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Graded PI-exponents of simple Lie superalgebras

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Abstract. We study $\mathbb{Z}_2$-graded identities of simple Lie superalgebras over a field of characteristic zero. We prove the existence of the graded PI-exponent for such algebras.

1. Introduction

Let A be an algebra over a field F with char $F=0$. A natural way of measuring the polynomial identities satisfied by A is by studying the asymptotic behaviour of its sequence of codimensions $\{c_n(A)\}$, $n=1,2,\dots$. If A is a finite dimensional algebra then the sequence $\{c_n(A)\}$ is exponentially bounded. In this case it is natural to ask the question about existence of the limit

$$\lim_{n\to\infty} \sqrt[n]{c_n(A)} \tag{1}$$

called the PI-exponent of A. Such question was first asked for associative algebras by Amitsur at the end of 1980's. A positive answer was given in [6]. Subsequently it was shown that the same problem has a positive solution for finite dimensional Lie algebras [14], for finite dimensional alternative and Jordan algebras [5] and for some other classes. Recently it was shown that in general the limit (1) does not exist even if $\{c_n(A)\}$ is exponentially bounded [15]. The counterexample constructed in [15] is infinite dimensional whereas for finite dimensional algebras the problem of the existence of the PI-exponent is still open. Nevertheless, if $\dim A<\infty$ and A is simple then the PI-exponent of A exists as it was proved in [8].

If in addition A has a group grading then graded identities, graded codimensions and graded PI-exponents can also be considered. In this paper we discuss

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graded codimensions behaviour for finite dimensional simple Lie superalgebras. Graded codimensions of finite dimensional Lie superalgebras were studied in a number of papers (see for example, [11] and [12]). In particular, in [11] an upper bound of graded codimension growth was found for one of the series of simple Lie superalgebras.

In the present paper we prove that the graded PI-exponent of any finite dimensional simple Lie superalgebra always exists. All details concerning numerical PI-theory can be found in [7].

2. Main constructions and definitions

Let $L=L_0\oplus L_1$ be a Lie superalgebra. Elements from the component L_0 are called *even* and elements from L_1 are called *odd*. Denote by $\mathcal{L}(X,Y)$ a free Lie superalgebra with infinite sets of even generators X and odd generators Y. A polynomial $f=f(x_1,\dots,x_m,y_1,\dots,y_n)\in\mathcal{L}(X,Y)$ is said to be a *graded identity* of Lie superalgebra $L=L_0\oplus L_1$ if $f(a_1,\dots,a_m,b_1,\dots,b_n)=0$ whenever $a_1,\dots,a_m\in L_0$, $b_1,\dots,b_n\in L_1$.

Denote by $\mathrm{Id}^{gr}(L)$ the set of all graded identities of L. Then $\mathrm{Id}^{gr}(L)$ is an ideal of $\mathcal{L}(X,Y)$. Given non-negative integers $0\le k\le n$, let $P_{k,n-k}$ be the subspace of all multilinear polynomials $f=f(x_1,\dots,x_k,y_1,\dots,y_{n-k})\in\mathcal{L}(X,Y)$ of degree k on even variables and of degree $n-k$ on odd variables. Then $P_{k,n-k}\cap\mathrm{Id}^{gr}(L)$ is the subspace of all multilinear graded identities of L of total degree n depending on k even variables and $n-k$ odd variables. Denote also by $P_{k,n-k}(L)$ the quotient

$$P_{k,n-k}(L)=\frac{P_{k,n-k}}{P_{k,n-k}\cap\mathrm{Id}^{gr}(L)}.$$

Then the *partial* graded $(k,n-k)$-codimension of L is

$$c_{k,n-k}(L)=\dim P_{k,n-k}(L)$$

and the *total* graded nth codimension of L is

$$c_n^{gr}(L)=\sum_{k=0}^{n}\binom{n}{k}c_{k,n-k}(L). \tag{2}$$

If the sequence $\{c_n^{gr}(L)\}_{n\ge1}$ is exponentially bounded then one can consider the related bounded sequence $\sqrt[n]{c_n^{gr}(L)}$. The latter sequence has the following lower and upper limits

$$\underline{\exp}^{gr}(L)=\liminf_{n\to\infty}\sqrt[n]{c_n^{gr}(L)}\quad\text{and}\quad\overline{\exp}^{gr}(L)=\limsup_{n\to\infty}\sqrt[n]{c_n^{gr}(L)}$$

called the *lower* and *upper* PI-exponents of L, respectively. If the ordinary limit exists, it is called the (ordinary) *graded* PI-*exponent* of L,

$$\exp^{gr}(L) = \lim_{n\to\infty} \sqrt[n]{c_n^{gr}(L)}.$$

Symmetric groups and their representations play an important role in the theory of codimensions. In particular, in the case of graded identities one can consider the $S_k \times S_{n-k}$-action on multilinear graded polynomials. Namely, the subspace $P_{k,n-k} \subseteq \mathcal{L}(X,Y)$ has a natural structure of $S_k \times S_{n-k}$-module where S_k acts on even variables $x_1,\dots,x_k$ while S_{n-k} acts on odd variables $y_1,\dots,y_{n-k}$. Clearly, $P_{k,n-k} \cap \mathrm{Id}^{gr}(L)$ is the submodule under this action and we get an induced $S_k \times S_{n-k}$-action on $P_{k,n-k}(L)$. The character $\chi_{k,n-k}(L) = \chi(P_{k,n-k}(L))$ is called $(k,n-k)$ *cocharacter* of L. Since char $F=0$, this character can be decomposed into the sum of irreducible characters

$$\chi_{k,n-k}(L) = \sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}} m_{\lambda,\mu}\chi_{\lambda,\mu} \tag{3}$$

where λ and μ are partitions of k and $n-k$, respectively. All details concerning representations of symmetric groups can be found in [9]. An application of S_n-representations in PI-theory can be found in [1], [3], [7].

Recall that an irreducible $S_k \times S_{n-k}$-module with the character $\chi_{\lambda,\mu}$ is the tensor product of S_k-module with the character χ_λ and S_{n-k}-module with the character χ_μ. In particular, the dimension $\deg \chi_{\lambda,\mu}$ of this module is the product $d_\lambda d_\mu$ where $d_\lambda = \deg \chi_\lambda$, $d_\mu = \deg \chi_\mu$. Taking into account multiplicities $m_{\lambda,\mu}$ in (3) we get the relation

$$c_{k,n-k}(L) = \sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}} m_{\lambda,\mu} d_\lambda d_\mu. \tag{4}$$

A number of irreducible components in the decomposition of $\chi_{k,n-k}(L)$, i.e. the sum

$$l_{k,n-k}(L) = \sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}} m_{\lambda,\mu}$$

is called the $(k,n-k)$-*colength* of L. The *total* graded colength $l_n^{gr}(L)$ is

$$l_n^{gr}(L) = \sum_{k=0}^{n} l_{k,n-k}(L).$$

Now let L be a finite dimensional Lie superalgebra, $\dim L = d$. Then

$$c_n^{gr}(L) \le d^n \tag{5}$$

by the results of [2] (see also [4]). On the other hand, there exists a polynomial φ such that

$$l_n^{gr} \le \varphi(n) \tag{6}$$

for all $n=1,2,\ldots$ as it was mentioned in [11]. Note also that $m_{\lambda,\mu}\neq 0$ in (3) only if $\lambda\vdash k, \mu\vdash n-k$ are partitions with at most d components, that is $\lambda=(\lambda_1,\ldots,\lambda_p)$, $\mu=(\mu_1,\ldots,\mu_q)$ and $p,q\le d=\dim L$.

Since all partitions under our consideration are of the height at most d, we will use the following agreement. If say, λ is a partition of k with $p<d$ components then we will write $\lambda=(\lambda_1,\ldots,\lambda_d)$ anyway, assuming that $\lambda_{p+1}=\ldots=\lambda_d=0$.

For studying asymptotic behaviour of codimensions it is convenient to use the following function defined on partitions. Let ν be a partition of m, $\nu=(\nu_1,\ldots,\nu_d)$. We introduce the following function of ν:

$$\Phi(\nu)=\frac{1}{(\frac{\nu_1}{m})^{\frac{\nu_1}{m}}\ldots(\frac{\nu_d}{m})^{\frac{\nu_d}{m}}}.$$

The values $\Phi(\nu)^m$ and $d_\nu=\deg\chi_\nu$ are very close in the following sense.

Lemma 2.1. [8, Lemma 1] *Let* $m\ge 100$. *Then*

$$\frac{\Phi(\nu)^m}{m^{d^2+d}}\le d_\nu\le m\Phi(\nu)^m.$$

Function Φ has also the following useful property. Let ν and ρ be two partitions of m with the corresponding Young diagrams D_ν, D_ρ. We say that D_ρ is obtained from D_ν by pushing down one box if there exist $1\le i<j\le d$ such that $\rho_i=\nu_i-1, \rho_j=\nu_j+1$ and $\rho_t=\nu_t$ for all remaining $1\le t\le d$.

Lemma 2.2. (see [8, Lemma 3], [16, Lemma 2]) *Let D_ρ be obtained from D_ν by pushing down one box. Then* $\Phi(\rho)\ge\Phi(\nu)$.

3. Existence of graded PI-exponents

Throughout this section let $L=L_0\oplus L_1$ be a finite dimensional simple Lie superalgebra, $\dim L=d$. Then by (5) its upper graded PI-exponent exists,

$$a=\overline{\exp}^{gr}(L)=\limsup_{n\to\infty}\sqrt[n]{c_n^{gr}(L)}.$$

Note that the even component L_0 of L is not solvable since L is simple (see [13, Chapter 3, §2, Proposition 2]).

We shall need the following fact.

Remark 3.1. Let G be a non-solvable finite dimensional Lie algebra over a field F of characteristic zero. Then the ordinary PI-exponent of G exists and is an integer not less than 2.

Proof. It is known that $c_n(G)$ is either polynomially bounded or it grows exponentially not slower that 2^n (see [10]). The first option is possible only if G is solvable. On the other hand $\exp(G)$ always exists and is an integer [14] therefore we are done. $\square$

By the previous remark $P_{n,0}(L) \gtrsim 2^n$ asymptotically and then

$$a \geq 2. \tag{7}$$

The following lemma is the key technical step in the proof of our main result.

Lemma 3.2. *For any $\varepsilon > 0$ and any $\delta > 0$ there exists an increasing sequence of positive integers $n_0, n_1, \ldots$ such that*

(i) $\sqrt[n]{c_n^{gr}(L)} > (1-\delta)(a-\varepsilon)$ *for all* $n = n_q$, $q = 1, 2, \ldots$,

(ii) $n_{q+1} - n_q \leq n_0 + d$.

Proof. Fix $\varepsilon, \delta > 0$. Since a is an upper limit there exist infinitely many indices n_0 such that

$$c_{n_0}^{gr}(L) > (a-\varepsilon)^{n_0}.$$

Fixing one of n_0 we can find an integer $0 \leq k_0 \leq n_0$ such that

$$\binom{n_0}{k_0} c_{k_0, n_0-k_0}(L) > \frac{1}{n_0+1}(a-\varepsilon)^{n_0} > \frac{1}{2n_0}(a-\varepsilon)^{n_0} \tag{8}$$

(see (2)). Relation (6) shows that

$$\sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} \leq \varphi(n)$$

for any $0 \leq k \leq n$ where $m_{\lambda,\mu}$ are taken from (3). Then (4) implies the existence of partitions $\lambda \vdash k_0, \mu \vdash n_0 - k_0$ such that

$$\binom{n_0}{k_0} d_\lambda d_\mu > \frac{1}{2n_0 \varphi(n_0)}(a-\varepsilon)^{n_0}. \tag{9}$$

The latter inequality means that there exists a multilinear polynomial

$$f = f(x_1, \ldots, x_{k_0}, y_1, \ldots, y_{n_0-k_0}) \in P_{k_0, n_0-k_0}$$

such that $F[S_{k_0}\times S_{n_0-k_0}]f$ is an irreducible $F[S_{k_0}\times S_{n_0-k_0}]$-submodule P_{k_0,n_0-k_0} with the character $\chi_{\lambda,\mu}$ and $f\notin \mathrm{Id}^{gr}(L)$. In particular, there exist $a_1,\dots,a_{k_0}\in L_0$, $b_1,\dots,b_{n_0-k_0}\in L_1$ such that

$$A=f(a_1,\dots,a_{k_0},b_1,\dots,b_{n_0-k_0})\neq 0$$

in L. First we will show how to find n_1,k_1 which are approximately equal to $2n_0,2k_0$, respectively, satisfying the same inequality as (8).

Since L is simple and $A\neq 0$ the ideal generated by A coincides with L. Clearly, every simple Lie superalgebra is centerless. Hence one can find $c_1,\dots,c_{d_1}\in L_0\cup L_1$ such that

$$[A,c_1,\dots,c_{d_1},A]\neq 0$$

and $d_1\le d-1$. Here we use the left-normed notation $[[a,b],c]=[a,b,c]$ for nonassociative products. It follows that a polynomial

$$[f_1,z_1,\dots,z_{d_1},f_2]=g_2\in P_{2k_0+p,2n_0-2k_0+r},\quad p+r=d_1,$$

is also a non-identity of L where $z_1,\dots,z_{d_1}\in X\cup Y$ are even or odd variables, whereas f_1 and f_2 are copies of f written on disjoint sets of indeterminates,

$$f_1=f\big(x_1^1,\dots,x_{k_0}^1,y_1^1,\dots,y_{n_0-k_0}^1\big),$$
$$f_2=f\big(x_1^2,\dots,x_{k_0}^2,y_1^2,\dots,y_{n_0-k_0}^2\big).$$

Consider the $S_{2k_0}\times S_{2n_0-2k_0}$-action on $P_{2k_0+p,2n_0-2k_0+r}$ where S_{2k_0} acts on $x_1^1,\dots,x_{k_0}^1,x_1^2,\dots,x_{k_0}^2$ and $S_{2n_0-2k_0}$ acts on $y_1^1,\dots,y_{n_0-k_0}^1,y_1^2,\dots,y_{n_0-k_0}^2$. Denote by M the $F[S_{2k_0}\times S_{2n_0-2k_0}]$-submodule generated by g_2 and examine its character. It follows from Richardson–Littlewood rule that

$$\chi(M)=\sum_{\substack{\nu\vdash 2k_0\\ \rho\vdash 2n_0-2k_0}} t_{\nu,\rho}\chi_{\nu,\rho}$$

where either $\nu=2\lambda=(2\lambda_1,\dots,2\lambda_d)$ or ν is obtained from 2λ by pushing down one or more boxes of $D_{2\lambda}$. Similarly, ρ is either equal to 2μ or ρ is obtained from 2μ by pushing down one or more boxes of $D_{2\mu}$. Then by Lemma 2.2 we have

$$\Phi(\nu)\ge\Phi(2\lambda)=\Phi(\lambda)\quad\text{and}\quad\Phi(\rho)\ge\Phi(2\mu)=\Phi(\mu).$$

By Lemma 2.1 and (9) we have

$$\binom{n_0}{k_0}\big(\Phi(\lambda)\Phi(\mu)\big)^{n_0}>\frac{1}{2n_0^3\varphi(n_0)}(a-\varepsilon)^{n_0}. \tag{10}$$

Now we present the lower bound for binomial coefficients in terms of function Φ. Clearly, the pair $(k,n-k)$ is a two-component partition of n if $k\geq n-k$. Otherwise $(n-k,k)$ is a partition of n. Since $x^{-x}y^{-y}=y^{-y}x^{-x}$ for all $x,y\geq 0, x+y=1$, we will use the notation $\Phi(\frac{k}{n},\frac{n-k}{n})$ in both cases $k\geq n-k$ or $n-k\geq k$. Then it easily follows from the Stirling formula that

$$\frac{1}{n}\Phi\left(\frac{k}{n},\frac{n-k}{n}\right)^n \leq \binom{n}{k} \leq n\Phi\left(\frac{k}{n},\frac{n-k}{n}\right)^n,$$

hence

$$\binom{qk_0}{qn_0} > \frac{1}{qn_0}\Phi\left(\frac{qk_0}{qn_0},\frac{qn_0-qk_0}{qn_0}\right)^{qn_0} = \frac{1}{qn_0}\Phi\left(\frac{k_0}{n_0},\frac{n_0-k_0}{n_0}\right)^{qn_0} \tag{11}$$

for all integers $q\geq 2$ and also

$$\left(\Phi\left(\frac{k_0}{n_0},\frac{n_0-k_0}{n_0}\right)\Phi(\lambda)\Phi(\mu)\right)^{n_0} > \frac{1}{2n_0^4\varphi(n_0)}(a-\varepsilon)^{n_0}, \tag{12}$$

by virtue of (10).

Recall that we have constructed earlier a multilinear polynomial $g_2=[f_1,z_1,\ldots,z_{d_1},f_2]$ which is not a graded identity of L and f_1,f_2 are copies of f. Applying the same procedure we can construct a non-identity of the type

$$g_q=[g_{q-1},w_1,\ldots,w_{d_{q-1}},f_q]$$

of total degree $n_{q-1}=n_{q-2}+n_0+w_1+\ldots+w_{d_{q-1}}$ where $d_{q-1}\leq d$ and f_q is again a copy of f for all $q\geq 2$.

As in the case $q=2$ the $F[S_{qk_0}\times S_{qn_0-qk_0}]$-submodule of $P_{k,n-k}(L)$ (where $n=n_{q-1}=qn_0+p', k=k_{q-1}=qk_0+p''$) contains an irreducible summand with the character $\chi_{\nu,\rho}$ where $\nu\vdash qk_0, \rho\vdash qn_0-qk_0$, $\Phi(\nu)\geq\Phi(\lambda), \Phi(\rho)\geq\Phi(\mu)$. Moreover, for $n=n_{q-1}$ we have

$$\begin{aligned} c_n^{gr}(L) &\geq \binom{qn_0}{qk_0}d_\nu d_\rho > \frac{1}{n^{2d^2+2d}}\binom{qn_0}{qk_0}\big(\Phi(\lambda)\Phi(\mu)\big)^{qn_0} \\ &> \frac{1}{n^{2d^2+2d+1}}\left(\Phi\left(\frac{k_0}{n_0},\frac{n_0-k_0}{n_0}\right)\Phi(\lambda)\Phi(\mu)\right)^{qn_0} \end{aligned}$$

by Lemma 2.1 and the inequality (11). Now it follows from (12) that

$$c_n^{gr}(L) > \frac{1}{n^{2d^2+2d+1}}\frac{1}{(2n_0^4\varphi(n_0))^q}(a-\varepsilon)^{qn_0}.$$

Note that $qn_0 \le n \le qn_0 + qd$. Hence $q/n \le 1/n_0$ and

$$(a-\varepsilon)^{qn_0} \ge \frac{(a-\varepsilon)^n}{a^{qd}}$$

since $a \ge 2$ (see (7)). Therefore

$$\sqrt[n]{c_n^{gr}(L)} > \frac{(a-\varepsilon)^n}{n^{\frac{2d^2+2d+1}{n}} (2a^d n_0^4 \varphi(n_0))^{\frac{1}{n_0}}}$$

for all $n = n_{q-1}$, $q = 1, 2, \ldots$. Finally note that the initial n_0 can be taken to be arbitrarily large. Hence we can suppose that

$$n^{-\frac{2d^2+2d+1}{n}} \left(2a^d n_0^4 \varphi(n_0)\right)^{-\frac{1}{n_0}} > 1-\delta$$

for all $n \ge n_0$. Hence the inequality

$$\sqrt[n]{c_n^{gr}(L)} > (1-\delta)(a-\varepsilon)^n$$

holds for all $n = n_q$, $q = 0, 1, \ldots$. The second inequality $n_{q+1} - n_q \le n_0 + d$ follows from the construction of the sequence $n_0, n_1, \ldots$, and we have thus completed the proof. $\square$

Now we are ready to prove the main result of the paper.

Theorem 3.3. *Let L be a finite dimensional simple Lie superalgebra over a field of characteristic zero. Then its graded PI-exponent*

$$\exp^{gr}(L) = \lim_{n\to\infty} \sqrt[n]{c_n^{gr}(L)}$$

exists an is less than or equal to $d = \dim L$.

Proof. First note that, given a multilinear polynomial $h = h(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \in P_{k,n-k}$, the linear span M of all its values in L is a L_0-module since

$$[h, z] = \sum_i h\big(x_1, \ldots, [x_i, z], \ldots, x_k, y_1, \ldots, y_{n-k}\big)$$
$$+ \sum_j h\big(x_1, \ldots, x_k, y_1, \ldots, [y_j, z], \ldots, y_{n-k}\big)$$

for any $z \in \mathcal{L}(X, Y)_0$. Hence $ML_1 \ne 0$ in L and $0 \equiv [h, w]$ is not an identity of L for odd variable w as soon as $h \notin \mathrm{Id}^{gr}(L)$. It follows that

$$c_{k,n-k+1}(L) \ge c_{k,n-k}(L)$$

and then

$$c_n^{gr}(L) \geq c_m^{gr}(L) \tag{13}$$

for $n \geq m$.

Fix arbitrary small $\varepsilon, \delta > 0$. By Lemma 3.2 there exists an increasing sequence $n_q, q=1,2,\ldots$, such that $c_n^{gr}(L) > ((1-\delta)(a-\varepsilon))^n$ for all $n=n_q, q=0,1,\ldots$, and $n_{q+1}-n_q \leq n_0+d$. Denote $b=(1-\delta)(a-\varepsilon)$ and take an arbitrary $n_q<n<n_{q+1}$. Then $c_n^{gr}(L) \geq c_{n_q}^{gr}(L)$ and $n-n_q \leq n_0+d$. Referring to (7) we may assume that $b>1$. Then $b^{n_q} \geq b^n \cdot b^{-(n_0+d)}$ and

$$c_n^{gr}(L) \geq \left(b^{1-\frac{n_0+d}{n}}\right)^n$$

for all $n_q \leq n \leq n_{q+1}$ and all $q=0,1,\ldots$, that is for all sufficiently large n. The latter inequality means that

$$\liminf_{n\to\infty} \sqrt[n]{c_n^{gr}(L)} \geq (1-\delta)b = (1-\delta)^2(a-\varepsilon).$$

Since ε, δ were chosen to be arbitrary, we have thus completed the proof of the theorem. □

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