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A note on the singularity category of an endomorphism ring

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Abstract. We propose the notion of partial resolution of a ring, which is by definition the endomorphism ring of a certain generator of the given ring. We prove that the singularity category of the partial resolution is a quotient of the singularity category of the given ring. Consequences and examples are given.

1. Introduction

Let A be a left coherent ring with a unit. Denote by A-mod the category of finitely presented left A-modules and by $\mathbf{D}^b(A\text{-mod})$ the bounded derived category. Following [3] and [15], the *singularity category* $\mathbf{D}_{\rm sg}(A)$ of A is the Verdier quotient category of $\mathbf{D}^b(A\text{-mod})$ with respect to the subcategory formed by perfect complexes. We denote by $q\colon \mathbf{D}^b(A\text{-mod})\to\mathbf{D}_{\rm sg}(A)$ the quotient functor. The singularity category measures the homological singularity of A.

Let ${}_AM$ be a finitely presented A-module. Denote by $\operatorname{add} M$ the full subcategory consisting of direct summands of finite direct sums of M. A *finite M-resolution* of an A-module X means an exact sequence $0\to N^{-n}\to N^{1-n}\to\ldots\to N^{-1}\to N^0\to X\to 0$ with each $N^{-i}\in\operatorname{add} M$, which remains exact after applying the functor $\operatorname{Hom}_A(M,-)$.

Recall that an A-module M is a *generator* if A lies in $\operatorname{add} M$. We consider the opposite ring of its endomorphism ring $\Gamma=\operatorname{End}_A(M)^{\rm op}$, and M becomes an A-Γ-bimodule. In particular, if Γ is left coherent, we have the functor $M\otimes_\Gamma -\colon \Gamma\text{-mod}\to A\text{-mod}$.

Let M be a generator with $\Gamma=\operatorname{End}_A(M)^{\rm op}$. Following the idea of [21], we call Γ a *partial resolution* of A if Γ is left coherent and any A-module X has a finite M-resolution, provided that it fits into an exact sequence $0\to X\to N^1\to N^2\to 0$ with $N^i\in\operatorname{add} M$. We mention that similar ideas might trace back to [1].

The following result justifies the terminology: the partial resolution Γ has "better" singularity than A. The result is inspired by [11, Theorem 1.2], where the case when A is an Artin algebra with finitely many indecomposable Gorenstein projective modules is studied.

Proposition 1.1. *Let A be a left coherent ring, and let Γ be a partial resolution of A as above. Then there is a triangle equivalence*

$$\mathbf{D}_{\rm sg}(\Gamma) \xrightarrow{\sim} \frac{\mathbf{D}_{\rm sg}(A)}{\langle q(M)\rangle}$$

induced by the functor $M\otimes_\Gamma -$.

Here, we identify the A-module M as the stalk complex concentrated on degree zero, and then $q(M)$ denotes the image in $\mathbf{D}_{\rm sg}(A)$. We denote by $\langle q(M)\rangle$ the smallest triangulated subcategory of $\mathbf{D}_{\rm sg}(A)$ that contains $q(M)$ and is closed under taking direct summands. Then $\mathbf{D}_{\rm sg}(A)/\langle q(M)\rangle$ is the corresponding Verdier quotient category.

The aim of this note is to prove Proposition 1.1 and discuss some related results and examples for Artin algebras.

2. The proof of Proposition 1.1

To give the proof, we collect in the following lemma some well-known facts. Throughout, A is a left coherent ring and ${}_AM$ is a generator with $\Gamma{=}\mathrm{End}_A(M)^{\rm op}$ being left coherent.

Recall the functor $M\otimes_\Gamma -\colon \Gamma\text{-mod}\to A\text{-mod}$. Denote by $\mathcal{N}$ the essential kernel of $M\otimes_\Gamma -$, i.e., the full subcategory of Γ-mod consisting of ${}_\Gamma Y$ such that $M\otimes_\Gamma Y\simeq 0$.

We will also consider the category A-Mod of arbitrary left A-modules. Note that the functor $M\otimes_\Gamma -\colon \Gamma\text{-Mod}\to A\text{-Mod}$ is left adjoint to $\mathrm{Hom}_A(M,-)\colon A\text{-Mod}\to \Gamma\text{-Mod}$. Denote by $\mathcal{N}'$ the essential kernel of $M\otimes_\Gamma -\colon \Gamma\text{-Mod}\to A\text{-Mod}$. The functor $\mathrm{Hom}_A(M,-)$ induces an equivalence $\mathrm{add}\, M\simeq\Gamma\text{-proj}$, where Γ-proj denotes the category of finitely generated projective Γ-modules.

For a class $\mathcal{S}$ of objects in a triangulated category $\mathcal{T}$, we denote by $\langle\mathcal{S}\rangle$ the smallest triangulated subcategory that contains $\mathcal{S}$ and is closed under taking direct summands. For example, the subcategory of $\mathbf{D}^b(A\text{-mod})$ formed by perfect complexes equals $\langle A\rangle$; here, we view a module as a stalk complex concentrated on degree zero.

Lemma 2.1. *Let the A-module M be a generator and $\Gamma{=}\mathrm{End}_A(M)^{\rm op}$. Then the following results hold*:

(1) *The functor* $\mathrm{Hom}_A(M,-)\colon A\text{-Mod}\to\Gamma\text{-Mod}$ *is fully faithful.*

(2) *The right* Γ*-module* M_Γ *is projective, and then the subcategory* $\mathcal{N}$ *of* Γ-mod *is a Serre subcategory, that is, it is closed under submodules, factor modules and extensions.*

(3) *The functor* $M\otimes_\Gamma -\colon \Gamma\text{-mod}\to A\text{-mod}$ *induces an equivalence* $\Gamma\text{-mod}/\mathcal{N}\xrightarrow{\sim} A\text{-mod}$, *where* $\Gamma\text{-mod}/\mathcal{N}$ *denotes the quotient category of* Γ-mod *with respect to the Serre subcategory* $\mathcal{N}$.

(4) *The functor* $M\otimes_\Gamma -\colon \Gamma\text{-mod}\to A\text{-mod}$ *induces a triangle equivalence*

$$\frac{\mathbf{D}^b(\Gamma\text{-mod})}{\langle\mathcal{N}\rangle}\xrightarrow{\sim}\mathbf{D}^b(A\text{-mod}).$$

Proof. (1) is contained in the Gabriel–Popescu theorem (see e.g. [20, Theorem X.4.1(i)]), and (2) is contained in [20, Proposition IV.6.7(i)]. In particular, the functor $M\otimes_\Gamma -$ is exact. Then we recall that the essential kernel of any exact functor between abelian categories is a Serre subcategory. We infer from (1) an equivalence $\Gamma\text{-Mod}/\mathcal{N}'\xrightarrow{\sim} A\text{-Mod}$; consult [6, Proposition I.1.3 and Section 2.5(d)]. Then (3) follows from [12, Proposition A.5]. Due to (3), the last statement follows from a general result [13, Theorem 3.2]. $\square$

The argument in the proof of the following result is essentially contained in the proof of [1, Section III.3, Theorem].

Lemma 2.2. *Keep the notation as above. Then any* Γ*-module in* $\mathcal{N}$ *has finite projective dimension if and only if* Γ *is a partial resolution of* A.

Proof. For the "if" part, assume that Γ is a partial resolution of A, and let ${}_\Gamma Y$ be a module in $\mathcal{N}$, that is, $M\otimes_\Gamma Y\simeq 0$. Next, take an exact sequence $0\to Y'\to P^1\xrightarrow{f} P^2\to Y\to 0$ in Γ-mod such that each P^i is projective. Recall the equivalence $\mathrm{Hom}_A(M,-)\colon \mathrm{add}\,M\to\Gamma\text{-proj}$. Then there is a map $g\colon N^1\to N^2$ with $N^i\in\mathrm{add}\,M$ such that $\mathrm{Hom}_A(M,g)$ is identified with f. Thus $M\otimes_\Gamma Y\simeq 0$ implies that g is epic. Moreover, if $X=\mathrm{Ker}\,g$, then $Y'\simeq\mathrm{Hom}_A(M,X)$. Hence by assumption X admits a finite M-resolution $0\to N^{-n}\to N^{1-n}\to\ldots\to N^{-1}\to N^0\to X\to 0$ with each $N^{-i}\in\mathrm{add}\,M$. Applying $\mathrm{Hom}_A(M,-)$ to it, we obtain a finite projective resolution of ${}_\Gamma Y'$. In particular, ${}_\Gamma Y$ has finite projective dimension. The "only if" part follows by reversing the argument. $\square$

Recall that $\langle A\rangle$ in $\mathbf{D}^b(A\text{-mod})$ equals the subcategory formed by perfect complexes. The singularity category is given by $\mathbf{D}_{\mathrm{sg}}(A)=\mathbf{D}^b(A\text{-mod})/\langle A\rangle$. We denote by $q\colon \mathbf{D}^b(A\text{-mod})\to\mathbf{D}_{\mathrm{sg}}(A)$ the quotient functor.

The following general result is due to [9, Proposition 3.3] in a slightly different setting. We include the proof for completeness.

Lemma 2.3. *Keep the notation as above. Then the functor* $M\otimes_\Gamma -\colon \Gamma\text{-mod}\to A\text{-mod}$ *induces a triangle equivalence* $\mathbf{D}_{\mathrm{sg}}(\Gamma)/\langle q(\mathcal{N})\rangle \xrightarrow{\sim} \mathbf{D}_{\mathrm{sg}}(A)/\langle q(M)\rangle$.

Proof. Recall from Lemma 2.1(4) the triangle equivalence $\mathbf{D}^b(\Gamma\text{-mod})/\langle\mathcal{N}\rangle \xrightarrow{\sim} \mathbf{D}^b(A\text{-mod})$ induced by $M\otimes_\Gamma -$. In particular, it sends Γ to M. Hence, it induces a triangle equivalence

$$\frac{\mathbf{D}^b(\Gamma\text{-mod})/\langle\mathcal{N}\rangle}{\langle\{\Gamma\}\cup\mathcal{N}\rangle/\langle\mathcal{N}\rangle} \xrightarrow{\sim} \frac{\mathbf{D}^b(A\text{-mod})}{\langle M\rangle}.$$

Since ${}_AM$ is a generator, we have $\langle A\rangle\subseteq\langle M\rangle\subseteq\mathbf{D}^b(A\text{-mod})$. It follows from [22, Section 2, Corollaire 4.3] that $\mathbf{D}^b(A\text{-mod})/\langle M\rangle=\mathbf{D}_{\mathrm{sg}}(A)/\langle q(M)\rangle$.

We apply [22, Section 2, Corollaire 4.3] to identify

$$\frac{\mathbf{D}^b(\Gamma\text{-mod})/\langle\mathcal{N}\rangle}{\langle\{\Gamma\}\cup\mathcal{N}\rangle/\langle\mathcal{N}\rangle} = \frac{\mathbf{D}^b(\Gamma\text{-mod})}{\langle\{\Gamma\}\cup\mathcal{N}\rangle}.$$

For the same reason, we identify

$$\frac{\mathbf{D}^b(\Gamma\text{-mod})}{\langle\{\Gamma\}\cup\mathcal{N}\rangle} = \frac{\mathbf{D}^b(\Gamma\text{-mod})/\langle\Gamma\rangle}{\langle\{\Gamma\}\cup\mathcal{N}\rangle/\langle\Gamma\rangle},$$

which equals $\mathbf{D}_{\mathrm{sg}}(\Gamma)/\langle q(\mathcal{N})\rangle$. Here, we recall that $\mathbf{D}_{\mathrm{sg}}(\Gamma)=\mathbf{D}^b(\Gamma\text{-mod})/\langle\Gamma\rangle$, and thus $\langle\{\Gamma\}\cup\mathcal{N}\rangle/\langle\Gamma\rangle=\langle q(\mathcal{N})\rangle$. In summary, we have the following identification

$$\frac{\mathbf{D}^b(\Gamma\text{-mod})/\langle\mathcal{N}\rangle}{\langle\{\Gamma\}\cup\mathcal{N}\rangle/\langle\mathcal{N}\rangle} = \frac{\mathbf{D}_{\mathrm{sg}}(\Gamma)}{\langle q(\mathcal{N})\rangle}.$$

Thus we are done with the required triangle equivalence. □

Proof of Proposition 1.1. Recall that for an A-module X, $q(X)\simeq 0$ in $\mathbf{D}_{\mathrm{sg}}(A)$ if and only if X has finite projective dimension. Hence by Lemma 2.2, all objects in $q(\mathcal{N})$ are isomorphic to zero. Thus the result follows from Lemma 2.3. □

We observe the following immediate consequence.

Corollary 2.4. *Let A be a left coherent ring and ${}_AM$ be a generator. Assume that $\Gamma=\mathrm{End}_A(M)^{\mathrm{op}}$ is left coherent and such that each finitely presented Γ-module has finite projective dimension. Then Γ is a partial resolution of A and* $\mathbf{D}_{\mathrm{sg}}(A)=\langle q(M)\rangle$.

In the situation of this corollary, we might even call Γ a *resolution* of A. Such a resolution always exists for any left artinian ring; see [1, Section III.3, Theorem].

Proof. The first statement follows from Lemma 2.2. By assumption, $\mathbf{D}_{\mathrm{sg}}(\Gamma)=0$ and thus the second statement follows from Proposition 1.1. □

3. Consequences and examples

We draw some consequences of Proposition 1.1 for Artin algebras. In this section, A will be an Artin algebra over a commutative artinian ring R.

3.1. Gorenstein projective generators

Recall that an A-module M is *Gorenstein projective* provided that M is reflexive and $\mathrm{Ext}^i_A(M,A)=0=\mathrm{Ext}^i_{A^{\mathrm{op}}}(M^*,A)$ for all $i\geq 1$. Here, $M^*=\mathrm{Hom}_A(M,A)$. Any projective module is Gorenstein projective.

Denote by A-Gproj the full subcategory of A-mod consisting of Gorenstein projective modules. It is closed under extensions and kernels of surjective maps. In particular, A-Gproj is a Frobenius exact category whose projective-injective objects are precisely projective A-modules. Denote by $A\text{-}\underline{\mathrm{Gproj}}$ the stable category; it is naturally triangulated by [7, Theorem I.2.8]. The Hom spaces in the stable category are denoted by $\underline{\mathrm{Hom}}$.

Recall from [3, Theorem 1.4.4] or [8, Theorem 4.6] that there is a full triangle embedding $F_A\colon A\text{-}\underline{\mathrm{Gproj}}\to\mathbf{D}_{\mathrm{sg}}(A)$ sending M to $q(M)$. Moreover, F_A is dense, and thus a triangle equivalence if and only if A is *Gorenstein*, that is, the injective dimension of the regular A-module has finite injective dimension on both sides.

A full subcategory $\mathcal{C}\subseteq A$-Gproj is *thick* if it contains all projective modules and for any short exact sequence $0\to M_1\to M_2\to M_3\to 0$ with terms in A-Gproj, all M_i lie in $\mathcal{C}$ provided that two of them lie in $\mathcal{C}$. In this case, the stable category $\underline{\mathcal{C}}$ is a triangulated subcategory of $A\text{-}\underline{\mathrm{Gproj}}$, and thus via F_A, a triangulated subcategory of $\mathbf{D}_{\mathrm{sg}}(A)$.

Proposition 3.1. *Let ${}_AM$ be a generator that is Gorenstein projective and such that* $\mathrm{add}\,M$ *is a thick subcategory of* A-Gproj. *Then* $\Gamma=\mathrm{End}_A(M)^{\mathrm{op}}$ *is a partial resolution of A and thus we have a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(\Gamma)\ \xrightarrow{\sim}\ \frac{\mathbf{D}_{\mathrm{sg}}(A)}{\underline{\mathrm{add}}\,M},$$

which is further triangle equivalent to $q(M)^{\perp}$.

Here, $q(M)^{\perp}$ denotes the *perpendicular subcategory* of $\mathbf{D}_{\rm sg}(A)$, which consists of objects X with $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(A)}(q(M), X[n])=0$ for all $n\in\mathbb{Z}$; it is a triangulated subcategory of $\mathbf{D}_{\rm sg}(A)$.

Proof. For any exact sequence $0\to X\to N^1\to N^2\to 0$ with $N^i\in\mathrm{add}\, M$, we have that X is Gorenstein projective, since A-Gproj is closed under kernels of surjective maps. Then by assumption, X lies in $\mathrm{add}\, M$. In particular, X admits a finite M-resolution. Therefore Γ is a partial resolution of A. We note that $\langle q(M)\rangle=\underline{\mathrm{add}}\, M$. Thus the first equivalence follows from Proposition 1.1 immediately.

For the second equivalence, we note by [15, Proposition 1.21] that for each object $X\in\mathbf{D}_{\rm sg}(A)$, $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(A)}(q(M), X)$ is a finite length R-module and then the cohomological functor $\mathrm{Hom}_{\mathbf{D}_{\rm sg}(A)}(q(M), -)\colon \underline{\mathrm{add}}\, M\to R\text{-mod}$ is representable. Hence, the triangulated subcategory $\underline{\mathrm{add}}\, M$ of $\mathbf{D}_{\rm sg}(A)$ is right admissible in the sense of [2]. Thus the result follows from [2, Proposition 1.6]. □

A special case is of independent interest: an algebra A is CM-*finite* provided that there exists a module M such that A-Gproj$=\mathrm{add}\, M$.

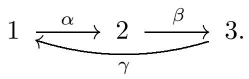
Corollary 3.2. [11, Theorem 1.2] *Let A be a* CM-*finite algebra such that* A-Gproj$=\mathrm{add}\, M$. *Set* $\Gamma=\mathrm{End}_A(M)^{\rm op}$. *Then Γ is a partial resolution of A, and there is a triangle equivalence*

$$\mathbf{D}_{\rm sg}(\Gamma) \xrightarrow{\sim} \frac{\mathbf{D}_{\rm sg}(A)}{A\text{-}\underline{\rm Gproj}}.$$

In particular, Γ has finite global dimension if and only if A is Gorenstein.

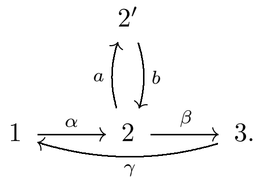
For the following example, we recall that any semisimple abelian category $\mathcal{A}$ has a unique (trivial) triangulated structure with the translation functor given by any auto-equivalence Σ. This triangulated category is denoted by $(\mathcal{A}, \Sigma)$.

Example 3.3. Let k be a field. Consider the algebra A given by the following quiver with relations $\{\beta\alpha\gamma\beta\alpha, \alpha\gamma\beta\alpha\gamma\beta\}$,



Here, we write the concatenation of arrows from right to left. The algebra A is a Nakayama algebra with admissible sequence $(5, 6, 6)$.

For each vertex i, we associate the simple A-module S_i. Consider the unique indecomposable module $S_2^{[3]}$ with top S_2 and length 3; it is the unique indecomposable non-projective Gorenstein projective A-module; see [5, Proposition 3.14(3)] or [16]. Set $M=A\oplus S_2^{[3]}$. Then add $M=A$-Gproj. The algebra $\Gamma=\mathrm{End}_A(M)^{\mathrm{op}}$ is given by the following quiver with relations $\{ab, \beta ba\alpha, ba-\alpha\gamma\beta\}$,



Then by Proposition 3.1 we have triangle equivalences

$$\mathbf{D}_{\mathrm{sg}}(\Gamma) \xrightarrow{\sim} \frac{\mathbf{D}_{\mathrm{sg}}(A)}{A\text{-}\underline{\mathrm{Gproj}}} \xrightarrow{\sim} q(S_2^{[3]})^{\perp}.$$

Here, we recall that $q(S_2^{[3]})^{\perp}=q(M)^{\perp}$ is the triangulated subcategory of $\mathbf{D}_{\mathrm{sg}}(A)$ consisting of objects X with $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(S_2^{[3]}), X[n])=0$ for all $n\in\mathbb{Z}$. The singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is triangle equivalent to the stable category A'-$\underline{\mathrm{mod}}$ for an elementary connected self-injective Nakayama algebra A' with admissible sequence $(4,4)$ such that $q(S_2^{[3]})$ corresponds to an A'-module of length 2; see [5, Corollary 3.11].

Then explicit calculation in A'-$\underline{\mathrm{mod}}$ yields that $q(S_2^{[3]})^{\perp}$ is equivalent to $k\times k$-mod; moreover, the translation functor Σ is induced by the algebra automorphism of $k\times k$ that switches the coordinates. In summary, we obtain a triangle equivalence

$$\mathbf{D}_{\mathrm{sg}}(\Gamma) \xrightarrow{\sim} (k\times k\text{-mod}, \Sigma).$$

We mention that by [11, Theorem 1.1] any Gorenstein projective Γ-module is projective. In particular, the algebra Γ is not Gorenstein.

3.2. Representations of quivers over dual numbers

Let k be a field and Q be a finite quiver without oriented cycles. Consider the path algebra kQ and the algebra $A=kQ[\varepsilon]=kQ\otimes_k k[\varepsilon]$ of dual numbers with coefficients in kQ, where $k[\varepsilon]$ is the algebra of dual numbers. Then A is Gorenstein. The stable category A-$\underline{\mathrm{Gproj}}$, which is equivalent to $\mathbf{D}_{\mathrm{sg}}(A)$, is studied in [17]; compare [4, Section 5]. We mention that the translation functor on A-$\underline{\mathrm{Gproj}}$ acts on objects as the identity; see [17, Proposition 4.10] or the following Lemma 3.4.

For an A-module Y, ε induces a kQ-module map $\varepsilon_Y\colon Y\to Y$ satisfying $\varepsilon_Y^2=0$. The *cohomology* of Y is defined as $H(Y)=\mathrm{Ker}\,\varepsilon_Y/\,\mathrm{Im}\,\varepsilon_Y$. This gives rise to a functor

$$H\colon A\text{-Gproj} \longrightarrow kQ\text{-mod},$$

which induces a cohomological functor $\underline{H}\colon A\text{-}\underline{\mathrm{Gproj}}\to kQ\text{-mod}$. We observe that kQ is a Gorenstein projective A-module, and that by [4, Lemma 5.3] there is an isomorphism of functors

$$\underline{H} \xrightarrow{\sim} \underline{\mathrm{Hom}}_A(kQ,-). \tag{1}$$

It follows from the cohomological property of $\underline{H}$ that for each subcategory $\mathcal{C}$ of kQ-mod, which is closed under extensions, kernels and cokernels, the corresponding subcategory $H^{-1}(\mathcal{C})$ of A-Gproj is thick.

Recall that the path algebra kQ is hereditary. For each kQ-module X, consider its minimal projective kQ-resolution $0\to P^{-1}\xrightarrow{d}P^0\to X\to 0$. Set $\eta(X)=P^{-1}\oplus P^0$; it is an A-module such that ε acts on P^{-1} by d and on P^0 as zero. If X is indecomposable, the A-module $\eta(X)$ is Gorenstein projective, which is indecomposable and non-projective. Observe that $H(\eta(X))\simeq X$. Indeed, if Y is an indecomposable non-projective Gorenstein projective A-module satisfying $H(Y)\simeq X$, then Y is isomorphic to $\eta(X)$. For two kQ-modules X and X', we have a natural isomorphism

$$\underline{\mathrm{Hom}}_A(\eta(X),\eta(X'))\simeq \mathrm{Hom}_{kQ}(X,X')\oplus \mathrm{Ext}^1_{kQ}(X,X'). \tag{2}$$

We refer the details to [17, Theorems 1 and 2].

Consider the Laurent polynomial algebra $kQ[x,x^{-1}]$ with coefficients in kQ; it is $\mathbb{Z}$-graded by means of $\deg kQ=0$, $\deg x=1$ and $\deg x^{-1}=-1$. We view $\mathbb{Z}$-graded algebras as differential graded algebras with trivial differential.

We denote by $\mathbf{D}(kQ[x,x^{-1}])$ the derived category of differential graded left $kQ[x,x^{-1}]$-modules and by $\mathrm{perf}(kQ[x,x^{-1}])$ the *perfect derived category* of $kQ[x,x^{-1}]$. Here, we recall that the perfect derived category $\mathrm{perf}(kQ[x,x^{-1}])$ is the full triangulated subcategory of $\mathbf{D}(kQ[x,x^{-1}])$ consisting of compact objects; see [10].

For a kQ-module X, we consider the graded $kQ[x,x^{-1}]$-module $X[x,x^{-1}]=kQ[x,x^{-1}]\otimes_{kQ}X$; it is viewed as a differential graded $kQ[x,x^{-1}]$-module with trivial differential.

The following result is implicitly contained in [4]; compare [17, Theorem 1].

Lemma 3.4. *Keep the notation as above. Then there is a triangle equivalence*

$$A\text{-}\underline{\mathrm{Gproj}} \xrightarrow{\sim} \mathrm{perf}(kQ[x,x^{-1}])$$

sending, for any kQ-module X, the A-module $\eta(X)$ to $X[x,x^{-1}]$.

Proof. We observe that the Laurent polynomial algebra $kQ[x,x^{-1}]$ is strongly graded and that kQ is hereditary. Then the triangle equivalence follows from [4, Propositions 5.5 and 5.4(2)].

We recall the complex P of A-modules in [4, Section 5.2]; it is a complete resolution of the A-module kQ. Then the triangle equivalence sends an A-module M to the Hom complex $\mathrm{Hom}_A(P,M)$. By [4, Lemma 3.4] the Hom complex $\mathrm{Hom}_A(P,M)$ is isomorphic to its cohomology $H(\mathrm{Hom}_A(P,M))$ in $\mathbf{D}(kQ[x,x^{-1}])$, which is isomorphic to $H(M)[x,x^{-1}]$; here, we use implicitly the isomorphism (1). Then we are done by recalling that $H(\eta(X))=X$. $\square$

Let E be an *exceptional* kQ-module, that is, an indecomposable kQ-module such that $\mathrm{Ext}^1_{kQ}(E,E)=0$. It is well known that the full subcategory $E^{\perp_{0,1}}$ of kQ-mod consisting of modules X with $\mathrm{Hom}_{kQ}(E,X)=0=\mathrm{Ext}^1_{kQ}(E,X)$ is equivalent to kQ'-mod for some finite quiver Q' without oriented cycles; moreover, the number of vertices of Q' is less than the number of vertices of Q by one; see [19, Theorem 2.3]. Take the minimal projective kQ-resolution $0\to P^{-1}\xrightarrow{\xi}P^0\to E\to 0$ of E. Then there is a universal localization $\theta\colon kQ\to B$ of algebras with respect to ξ such that B is Morita equivalent to kQ'; compare [18, Chapter 4] and [4, Proposition 3.1(2)].

Proposition 3.5. *Keep the notation as above. Set* $M=A\oplus\eta(E)$ *and* $\Gamma=\mathrm{End}_A(M)^{\mathrm{op}}$. *Then the following statements hold*:

(1) $\mathrm{add}\,M$ *is a thick subcategory of* A-Gproj, *and the corresponding stable category* $\underline{\mathrm{add}}\,M$ *is triangle equivalent to* $(k\text{-mod},\mathrm{Id}_{k\text{-mod}})$.

(2) *The perpendicular subcategory* $q(M)^{\perp}$ *of* $\mathbf{D}_{\mathrm{sg}}(A)$ *is triangle equivalent to the stable category* $kQ'[\varepsilon]$-$\underline{\mathrm{Gproj}}$.

(3) *There is a triangle equivalence* $\mathbf{D}_{\mathrm{sg}}(\Gamma)\xrightarrow{\sim}kQ'[\varepsilon]$-$\underline{\mathrm{Gproj}}$.

Proof. For (1), recall that $\mathrm{End}_{kQ}(E)\simeq k$. It follows that the subcategory $\mathrm{add}\,E$ of kQ-mod is closed under extensions, kernels and cokernels. Observe that $\mathrm{add}\,M=H^{-1}(\mathrm{add}\,E)$. Then the thickness of $\mathrm{add}\,M$ follows. The stable category $\underline{\mathrm{add}}\,M$ is given by $\mathrm{add}\,\eta(E)$; moreover, by the natural isomorphism (2) we have $\underline{\mathrm{End}}_A(\eta(E))\simeq k$. We recall that the translation functor acts on $\mathrm{add}\,\eta(E)$ as the identity. Then we infer (1).

For (2) we identify $\mathbf{D}_{\mathrm{sg}}(A)$ with A-$\underline{\mathrm{Gproj}}$. By Lemma 3.4 we identify $q(M)^{\perp}=q(\eta(E))^{\perp}$ with the perpendicular subcategory $E[x,x^{-1}]^{\perp}$ in $\mathrm{perf}(kQ[x,x^{-1}])$.

Consider the exact sequence of graded $kQ[x,x^{-1}]$-modules

$$0\longrightarrow P^{-1}[x,x^{-1}]\xrightarrow{\xi[x,x^{-1}]}P^0[x,x^{-1}]\longrightarrow E[x,x^{-1}]\longrightarrow 0.$$

We observe that the homomorphism $kQ[x,x^{-1}]\to B[x,x^{-1}]$ of graded algebras, induced by θ, is a *graded universal localization* with respect to the morphism $\xi[x,x^{-1}]$; see [4, Section 3]. Indeed, the algebra homomorphism $kQ[x,x^{-1}]\to B[x,x^{-1}]$ is an ungraded universal localization with respect to $\xi[x, x^{-1}]$, and thus a graded universal localization.

Then it follows from [4, Proposition 3.5] and [14, Theorem 2.1] that the perpendicular subcategory $E[x, x^{-1}]^{\perp}$ is triangle equivalent to $\mathrm{perf}(B[x, x^{-1}])$, the perfect derived category of $B[x, x^{-1}]$.

Since the algebras B and kQ' are Morita equivalent, it follows that as differential graded algebras, $B[x, x^{-1}]$ and $kQ'[x, x^{-1}]$ are derived equivalent; in particular, $\mathrm{perf}(B[x, x^{-1}])$ is triangle equivalent to $\mathrm{perf}(kQ'[x, x^{-1}])$, which is further equivalent to $kQ'[\varepsilon]\text{-}\underline{\mathrm{Gproj}}$ by Lemma 3.4. In summary, we obtain a triangle equivalence between $q(M)^{\perp}$ and $kQ'[\varepsilon]\text{-}\underline{\mathrm{Gproj}}$. We mention that this equivalence might also be deduced from [17, Theorem 1].

The last statement follows from (2) and Proposition 3.1. $\square$

We conclude with an example of Proposition 3.5.

Example 3.6. Let Q be the quiver $1 \xrightarrow{\alpha} 2$ and let $A = kQ[\varepsilon]$. Denote by S_1 the simple kQ-module corresponding to the vertex 1; it is an exceptional kQ-module. Then $\eta(S_1) = kQ$ such that ε acts as the multiplication of α from the right. The algebra $\Gamma = \mathrm{End}_A(A \oplus \eta(S_1))^{\mathrm{op}}$ is given by the following quiver with relations $\{\beta\alpha, \alpha\delta - \gamma\beta\}$,

$$1 \underset{\delta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 3.$$

The corresponding quiver Q' in Proposition 3.5 is a single vertex, and then the stable category $kQ'[\varepsilon]\text{-}\underline{\mathrm{Gproj}}$ is triangle equivalent to $(k\text{-mod}, \mathrm{Id}_{k\text{-mod}})$. Hence, by Proposition 3.5(3), we obtain a triangle equivalence

$$\mathbf{D}_{\mathrm{sg}}(\Gamma) \xrightarrow{\sim} (k\text{-mod}, \mathrm{Id}_{k\text{-mod}}).$$

We mention that the simple Γ-module corresponding to the vertex 3 is localizable, and whose corresponding left retraction $L(\Gamma)$ is an elementary connected Nakayama algebra with admissible sequence $(3, 4)$; see [5]. Then by [5, Lemma 3.12(2) and Proposition 2.6] any Gorenstein projective Γ-module is projective; in particular, Γ is not Gorenstein. The above triangle equivalence might also be deduced from [5, Proposition 2.13 and Corollary 3.11].

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