}(x,\eta),\xi-\eta\rangle \geq \alpha(|\xi|^2+|\eta|^2)^{(p-2)/2}|\xi-\eta|^2$$

for every $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{(0,0)\}$ and a.e. $x\in\mathbb{R}^n$. Here α and β are positive constants.

Under a capacitary density condition on Ω, for $2-1/n<p\leq n$ we show in this paper the following integral gradient bound

$$(1.4)\qquad \int_\Omega |\nabla u|^q\,dx \leq C\int_{\mathbb{R}^n} \mathcal{M}_1(\chi_\Omega|\mu|)^{q/(p-1)}\,dx,$$

where q lies below or near the natural exponent p, i.e., $0<q<p+\varepsilon$ for some small $\varepsilon>0$ depending only on n, p, α, β, and Ω. In (1.4), χ_Ω is the characteristic function of Ω and $\mathcal{M}_1$ is the fractional maximal function defined for each nonnegative locally finite measure ν in $\mathbb{R}^n$ by

$$\mathcal{M}_1(\nu)(x)=\sup_{r>0}\frac{r\nu(B_r(x))}{|B_r(x)|},\qquad x\in\mathbb{R}^n.$$

By a capacitary density condition on Ω we mean in this paper that the complement $\mathbb{R}^n\setminus\Omega$ is *uniformly p-thick*, i.e. there exist constants $c_0, r_0>0$ such that for all $0<t\leq r_0$ and all $x\in\mathbb{R}^n\setminus\Omega$ it holds that

$$(1.5)\qquad \operatorname{cap}_p(\overline{B_t(x)}\cap(\mathbb{R}^n\setminus\Omega), B_{2t}(x))\geq c_0\operatorname{cap}_p(\overline{B_t(x)}, B_{2t}(x)).$$

Here for a compact set $K\subset B_{2t}(x)$ we define its p-capacity by

$$\operatorname{cap}_p(K, B_{2t}(x))=\inf\left\{\int_{B_{2t}(x)}|\nabla\varphi|^p\,dy:\varphi\in C_0^\infty(B_{2t}(x)) \text{ and } \varphi\geq\chi_K\right\}.$$

It is easy to see that domains satisfying (1.5) include those with Lipschitz boundaries or even those that satisfy a uniform exterior corkscrew condition, where the latter means that there exist constants $c_0, r_0>0$ such that for all $0<t\leq r_0$ and all $x\in\mathbb{R}^n\setminus\Omega$, there is $y\in B_t(x)$ such that $B_{t/c_0}(y)\subset\mathbb{R}^n\setminus\Omega$.

The restriction $q<p+\varepsilon$ for a small $\varepsilon>0$ is a natural one in order to obtain (1.4). For one reason by now it is well known that, in general, the structural assumptions (1.2) and (1.3) on the nonlinearity $\mathcal{A}(x,\xi)$ are not enough to ensure higher integrability even locally for gradients of solutions to (1.1) (see, e.g., [21]). For another reason our condition on the domain Ω allows all domains with Lipschitz boundaries, whereas an example given in [15] (see also [20]) makes it clear that global $W^{1,q}$ regularity, $q>2$, fails in general even for solutions to Laplace equations ($p=2$) over polygonal domains.

We should mention that, at least in the case $2\leq p\leq n$, a local version of inequality (1.4) has already been obtained by G. Mingione for the first time in [24] and the possibility of extending such local results to global ones was also mentioned in the same paper. Some of the key ideas in [24] are borrowed in this work in order to obtain (1.4), but technically our presentation is somewhat different from that of [24].

A solution u to the boundary value problem (1.1) is understood in the following sense. For each integer $k>0$ the truncation

$$T_k(u) := \max\{-k, \min\{k, u\}\}$$

belongs to $W_0^{1,p}(\Omega)$ and satisfies

$$-\mathrm{div}\, \mathcal{A}(x, \nabla T_k(u)) = \mu_k$$

in the sense of distributions in Ω for a finite measure μ_k in Ω. Moreover, if we extend both μ and μ_k by zero to $\mathbb{R}^n \setminus \Omega$ then μ_k^+ and μ_k^- converge respectively to μ^+ and μ^- weakly as measures in $\mathbb{R}^n$. Here for a (signed) measure ν, ν^+ and ν^- stand for its positive and negative parts respectively, i.e., $\nu=\nu^+-\nu^-$. The existence of such solutions to the measure datum problem (1.1) is now well known (see, e.g., [7]). Alternatively, one can also adopt the notion of *solutions obtained by limit of approximations* (SOLA) (see [3], [4], and [8]) as having been employed, e.g., in [11] and [25].

It is not hard to see that for a nonnegative locally finite measure ν in $\mathbb{R}^n$ we have

$$\mathcal{M}_1(\nu)(x) \le c(n)\mathbf{I}_1(\nu)(x) := c(n) \int_{\mathbb{R}^n} \frac{d\nu(y)}{|x-y|^{n-1}}, \quad x \in \mathbb{R}^n.$$

Thus inequality (1.4) can be viewed as an integral potential bound for gradients of solutions to (1.1). In fact, by a well-known result of Muckenhoupt and Wheeden [26] it is equivalent to use the first order Riesz potential $\mathbf{I}_1$ in place of $\mathcal{M}_1$ on the right-hand side of (1.4).

Inequality (1.4) holds also in the setting of Lorentz spaces. Recall that the Lorentz space $L^{s,t}(\Omega)$, with $0<s<\infty$ and $0<t\le\infty$, is the set of measurable functions g on Ω such that

$$\|g\|_{L^{s,t}(\Omega)} := \left(s \int_0^\infty (\alpha^s |\{x \in \Omega : |g(x)| > \alpha\}|)^{t/s} \frac{d\alpha}{\alpha} \right)^{1/t} < \infty$$

when $t\neq\infty$; for $t=\infty$ the space $L^{s,\infty}(\Omega)$ is the weak L^s or Marcinkiewicz space with quasinorm

$$\|g\|_{L^{s,\infty}(\Omega)} := \sup_{\alpha>0} \alpha |\{x \in \Omega : |g(x)| > \alpha\}|^{1/s}.$$

It is easy to see that when $t=s$ the Lorentz space $L^{s,s}(\Omega)$ is nothing but the Lebesgue space $L^s(\Omega)$.

We are now ready to state the main result of the paper.

Theorem 1.1. *Let $2-1/n<p\leq n$ and suppose that $\Omega\subset\mathbb{R}^n$ is a bounded domain whose complement $\mathbb{R}^n\setminus\Omega$ is uniformly p-thick with constants $c_0, r_0>0$. Then there exists $\varepsilon=\varepsilon(n,p,\alpha,\beta,c_0)>0$ such that for any $0<q<p+\varepsilon$ and $0<t\leq\infty$, and for any solution u to (1.1) with a finite measure μ it holds that*

$$\|\nabla u\|_{L^{q,t}(\Omega)}\leq C\|\mathcal{M}_1(\chi_\Omega|\mu|)^{1/(p-1)}\|_{L^{q,t}(\mathbb{R}^n)}. \tag{1.6}$$

Here the constant C depends only on n, p, q, t, c_0, and $\mathrm{diam}(\Omega)/r_0$.

Remark 1.2. The space $L^{q,t}(\mathbb{R}^n)$ appearing on the right-hand side of (1.6) can be replaced by $L^{q,t}(B_0)$ for any ball B_0 of radius, say, $R_0\leq 2\,\mathrm{diam}(\Omega)$ that contains Ω. Moreover, it can also be replaced by the space $L^{q,t}(\Omega)$ provided the domain Ω satisfies an additional interior density condition: there exist constants $c_1, r_1>0$ such that for all $0<t\leq r_1$ and all $x\in\Omega$ it holds that

$$|B_t(x)\cap\Omega|\geq c_1|B_t(x)|.$$

In particular, (1.6) with $L^{q,t}(\Omega)$ in place of $L^{q,t}(\mathbb{R}^n)$ holds on any Lipschitz domain Ω.

By the boundedness property of the first order fractional maximal function on Lorentz spaces we obtain the following corollary.

Corollary 1.3. *Let $2-1/n<p\leq n$ and $0<t\leq\infty$, and suppose that Ω is a bounded domain in $\mathbb{R}^n$ whose complement $\mathbb{R}^n\setminus\Omega$ is uniformly p-thick with constants $c_0, r_0>0$. Assume that $1<\gamma<n(p+\varepsilon)/(n(p-1)+p+\varepsilon)$, where $\varepsilon=\varepsilon(n,p,\alpha,\beta,c_0)>0$ is as in Theorem 1.1. Then for any solution u to (1.1) with $\mu=f\in L^{\gamma,t}(\Omega)$ it holds that*

$$\left\||\nabla u|^{p-1}\right\|_{L^{n\gamma/(n-\gamma),t}(\Omega)}\leq C\,\|f\|_{L^{\gamma,t}(\Omega)}\,.$$

Here the constant C depends only on n, p, q, t, c_0, and $\mathrm{diam}(\Omega)/r_0$.

Remark 1.4. For $\mu=f\in L^{\gamma,\gamma}(\Omega)=L^{\gamma}(\Omega)$ with $1<\gamma<np/(n(p-1)+p)$, Boccardo and Gallouët obtained in [4] the solvability of equation (1.1) with a (unique) solution $u\in W_0^{1,n\gamma(p-1)/(n-\gamma)}(\Omega)$ only under the assumption that Ω is bounded. For $1<\gamma<np/(n(p-1)+p)$, see also the papers [1], [9], and [16]. On the other hand, the Lorentz space borderline case $\gamma=np/(n(p-1)+p)$, with $p<n$, was first obtained by Mingione [24] in the local setting. The possibility of extending such local results to global ones was also mentioned without proof in the same paper. Note that since

$\varepsilon>0$ we have

$$\frac{np}{n(p-1)+p} < \frac{n(p+\varepsilon)}{n(p-1)+p+\varepsilon}.$$

We next take this opportunity to discuss a Calderón–Zygmund type estimate below the natural exponent p for $\mathcal{A}$-superharmonic functions in the whole space $\mathbb{R}^n$. For the notion of $\mathcal{A}$-superharmonicity see [13], [17], and [18]. Suppose now that u is an $\mathcal{A}$-superharmonic solution to the equation

$$\operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} F \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{1.7}$$

In a recent paper [27] we show that, for $2-1/n<p\leq n$ and $\max\{1,p-1\}<q<p$, and under a BMO type smallness condition on the nonlinearity $\mathcal{A}$, it holds that

$$\|\nabla u\|_{L^q(\mathbb{R}^n)} \leq C\, \|F\|^{1/(p-1)}_{L^{q/(p-1)}(\mathbb{R}^n)}, \tag{1.8}$$

provided that $\|\nabla u\|_{L^q(\mathbb{R}^n)}<\infty$. The following theorem shows that the norm $\|\nabla u\|_{L^q(\mathbb{R}^n)}$ is in fact finite as long as $\nabla u\in L^1(\mathbb{R}^n,\mathbb{R}^n)$. This answers a question raised by the author in [27, Remark 3.3].

Theorem 1.5. *Let* $2-1/n<p\leq n$, $\max\{1,p-1\}<q<p$ *and* $0<t\leq\infty$, *and suppose that* F *is a vector field in* $L^{q/(p-1),t/(p-1)}(\mathbb{R}^n,\mathbb{R}^n)$. *Assume that* u *is an entire* $\mathcal{A}$*-superharmonic solution of* (1.7) *such that* $\nabla u\in L^1(\mathbb{R}^n,\mathbb{R}^n)$. *Then one has the estimate*

$$\|\nabla u\|_{L^{q,t}(\mathbb{R}^n)} \leq C\, \|\nabla u\|_{L^1(\mathbb{R}^n)} + C\, \|F\|^{1/(p-1)}_{L^{q/(p-1),t/(p-1)}(\mathbb{R}^n)}, \tag{1.9}$$

where $C=C(n,p,q,t,\alpha,\beta)$.

It is worth mentioning that estimate (1.8), with $\max\{1,p-1\}<q<p$, was conjectured by T. Iwaniec in [14] to hold for all distributional solutions to (1.7). Thus Theorem 1.5 provides a solution to this conjecture when the solution u belongs to the class of $\mathcal{A}$-superharmonic functions. Here the assumption $q>1$ is essential in our approach to (1.9). As mentioned above the first term in the right-hand side of (1.9) can be dropped if $\mathcal{A}$ satisfies an additional smallness condition of BMO type. In general, we have the following existence result where the exponent q may go below 1.

Theorem 1.6. *Let $2-1/n<p<n$, $p-1<q\leq p$, and $0<t\leq\infty$. Suppose that $F\in L^{q/(p-1),t/(p-1)}(\mathbb{R}^n,\mathbb{R}^n)$ with $-\operatorname{div} F\geq 0$ in $\mathcal{D}'(\mathbb{R}^n)$. Then there exists an entire nonnegative $\mathcal{A}$-superharmonic solution of (1.7) such that*

$$\|u\|_{L^{nq/(n-q),t}(\mathbb{R}^n)}+\|\nabla u\|_{L^{q,t}(\mathbb{R}^n)}\leq C\,\|F\|^{1/(p-1)}_{L^{q/(p-1),t/(p-1)}(\mathbb{R}^n)},$$

where $C=C(n,p,q,t,\alpha,\beta)$.

Remark 1.7. If $p-1<q\leq n(p-1)/(n-1)$ then by [28, Theorem 3.1] we have $\operatorname{div} F=0$. Thus in this case the solution u obtained in Theorem 1.6 is identically zero. This also implies that Theorem 1.6 holds as well in the case $p=n$, with $u\equiv 0$ being a valid nonnegative solution.

The proofs of Theorems 1.1, 1.5, and 1.6 will be presented in Section 4.

2. Interior and boundary comparison estimates

Following Mingione [24], in order to prove Theorem 1.1 we need to obtain certain local interior and boundary comparison estimates. First let us consider the interior ones. With $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$, for each ball $B_{2R}=B_{2R}(x_0)\Subset\Omega$ we defined $w\in u+W^{1,p}_0(B_{2R})$ as the unique solution to the Dirichlet problem

$$\begin{cases}\operatorname{div}\mathcal{A}(x,\nabla w)=0 & \text{in } B_{2R},\\ w=u & \text{on } \partial B_{2R}.\end{cases}\tag{2.1}$$

Then a well-known version of Gehring's lemma applied to the function w defined above yields the following result (see [12, Theorem 6.7] and [12, Remark 6.12]).

Lemma 2.1. *With $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$, let w be as in (2.1). Then there exists a constant $\theta_0=\theta_0(n,p,\alpha,\beta)>1$ such that for any $t\in(0,p]$ the reverse Hölder type inequality*

$$\left(\frac{1}{|B_{\rho/2}(z)|}\int_{B_{\rho/2}(z)}|\nabla w|^{\theta_0 p}\,dx\right)^{1/\theta_0 p}\leq C\left(\frac{1}{|B_\rho(z)|}\int_{B_\rho(z)}|\nabla w|^t\,dx\right)^{1/t}$$

holds for all balls $B_\rho(z)\subset B_{2R}(x_0)$ with a constant C depending only on n, p, α, β, and t.

It is worth mentioning that the approach of using this kind of reverse Hölder's inequalities with arbitrarily small exponents in the context of measure datum problems has been first implemented by Mingione in the paper [23].

The following important comparison lemma involving an estimate "below the natural growth exponent" was also established in [23] (see also [11, Lemma 3.3]) for the degenerate case $p\geq 2$. This lemma was later obtained in [10, Lemma 4.2] for the singular case $2-1/n<p<2$.

Lemma 2.2. *With $p>2-1/n$, let $u\in W^{1,p}_{\rm loc}(\Omega)$ be a solution of (1.1) and let w be as in (2.1). Then there is a constant $C=C(n,p,\alpha,\beta)$ such that*

$$\frac{1}{|B_{2R}|}\int_{B_{2R}}|\nabla u-\nabla w|\,dx \leq C\left(\frac{|\mu|(B_{2R})}{R^{n-1}}\right)^{1/(p-1)}+C\frac{|\mu|(B_{2R})}{R^{n-1}}\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}|\nabla u|\,dx\right)^{2-p}.$$

Moreover, when $p\geq 2$ the second term on the right-hand side can be dropped.

Next we consider the counterparts of Lemmas 2.1 and 2.2 up to the boundary. As $\mathbb{R}^n\setminus\Omega$ is uniformly p-thick with constants $c_0, r_0>0$, there exists $1<p_0=p_0(n,p,c_0)<p$ such that $\mathbb{R}^n\setminus\Omega$ is uniformly p_0-thick with constants $c_*=c(n,p,c_0)$ and r_0. This is by now a classical result due to Lewis [19] (see also [22]). Moreover, p_0 can be chosen near p so that $p_0\in(np/(n+p),p)$. Thus, since $p_0<n$, we have

$$\operatorname{cap}_{p_0}(\overline{B_t(x)}\cap(\mathbb{R}^n\setminus\Omega), B_{2t}(x))\geq c_*\operatorname{cap}_{p_0}(\overline{B_t(x)}, B_{2t}(x))\geq C(n,p,c_0)t^{n-p_0} \tag{2.2}$$

for all $0<t\leq r_0$ and all $x\in\mathbb{R}^n\setminus\Omega$.

Now let $x_0\in\partial\Omega$ be a boundary point and for $0<2R\leq r_0$ we set $\Omega_{2R}=\Omega_{2R}(x_0)=B_{2R}(x_0)\cap\Omega$. For $u\in W^{1,p}_0(\Omega)$ we consider the unique solution $w\in u+W^{1,p}_0(\Omega_{2R})$ to the equation

$$\begin{cases}\operatorname{div}\mathcal{A}(x,\nabla w)=0 & \text{in } \Omega_{2R},\\ w=u & \text{on } \partial\Omega_{2R}.\end{cases} \tag{2.3}$$

In what follows we extend μ and u by zero to $\mathbb{R}^n\setminus\Omega$ and then extend w by u to $\mathbb{R}^n\setminus\Omega_{2R}$.

Lemma 2.3. *With $u\in W_0^{1,p}(\Omega)$, let w be as in (2.3). Then there exists a constant $\theta_0=\theta_0(n,p,\alpha,\beta,c_0)>1$ such that the reverse Hölder type inequality*

$$\left(\frac{1}{|B_{\rho/2}(z)|}\int_{B_{\rho/2}(z)}|\nabla w|^{\theta_0 p}\,dx\right)^{1/\theta_0}\le\frac{C}{|B_{11\rho/4}(z)|}\int_{B_{11\rho/4}(z)}|\nabla w|^p\,dx$$

holds for all balls $B_{11\rho/4}(z)\subset B_{2R}(x_0)$ with a constant C depending only on n, p, α, β, and c_0.

Proof. By Gehring's lemma it is enough to show that there exists $\varepsilon\in(0,1)$ such that

$$\left(\frac{1}{|B_{\rho/2}(z)|}\int_{B_{\rho/2}(z)}|\nabla w|^p\,dx\right)^{\varepsilon}\le\frac{C}{|B_{11\rho/4}(z)|}\int_{B_{11\rho/4}(z)}|\nabla w|^{\varepsilon p}\,dx \tag{2.4}$$

for all balls $B_{11\rho/4}(z)\subset B_{2R}(x_0)$.

Inequality (2.4) obviously holds when $B_\rho(z)\subset\mathbb{R}^n\setminus\Omega$. Next we suppose that $B_\rho(z)\subset\Omega$. Let $\varphi\in C_0^\infty(B_\rho(z))$ be such that $0\le\varphi\le1$, $\varphi\equiv1$ in $B_{\rho/2}(z)$ and $|\nabla\varphi|\le c/\rho$. Then using

$$\phi=(w-\overline{w}_{B_\rho(z)})\varphi^p,\quad\text{with } \overline{w}_{B_\rho(z)}=\frac{1}{|B_\rho(z)|}\int_{B_\rho(z)}w\,dy,$$

as a test function for (2.3) we find that

$$\int_{B_\rho(z)}|\nabla w|^p\varphi^p\,dx\le C\int_{B_\rho(z)}|\nabla w|^{p-1}|\nabla\varphi|\varphi^{p-1}|w-\overline{w}_{B_\rho(z)}|\,dx.$$

Thus by Hölder's inequality we get that

$$\int_{B_{\rho/2}(z)}|\nabla w|^p\,dx\le\frac{C}{\rho^p}\int_{B_\rho(z)}|w-\overline{w}_{B_\rho(z)}|^p\,dx.$$

This yields

$$\left(\frac{1}{|B_{\rho/2}(z)|}\int_{B_{\rho/2}(z)}|\nabla w|^p\,dx\right)^{1/p}\le C\left(\frac{1}{|B_\rho(z)|}\int_{B_\rho(z)}|\nabla w|^{mp}\,dx\right)^{1/mp}$$

by the Poincaré–Sobolev inequality, where

$$m=\begin{cases}n/(n+p), & \text{if } np/(n+p)\ge1,\\ 1/p, & \text{if } np/(n+p)<1.\end{cases}$$

Hence, we obtain (2.4) with $\varepsilon=m$.

Finally, we consider the case $B_\rho(z)\cap\partial\Omega\neq\varnothing$. In this case we choose $z_0\in\partial\Omega$ such that $|z-z_0|=\operatorname{dist}(z,\partial\Omega)$. Then $|z-z_0|<\rho$ and thus

$$B_{\rho/2}(z)\subset B_{3\rho/2}(z_0)\subset B_{7\rho/4}(z_0)\subset B_{11\rho/4}(z)\subset B_{2R}(x_0).$$

Let $\varphi\in C_0^\infty(B_{7\rho/4}(z_0))$ be such that $0\le\varphi\le1$, $\varphi\equiv1$ in $B_{3\rho/2}(z_0)$ and $|\nabla\varphi|\le c/\rho$. Using $\phi=w\varphi^p$ as a test function for (2.3) we find that

$$\int_{B_{3\rho/2}(z_0)}|\nabla w|^p\,dx\le\frac{C}{\rho^p}\int_{B_{7\rho/4}(z_0)}|w|^p\,dx.$$

Recall now that $\mathbb{R}^n\setminus\Omega$ is uniformly p_0-thick for some $p_0\in(np/(n+p),p)$. Thus $p<p_0n/(n-p_0)$ and by Hölder's inequality we get

$$\begin{aligned}\biggl(\frac{1}{|B_{\rho/2}(z)|}\int_{B_{\rho/2}(z)}|\nabla w|^p\,dx\biggr)^{1/p}&\\ \le\frac{C}{\rho}\biggl(\frac{1}{|B_{7\rho/4}(z_0)|}&\int_{B_{7\rho/4}(z_0)}|w|^p\,dx\biggr)^{1/p}\\ \le\frac{C}{\rho}\biggl(\frac{1}{|B_{7\rho/4}(z_0)|}&\int_{B_{7\rho/4}(z_0)}|w|^{np_0/(n-p_0)}\,dx\biggr)^{(n-p_0)/np_0}.\end{aligned}$$

On the other hand, with $K=\{x\in\overline{B}_{7\rho/4}(z_0):w(x)=0\}$, by a Sobolev type inequality (see Lemma 8.11 and Remark 8.14 in [22])

$$\begin{aligned}\biggl(\frac{1}{|B_{7\rho/4}(z_0)|}\int_{B_{7\rho/4}(z_0)}|w|^{np_0/(n-p_0)}\,dx\biggr)^{(n-p_0)/np_0}&\\ \le C\biggl(\frac{1}{\operatorname{cap}_{p_0}(K,B_{7\rho/2}(z_0))}&\int_{B_{7\rho/4}(z_0)}|\nabla w|^{p_0}\,dx\biggr)^{1/p_0}\\ \le C\biggl(\rho^{p_0}\frac{1}{|B_{7\rho/4}(z_0)|}&\int_{B_{7\rho/4}(z_0)}|\nabla w|^{p_0}\,dx\biggr)^{1/p_0},\end{aligned}$$

where we used (2.2) in the last inequality which is valid since $7\rho/4<11\rho/4\le2R\le r_0$. These inequalities yield

$$\biggl(\frac{1}{|B_{\rho/2}(z)|}\int_{B_{\rho/2}(z)}|\nabla w|^p\,dx\biggr)^{1/p}\le C\biggl(\frac{1}{|B_{11\rho/4}(z)|}\int_{B_{11\rho/4}(z)}|\nabla w|^{p_0}\,dx\biggr)^{1/p_0},$$

and thus we get (2.4) with $\varepsilon=p_0/p\in(0,1)$. □

On the other hand, arguing as in [12, Remark 6.12] (see also [10, Lemma 3.2]) we have following lemma.

Lemma 2.4. *Let $A\subset\mathbb{R}^n$ be an open set and let $f\colon A\to\mathbb{R}$ be an integrable function such that*

$$\left(\frac{1}{|B_\rho|}\int_{B_\rho}|f|^{\theta_0}\,dx\right)^{1/\theta_0}\leq\frac{C}{|B_{11\rho/2}|}\int_{B_{11\rho/2}}|f|\,dx$$

for all concentric balls $B_\rho\subset B_{11\rho/2}\subset A$, where $\theta_0>1$ and $C\geq 0$. Then for every $t\in(0,1]$ and $\theta\in(0,\theta_0]$ there exists a constant $C_0=C_0(n,C,t)$ such that

$$\left(\frac{1}{|B_\rho|}\int_{B_\rho}|f|^{\theta}\,dx\right)^{1/\theta}\leq C_0\left(\frac{1}{|B_{6\rho}|}\int_{B_{6\rho}}|f|^{t}\,dx\right)^{1/t}$$

for all concentric balls $B_\rho\subset B_{6\rho}\subset A$.

Thus combining the last two lemmas we obtain the following reverse Hölder type inequality, a version of Lemma 2.1 up to the boundary.

Lemma 2.5. *With $u\in W_0^{1,p}(\Omega)$, let w be as in (2.3). Then there exists a constant $\theta_0=\theta_0(n,p,\alpha,\beta,c_0)>1$ such that for every $t\in(0,p]$ the reverse Hölder type inequality*

$$\left(\frac{1}{|B_{\rho/2}(z)|}\int_{B_{\rho/2}(z)}|\nabla w|^{\theta_0 p}\,dx\right)^{1/\theta_0 p}\leq C\left(\frac{1}{|B_{3\rho}(z)|}\int_{B_{3\rho}(z)}|\nabla w|^{t}\,dx\right)^{1/t}$$

holds for all balls $B_{3\rho}(z)\subset B_{2R}(x_0)$ with a constant $C=C(n,p,t,\alpha,\beta,c_0)$.

We also have a counterpart of Lemma 2.2 up to the boundary.

Lemma 2.6. *With $p>2-1/n$, let $u\in W_0^{1,p}(\Omega)$ be a solution of (1.1) and let w be as in (2.3). Then there is a constant $C=C(n,p,\alpha,\beta)$ such that*

$$\frac{1}{|B_{2R}|}\int_{B_{2R}}|\nabla u-\nabla w|\,dx$$
$$\leq C\left(\frac{|\mu|(B_{2R})}{R^{n-1}}\right)^{1/(p-1)}+C\frac{|\mu|(B_{2R})}{R^{n-1}}\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}|\nabla u|\,dx\right)^{2-p}.$$

Moreover, when $p\geq 2$ the second term on the right-hand side can be dropped.

Proof. A proof of this lemma can be obtained by the method of [23], [11], [10], and [25] that was implemented for the interior situation, i.e., Lemma 2.2. Here, to avoid a scaling argument, we choose to present a slightly different approach based on a technique in [2]. Note that u, w, and μ are all zero outside Ω. Since both u and w are solutions we find that

$$\int_{\Omega_{2R}} \langle \mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla w),\nabla\varphi\rangle\,dx=\int_{\Omega_{2R}}\varphi\,d\mu \tag{2.5}$$

for every $\varphi\in W_0^{1,p}(\Omega_{2R})$. Thus choosing $\varphi=T_k(u-w)$, $k>0$, in (2.5) we have

$$\int_{\{x\in\Omega_{2R}:|u(x)-w(x)|<k\}} g(u,w)(x)\,dx\le ck|\mu|(\Omega_{2R}), \tag{2.6}$$

where we set

$$g(u,w)=(|\nabla u|^2+|\nabla w|^2)^{(p-2)/2}|\nabla(u-w)|^2.$$

For $k,\lambda\ge 0$ we now put

$$\Phi(k,\lambda)=|\{x\in\Omega_{2R}:|u(x)-w(x)|>k \text{ and } g(u,w)(x)>\lambda\}|.$$

As the map $\lambda\mapsto\Phi(k,\lambda)$ is nonincreasing we find that

$$\Phi(0,\lambda)\le\frac{1}{\lambda}\int_0^\lambda\Phi(0,s)\,ds\le\Phi(k,0)+\frac{1}{\lambda}\int_0^\lambda[\Phi(0,s)-\Phi(k,s)]\,ds.$$

Thus

$$\begin{aligned}\Phi(0,\lambda)&\le|\{x\in\Omega_{2R}:|u(x)-w(x)|>k\}|\\&\qquad+\frac{1}{\lambda}\int_0^\lambda|\{x\in\Omega_{2R}:|u(x)-w(x)|<k \text{ and } g(u,w)(x)>s\}|\,ds\\&\le|\{x\in\Omega_{2R}:|u(x)-w(x)|>k\}|+\frac{1}{\lambda}\int_{\{x\in\Omega_{2R}:|u(x)-w(x)|<k\}}g(u,w)\,dx\\&\le|\{x\in\Omega_{2R}:|u(x)-w(x)|>k\}|+\frac{1}{\lambda}ck|\mu|(\Omega_{2R}),\end{aligned}$$

where (2.6) was used in the last inequality. Using the Sobolev inequality this gives that

$$\Phi(0,\lambda)\le ck^{-n/(n-1)}\,\|\nabla u-\nabla w\|_{L^1(\Omega_{2R})}^{n/(n-1)}+\frac{1}{\lambda}ck|\mu|(\Omega_{2R})$$

which holds for all $k>0$. Choosing

$$k=\frac{[\lambda\,\|\nabla u-\nabla w\|_{L^1(\Omega_{2R})}^{n/(n-1)}/|\mu|(\Omega_{2R})]^{(n-1)}}{2n-1}$$

in the above inequality we arrive at

$$\lambda^{n/(2n-1)}\Phi(0,\lambda)\le c|\mu|(\Omega_{2R})^{n/(2n-1)}\,\|\nabla u-\nabla w\|_{L^1(\Omega_{2R})}^{n/(2n-1)}.$$

Letting $\lambda=s^p$ this yields

$$\|g(u,w)^{1/p}\|_{L^{np/(2n-1),\infty}(\Omega_{2R})}\leq c|\mu|(\Omega_{2R})^{1/p}\,\|\nabla u-\nabla w\|^{1/p}_{L^1(\Omega_{2R})},$$

and by Hölder's inequality

$$\|g(u,w)^{1/p}\|_{L^1(\Omega_{2R})}\leq c|\mu|(\Omega_{2R})^{1/p}|\Omega_{2R}|^{1-(2n-1)/np}\,\|\nabla u-\nabla w\|^{1/p}_{L^1(\Omega_{2R})}, \tag{2.7}$$

where we used the fact that $p>2-1/n$.

We next consider separately the case $p\geq 2$ and the case $2-1/n<p<2$. For $p\geq 2$ using the pointwise bound

$$|\nabla u-\nabla w|\leq g(u,w)^{1/p}$$

coupled with inequality (2.7) we easily obtain the desired result. For $2-1/n<p<2$ we write

$$\begin{aligned}|\nabla u-\nabla w| &= g(u,w)^{1/2}(|\nabla u|^2+|\nabla w|^2)^{(2-p)/4}\\ &\leq cg(u,w)^{1/2}(|\nabla u-\nabla w|^{(2-p)/2}+|\nabla u|^{(2-p)/2})\\ &\leq cg(u,w)^{1/p}+\tfrac{1}{2}|\nabla u-\nabla w|+cg(u,w)^{1/2}|\nabla u|^{(2-p)/2},\end{aligned}$$

and thus

$$|\nabla u-\nabla w|\leq cg(u,w)^{1/p}+cg(u,w)^{1/2}|\nabla u|^{(2-p)/2}.$$

Using this and Hölder's inequality we get that

$$\|\nabla u-\nabla w\|_{L^1(\Omega_{2R})}\leq c\|g(u,w)^{1/p}\|_{L^1(\Omega_{2R})}+c\|g(u,w)^{1/p}\|^{p/2}_{L^1(\Omega_{2R})}\,\|\nabla u\|^{(2-p)/2}_{L^1(\Omega_{2R})}.$$

By (2.7) this yields

$$\begin{aligned}\|\nabla u-\nabla w\|_{L^1(\Omega_{2R})}\leq{}& c|\mu|(\Omega_{2R})^{1/p}|\Omega_{2R}|^{1-(2n-1)/np}\,\|\nabla u-\nabla w\|^{1/p}_{L^1(\Omega_{2R})}\\ &+c|\mu|(\Omega_{2R})^{1/2}|\Omega_{2R}|^{(1-(2n-1)/np)p/2}\\ &\times\|\nabla u-\nabla w\|^{1/2}_{L^1(\Omega_{2R})}\,\|\nabla u\|^{(2-p)/2}_{L^1(\Omega_{2R})},\end{aligned}$$

or

$$\begin{aligned}\|\nabla u-\nabla w\|^{1/2}_{L^1(\Omega_{2R})}\leq{}& c|\mu|(\Omega_{2R})^{1/p}|\Omega_{2R}|^{1-(2n-1)/np}\,\|\nabla u-\nabla w\|^{1/p-1/2}_{L^1(\Omega_{2R})}\\ &+c|\mu|(\Omega_{2R})^{1/2}|\Omega_{2R}|^{(1-(2n-1)/np)p/2}\,\|\nabla u\|^{(2-p)/2}_{L^1(\Omega_{2R})}.\end{aligned}$$

Thus using Young's inequality for the first term on the right-hand side we get that

$$\begin{aligned}\|\nabla u-\nabla w\|^{1/2}_{L^1(\Omega_{2R})}\leq{}& c|\mu|(\Omega_{2R})^{1/2(p-1)}|\Omega_{2R}|^{(1-(2n-1)/np)p/2(p-1)}\\ &+c|\mu|(\Omega_{2R})^{1/2}|\Omega_{2R}|^{(1-(2n-1)/np)p/2}\,\|\nabla u\|^{(2-p)/2}_{L^1(\Omega_{2R})}.\end{aligned}$$

The desired result is easily seen to follow from the last inequality. □

3. Applications of comparison estimates

Our approach to Theorem 1.1 is based on the following technical lemma which allows one to work with balls instead of cubes. A version of this lemma appeared for the first time in [31]. It can be viewed as a version of the Calderón–Zygmund–Krylov–Safonov decomposition that has been used in [6] and [24]. A proof of this lemma, which uses Lebesgue's differentiation theorem and the standard Vitali covering lemma, can be found in [5] with obvious modifications to fit the setting here.

Lemma 3.1. *Assume that $A\subset\mathbb{R}^n$ is a measurable set for which there exist $c_1, r_1>0$ such that*

$$|B_t(x)\cap A|\geq c_1|B_t(x)| \tag{3.1}$$

holds for all $x\in A$ and $0<t\leq r_1$. Fix $0<r\leq r_1$ and let $C\subset D\subset A$ be measurable sets for which there exists $0<\varepsilon<1$ such that

(1) $|C|<\varepsilon r^n|B_1|$;

(2) *for all $x\in A$ and $\rho\in(0,r]$, if $|C\cap B_\rho(x)|\geq\varepsilon|B_\rho(x)|$, then $B_\rho(x)\cap A\subset D$.*

Then we have the estimate

$$|C|\leq\frac{\varepsilon}{c_1}|D|.$$

We now recall that for a function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the Hardy–Littlewood maximal function of f is defined by

$$\mathcal{M}(f)(x)=\sup_{r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|\,dy.$$

In order to apply Lemma 3.1 we need the following proposition, whose proof relies essentially on the comparison estimates obtained in the previous section.

Proposition 3.2. *There exist constants $A,\theta_0>1$, depending only on n, p, α, β, and c_0, so that the following holds for any $T>1$ and any $\lambda>0$. Suppose that $u\in W^{1,p}_0(\Omega)$ is a weak solution of (1.1) with $\mathcal{A}$ satisfying (1.2) and (1.3). Assume that for some ball $B_\rho(y)$ with $16\rho\leq r_0$ we have*

$$\{x\in B_\rho(y):\mathcal{M}(\chi_\Omega|\nabla u|)(x)\leq\lambda \text{ and } \mathcal{M}_1(\chi_\Omega|\mu|)(x)^{1/(p-1)}\leq\varepsilon(T)\lambda\}\neq\varnothing,$$

where $\varepsilon(T)$ is defined by

$$\varepsilon(T)=\begin{cases}T^{-p\theta_0+1}, & \text{if } p\geq 2,\\ T^{(-p\theta_0+1)/(p-1)}, & \text{if } 2-1/n<p<2.\end{cases} \tag{3.2}$$

Then

$$|\{x\in B_\rho(y):\mathcal{M}(\chi_\Omega|\nabla u|)(x)>AT\lambda\}|<T^{-p\theta_0}|B_\rho(y)|. \tag{3.3}$$

Proof. By hypothesis, there exists $x_0 \in B_\rho(y)$ such that for any $r>0$,

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \chi_\Omega |\nabla u|\, dz \le \lambda \quad \text{and} \quad \frac{r}{|B_r(x_0)|}\int_{B_r(x_0)} \chi_\Omega \, d|\mu| \le [\varepsilon(T)\lambda]^{p-1}. \tag{3.4}$$

We first claim that for $x\in B_\rho(y)$ we have

$$\mathcal{M}(\chi_\Omega|\nabla u|)(x) \le \max\{\mathcal{M}(\chi_{B_{2\rho}(y)\cap\Omega}|\nabla u|)(x), 3^n\lambda\}. \tag{3.5}$$

Indeed, for $r\le\rho$ we have $B_r(x)\cap\Omega\subset B_{2\rho}(y)\cap\Omega$ and thus

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} \chi_\Omega|\nabla u|\, dz = \frac{1}{|B_r(x)|}\int_{B_r(x)} \chi_{B_{2\rho}(y)\cap\Omega}|\nabla u|\, dz,$$

whereas for $r>\rho$ we have $B_r(x)\subset B_{3r}(x_0)$ from which by (3.4) yields

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} \chi_\Omega|\nabla u|\, dz \le 3^n \frac{1}{|B_{3r}(x_0)|}\int_{B_{3r}(x_0)} \chi_\Omega|\nabla u|\, dz \le 3^n\lambda.$$

In view of (3.5) we see that (3.3) trivially holds provided that $A\ge 3^n$ and $B_{4\rho}(y)\subset\mathbb{R}^n\setminus\Omega$. Thus it is enough to consider the case $B_{4\rho}(y)\subset\Omega$ and the case $B_{4\rho}(y)\cap\partial\Omega\ne\varnothing$.

First we consider the case when $B_{4\rho}(y)\subset\Omega$. Let $w\in u+W_0^{1,p}(B_{4\rho}(y))$ be the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}\mathcal{A}(x,\nabla w)=0 & \text{in } B_{4\rho}(y), \\ w=u & \text{on } \partial B_{4\rho}(y). \end{cases} \tag{3.6}$$

By weak type $(1,1)$ estimates for the maximal function we have

$$\begin{aligned} &|\{x\in B_\rho(y): \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|)(x) > AT\lambda\}| \\ &\quad\le |\{x\in B_\rho(y): \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla w|)(x) > AT\lambda/2\}| \\ &\qquad + |\{x\in B_\rho(y): \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u-\nabla w|)(x) > AT\lambda/2\}| \\ &\quad\le C(AT\lambda)^{-p\theta_0}\int_{B_{2\rho}(y)} |\nabla w|^{p\theta_0}\, dx + C(AT\lambda)^{-1}\int_{B_{2\rho}(y)} |\nabla u-\nabla w|\, dx. \end{aligned}$$

Note that by Lemma 2.1 we have

$$\begin{aligned} \left(\frac{1}{|B_{2\rho}(y)|}\int_{B_{2\rho}(y)} |\nabla w|^{p\theta_0}\, dx\right)^{1/p\theta_0} &\le \frac{C}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)} |\nabla w|\, dx \\ &\le \frac{C}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)} |\nabla u|\, dx \\ &\quad + \frac{C}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)} |\nabla u-\nabla w|\, dx \end{aligned}$$

and thus

$$(3.7)\quad |\{x\in B_\rho(y):\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|)(x)>AT\lambda\}|$$
$$\leq C(AT\lambda)^{-p\theta_0}|B_\rho(y)|\left(\frac{1}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)}|\nabla u|\,dx\right)^{p\theta_0}$$
$$+C(AT\lambda)^{-p\theta_0}|B_\rho(y)|\left(\frac{1}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)}|\nabla u-\nabla w|\,dx\right)^{p\theta_0}$$
$$+C(AT\lambda)^{-1}|B_\rho(y)|\frac{1}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)}|\nabla u-\nabla w|\,dx.$$

On the other hand, by Lemma 2.2 we have that

$$(3.8)\quad \frac{1}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)}|\nabla u-\nabla w|\,dx$$
$$\leq C\left(\frac{|\mu|(B_{5\rho}(x_0))}{\rho^{n-1}}\right)^{1/(p-1)}+C\frac{|\mu|(B_{5\rho}(x_0))}{\rho^{n-1}}\left(\frac{1}{|B_{5\rho}(x_0)|}\int_{B_{5\rho}(x_0)}|\nabla u|\,dx\right)^{2-p},$$

where the last term should be dropped when $p\geq 2$. Thus by (3.4) and the definition of $\varepsilon(T)$ we get that

$$\frac{1}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)}|\nabla u-\nabla w|\,dx\leq CT^{-p\theta_0+1}\lambda$$

if $p\geq 2$, and

$$\frac{1}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)}|\nabla u-\nabla w|\,dx\leq CT^{(-p\theta_0+1)/(p-1)}\lambda+CT^{-p\theta_0+1}\lambda$$

if $2-1/n<p<2$. In any case, since $T>1$, we have

$$(3.9)\qquad \frac{1}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)}|\nabla u-\nabla w|\,dx\leq CT^{-p\theta_0+1}\lambda.$$

At this point combining (3.7), (3.9) and using that $T>1$ we find that

$$|\{x\in B_\rho(y):\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|)(x)>AT\lambda\}|\leq (CA^{-p\theta_0}+CA^{-1})T^{-p\theta_0}|B_\rho(y)|.$$

We now choose A so that $A\geq 3^n$ and $2CA^{-1}\leq\frac{1}{2}$, i.e., $A\geq\max\{3^n,4C\}$. Then we have that

$$|\{x\in B_\rho(y):\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|)(x)>AT\lambda\}|\leq \tfrac{1}{2}T^{-p\theta_0}|B_\rho(y)|,$$

which in view of (3.5) yields (3.3).

Finally, we consider the case when $B_{4\rho}(y)\cap\partial\Omega\neq\varnothing$. Let $y_0\in\partial\Omega$ be a boundary point such that $|y-y_0|=\mathrm{dist}(y,\partial\Omega)$. Define $w\in u+W_0^{1,p}(\Omega_{16\rho}(y_0))$ as the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}\mathcal{A}(x,\nabla w)=0 & \text{in } \Omega_{16\rho}(y_0),\\ w=u & \text{on } \partial\Omega_{16\rho}(y_0).\end{cases}$$

Here we also extend u by zero to $\mathbb{R}^n\setminus\Omega$ and then extend w by u to $\mathbb{R}^n\setminus\Omega_{16\rho}(y_0)$. As in (3.7) in this case we have

$$\begin{aligned}(3.10)\quad &|\{x\in B_\rho(y):\mathcal{M}(\chi_{\Omega_{2\rho}(y)}|\nabla u|)(x)>AT\lambda\}|\\ &\quad\le C(AT\lambda)^{-p\theta_0}|B_\rho(y)|\bigg(\frac{1}{|B_{12\rho}(y)|}\int_{B_{12\rho}(y)}|\nabla u|\,dx\bigg)^{p\theta_0}\\ &\quad\quad+C(AT\lambda)^{-p\theta_0}|B_\rho(y)|\bigg(\frac{1}{|B_{12\rho}(y)|}\int_{B_{12\rho}(y)}|\nabla u-\nabla w|\,dx\bigg)^{p\theta_0}\\ &\quad\quad+C(AT\lambda)^{-1}|B_\rho(y)|\frac{1}{|B_{12\rho}(y)|}\int_{B_{12\rho}(y)}|\nabla u-\nabla w|\,dx,\end{aligned}$$

where Lemma 2.5 is used in stead of Lemma 2.1. Since

$$B_{12\rho}(y)\subset B_{16\rho}(y_0)\subset B_{20\rho}(y)\subset B_{21\rho}(x_0)$$

by Lemma 2.6, as in (3.9), we find that

$$(3.11)\qquad \frac{1}{|B_{12\rho}(y)|}\int_{B_{12\rho}(y)}|\nabla u-\nabla w|\,dx\le CT^{-p\theta_0+1}\lambda.$$

Inequalities (3.10) and (3.11) and the fact that $T>1$ now yield

$$|\{x\in B_\rho(y):\mathcal{M}(\chi_{\Omega_{2\rho}(y)}|\nabla u|)(x)>AT\lambda\}|\le(CA^{-p\theta_0}+CA^{-1})T^{-p\theta_0}|B_\rho(y)|,$$

and thus we arrive at

$$|\{x\in B_\rho(y):\mathcal{M}(\chi_{\Omega_{2\rho}(y)}|\nabla u|)(x)>AT\lambda\}|\le\tfrac{1}{2}T^{-p\theta_0}|B_\rho(y)|$$

provided $A\ge\max\{3^n,4C\}$. The last bound and (3.5) yield (3.3) as desired. □

Remark 3.3. By approximation Proposition 3.2 continues to hold without assuming that $u\in W_0^{1,p}(\Omega)$. To this end, let $u_k=T_k(u)$ for each integer $k>0$. Then by our notion of solutions $u_k\in W_0^{1,p}(\Omega)$ solves

$$(3.12)\qquad -\operatorname{div}\mathcal{A}(x,\nabla u_k)=\mu_k$$

for a finite measure μ_k in Ω. Moreover, if we extend both μ and μ_k by zero to $\mathbb{R}^n\setminus\Omega$ then μ_k^+ and μ_k^- converge respectively to μ^+ and μ^- weakly as measures in $\mathbb{R}^n$. This implies in particular that

$$\limsup_{k\to\infty}|\mu_k|(B_r(z))\leq|\mu|(\overline{B_r(z)}) \tag{3.13}$$

for any ball $B_r(z)\subset\mathbb{R}^n$. To show (3.3) it is enough to consider the case $B_{4\rho}(y)\subset\Omega$ as the case $B_{4\rho}(y)\cap\partial\Omega\neq\varnothing$ is just similar. By working with (3.12) then instead of (3.8) we have

$$\begin{aligned}\frac{1}{|B_{4\rho}(y)|}&\int_{B_{4\rho}(y)}|\nabla u_k-\nabla w_k|\,dx\\&\leq C\biggl(\frac{|\mu_k|(B_{5\rho}(x_0))}{\rho^{n-1}}\biggr)^{1/(p-1)}+C\frac{|\mu_k|(B_{5\rho}(x_0))}{\rho^{n-1}}\biggl(\frac{1}{|B_{5\rho}(x_0)|}\int_{B_{5\rho}(x_0)}|\nabla u_k|\,dx\biggr)^{2-p},\end{aligned}$$

where the last term should be dropped when $p\geq2$. Here w_k is the solution of (3.6) with u_k in place of u. Thus using (3.4) and (3.13) we have the following analogue of (3.9)

$$\limsup_{k\to\infty}\frac{1}{|B_{4\rho}(y)|}\int_{B_{4\rho}(y)}|\nabla u_k-\nabla w_k|\,dx\leq CT^{-p\theta_0+1}\lambda,$$

from which we obtain, for large enough A, that

$$\limsup_{k\to\infty}|\{x\in B_\rho(y):\mathcal{M}(\chi_\Omega|\nabla u_k|)(x)>AT\lambda\}|\leq\tfrac{1}{2}T^{-p\theta_0}|B_\rho(y)|. \tag{3.14}$$

In equality (3.3) (with $2A$ in place of A) follows from (3.14) by writing

$$\begin{aligned}|\{x\in B_\rho(y):&\mathcal{M}(\chi_\Omega|\nabla u|)(x)>2AT\lambda\}|\\&\leq|\{x\in B_\rho(y):\mathcal{M}(\chi_\Omega|\nabla u_k|)(x)>AT\lambda\}|\\&\quad+|\{x\in B_\rho(y):\mathcal{M}(\chi_\Omega|\nabla u-\nabla u_k|)(x)>AT\lambda\}|\end{aligned}$$

and using the weak type $(1,1)$ bound of the maximal function.

We remark that the above argument works equally well for *solutions obtained by limit of approximations* (SOLA) as property (3.13) holds also for the approximating measures in that case (see, e.g., [11, Section 5]).

Proposition 3.2 can be restated as follows.

Proposition 3.4. *There exist constants $A, \theta_0 > 1$, depending only on n, p, α, β, and c_0, so that the following holds for any $T > 1$ and any $\lambda > 0$. Let u be a solution of (1.1) with $\mathcal{A}$ satisfying (1.2) and (1.3). Suppose that for some ball $B_\rho(y)$ with $16\rho \leq r_0$ we have*

$$|\{x \in B_\rho(y) : \mathcal{M}(\chi_\Omega |\nabla u|)(x) > AT\lambda\}| \geq T^{-p\theta_0} |B_\rho(y)|.$$

Then

$$B_\rho(y) \subset \{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega |\nabla u|)(x) > \lambda \text{ or } \mathcal{M}_1(\chi_\Omega |\mu|)(x)^{1/(p-1)} > \varepsilon(T)\lambda\},$$

where $\varepsilon(T)$ is as defined in (3.2).

We can now apply Lemma 3.1 and the last proposition to get the following result.

Lemma 3.5. *There exist constants $A, \theta_0 > 1$, depending only on n, p, α, β, and c_0, so that the following holds for any $T > 1$. Let u be a solution of (1.1) with $\mathcal{A}$ satisfying (1.2) and (1.3). Let B_0 be a ball of radius R_0. Fix a real number $0 < r \leq \min\{r_0, 2R_0\}/16$ and suppose that there exists $N > 0$ such that*

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega |\nabla u|)(x) > N\}| < T^{-p\theta_0} r^n |B_1|. \tag{3.15}$$

Then for any integer $k \geq 0$ it holds that

$$\begin{aligned} 4|\{x \in B_0 : \mathcal{M}(\chi_\Omega |\nabla u|)(x) > N(AT)^{k+1}\}| \\ \leq c(n) T^{-p\theta_0} |\{x \in B_0 : \mathcal{M}(\chi_\Omega |\nabla u|)(x) > N(AT)^k\}| \\ + c(n) |\{x \in B_0 : \mathcal{M}_1(\chi_\Omega |\mu|)(x)^{1/(p-1)} > \varepsilon(T) N(AT)^k\}|, \end{aligned}$$

where $\varepsilon(T)$ is as defined in (3.2).

Proof. Let A and $\theta_0 > 1$ be as in Proposition 3.4 and set

$$C = \{x \in B_0 : \mathcal{M}(\chi_\Omega |\nabla u|)(x) > N(AT)^{k+1}\},$$

and

$$D = \{x \in B_0 : \mathcal{M}(\chi_\Omega |\nabla u|(x)) > N(AT)^k \text{ or } \mathcal{M}_1(\chi_\Omega |\mu|)(x)^{1/(p-1)} > \varepsilon(T) N(AT)^k\}.$$

Since $AT > 1$ the assumption (3.15) implies that $|C| < T^{-p\theta_0} r^n |B_1|$. Moreover, if $x \in B_0$ and $\rho \in (0, r]$ are such that $|C \cap B_\rho(x)| \geq T^{-p\theta_0} |B_\rho(x)|$, then $16\rho \leq r_0$ and thus by using Proposition 3.4 with $\lambda = N(AT)^k$ we have

$$B_\rho(x) \cap B_0 \subset D.$$

Hence the hypotheses of Lemma 3.1 are satisfied with $A{=}B_0$ and $\varepsilon{=}T^{-p\theta_0}$ (note that condition (3.1) holds for all $0{<}t{\le}2R_0$). Since $T{>}1$, this yields

$$|C| \le c(n)T^{-p\theta_0}|D| \le c(n)T^{-p\theta_0}|\{x\in B_0 : \mathcal{M}(\chi_\Omega|\nabla u|)(x) > N(AT)^k\}| \\ + c(n)|\{x\in B_0 : \mathcal{M}_1(\chi_\Omega|\mu|)(x)^{1/(p-1)} > \varepsilon(T)N(AT)^k\}|.$$

The proof of the lemma is then complete. □

Remark 3.6. From its proof we see that Lemma 3.5 also holds if B_0 is replaced by Ω provided that $A{=}\Omega$ satisfies (3.1) with some constants $c_1, r_1{>}0$. Of course, in this case r should be chosen so that $0{<}r{\le}\min\{r_0, r_1\}/16$.

4. Global Lorentz estimates

We are now ready to prove the main theorem of the paper.

Proof of Theorem 1.1. Let B_0 be a ball of radius $R_0{\le}2\,\mathrm{diam}(\Omega)$ that contains Ω. Note then that $\mathrm{diam}(\Omega){\le}2R_0$. As usual we set u and μ to be zero in $\mathbb{R}^n\setminus\Omega$. We are planning to show that

$$\|\nabla u\|_{L^{q,t}(\Omega))} \le C\|\mathcal{M}_1(|\mu|)^{1/(p-1)}\|_{L^{q,t}(B_0)}, \tag{4.1}$$

where $0{<}q{<}p{+}\varepsilon$ and $0{<}t{\le}\infty$. Here $\varepsilon{>}0$ is a small number depending only on n, p, α, β, and c_0. In what follows we consider only the case $t{\ne}\infty$ as for $t{=}\infty$ the proof is similar. Moreover, to prove (4.1) we may assume that

$$\|\nabla u\|_{L^1(\Omega)} \ne 0.$$

Let $r{=}\min\{r_0, \mathrm{diam}(\Omega)\}/16$. For $T{>}1$ to be determined, we claim that there exists $N{>}0$ such that

$$|\{x\in\mathbb{R}^n : \mathcal{M}(|\nabla u|)(x) > N\}| < T^{-p\theta_0}r^n|B_1|.$$

To see this, we first use the weak type $(1,1)$ estimate for the maximal function to get that

$$|\{x\in\mathbb{R}^n : \mathcal{M}(|\nabla u|)(x) > N\}| < \frac{C(n)}{N}\int_\Omega |\nabla u|\,dx.$$

Then we choose $N{>}0$ so that

$$\frac{C(n)}{N}\int_\Omega |\nabla u|\,dx = T^{-p\theta_0}r^n|B_1|. \tag{4.2}$$

Let $A, \theta_0 > 1$ be as in Lemma 3.5 and let $\varepsilon(T)$ be as in (3.2). For $0 < t < \infty$ we now consider the sum

$$S = \sum_{k=1}^{\infty} [(AT)^{qk} |\{x \in B_0 : \mathcal{M}(|\nabla u|)(x) > N(AT)^k\}|]^{t/q}.$$

Note that we have

$$C^{-1} S \leq \|\mathcal{M}(|\nabla u/N|)\|^t_{L^{q,t}(B_0)} \leq C(|B_0|^{t/q} + S). \tag{4.3}$$

By Lemma 3.5 we find that

$$\begin{aligned} S &\leq C \sum_{k=1}^{\infty} [(AT)^{qk} T^{-p\theta_0} |\{x \in B_0 : \mathcal{M}(|\nabla u|)(x) > N(AT)^{k-1}\}|]^{t/q} \\ &\quad + C \sum_{k=1}^{\infty} [(AT)^{qk} |\{x \in B_0 : \mathcal{M}_1(|\mu|)(x)^{1/(p-1)} > \varepsilon(T) N(AT)^{k-1}\}|]^{t/q} \\ &\leq C[(AT)^q T^{-p\theta_0}]^{t/q} (S + |B_0|^{t/q}) + C_1 \|\mathcal{M}_1(|\mu|/N^{p-1})^{1/(p-1)}\|^t_{L^{q,t}(B_0)}. \end{aligned}$$

Thus for $q < p\theta_0$, i.e., $q < p + \varepsilon$ with $\varepsilon = p(\theta_0 - 1)$, and T sufficiently large we have

$$S \leq C(|B_0|^{t/q} + \|\mathcal{M}_1(|\mu|/N^{p-1})^{1/(p-1)}\|^t_{L^{q,t}(B_0)}).$$

By (4.3) this yields

$$\|\nabla u/N\|_{L^{q,t}(\Omega)} \leq C(|B_0|^{1/q} + \|\mathcal{M}_1(|\mu|/N^{p-1})^{1/(p-1)}\|_{L^{q,t}(B_0)}),$$

and thus

$$\|\nabla u\|_{L^{q,t}(\Omega)} \leq C(|B_0|^{1/q} N + \|\mathcal{M}_1(|\mu|)^{1/(p-1)}\|_{L^{q,t}(B_0)}). \tag{4.4}$$

On the other hand, by (4.2) and the condition $p > 2 - 1/n$ we get

$$\begin{aligned} N &\leq C r^{-n} \|\nabla u\|_{L^1(\Omega)} \\ &\leq C \min\{r_0, \operatorname{diam}(\Omega)\}^{-n} |\Omega|^{1-(n-1)/n(p-1)} |\mu|(\Omega)^{1/(p-1)} \\ &\leq C \min\{r_0, \operatorname{diam}(\Omega)\}^{-n} \operatorname{diam}(\Omega)^n \left(\frac{|\mu|(\Omega)}{\operatorname{diam}(\Omega)^{n-1}} \right)^{1/(p-1)}, \end{aligned}$$

where the second inequality follows from standard estimates for equations with measure data (see, e.g., [2] and [7]). Thus for any $x \in B_0$ we have

$$N \leq C(n, p, \operatorname{diam}(\Omega)/r_0) \mathcal{M}_1(|\mu|)(x)^{1/(p-1)},$$

which holds since $R_0 \leq 2 \operatorname{diam}(\Omega)$. Combining the last inequality with (4.4) we obtain (4.1) as desired. $\square$

Next, we present the proof of Theorem 1.5.

Proof of Theorem 1.5. Since u is $\mathcal{A}$-superharmonic there is a nonnegative measure $\mu[u]$ such that

$$-\text{div}\,\mathcal{A}(x,\nabla u)=-\text{div}\,F=\mu[u] \tag{4.5}$$

in the sense of distributions in $\mathbb{R}^n$. Moreover, for each integer $k>0$ the function $u_k=T_k(u)\in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ is also $\mathcal{A}$-superharmonic and satisfies $\mu[u_k]\to\mu[u]$ weakly as measures in $\mathbb{R}^n$. Here $\mu[u_k]$ is the nonnegative measure generated by the $\mathcal{A}$-superharmonic function u_k.

Thus it is easily seen that Lemma 3.5 holds also for solutions of (4.5) with $\Omega=B_0=\mathbb{R}^n$ and, say, with $r=1$. More precisely, there exist constants $A,\theta_0>1$, depending only on n, p, α, and β, such that the following holds for any $T>1$. Suppose that u is an $\mathcal{A}$-superharmonic solution of (4.5) such that

$$|\{x\in\mathbb{R}^n:\mathcal{M}(|\nabla u|)(x)>N\}|<T^{-p\theta_0}|B_1| \tag{4.6}$$

for some $N>0$. Then for any integer $k\ge 0$, and with $\varepsilon(T)$ as in (3.2),

$$\begin{aligned}|\{x\in\mathbb{R}^n:\mathcal{M}(|\nabla u|)(x)>N(AT)^{k+1}\}|&\\ \le c(n)T^{-p\theta_0}|\{x\in\mathbb{R}^n:\mathcal{M}(|\nabla u|)(x)>N(AT)^k\}|&\\ +c(n)|\{x\in\mathbb{R}^n:\mathcal{M}_1(\mu[u])(x)^{1/(p-1)}>\varepsilon(T)N(AT)^k\}|&.\end{aligned} \tag{4.7}$$

To continue, for $T>1$ to be chosen later, we now take

$$N=\frac{C(n)}{T^{-p\theta_0}|B_1|}\,\|\nabla u\|_{L^1(\mathbb{R}^n)}>0 \tag{4.8}$$

with $C(n)$ large enough so that condition (4.6) holds true.

For $0<t<\infty$ (the case $t=\infty$ is similar) we next consider the sums

$$S^+=\sum_{k=1}^{\infty}[(AT)^{qk}|\{x\in\mathbb{R}^n:\mathcal{M}(|\nabla u|)(x)>N(AT)^k\}|]^{t/q}$$

and

$$S^-=\sum_{k=-\infty}^{0}[(AT)^{qk}|\{x\in\mathbb{R}^n:\mathcal{M}(|\nabla u|)(x)>N(AT)^k\}|]^{t/q}.$$

By (4.7) we find that

$$S^{+} \le C\sum_{k=1}^{\infty}[(AT)^{qk}T^{-p\theta_0}|\{x\in\mathbb{R}^n:\mathcal{M}(|\nabla u|)(x)>N(AT)^{k-1}\}|]^{t/q}$$
$$+C\sum_{k=1}^{\infty}[(AT)^{qk}|\{x\in\mathbb{R}^n:\mathcal{M}_1(\mu[u])(x)^{1/(p-1)}>\varepsilon(T)N(AT)^{k-1}\}|]^{t/q}$$
$$\le C[(AT)^{q}T^{-p\theta_0}]^{t/q}(S^{+}+|\{x\in\mathbb{R}^n:\mathcal{M}(|\nabla u|)(x)>N\}|^{t/q})$$
$$+C_1\|\mathcal{M}_1(\mu[u]/N^{p-1})^{1/(p-1)}\|^{t}_{L^{q,t}(\mathbb{R}^n)}.$$

Thus for $q<p\theta_0$, i.e., $q<p+\varepsilon$ with $\varepsilon=p(\theta_0-1)$, and T sufficiently large we have

$$S^{+} \le C|\{x\in\mathbb{R}^n:\mathcal{M}(|\nabla u|)(x)>N\}|^{t/q}+C\|\mathcal{M}_1(\mu[u]/N^{p-1})^{1/(p-1)}\|^{t}_{L^{q,t}(\mathbb{R}^n)}$$
$$\le C(S^{-}+\|\mathcal{M}_1(\mu[u]/N^{p-1})^{1/(p-1)}\|^{t}_{L^{q,t}(\mathbb{R}^n)}).$$

On the other hand, by the weak type $(1,1)$ bound for the maximal function and (4.8) we get

$$S^{-} \le \sum_{k=-\infty}^{0}\left((AT)^{qk}\frac{C(n)}{N(AT)^{k}}\int_{\Omega}|\nabla u|\,dx\right)^{t/q} \tag{4.9}$$
$$=\sum_{k=-\infty}^{0}[(AT)^{k(q-1)}T^{-p\theta_0}|B_1|]^{t/q}\le C(q,t,p,\theta_0,A,T),$$

where the last inequality follows since $q>1$. Note that we have

$$C^{-1}(S^{+}+S^{-})\le\|\mathcal{M}(|\nabla u/N|)\|^{t}_{L^{q,t}(\mathbb{R}^n)}\le C(S^{+}+S^{-}),$$

and thus by (4.9) and (4.9) this yields

$$\|\nabla u/N\|_{L^{q,t}(\mathbb{R}^n)}\le C(1+\|\mathcal{M}_1(\mu[u]/N^{p-1})^{1/(p-1)}\|_{L^{q,t}(\mathbb{R}^n)}).$$

We therefore have that

$$\|\nabla u\|_{L^{q,t}(\Omega)}\le C(N+\|\mathcal{M}_1(\mu[u])^{1/(p-1)}\|_{L^{q,t}(\mathbb{R}^n)})$$
$$\le C(\|\nabla u\|_{L^{1}(\mathbb{R}^n)}+\|\mathcal{M}_1(\mu[u])^{1/(p-1)}\|_{L^{q,t}(\mathbb{R}^n)})$$
$$\le C(\|\nabla u\|_{L^{1}(\mathbb{R}^n)}+\|\mathbf{I}_1(\mu[u])^{1/(p-1)}\|_{L^{q,t}(\mathbb{R}^n)}).$$

Notice that, by the second equality in (4.5), the equality

$$\mathbf{I}_1(\mu[u])=c(n)\sum_{j=1}^{n}R_jf_j \tag{4.10}$$

holds a.e. in $\mathbb{R}^n$, where $F=(f_1, f_2, ..., f_n)$ and $R_j f_j$ denotes the jth Riesz transform of the function f_j (see [28], p. 1580). Since $q>p-1$ this yields

$$\|\mathbf{I}_1(\mu[u])^{1/(p-1)}\|_{L^{q,t}(\mathbb{R}^n)} = \|\mathbf{I}_1(\mu[u])\|^{1/(p-1)}_{L^{q/(p-1),t/(p-1)}(\mathbb{R}^n)} \le C\|F\|^{1/(p-1)}_{L^{q/(p-1),t/(p-1)}(\mathbb{R}^n)},$$

and the desired estimate follows. □

Finally, we prove Theorem 1.6.

Proof of Theorem 1.6. Since $-\operatorname{div} F\ge 0$ in $\mathcal{D}'(\mathbb{R}^n)$ there is a nonnegative measure μ in $\mathbb{R}^n$ such that

$$-\operatorname{div} F=\mu.$$

For each integer $m>0$ let B_m denote the ball of radius m centered at the origin of $\mathbb{R}^n$. Also, let μ_{B_m} be the restriction of μ to the ball B_m. Then there exists a nonnegative $\mathcal{A}$-superharmonic function u_m in B_m such that

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_m)=\mu_{B_m} & \text{in } B_m \\ u_m=0 & \text{on } \partial B_m. \end{cases} \tag{4.11}$$

By Theorem 1.1 we have that

$$\|\nabla u_m\|_{L^{q,t}(B_m)} \le C\|\mathcal{M}_1(\mu)^{1/(p-1)}\|_{L^{q,t}(\mathbb{R}^n)} \le C\|\mathbf{I}_1(\mu)^{1/(p-1)}\|_{L^{q,t}(\mathbb{R}^n)}, \tag{4.12}$$

where C is independent of m. Thus for $q>1$ the Sobolev inequality on Lorentz spaces (see [32, Theorem 2.10.2]) yields

$$\|u_m\|_{L^{nq/(n-q),t}(B_m)} \le C\|\mathbf{I}_1(\mu)\|^{1/(p-1)}_{L^{q/(p-1),t/(p-1)}(\mathbb{R}^n)}. \tag{4.13}$$

Inequality (4.13) holds also for $p-1<q\le 1$. To see this first note that by [29, Theorem 2.1] (see also [18], [22], and [30]) we have a pointwise bound

$$u_m(x) \le C\mathbf{W}_{1,p}(\mu)(x), \quad x\in\mathbb{R}^n, \tag{4.14}$$

where $C=C(n, p, \alpha, \beta)$ and

$$\mathbf{W}_{1,p}(\mu)(x) := \int_0^\infty \left(\frac{\mu(B_t(x))}{t^{n-p}}\right)^{1/(p-1)} \frac{dt}{t}$$

is the Wolff potential of μ. Since $1/(p-1)>1$ we find that

$$\mathbf{W}_{1,p}(\mu)(x) \leq C\left(\int_0^\infty \frac{\mu(B_t(x))}{t^{n-p}}\,\frac{dt}{t}\right)^{1/(p-1)}$$
$$= C[\mathbf{I}_p(\mu)(x)]^{1/(p-1)} = C(\mathbf{I}_{p-1}[\mathbf{I}_1(\mu)](x))^{1/(p-1)}.$$

Here for $0<\alpha<n$ and a nonnegative measure ν, $\mathbf{I}_\alpha(\nu)$ is the (unnormalized) Riesz potential of ν defined by

$$\mathbf{I}_\alpha(\nu)(x) := \int_{\mathbb{R}^n} \frac{d\nu(y)}{|x-y|^{n-\alpha}}, \quad x \in \mathbb{R}^n.$$

Thus by (4.14) and the Sobolev inequality [32, Theorem 2.10.2] we have that

$$\|u_m\|_{L^{nq/(n-q),t}(B_m)} \leq C\|\mathbf{I}_{p-1}[\mathbf{I}_1(\mu)]\|^{1/(p-1)}_{L^{nq/(n-q)(p-1),t/(p-1)}(\mathbb{R}^n)}$$
$$\leq C\|\mathbf{I}_1(\mu)\|^{1/(p-1)}_{L^{q/(p-1),t/(p-1)}(\mathbb{R}^n)}$$

as claimed.

At this point we use [17, Theorem 1.17] to find a subsequence $\{u_{m_j}\}_{j=1}^\infty$ of $\{u_m\}_{m=1}^\infty$ and an $\mathcal{A}$-superharmonic function u in $\mathbb{R}^n$ such that

$$u(x) = \lim_{j\to\infty} u_{m_j}(x)$$

a.e. in $\mathbb{R}^n$ and that $\nabla u_{m_j} \to \nabla u$ a.e. in the set $\{x\in\mathbb{R}^n : u(x)<\infty\}$. Note that by (4.13) and Fatou's lemma, u is finite a.e. and

$$\|u\|_{L^{nq/(n-q),t}(\mathbb{R}^n)} \leq C\|\mathbf{I}_1(\mu)\|^{1/(p-1)}_{L^{q/(p-1),t/(p-1)}(\mathbb{R}^n)}.$$

Likewise, it follows from (4.10), (4.12) and Fatou's lemma that

$$\|\nabla u\|_{L^{q,t}(\mathbb{R}^n)} \leq C\,\|\mathbf{I}_1(\mu)\|^{1/(p-1)}_{L^{q/(p-1),t/(p-1)}(\mathbb{R}^n)} \leq C\|F\|^{1/(p-1)}_{L^{q/(p-1),t/(p-1)}(\mathbb{R}^n)}.$$

Finally, (4.11) and the weak continuity result of [30] imply that u is a solution of (1.7). □

References

1. Alvino, A., Ferone, V. and Trombetti, G., Estimates for the gradient of solutions of nonlinear elliptic equations with L^1-data, *Ann. Mat. Pura Appl.* **178** (2000), 129–142.
2. Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M. and Vazquez, J. L., An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **22** (1995), 241–273.

3. Boccardo, L. and Gallouët, T., Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.* **87** (1989), 149–169.
4. Boccardo, L. and Gallouët, T., Nonlinear elliptic equations with right-hand side measures, *Comm. Partial Differential Equations* **17** (1992), 641–655.
5. Byun, S. and Wang, L., Elliptic equations with BMO coefficients in Reifenberg domains, *Comm. Pure Appl. Math.* **57** (2004), 1283–1310.
6. Caffarelli, L. and Peral, I., On $W^{1,p}$ estimates for elliptic equations in divergence form, *Comm. Pure Appl. Math.* **51** (1998), 1–21.
7. Dal Maso, G., Murat, F., Orsina, A. and Prignet, A., Renormalized solutions of elliptic equations with general measure data, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **28** (1999), 741–808.
8. Dall'Aglio, A., Approximated solutions of equations with L^1 data. Applications to the H-convergence of quasilinear parabolic equations, *Ann. Mat. Pura Appl.* **170** (1996), 207–240.
9. Del Vecchio, T., Nonlinear elliptic equations with measure data, *Potential Anal.* **4** (1995), 185–203.
10. Duzaar, F. and Mingione, G., Gradient estimates via linear nonlinear potentials, *J. Funct. Anal.* **259** (2010), 2961–2998.
11. Duzaar, F. and Mingione, G., Gradient estimates via non-linear potentials, *Amer. J. Math.* **133** (2011), 1093–1149.
12. Giusti, E., *Direct Methods in the Calculus of Variations*, World Scientific, River Edge, NJ, 2003.
13. Heinonen, J., Kilpeläinen, T. and Martio, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford University Press, Oxford, 1993.
14. Iwaniec, T., Projections onto gradient fields and L^p-estimates for degenerated elliptic operators, *Studia Math.* **75** (1983), 293–312.
15. Jerison, D. and Kenig, C., The inhomogeneous Dirichlet problem in Lipschitz domains, *J. Funct. Anal.* **130** (1995), 161–219.
16. Kilpeläinen, T. and Li, G., Estimates for p-Poisson equations, *Differential Integral Equations* **13** (2000), 791–800.
17. Kilpeläinen, T. and Malý, J., Degenerate elliptic equations with measure data and nonlinear potentials, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **19** (1992), 591–613.
18. Kilpeläinen, T. and Malý, J., The Wiener test and potential estimates for quasilinear elliptic equations, *Acta Math.* **172** (1994), 137–161.
19. Lewis, J. L., Uniformly fat sets, *Trans. Amer. Math. Soc.* **308** (1988), 177–196.
20. Mengesha, T. and Phuc, N. C., Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains, *J. Differential Equations* **250** (2011), 2485–2507.
21. Meyers, N. G., An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **17** (1963), 189–206.
22. Mikkonen, P., On the Wolff potential and quasilinear elliptic equations involving measures, *Ann. Acad. Sci. Fenn. Math. Diss.* **104** (1996), 71.
23. Mingione, G., The Calderón–Zygmund theory for elliptic problems with measure data, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **6** (2007), 195–261.
24. Mingione, G., Gradient estimates below the duality exponent, *Math. Ann.* **346** (2010), 571–627.

25. Mingione, G., Nonlinear measure data problems, *Milan J. Math.* **79** (2011), 429–496.
26. Muckenhoupt, B. and Wheeden, R., Weighted norm inequalities for fractional integrals, *Trans. Amer. Math. Soc.* **192** (1974), 261–274.
27. Phuc, N. C., On Calderón–Zygmund theory for p- and $\mathcal{A}$-superharmonic functions, *Calc. Var. Partial Differential Equations* **46** (2013), 165–181.
28. Phuc, N. C. and Torres, M., Characterizations of the existence and removable singularities of divergence measure vector fields, *Indiana Univ. Math. J.* **57** (2008), 1573–1597.
29. Phuc, N. C. and Verbitsky, I. E., Quasilinear and Hessian equations of Lane–Emden type, *Ann. of Math.* **168** (2008), 859–914.
30. Trudinger, N. S. and Wang, X.-J., On the weak continuity of elliptic operators and applications to potential theory, *Amer. J. Math.* **124** (2002), 369–410.
31. Wang, L., A geometric approach to the Calderón–Zygmund estimates, *Acta Math. Sin.* (*Engl. Ser.*) **19** (2003), 381–396.
32. Ziemer, W. P., *Weakly Differentiable Functions*, Grad. Texts in Math. **120**, Springer, New York, 1989.

Nguyen Cong Phuc
Department of Mathematics
Louisiana State University
303 Lockett Hall
Baton Rouge, LA 70803
U.S.A.
pcnguyen@math.lsu.edu

Received June 28, 2012
published online February 1, 2013