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Distribution of random Cantor sets on tubes

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Abstract. We show that there exist $(d-1)$-Ahlfors regular compact sets $E \subset \mathbb{R}^d$, $d \geq 2$ such that for any $t < d-1$, we have

$$\sup_T \frac{\mathcal{H}^{d-1}(E \cap T)}{w(T)^t} < \infty$$

where the supremum is over all tubes T with width $w(T) > 0$. This settles a question of T. Orponen. The sets we construct are random Cantor sets, and the method combines geometric and probabilistic estimates on the intersections of these random Cantor sets with affine subspaces.

1. Introduction

A set $E \subset \mathbb{R}^d (d \geq 2)$ is called tube null if for any $\varepsilon > 0$, there exist countable many tubes $\{T_i\}$ covering E and $\sum_i w(T_i)^{d-1} < \varepsilon$. Here and in what follows, a tube T with width $w = w(T) > 0$ is the $w/2$-neighborhood of some line in $\mathbb{R}^d$. We always assume that our tubes have positive width.

This notion comes from the study of the localization problem of the Fourier transform in dimension $d \geq 2$ (this problem can be regarded as looking for the analogues of Riemann's localization principle in higher dimensions). In [2], they proved that if $E \subset B$ (here B is the unit ball of $\mathbb{R}^d$) is tube null, then E is a Set of Divergence for the Localization Problem ($SDLP$). It's an open problem whether every $SDLP$ is tube null, for more details see [2].

It's easy to see that a set $E \subset \mathbb{R}^d$ with $\mathcal{H}^{d-1}(E) = 0$ is tube null. Indeed, [2, Proposition 7] claims that if $E \subset \mathbb{R}^d$ with $0 < \mathcal{H}^{d-1}(E) < \infty$, then E is tube null. This implies

$$\sup_T \frac{\mathcal{H}^{d-1}(E \cap T)}{w(T)^{d-1}} = \infty. \tag{1}$$

Since if there is a positive constant C such that $\mathcal{H}^{d-1}(E \cap T) \leq C w(T)^{d-1}$ for all tubes T, then for any countable family of tubes $\{T_i\}$ which cover E, we have

$$\sum_i w(T_i)^{d-1} \geq C^{-1} \sum_i \mathcal{H}^{d-1}(E \cap T_i) \geq C^{-1} \mathcal{H}^{d-1}(E),$$

which would contradict the tube nullity of E. Thus (1) holds. In [5], they showed that the Von Koch curve is tube null. For more tube null examples, see [2].

For the sets which are not tube null, in [2] they showed that for any $s \in (d-1/2, d)$, there exists set E with $\dim_H(E) = s$ and E is not tube null. The sharp low bound of above s was obtained in [10], they proved that there exist set with Hausdorff dimension $d-1$ which are not tube null (thus answered the question of [2]).

Motivated by [1, Proposition 1], Carbery asks to determine which pairs $(s, t) \in [0, d] \times [0, d]$ are admissible in the sense that there exists a set $E \subset \mathbb{R}^d$ with $0 < \mathcal{H}^s(E) < \infty$ and satisfies

$$(2) \qquad \sup_T \frac{\mathcal{H}^s(E \cap T)}{w(T)^t} < \infty.$$

This problem can be regarded as to concern the distribution of sets on tubes. By the works of [1], [2], [7] and [10] (different contributions), we know that all the pairs (s, t) with $t \leq \min\{d-1, s\}$ except $(d-1, d-1)$ are admissible. In [7], Orponen raised the following question: is it possible to construct a set $E \subset \mathbb{R}^d$ with $0 < \mathcal{H}^{d-1}(E) < \infty$ such that for every $t < d-1$,

$$(3) \qquad \sup_T \frac{\mathcal{H}^{d-1}(E \cap T)}{w(T)^t} < \infty?$$

We are able to settle this question.

Theorem 1.1. *There exists a $(d-1)$-Ahlfors regular compact set $E \subset \mathbb{R}^d$, such that for every $t < d-1$,*

$$(4) \qquad \sup_T \frac{\mathcal{H}^{d-1}(E \cap T)}{w(T)^t} < \infty.$$

Recall that $E \subset \mathbb{R}^d$ is called Q-Ahlfors regular for $0 < Q \leq d$, if there exist positive constant C such that $r^Q/C \leq \mathcal{H}^Q(E \cap B(x, r)) \leq C r^Q$ for all $x \in E$ and $0 < r < \mathrm{diam}(E)$, where $\mathrm{diam}(E)$ denotes the diameter of E.

The paper is organized as follows. The random Cantor sets are introduced in Section 2 together with the required notations, definitions and results. In Section 3 we present some geometric lemmas. Section 4 contains the main probabilistic argument.

2. Random Cantor sets and their projections

In this section, we define the random Cantor sets and state our results for them. Closely related random models have been consider in [3] and [10].

Let (M_n) and (N_n) be sequences of integers with $1 \leq N_n \leq M_n^d$ ($M_n \geq 2$) for all n. Denote $r_n = \prod_{k=1}^n M_k^{-1}$, and $P_n = \prod_{k=1}^n N_k$. We decompose the unit cube $[0,1]^d$ into M_1^d interior disjoint M_1-adic closed subcubes and randomly choose interior disjoint $N_1 \leq M_1^d$ of these closed subcubes such that each of the closed subcubes has the same probability (i.e. N_1/M_1^d) of being chosen, and denote their union by E_1. Given E_n, a random collection of P_n interior disjoint r_n-adic closed subcubes of $[0,1]^d$, independently inside each of these closed cubes we choose N_{n+1} interior disjoint (r_{n+1})-adic closed subcubes such that each of these closed subcubes has the same probability (i.e. N_{n+1}/M_{n+1}^d) of being chosen. Let E_{n+1} be the union of the chosen closed cubes. Denote by ω the element in the probability space Ω induced by the construction described above. Let $E = E^\omega$ be the random limit set

$$E = \bigcap_{n=1}^\infty E_n.$$

We also denote the random limit set by $E(M_n, N_n)$ when we want to stress the connection to the deterministic sequences M_n and N_n.

Remark 2.1. One natural way to choose subcubes is that we first randomly choose one such that every subcube has the same probability of being chosen. Then we choose the second subcube from the remaining subcubes such that every subcubes has the same probability of being chosen, and go on this way. But in fact, the above model contains more general random Cantor sets. For two specific examples see Example 2.4.

Important assumption: In this paper, we assume that M_n is uniformly bounded which means that there exists $M \in \mathbb{N}$, such that $M_n \leq M$ for every $n \in \mathbb{N}$. Then it's easy to see that all the Cantor sets $E(M_n, N_n)$ have Hausdorff dimension s, where

$$(5) \qquad\qquad s = \liminf_{n \to \infty} \frac{\log P_n}{-\log r_n}.$$

Let $G(d,m)$ denote the family of all m-dimensional linear subspaces of $\mathbb{R}^d$ and $A(d,m)$ denote the family of all m-dimensional planes of $\mathbb{R}^d$ that intersect the cube $[0,1]^d$. For every $V \in A(d,m)$, denote by π_V the orthogonal projection onto V and by $\dim_H F$ the Hausdorff dimension of a set F. Recall the classical Marstand–Mattila projection theorem (see e.g. [4] and [6]): Let $F \subset \mathbb{R}^d$ $(d \geq 2)$ be a Borel set with Hausdorff dimension s. If $s \leq k$, then the orthogonal projection of F onto almost all k-planes has Hausdorff dimension s; if $s > k$, then the orthogonal projection of F onto almost all k-planes has positive k-dimensional Lebesgue measure.

Recently, there has been a growing interest in showing that for various random fractals there are a.s. no exceptional directions in the projection theorem. We will prove the following projection theorem for the above random Cantor sets.

Theorem 2.2. *If $s \leq k$, then almost surely $\dim_H \pi_V(E) = s$ for all $V \in G(d,k)$.*

For other random sets, same kind of results have been recently obtained e.g. in [3], [8]–[11] and [12]. For $V \in G(d,k)$ such that $V^\perp$ is not parallel to any coordinate hyperplane, the claim of Theorem 2.2 follows from [11, Theorem 10.1]. In this paper, we give a direct proof for Theorem 2.2 without relying on the theory of general spatially independent martingales developed in [11]. In particular, we verity in detail the claim of [11, Remark 10.3(ii)] for the model at hand.

We consider the natural random measure on the random Cantor set. We denote by $D_n = D_n([0,1]^d)$, $n \in \mathbb{N}$ all the r_n-adic closed subcubes of the unit cube $[0,1]^d$. Let $E = \bigcap_{n=1}^{\infty} E_n$ be a realization. For any n and $Q \in D_n$, define

$$\mu_0(Q) = \begin{cases} P_n^{-1} & \text{if } Q \subset E_n, \\ 0 & \text{otherwise.} \end{cases}$$

By Kolmogorov's extension theorem, there is a unique measure μ on $[0,1]^d$ such that $\mu(Q) = \mu_0(Q)$ for any $Q \in D_n, n \in \mathbb{N}$.

In the following, tubular neighborhoods of the elements in $A(d,m)$ are called strips $(1 \leq m \leq d-1)$. More precisely, a strip S of width $w(S) = \delta > 0$, defined by an element $W \in A(d,m)$, is the set

$$S = \big\{ x \in \mathbb{R}^d \, | \, \mathrm{dist}(x, W) < \delta/2 \big\}$$

where dist is the Euclidean distance. We also denote this strip by $S(W)$ when it was induced by W. Denote by $S(d,m)$ all the strips induced by the element of $A(d,m)$ as above. Notice that we call the strips in $S(d,1)$ tubes.

Theorem 2.2 is easily deduced from the following estimate for the projections of the measure μ.

Lemma 2.3. *If $s \leq k$, then almost surely for any $0 < t < s$,*

$$(6) \qquad \sup_{S \in S(d, d-k)} \frac{\mu(E \cap S)}{w(S)^t} < \infty.$$

Lemma 2.3 will be proved in Section 4. Next we prove Theorem 2.2 and Theorem 1.1 assuming that Lemma 2.3 holds.

Proof of Theorem 2.2. Clearly $\dim_H \pi_V(E) \leq \dim_H(E) \leq s$ for all $V \in G(d, k)$, so it remains to verify the lower bound.

Using Lemma 2.3 we see that, almost surely, the estimate

$$\frac{(\pi_V)_* \mu(B(x, r))}{(2r)^t} = \frac{\mu(E \cap S')}{(2r)^t} \leq \sup_{S \in S(d, d-k)} \frac{\mu(E \cap S)}{w(S)^t} < \infty$$

holds for all $V \in G(d, k)$, $x \in V$ and r and simultaneously for all $t < s$, where $(\pi_V)_* \mu$ is the image measure of μ under the orthogonal projection of π_V and S' is the strip with width $2r$ induced by orthogonal complement of V at the point x. Thus with full probability $\dim_H \pi_V(E) \geq t$ holds for all $V \in G(d, k)$ (see e.g. [4, Chapter 4]). Approaching s along a sequence gives, almost surely for all $V \in G(d, k)$, the lower bound $\dim_H E \geq s$. □

We prove Theorem 1.1 by choosing $M_n = 2$ and $N_n = 2^{d-1}$ for all $n \in \mathbb{N}$ in the above random construction.

Proof of Theorem 1.1. Let $M_n = 2$ and $N_n = 2^{d-1}$ for all $n \in \mathbb{N}$. Then for every $E \in E(2, 2^{d-1})$ and for the natural measure μ on E, we have that

$$(7) \qquad \mu\big(B(x, r)\big) \asymp r^{d-1}$$

for $x \in E$ and $0 < r < 1$ where the symbol $\asymp$ means that the ratio of both sides is bounded above and below by positive and finite constants which does not depend on x and r. Thus we have that $\mu \asymp \mathcal{H}^{d-1}|_E$ (see e.g. [6, Chapter 6]), and so we can replace μ by $\mathcal{H}^{d-1}$ in (6). It implies that almost surely for any $t < d-1$, we have

$$(8) \qquad \sup_{S \in S(d, d-k)} \frac{\mathcal{H}^{d-1}(E \cap S)}{w(S)^t} < \infty.$$

Since μ is a probability measure, it follows that $0 < \mathcal{H}^{d-1}(E) < \infty$. By (7) all the sets $E(2, 2^{d-1})$ are $(d-1)$-Ahlfors regular. Thus we complete the proof. □

Now we present two concrete examples of random Cantor sets on $\mathbb{R}^2$ that fit into our general frame work.
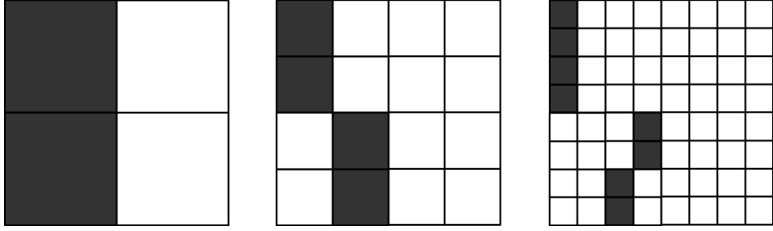
Figure 1. The first three steps in the construction of E.
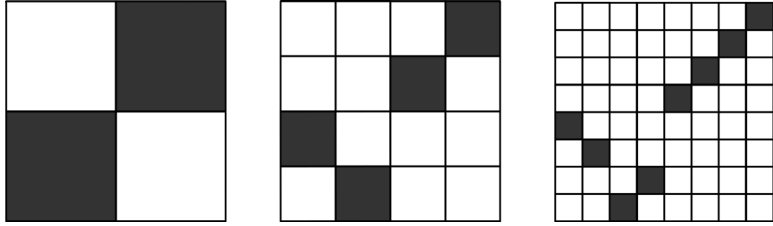
Figure 2. The first three steps in the construction of F.

Example 2.4. Consider the unit cube $[0,1]^2$. Let $N_n=M_n=2$ for all $n\in\mathbb{N}$. Let $Q_1=[0,1/2]\times[0,1/2]$, $Q_2=[1/2,1]\times[0,1/2]$, $Q_3=[1/2,1]\times[1/2,1]$ and $Q_4=[0,1/2]\times[1/2,1]$. Let $L=\{Q_1,Q_4\}$, $R=\{Q_2,Q_3\}$, $\tilde{L}=\{Q_1,Q_3\}$, and $\widetilde{D}=\{Q_2,Q_4\}$ corresponding to 'left', 'right', 'bottom left and top right', and 'bottom right and top left' subcubes of the unit cube.

Let $E_1=L$ or R with the same probability $1/2$. Note that then every subcube has the same probability $1/2$ of being chosen. Given E_n, a random collection of 2^n interior disjoint 2^n-adic closed subcubes of $[0,1]^2$, independently inside each of these cubes we choose the 'left' or 'right' column of the subcubes in the same way as $E_1\subset[0,1]^2$. Let E_{n+1} be the union of the chosen cubes. In the end we have the limit set (for an example see Figure 1)

$$E = \bigcap_{n=1}^{\infty} E_n.$$

If, on the other hand, we define another random process by changing L and R in the above construction to $\tilde{L}$ and $\widetilde{Q}$, we end up with another random set, denoted by $F(2,2)$. For an example see Figure 2.

Note that the construction of both random sets $E(2,2)$ and $F(2,2)$ are special cases of our random Cantor sets model which we described at the beginning of this

section. Note that both of these constructions give rise to random sets as used in the proof of Theorem 1.1.

In the end, we will show that $E(2,2)$ and $F(2,2)$ are "different". Indeed for every element E of $E(2,2)$, we have $\pi_y(E) = \{0\} \times [0,1]$ and $\mathcal{H}^1(\pi_x(E)) \leq 1/2$, where π_x, π_y are projections onto x-axis, y-axis respectively. But $\pi_x(F) = [0,1] \times \{0\}$ and $\pi_y(F) = \{0\} \times [0,1]$ for all $F \in F(2,2)$.

3. Geometric part

In this section, we present some geometric lemmas. The following results are adapted from [10] to our setting. In [10], Corollary 3.2 is proved for lines. Here we give the detailed proof for general affine subspaces of any dimension.

We are going to define the angle between a plane $W \in A(d,m)$ and a hyperplane $H \in A(d, d-1)$. We assume $W \in G(d,m)$ and $H \in G(d, d-1)$ first. We say that they have zero angle if $W \subset H$. Otherwise we have $H + W = \mathbb{R}^d$ where

$$(9) \qquad H + W := \{h + w : h \in H, w \in W\}.$$

Applying the basic dimension formula in linear algebra for H and W, we have that $\dim(H \cap W) = m - 1$. Thus for any $x \in H \cap W$, there is unique affine line $\ell_x \subset W, x \in \ell_x$, $\ell_x \perp (H \cap W)$. We choose an affine unit vector $e(x) \in \ell_x$ such that the root of $e(x)$ is x. Let

$$\theta(H, W) := \theta(H, \ell_x)$$

for some $x \in H \cap W$ (there is only one point in $H \cap W$ when $m = 1$), where $\theta(H, \ell)$ is the angle between the line ℓ and the plane H defined in the usual manner. Since ℓ_x and ℓ_y are parallel for any $x, y \in H \cap W$, the angle $\theta(H, W)$ doesn't depend on the choice of x. For the case that $W \in A(d,m)$ and $H \in A(d, d-1)$, there are unique subspaces $W' \in G(d,m)$ and $H' \in G(d, d-1)$ parallel to W and H, respectively. We define $\theta(H, W) := \theta(H', W')$.

Let $H_i = \{x \in \mathbb{R}^d : x_i = 0\}$ for $1 \leq i \leq d$. Define
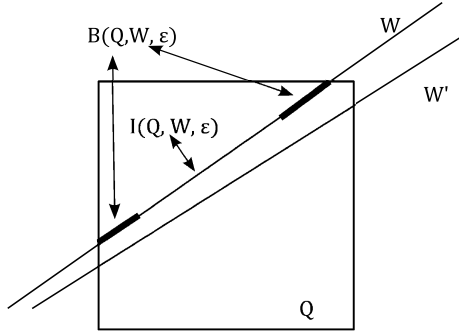
$$\Gamma_n(d, m) = \left\{ W \in A(d, m) : \min_{1 \leq i \leq d} \theta(W, H_i) \geq r_n^d \right\}$$

and $\Gamma(d, m) = \cup_{n \in \mathbb{N}} \Gamma_n(d, m)$. In the following we use $C(d)$ to represent constants which don't depend on n. We use $\#J$ to denote the cardinality of a set J.

Lemma 3.1. *For any $n \in \mathbb{N}$, there is $\Gamma'_n(d, m) \subset \Gamma(d, m)$ such that for any $W \in \Gamma_n(d, m)$, there exists $W' \in \Gamma'_n(d, m)$ with*

$$\mathcal{H}^m(W \cap Q) \leq \mathcal{H}^m(W' \cap Q) + C(d) r_n^{d+m}$$

for all $Q \in D_n$. Furthermore $\#\Gamma'_n(d, m) < r_n^{-C(d)}$.

Figure 3. For $d=2$, $m=1$.

Proof. Define a metric among $\Gamma(d, m)$ by setting

$$\rho(V, W) = \sup_{x \in [0,1]^d} \left| \pi_V(x) - \pi_W(x) \right|.$$

Let $\alpha = r_n^d$, and $\varepsilon = r_n^{2d+1}$. Let $\Gamma_n'(d, m)$ be an ε-dense subset of $\Gamma(d, m)$ in the ρ-metric. There is such an $\Gamma_n'(d, m)$ with $\#\Gamma_n'(d, m) < \varepsilon^{-C(d)}$.

Let $W \in \Gamma_n'(d, m)$, then we choose $W' \in A^1$ such that $\rho(W, W') \leq \varepsilon$. For any r_n-adic cube Q of D_n with $Q \cap W \neq \varnothing$, denote by

$$B(Q, W, \varepsilon) = \left\{ x \in Q \cap W : \operatorname{dist}(x, \partial Q) \leq \varepsilon \right\}$$

the "boundary part" of $Q \cap W$ and by $I(Q, W, \varepsilon) = (Q \cap W) \setminus B(Q, W, \varepsilon)$ the "interior part" of $Q \cap W$, see Figure 3. We have that

$$(10) \qquad\qquad Q \cap W = I(Q, W, 2\varepsilon) \cup B(Q, W, 2\varepsilon).$$

Now we will provide that $I(Q, W, 2\varepsilon) \subset \pi_W(W' \cap Q)$. For every $x \in I(Q, W, 2\varepsilon)$ there is a unique $y \in W'$ such that $\pi_W(y) = x$. Since

$$\operatorname{dist}(x, y) = \operatorname{dist}\left(\pi_W(y), \pi_{W'}(y) \right) \leq \varepsilon$$

and $\operatorname{dist}(x, \partial(Q)) > 2\varepsilon$, we have $y \in B(x, \frac{3\varepsilon}{2}) \subset Q$. It follows that $I(Q, W, 2\varepsilon) \subset \pi_W(W' \cap Q)$ and then

$$(11) \qquad\qquad \mathcal{H}^m\left(I(Q, W, 2\varepsilon) \right) \leq \mathcal{H}^m\left(Q \cap W' \right).$$

Now we intend to provide that $\mathcal{H}^m(B(Q, W, 2\varepsilon)) \leq C(d) r_n^{d+m}$. For any $x \in W \cap \partial Q$, there exists at least one face of Q which contains x. Then choose any such face and denote it by $F(x)$. Let $H(x)$ be the hyperplane which contains $F(x)$. Then there is a local orthogonal basis at x, $\{e_1(x), e_2(x), ..., e_m(x)\}$ of W, such that

$e_m(x) \perp (W \cap H(x))$ and we denote by $(x_1, x_2, ..., x_m)$ the co-ordinates of x with respect to this basis. Then

$$B(Q, W, 2\varepsilon) \subset \left\{ |x_m| \leq \frac{2\varepsilon}{\sin \alpha} \text{ in the above local coordinates} \right\}.$$

Note that once the face is fixed, $|x_m|$ does not depend on the choice of these local coordinates. Thus

$$\mathcal{H}^m \big(B(Q, W, 2\varepsilon) \big) \leq (2\sqrt{m} r_n)^{m-1} \frac{2\varepsilon}{\sin \alpha}.$$

There exists a constant $M \in \mathbb{N}$, such that for $n \geq M$ imply $\sin(r_n) > \frac{1}{2} r_n$. Thus we can choose a large constant $C(d)$ such that

$$(12) \qquad \mathcal{H}^m \big(B(Q, W, 2\varepsilon) \big) \leq C(d) r_n^{d+m}$$

for all $n \in \mathbb{N}$. Applying the estimates (10), (11) and (12), we have

$$\mathcal{H}^m (W \cap Q) \leq \mathcal{H}^m \big(W' \cap Q \big) + C(d) r_n^{d+m}.$$

Thus we complete the proof. $\square$

Let $m = d - k$ in Lemma 3.1 and recall that the number of r_n^{-1}-adic subcubes of E_n is at most r_n^{-d}. Let $\Gamma_n := \Gamma_n(d, d-k)$ and $\Gamma := \Gamma(d, d-k)$. Let A, B be two subset of $\mathbb{R}^d$. Define

$$|A \cap B| := \mathcal{H}^{d-k}(A \cap B).$$

We have the following easy corollary.

Corollary 3.2. *For any $n \in \mathbb{N}$, there is $\Gamma'_n \subset \Gamma$ such that for any $W \in \Gamma_n$, there exists $W' \in \Gamma'_n$ with*

$$|W \cap E_n| \leq |W' \cap E_n| + C(d) r_n^{d-k}$$

for any realization E_n. Further more $\#\Gamma'_n \leq r_n^{-C(d)}$.

Proof. By Lemma 3.1, we have that for any $W \in \Gamma_n$, there exist $W' \in \Gamma'_n$ such that

$$(13) \qquad |W \cap Q| \leq |W' \cap Q| + C(d) r_n^{2d-k}$$

for each $Q \in D_n$ and $\#\Gamma'_n < r_n^{-C(d)}$. For any realization E_n, we sum the two sides of (13) over $Q \in D_n$ such that $Q \subset E_n$:

$$(14) \qquad \sum_{Q \subset E_n} |W \cap Q| \leq \sum_{Q \subset E_n} |W' \cap Q| + C(d) r_n^{d-k}.$$

By the definition of $|W \cap E_n|$, we arrive at the required estimate. $\square$

For a strip $S \in S(d, d-k)$, denote

$$Z(S, n) = \#\{Q \text{ is an } r_n \text{ adic cube} : Q \cap E_n \cap S \neq \varnothing\}.$$

For later use in Corollary 4.4, we state the following easy fact as a lemma.

Lemma 3.3. *If $|W \cap E_n| \leq h$ for all $W \in \Gamma_n$, then for any strip $S \in S(d, d-k)$ with width $0 < w(S) \leq r_n$, we have*

$$Z(S, n) \leq C(d) r_n^{k-d} h.$$

Proof. We assume $S(V) \in S(d, d-k)$ with $V \in \Gamma_n$ first. Let $V^\perp \in G(d, k)$ be the orthogonal complement of V and $z := V^\perp \cap V$. Let $B_{V^\perp}(y, r)$ be the ball of $V^\perp$ with center y and radius r. Let $t_n := r_n/2 + \sqrt{d} r_n$. Since

$$\{Q \text{ is an } r_n \text{ adic cube} : Q \cap E_n \cap S \neq \varnothing\} \subset V(t_n)$$

where $V(t_n)$ is the t_n neighborhood of V in $\mathbb{R}^d$, we have

(15) $$Z(S, n) r_n^d \leq \mathcal{H}^d\big(E_n \cap V(t_n)\big).$$

Using Fubini's theorem and the condition $|W \cap E_n| \leq h$ for all $W \in \Gamma_n$, we obtain that

(16) $$\mathcal{H}^d\big(E_n \cap V(t_n)\big) = \int_{B_{V^\perp}(z, t_n)} \mathcal{H}^{d-k}\big(E_n \cap P_{V^\perp}^{-1}(x)\big) \, d\mathcal{H}^k(x) \leq (t_n)^k h.$$

By (15) and (16) we have $Z(S, n) r_n^d \leq (t_n)^k h$. Let $C_1(d) := (1 + 2\sqrt{d})^k$. We get that $Z(S, n) \leq C_1(d) r_n^{k-d} h$.

For a strip $S(V)$ with $V \in \Gamma_n^c$ (Γ_n^c is the complement of Γ_n that is $A(d, d-k) \backslash \Gamma_n$) and $w(S) \leq r_n$, there is a strip

$$\widetilde{S} = \big\{x \in \mathbb{R}^d : \text{dist}(x, \widetilde{V}) \leq 5r_n\big\}$$

and $S \subset \widetilde{S}$. Thus

$$Z(S, n) \leq Z(\widetilde{S}, n) \leq 10 C_1(d) r_n^{k-d} h.$$

Let $C(d) = 10 C_1$. Thus the proof is completed. $\square$

Note that the constant $C(d)$ may be different in different places of this section. For the convenience in what follows we fix a constant $C(d)$ such that the statements of all the lemmas and corollaries hold with this constant.

4. Probabilistic part

We use a similar method as in [3] to estimate the intersections of our sets with affine planes. The random Cantor sets studied in [3] are different from the ones considered here. We choose interior disjoint closed subcubes at every step of our constructions, while in [3] overlaps are allowed. Since we assume that M_n are uniformly bounded, the proof here will be simpler than that of [3]. On the other hand, we give here the detailed proof for general d and m while in the main part of [3], it is assumed that $d=2$, $m=1$.

We fix a number $t<s\leq k$ and let $0<5\varepsilon\leq s-t$. Recall that by (5), there exists $n_0\in\mathbb{N}$ such that

$$(17) \qquad r_m^{-t-4\varepsilon} \leq r_m^{-s+\varepsilon} \leq P_m \leq r_m^{-s-\varepsilon}$$

holds for all $m\geq n_0$, $m\in\mathbb{N}$. For this n_0, there is a constant R_0 such that

$$(18) \qquad |W\cap E_{n_0}| \leq R_0 P_{n_0} r_{n_0}^{t+d-k},$$

holds for all $W\in A(d,d-k)$ and any realization E_{n_0}. Let $W\in\Gamma$, $n\in\mathbb{N}$. Define

$$Y_n^W = \left(P_n r_n^d\right)^{-1} |W\cap E_n|.$$

Denote by $\mathbb{P}(\cdot|A)$ the conditional probability conditioned on the event A.

Lemma 4.1. *Let $n>n_0, n\in\mathbb{N}$ and $W\in\Gamma$. Then for any positive λ and λ_0 with $\lambda(2\sqrt{d}r_{n-1})^{d-k}(P_n r_n^d)^{-1}\leq\lambda_0\leq 1$, we have*

$$(19) \qquad \mathbb{E}\left(e^{\lambda Y_n^W}\big| E_{n-1}\right) \leq e^{(1+\lambda_0)\lambda Y_{n-1}^W}.$$

Proof. Let $Q_1, Q_2, ..., Q_K$ be the cubes in E_{n-1} hitting W. For each $1\leq i\leq K$, consider the random variable

$$(20) \qquad X_i = \left(P_n r_n^d\right)^{-1}\mathcal{H}^{d-k}(W\cap E_n\cap Q_i).$$

Thus we have $Y_n^W=\sum_{i=1}^K X_i$. For each $1\leq i\leq K$, we have that

$$(21) \qquad \mathbb{E}(X_i|E_{n-1}) = \left(P_{n-1} r_{n-1}^d\right)^{-1}|W\cap Q_i|.$$

Conditional on E_{n-1}, recall that the cubes forming E_n are chosen independently inside each Q_i, $1\leq i\leq K$. Thus the random variables X_i, $1\leq i\leq K$ are independent. And so $e^{\lambda X_i}$ are also independent. This gives

$$(22) \qquad \mathbb{E}\left(e^{\lambda Y_n^W}\big| E_{n-1}\right) = \prod_{i=1}^K \mathbb{E}\left(e^{\lambda X_i}\big| E_{n-1}\right).$$

For all $|x| \leq \rho \leq 1$, we use the fact $e^x \leq 1 + (1+\rho)x$ and

$$\lambda X_i \leq \lambda (2\sqrt{d}r_{n-1})^{d-k} (P_n r_n^d)^{-1} \leq \lambda_0 \leq 1$$

for $1 \leq i \leq K$, to obtain

$$(23) \hspace{4cm} e^{\lambda X_i} \leq 1 + (1+\lambda_0)\lambda X_i.$$

Thus by (21) and the trivial inequality $1 + x \leq e^x$, we have

$$(24) \hspace{2cm} \mathbb{E}\big(e^{\lambda X_i} \big| E_{n-1}\big) \leq \exp\big((1+\lambda_0)\lambda \big(P_{n-1} r_{n-1}^d\big)^{-1} |W \cap Q_i|\big).$$

Combing this with (22) and the definition of Y_{n-1}^W, we finish the proof. $\quad\square$

Let $R > 2R_0 C(n_0)$ be a constant where

$$(25) \hspace{3cm} C(n_0) := (2\sqrt{d}M)^{d-k} \prod_{i=n_0+1}^{\infty} \big(1 + r_i^{\varepsilon}\big).$$

By applying Lemma 4.1 and the total expectation formula, we have the following estimate.

Lemma 4.2. *For any $n > n_0$, $n \in \mathbb{N}$ and $W \in \Gamma$, we have the bound*

$$(26) \hspace{3cm} \mathbb{P}\big(Y_n^W > R r_n^{t-k}\big) \leq \exp\big(-r_n^{-\varepsilon}\big).$$

Proof. Let $\lambda = C(n_0)^{-1} P_n r_n^{k+3\varepsilon}$. We apply Markov's inequality to the random variable $e^{\lambda Y_n^W}$. This gives

$$(27) \hspace{3cm} \mathbb{P}\big(Y_n^W > R r_n^{t-k}\big) \leq e^{-\lambda R r_n^{t-k}} \mathbb{E}\big(e^{\lambda Y_n^W}\big).$$

Now we are going to estimate $\mathbb{E}(e^{\lambda Y_n^W})$. By the choice of λ, we have

$$\lambda (2\sqrt{d}r_{n-1})^{d-k} \big(P_n r_n^d\big)^{-1} \leq r_n^{\varepsilon}.$$

Applying Lemma 4.1 we have

$$(28) \hspace{3cm} \mathbb{E}\big(e^{\lambda Y_n^W} \big| E_{n-1}\big) \leq e^{(1+r_n^{\varepsilon})\lambda Y_{n-1}^W}.$$

The total expectation formula and estimate (28) imply

$$(29) \hspace{2cm} \mathbb{E}\big(e^{\lambda Y_n^W}\big) = \mathbb{E}\big(\mathbb{E}\big(e^{\lambda Y_n^W} \big| E_{n-1}\big)\big) \leq \mathbb{E}\big(e^{(1+r_n^{\varepsilon})\lambda Y_{n-1}^W}\big).$$

By the choice of $\lambda, n \geq n_0$ and estimate (17), we see that

$$\frac{\lambda(2\sqrt{d}r_{j-1})^{d-k}\prod_{i=j+1}^{n}(1+r_i^\varepsilon)}{P_j r_j^d} \leq \frac{P_n r_n^{k+3\varepsilon} r_j^{d-k}}{P_j r_j^d} \leq r_j^\varepsilon,$$

holds for all $n_0 < j < n$. Applying (29) inductively, we have

$$(30) \qquad \mathbb{E}\big(e^{\lambda Y_n^W}\big) \leq \mathbb{E}\big(e^{\lambda Y_{n_0}^W \prod_{i=n_0+1}^{n}(1+r_i^\varepsilon)}\big) \leq \exp\big(P_n r_n^{k+3\varepsilon} R_0 r_{n_0}^{t-k}\big).$$

The last inequality holds by our choice of λ and the condition that

$$(31) \qquad |W \cap E_{n_0}| \leq R_0 P_{n_0} r_{n_0}^{t+d-k}$$

holds for all $W \in \Gamma$. Combining the estimates (30) and (27), we have

$$\mathbb{P}\big(Y_n^W > Rr_n^{t-k}\big) \leq \exp\big(-\lambda R r_n^{t-k} + P_n r_n^{k+3\varepsilon} R_0 r_{n_0}^{t-k}\big)$$

$$= \exp\big(-P_n r_n^{k+3\varepsilon} r_n^{t-k}\big(C(n_0)^{-1}R - R_0 r_{n_0}^{t-k} r_n^{k-t}\big)\big)$$

$$(32) \qquad\qquad \leq \exp\big(-r_n^{-\varepsilon}\big).$$

The last inequality holds since $P_n r_n^t \geq r_n^{-4\varepsilon}$ and $R > 2R_0 C(n_0)$ (we also ask that $R_0 > 1$). $\quad\square$

Let $n \in \mathbb{N}$ and $W \in \Gamma$. Denoted by $G_n(W)$ the (good) event

$$|W \cap E_n| \leq RP_n r_n^{t+d-k} + C(d)r_n^{d-k}.$$

Let G_n be the event that $G_n(W)$ holds for all $W \in \Gamma_n$. Applying Corollary 3.2 and Lemma 4.2, we have the following result.

Corollary 4.3. *We have* $\mathbb{P}(\cup_{k=1}^{\infty} \cap_{n=k}^{\infty} G_n) = 1$.

Proof. By Corollary 3.2 we know that if the estimate

$$\big|W' \cap E_n^\omega\big| \leq RP_n r_n^{t+1}$$

holds for all $W' \in \Gamma_n'$, then the estimate

$$\big|W \cap E_n^\omega\big| \leq RP_n r_n^{t+1} + C(d)r_n^{d-k},$$

holds for any $W \in \Gamma_n$. Thus $\omega \in G_n$. Let $n \geq N_0, n \in \mathbb{N}$. Then by the above argument we have

$$\mathbb{P}\big(G_n^c\big) \leq \mathbb{P}\big(\big|W \cap E_n^\omega\big| > P_n R_n r_n^{t+d-k} \text{ for some } W' \in \Gamma_n'\big)$$

$$\leq r_n^{-C(d)} \exp\big(-r_n^{-\varepsilon}\big),$$

where G_n^c is the complement of G_n. The last inequality holds by Lemma 4.2. Note that there are at most $r_n^{-C(d)}$ elements in Γ_n'.

Since the series $\sum_{n=1}^{\infty} r_n^{-C(d)} \exp(-r_n^{-\varepsilon})$ converges, the Borel–Cantelli lemma implies $\mathbb{P}(\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} G_n^c) = 0$. $\square$

Now we are going to estimate the distribution of the natural measure $\mu = \mu^{\omega}$ on strips.

Corollary 4.4. *Let* $\omega \in \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} G_n$. *Then*

$$(33) \qquad \sup_{S \in S(d,d-k)} \frac{\mu^{\omega}(E^{\omega} \cap S)}{w(S)^t} < \infty.$$

Proof. Since $\omega \in \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} G_n$, there exits n_{ω} such that $E^{\omega} \in G_n$ for all $n \geq n_{\omega}$. It means the estimate

$$|W \cap E_n| \leq RP_n r_n^{t+d-k} + C(d) r_n^{d-k} \leq 2RP_n r_n^{t+d-k}$$

holds for all $n \geq n_{\omega}$ and all $W \in \Gamma_n$. The last inequality holds by choosing large R.

Let $S(W) \in S(d, d-k)$ with width $w(S)$. We assume $w(S) \leq r_{n_{\omega}}$ first. There exists $n \geq n_{\omega}$ such that $r_{n+1} < w(S) \leq r_n$. By Lemma 3.3, we have $Z(S, n) \leq 2C(d)RP_n r_n^t$. Thus

$$\mu(S) \leq 2C(d)Rr_n^t \leq 2C(d)RM^t w(S)^t.$$

In the case that $w(S) > r_{n_{\omega}}$, it's trivial to see that

$$(34) \qquad \sup_{\substack{S \in S(d,d-k) \\ w(S) > r_{n_{\omega}}}} \frac{\mu(E \cap S)}{w(S)^t} \leq r_{n_{\omega}}^{-1},$$

since μ is a probability measure. Thus we complete the proof. $\square$

Notice that all the above claims hold for any $t < s$. Letting $t \to s$ through a countable sequence gives the proof of Lemma 2.3:

Proof of Lemma 2.3. Let $0 < t_1 < t_2 < \dots$ such that $t_k \nearrow s$. For every t_k, we denote by Ω_k the event

$$(35) \qquad \sup_{S \in S(d,d-k)} \frac{\mu(E \cap S)}{w(S)^{t_k}} < \infty.$$

By Corollary 4.3 and Corollary 4.4, we have $\mathbb{P}(\Omega_k) = 1$. So $\mathbb{P}(\cap_{k=1}^{\infty} \Omega_k) = 1$ as well. Let $\omega \in \cap_{k=1}^{\infty} \Omega_k$, then μ^{ω} satisfies (35) for every t_k.

For any $t<s$, there is t_k, such that $t<t_k<s$. We have $w(S)^t \geq w(S)^{t_k}$ when $w(S) \leq 1$. Thus

$$(36) \qquad \sup_{\substack{S \in S(d,d-k) \\ w(S) \leq 1}} \frac{\mu(E \cap S)}{w(S)^t} \leq \sup_{\substack{S \in S(d,d-k) \\ w(S) \leq 1}} \frac{\mu(E \cap S)}{w(S)^{t_k}} < \infty.$$

Again since μ is a probability measure we have

$$(37) \qquad \sup_{\substack{S \in S(d,d-k) \\ w(S) > 1}} \frac{\mu(E \cap S)}{w(S)^t} \leq 1.$$

Combing this with the estimate (36), the claim follows. $\qquad \square$

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