Ark. Mat., **55** (2017), 185–198
DOI: 10.4310/ARKIV.2017.v55.n1.a9

A four-dimensional Neumann ovaloid

Lavi Karp and Erik Lundberg

Abstract. What is the shape of a uniformly massive object that generates a gravitational potential equivalent to that of two equal point-masses? If the weight of each point-mass is sufficiently small compared to the distance between the points then the answer is a pair of balls of equal radius, one centered at each of the two points, but otherwise it is a certain domain of revolution about the axis passing through the two points. The existence and uniqueness of such a domain is known, but an explicit parameterization is known only in the plane where the region is referred to as a Neumann oval. We construct a four-dimensional "Neumann ovaloid", solving explicitly this inverse potential problem.

1. Introduction

A domain $\Omega \subset \mathbb{R}^n$ is called a *quadrature domain* if it admits a formula for the integration of any harmonic and integrable function u in Ω,

$$\int_\Omega u\, dV = \langle T, u \rangle, \tag{1}$$

where T is a distribution (independent of u) such that $T\big|_{\mathbb{R}^n \setminus \Omega} = 0$. In particular, when T is a measure, then by applying (1) to the Newtonian kernel it results that the external potential of the body Ω with density one is equal to the potential of the measure T. If T is a finitely-supported distribution of finite-order (so the right-hand-side of (1) is a finite sum of weighted point evaluations of u and its partial derivatives), then Ω is referred to as a *quadrature domain in the classical sense*. There are many examples of quadrature domains in the classical sense in the plane, where conformal mappings can be used to construct them (see e.g. [1]). But in higher dimensional spaces, very few explicit examples are known [2] and [6],

Key words and phrases: quadrature domain, Schwarz function, Neumann oval, inverse potential problem, elliptic integral.
2010 Mathematics Subject Classification: Primary 31A35, 30C20; secondary 35R35.

and the only known explicit example involving simple point evaluations (with no partial derivatives appearing) is the ball. There are however existence results for such quadrature domains [4] and [10].

1.1. Neumann's oval

One of the simplest non-trivial examples of a quadrature domain (in the classical sense) in the plane is the region whose boundary is described by the real algebraic curve (excluding the origin):

$$(x^2+y^2)^2 = \alpha^2(x^2+y^2)+4\varepsilon^2x^2. \tag{2}$$

This curve is referred to as Neumann's oval (it is also known as the hippopede of Booth). Denoting this region by Ω, the quadrature formula is a sum of two point evaluations:

$$\int_\Omega u\,dA = \pi B\cdot u(-\varepsilon,0)+\pi B\cdot u(\varepsilon,0),$$

where the coefficient B is a function of α and ε. This quadrature identity was discovered by C. Neumann in 1908 [9], see [1, Ch. 5, 14] or [11, Ch. 3] for details.

1.2. A four-dimensional Neumann ovaloid

More generally, we refer to a domain $\Omega\subset\mathbb{R}^n$ as a *Neumann ovaloid* if it admits a quadrature formula having two quadrature nodes with equal weights. We will consider the case of a four-dimensional Neumann ovaloid $\Omega\subset\mathbb{R}^4$ satisfying:

$$\int_\Omega u\,dV = \pi^2 A\left\{u(-\varepsilon,0,0,0)+u(\varepsilon,0,0,0)\right\}, \tag{3}$$

for some positive constant A.

As one should expect, Ω is axially-symmetric (see the next paragraph below). However, the axially-symmetric domain in $\mathbb{R}^4$ generated by the rotation of a two-dimensional Neumann oval is not a Neumann ovaloid, and in fact, it is not even a quadrature domain in the classical sense. Instead, as the first author showed in [6], it has a quadrature formula supported on the whole segment joining the original quadrature points.

Concerning uniqueness, suppose that the distribution T of (1) is a non-negative measure with compact support in a hyperplane, as in the case of interest (3). If Ω is a bounded quadrature domain for T, then Ω is symmetric with respect to the same hyperplane and the complement of Ω is connected. This result was proved by Sakai in the plane [10, §14], and for the n-dimensional case see [5] or [11, §4]. We may apply this to every hyperplane passing through the two quadrature points in order to conclude that the Neumann ovaloid Ω is axially symmetric.

Since the Neumann ovaloid Ω is axially symmetric and has connected complement, it must be generated by rotation of a simply connected planar domain D_p. Invoking the Riemann mapping theorem, there is a conformal map f from the unit disk $\mathbb{D}$ to D_p, and f is unique once the value of $f(0)$ and $\arg f'(0)$ are prescribed. In order to construct Ω, it thus suffices to determine explicitly the conformal map f.

Theorem 1. *Let $\Omega\subset\mathbb{R}^4$ be the quadrature domain that satisfies the formula (3), and let D_p denote the simply connected domain that generates Ω by rotation. Let f be the conformal map from the unit disk $\mathbb{D}$ to D_p such that $f(0)=0$ and $f'(0)>0$. Then f is given by:*

$$f(\zeta)=\frac{C}{2\pi i}\int_{\{|w|=1\}}\frac{1}{w-\zeta}\,\frac{(w^2-1)\sqrt{(w^2+a^2)(1+a^2w^2)}}{(w^2-b^2)(1-b^2w^2)}\,dw, \tag{4}$$

for some real-valued positive constants a, b and C.

Remark 2. There are existence results for point mass quadrature domains [3], [4] and [10]; the important attribute of Theorem 1 is the explicit formula for the domain. For each ε and A, the Neumann ovaloid (3) is unique, this follows from the uniqueness of bounded quadrature domains for non-negative measures with compact support in a hyperplane (see [5], [10] and [11]).

In Section 2, we review some essential background on axially symmetric quadrature domains in $\mathbb{R}^4$ following [2] and [6]. We prove Theorem 1 in Section 3. The proof is based on the approach introduced in [2] where A. Eremenko and the second author used it to give a negative answer to the question of H. S. Shapiro [11] on the algebraicity of quadrature domains in $n>2$ dimensions. In Section 4 we describe how a and b of formula (4) are related when $C=1$ is fixed, and we show that the quadrature formula for the ball is recovered from (3) in the limit as a and b tend to zero. In Section 5, we present some graphics based on numerical implementation of Theorem 1.

2. Preliminaries

Here we review an algebraic technique that applies to axially symmetric quadrature domains in $\mathbb{R}^4$. For further details see [2] and [6]. An equivalent definition of the quadrature domain (1) is by the free boundary problem

$$\begin{cases}\Delta u=T, & \text{in } \Omega\\ u=\frac{1}{2}|x|^2,\nabla u=x & \text{on } \partial\Omega\end{cases}. \tag{5}$$

The two notions are equivalent when the distribution T is compactly supported in Ω. A solution u to the overdetermined system (5) is called the *Schwarz potential* of Ω ([7] and [11]).

In the case of the Neumann ovaloid (3), the distribution T, as well as the domain Ω has axial symmetry about the x_1-axis. Hence the Schwarz potential u can be represented by a function of two variables, namely

$$u(x_1,x_2,x_3,x_4)=U(X,Y),$$

where $X=x_1$, $Y=\sqrt{x_2^2+x_3^2+x_4^2}$ and U extended to the lower plane by $U(X,Y)=U(X,-Y)$. Then the function U satisfies the free boundary problem

$$\begin{cases} \Delta U+2Y^{-1}\frac{\partial U}{\partial Y}=T, & \text{in } D_p \\ U=\frac{1}{2}(X^2+Y^2), \nabla U=(X,Y), & \text{on } \partial D_p \end{cases}, \tag{6}$$

where $D_p\subset\mathbb{R}^2$ is the domain whose rotation generates Ω.

The main idea is to set $V(X,Y)=YU(X,Y)$. Then V satisfies the Cauchy problem

$$\begin{cases} \Delta V=0, & \text{near } \partial D_p \\ V=\frac{Y}{2}(X^2+Y^2), & \text{on } \partial D_p \\ \nabla V=(XY,\frac{1}{2}(X^2+3Y^2)), & \text{on } \partial D_p \end{cases}. \tag{7}$$

This enables us to solve the free boundary problem (6) by means of the Schwarz function. Indeed, letting $z=X+iY$ and $V_z=\frac{1}{2}\left(\frac{\partial V}{\partial X}-i\frac{\partial V}{\partial Y}\right)$, then on the boundary ∂D_p,

$$V_z(X,Y)=\frac{1}{2}XY-i\frac{1}{4}(X^2+3Y^2)=\frac{i}{4}(\bar{z}^2-2\bar{z}z),$$

where $\bar{z}=X-iY$. Next, we replace $\bar{z}$ by $S(z)$ on the boundary ∂D_p, and we obtain that

$$V(z)=2\mathrm{Re}\left(\int V_z(z,S(z))\,dz\right) \tag{8}$$

is the solution to the Cauchy problem (7) (see [6, Lemma 2.3]). The representation of the solution V by (8) implies that the singularities of $V_z(z,S(z))=\frac{i}{4}\left(S^2(z)-2zS(z)\right)$ determine the distribution T in (5). Since we are interested only in the structure of the distribution T, we may perturb this expression by a holomorphic function. Thus we conclude that the singularities of the expression

$$\frac{i}{4}\left(S^2(z)-2zS(z)\right)+\frac{i}{4}z^2=\frac{i}{4}\left(S(z)-z\right)^2 \tag{9}$$

govern the distribution T.

Due to the axial symmetry, the support of the distribution T is on the $x_1 = X$-axis. So if the expression in (9) has non-vanishing residue at the points $(\pm\varepsilon, 0)$, then in the integration (8) causes to a logarithmic term and consequently the support of T is on a segment joining these two points. This implies that Ω will not be point masses quadrature domain. For example, if D_p is the Neumann oval (2), then

$$\frac{i}{4}\left(S(z)-z\right)^2 = i\frac{(\alpha^2+2\varepsilon^2)^2}{16}\left(\frac{1}{(z-\varepsilon)^2}+\frac{1}{(z+\varepsilon)^2}\right)+i\frac{\alpha^4}{16\varepsilon}\left(\frac{1}{z-\varepsilon}-\frac{1}{z+\varepsilon}\right)+h(z),$$

where h is holomorphic in D_p.

However, if we require that

$$\frac{1}{2}\left(S(z)-z\right)^2 = \frac{A}{(z-\varepsilon)^2}+\frac{A}{(z+\varepsilon)^2}+h(z), \tag{10}$$

then

$$\begin{aligned} V(X,Y) &= 2\mathrm{Re}\left(\int V_z(z,S(z))\,dz\right) \\ &= \frac{-AY}{(X-\varepsilon)^2+Y^2}+\frac{-AY}{(X+\varepsilon)^2+Y^2}+H(X,Y), \end{aligned}$$

with H harmonic in D_p. Recalling that $U(X,Y)=Y^{-1}V(X,Y)$, we see that the singular part of the Schwarz potential (5) comprises the expression $\frac{A}{(x_1\pm\varepsilon)^2+x_2^2+x_3^2+x_4^2}$, which is the fundamental solution of the Laplacian in $\mathbb{R}^4$ at the points $(\pm\varepsilon,0,0,0)$. Hence

$$T = \pi^2 A\left(\delta_{(-\varepsilon,0,0,0)}+\delta_{(+\varepsilon,0,0,0)}\right)$$

is the distribution of the Schwarz potential (5) and consequently the rotation of D_p yields the Neumann Ovaloid (3), here δ denote the Dirac measure.

3. Construction of the Neumann ovaloid

As in the statement of Theorem 1, take f to be the conformal map from the unit disk $\mathbb{D}$ into D_p such that $f(0)=0$ and $f'(0)>0$ is real. With this normalization, the conformal map is unique. By symmetry of the domain D_p under complex conjugation, and uniqueness of the conformal map, f is real, i.e., $f^*(w):=\overline{f(\overline{w})}=f(w)$.

Let S be the Schwarz function of the boundary ∂D_p. Following [2], we consider the pullback of (10) under the conformal map f. Using the relation $S(f(w))=f^*(1/w)=f(1/w)$, we have

$$\frac{1}{2}\left(f(w)-f\left(\frac{1}{w}\right)\right)^2 = \frac{A}{(f(w)-\varepsilon)^2}+\frac{A}{(f(w)+\varepsilon)^2}+h(f(w)). \tag{11}$$

Setting

$$g(w)=\left(f(w)-f\left(\frac{1}{w}\right)\right)^2, \tag{12}$$

then obviously $g(1/w)=g(w)$. Using this fact, we can use (11) to analytically continue $g(w)$ to the entire plane as a meromorphic (and in fact rational) function. Namely, we prove the following:

Lemma 3. *The function g is a rational function of degree exactly 8 and takes the form*:

$$g(w)=\frac{c(w^2-1)^2(w^2+a^2)(1+a^2w^2)}{(w^2-b^2)^2(1-b^2w^2)^2}, \tag{13}$$

with a, b, c real constants and $a,b\neq\pm 1$.

Proof of Lemma 3. The points $\pm b\in\mathbb{D}$ are the preimages of $\pm\varepsilon\in D_p$ under f. From (11) g has a pole of order two at $\pm b$ and no other poles in $\mathbb{D}$. Since $g\left(\frac{1}{w}\right)=g(w)$, g has also poles of order two at $\pm\frac{1}{b}$. There are no other poles in $\mathbb{C}\setminus\mathbb{D}$, since otherwise we would have a contradiction to (10). Thus, g is a meromorphic function in the entire plane with exactly four poles of order two. If $g(0)=0$, then the conformal map satisfies $f(w)\to\infty$ as $w\to\infty$, and that would imply that the Schwarz function has a pole at the origin, which contradicts (10). This implies that g is a rational function of degree exactly 8.

Having determined the location of the poles, we have

$$g(w)=\frac{P(w)}{(w^2-b^2)^2(1-b^2w^2)^2},$$

where P is a polynomial of degree 8. From (12) we see that $g(\pm 1)=0$ and the zeros have order two. Hence g has additional four zeros. So suppose w_0 is a zero of g, then $w_0\neq 0$ since $g(0)\neq 0$. Note that $f(w)=-f(-w)$, which follows from the symmetry of D_p with respect to the real and imaginary axes, hence $-w_0$ is a zero of g, and by (12) g vanishes $\pm 1/w_0$. Thus

$$P(w)=c(w^2-1)^2(w^2-w_0^2)(w^2-1/w_0^2).$$

Now set $w=x+iy$ and $w_0=ae^{i\theta}$ for some positive a, then

$$(x^2-w_0^2)(x^2-1/w_0^2)=x^4-x^2(a^2e^{i2\theta}+a^{-2}e^{-i2\theta})+1.$$

Since $f(w)=f^*(w)=\overline{f(\overline{w})}$, g is non-negative on the real axis. Hence the above expression is real, and therefore

$$a^2e^{i2\theta}+a^{-2}e^{-i2\theta}=\overline{a^2e^{i2\theta}+a^{-2}e^{-i2\theta}}=a^2e^{-i2\theta}+a^{-2}e^{i2\theta},$$

which results in the identity

$$a^2 \sin 2\theta = a^{-2} \sin 2\theta.$$

Thus either $a=1$ or $\theta=0$ or $\theta=\frac{\pi}{2}$. Note that $a=1$ is impossible since then $\sqrt{g}$ would have branch points on the unit circle and consequently f is not single valued. If $\theta=0$, then

$$\begin{aligned}(x^2-w_0^2)(x^2-1/w_0)^2 &= x^4-x^2(a^2+a^{-2})+1\\ &= \left(x^2-\frac{a^2+a^{-2}}{2}\right)^2+1-\left(\frac{a^2+a^{-2}}{2}\right)^2\end{aligned}$$

is negative for some x when $a\neq\pm1$. Hence $\theta=\frac{\pi}{2}$ and then

$$(x^2-w_0^2)(x^2-1/w_0)^2 = x^4+x^2(a^2+a^{-2})+1 \geq x^4+2x^2+1 = (x^2+1)^2 > 0.$$

Thus $w_0=ia$, and we have

$$P(w) = c(w^2-1)^2(w^2+a^2)(1+a^2w^2)$$

and we have obtained (13). $\square$

Applying Lemma 3, and taking the square root in (12) we have:

$$f(w)-f\left(\frac{1}{w}\right) = \sqrt{g(w)} := \frac{C(w^2-1)\sqrt{(w^2+a^2)(1+a^2w^2)}}{(w^2-b^2)(1-b^2w^2)},$$

where $C=\sqrt{c}$. Multiplying by $(2\pi i(w-\zeta))^{-1}$ and integrating with respect to dw along the contour $\{|w|=1\}$, we obtain

$$\frac{1}{2\pi i}\int_{\{|w|=1\}} \frac{f(w)-f(1/w)}{w-\zeta}\,dw = \frac{1}{2\pi i}\int_{\{|w|=1\}} \frac{\sqrt{g(w)}}{w-\zeta}\,dw.$$

The term $\frac{f(1/w)}{w-\zeta}$ integrates to zero since it is analytic in $\mathbb{C}\setminus\mathbb{D}$ and

$$\frac{f(1/w)}{w-\zeta} = O(1/|w|^2), \quad \text{as } w \longrightarrow \infty.$$

Since $f(w)$ is analytic in $\mathbb{D}$, we may apply the Cauchy integral formula:

$$f(\zeta) = \frac{1}{2\pi i}\int_{\{|w|=1\}} \frac{f(w)}{w-\zeta}\,dw = \frac{1}{2\pi i}\int_{\{|w|=1\}} \frac{\sqrt{g(w)}}{w-\zeta}\,dw.$$

Thus we have:

$$f(\zeta) = \frac{C}{2\pi i}\int_{\{|w|=1\}} \frac{1}{w-\zeta}\,\frac{(w^2-1)\sqrt{(w^2+a^2)(1+a^2w^2)}}{(w^2-b^2)(1-b^2w^2)}\,dw, \tag{14}$$

and this completes the proof of Theorem 1.

4. The relation between *a* and *b*

Notice that the quadrature domain depends on only two parameters, ε and A, while the conformal map f appears to depend on three real parameters, $a\geq 0$, $b\geq 0$, and $C>0$. However, fixing $C>0$, then the value of a is determined by the value of b. In this section, we fix $C=1$, which gives that f is the identity map when $a=b=0$. We will show how the parameters a and b are related, and that as a and b tend to zero, then the quadrature formula for the unit ball is recovered.

Notice that (10) provides a constraint relating a and b. Namely, we observe that the residue of $(S(z)-z)^2$ at $z=\pm\varepsilon$ vanishes, which is equivalent to $\operatorname{Res}(gf'\colon\pm b)=0$.

Since the pole is of order two, we have:

$$\operatorname{Res}(gf':b)=\frac{d}{d\zeta}\left((\zeta-b)^2 g(\zeta)f'(\zeta)\right)\big|_{\zeta=b}=0. \tag{15}$$

Setting

$$H(\zeta,a,b)=(\zeta-b)^2 g(\zeta)=\frac{(w^2-1)^2(w^2+a^2)(1+w^2a^2)}{(w+b)^2(1-w^2b^2)^2},$$

then by the formula of f (4),

$$\begin{aligned}\operatorname{Res}(gf':b)&=\frac{1}{2\pi i}\int_{\{|w|=1\}}\left[\frac{\frac{\partial H}{\partial\zeta}(b,a,b)}{(w-b)^2}+2\frac{H(b,a,b)}{(w-b)^3}\right]\sqrt{g(w)}\,dw\\&=:F(a,b).\end{aligned} \tag{16}$$

Thus we conclude that the rotation of the domain D_p generates the Neumann ovaloid (3) if and only if $F(a,b)=0$ for some positive a and b. We shall use the implicit function theorem in order to justify that identity (16) determines the parameter a as a function of b.

To this end, the expressions $\frac{\partial H}{\partial\zeta}(b,a,b)$ and $H(b,a,b)$ can be computed using symbolic computation software:

$$\frac{\partial H}{\partial\zeta}(\zeta,a,b)\Big|_{\zeta=b}=\frac{a^2+b^2(-4a^4+4a^2-1)+4b^4(a^4-a^2+1)+b^6(a^4+4a^2+4)+3b^8a^2}{4b^{11}+8b^9-8b^5-4b^3}, \tag{17}$$

$$H(b,a,b)=\frac{a^2+b^2(a^4+1)+b^4a^2}{4b^6+8b^4+4b^2}. \tag{18}$$

It follows from the above computations that the functional $F(a,b)$ is not continuous when $b=0$. Hence, we set $a=\sqrt{\delta}$ and

$$G(\delta,b)=b^3F(\sqrt{\delta},b).$$

Then obviously $F(a,b)=0$ for $(a,b)\in\mathbb{R}_+^2$ if and only if $G(\delta,b)=0$. We shall now compute the Taylor expansion of G near the origin. From (17) and (18) we see that

$$G(\delta,0)=\frac{1}{2\pi i}\int_{\{|w|=1\}}\left(-\frac{\delta}{4}\right)\frac{(w^2-1)\sqrt{(w^2+\delta)(1+\delta w^2)}}{w^4}\,dw.$$

Hence $G(0,0)=0$ and

$$\begin{aligned}\frac{\partial G}{\partial\delta}(0,0)&=\lim_{\delta\to0}\frac{1}{\delta}G(\delta,0)\\&=\lim_{\delta\to0}\frac{1}{2\pi i}\int_{\{|w|=1\}}\left(-\frac{1}{4}\right)\frac{(w^2-1)\sqrt{(w^2+\delta)(1+\delta w^2)}}{w^4}\,dw\\&=\frac{1}{2\pi i}\int_{\{|w|=1\}}\left(-\frac{1}{4}\right)\frac{(w^2-1)w}{w^4}\,dw=-\frac{1}{4}.\end{aligned}$$

Since both $b^3\frac{\partial H}{\partial\zeta}(b,0,b)$ and $b^3H(b,0,b)$ are of order $O(b^2)$ for b near zero, $\frac{\partial G}{\partial b}(0,0)=0$ and

$$\begin{aligned}\frac{\partial G}{\partial b}(0,b)=\frac{1}{2\pi i}\int_{\{|w|=1\}}&\left[\left(\frac{\frac{\partial^2}{\partial b\partial\zeta}\left(b^3H(b,0,b)\right)}{(w-b)^2}+2\frac{\frac{\partial}{\partial b}\left(b^3H(b,0,b)\right)}{(w-b)^3}\right)\right.\\&\left.\times\frac{(w^2-1)w}{(w^2-b^2)(1-b^2w^2)}\right]dw+O(b^2).\end{aligned}$$

From (17) and (18) we see that $\frac{\partial^2}{\partial b\partial\zeta}\left(b^3H(b,0,b)\right)=\frac{b}{2}+O(b^2)$ and $\frac{\partial}{\partial b}\left(b^3H(b,0,b)\right)=O(b^2)$, hence

$$\frac{\partial^2G}{\partial b^2}(0,0)=\lim_{b\to0}\frac{1}{b}\frac{\partial G}{\partial b}(0,b)=\frac{1}{2\pi i}\int_{\{|w|=1\}}\left(\frac{1}{2}\right)\frac{(w^2-1)w}{w^4}\,dw=\frac{1}{2}.$$

In a similar manner we have computed $\frac{\partial^2G}{\partial b\partial\delta}(0,0)=\frac{\partial^2G}{\partial\delta^2}(0,0)=0$ (see Appendix A), hence

$$G(\delta,b)=\frac{1}{4}\left(-\delta+b^2\right)+O(\delta^3)+O(b^3). \tag{19}$$

Thus by the implicit function theorem there is a function $\delta(b)$ such that $G(\delta(b),b)=0$ for b near zero and positive. Moreover, we see from (19) that $\delta=\delta(b)=b^2+O(b^3)$. Hence the functional $F(a,b)$, which is defined by (16), vanishes near the origin on a curve, in which $a\approx b$, or more precisely

$$a^2 = b^2 + O(b^3). \tag{20}$$

The coefficient A of the quadrature identity (3) can be computed as follows. When $F(a,b)=0$, then (10) holds and therefore by (11)

$$\begin{aligned} A &= \frac{1}{4\pi i}\int_{\{|z-\varepsilon|=\rho\}} (S(z)-z)^2\,(z-\varepsilon)\,dz \\ &= \frac{1}{4\pi i}\int_{\{|w-b|=\tilde\rho\}} g(w)\,(f(w)-f(b))\,f'(w)\,dw. \end{aligned}$$

Writing $f(w)-f(b)=(w-b)\,(f'(b)+\Phi(b))$, where $\Phi(b)=0$, then by Lemma 3,

$$\begin{aligned} A = A(b) &= \frac{1}{4\pi i}\int_{\{|w-b|=\tilde\rho\}} \frac{(w^2-1)^2(w^2+a^2)(1+w^2a^2)\,(f'(b)+\Phi(b))\,f'(w)}{(w+b)^2(1-w^2b^2)^2}\,\frac{dw}{(w-b)} \\ &= \frac{1}{2}\,\frac{(b^2-1)^2(b^2+a^2)(1+b^2a^2)}{(2b)^2(1-b^4)^2}\,(f'(b))^2\,. \end{aligned}$$

The estimate (20) enables us to examine the asymptotic behavior as b goes to zero. We see that

$$\lim_{b\to 0} A(b) = \frac{1}{4}(f'(0))^2,$$

and since when $b\to 0$, the conformal mapping f tends to the identity, we conclude that $\lim_{b\to 0} A(b)=A(0)=\frac{1}{4}$. On the other hand, letting ε tend to zero in (3), then the right-hand side goes to $2\pi^2 A(0)u(0,0,0,0)=\frac{\pi^2}{2}u(0,0,0,0)$, and the domain Ω tends to the unit ball. Hence the limit coincides with the mean value property of harmonic functions.

5. Numerics

Numerical implementation of Theorem 1 requires determining appropriate choices of parameters. The parameter C simply scales the domain, so that there is only a one-parameter family of different shapes. While varying the choice of b, we choose a to satisfy the relation (15). This is done numerically using Matlab. We then use the condition $f(b)=\varepsilon$ (where C appears as a scalar) in order to choose C so that $\varepsilon=1$. This leads to a one-parameter family of different shapes having the same "foci" $\pm\varepsilon=\pm 1$. We note that this family can also be interpreted as a four-dimensional Hele-Shaw flow with two point sources (located at the foci).

We used Matlab to perform numerical integration of the Cauchy transform appearing in (4). This requires some care in checking that the branch of the square

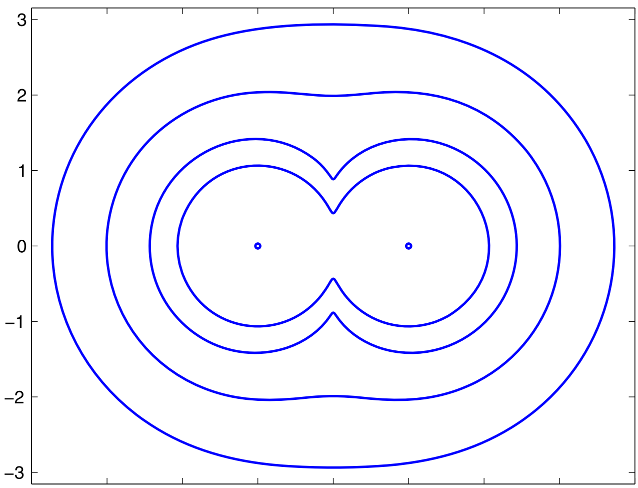

Figure 1. Profiles of some confocal ovaloids plotted using Matlab. Each curve is the image of the unit circle under a conformal mapping f obtained as a numerical Cauchy transform.

root is defined appropriately. The radius of the contour of integration should also be increased slightly (without crossing any singularities of the integrand); this is a convenient way to avoid numerically integrating through the simple pole presented by the fact that ζ is on the unit circle.

Having carried out these steps, we display some images of the profile curves for the resulting confocal Neumann ovaloids in Figure 1.

A. Appendix

Here we shall provide further details for the computations of the second order derivatives $\frac{\partial^2 G}{\partial b \partial \delta} G(0,0)$ and $\frac{\partial^2 G}{\partial^2 \delta} G(0,0)$. From the expressions (17) and (18) we see that

$$\begin{aligned}\frac{\partial G}{\partial \delta}(\delta,b) = \frac{1}{2\pi i}\int_{\{|w|=1\}} &\left[\left(\frac{\delta + b^2(-4\delta^2+4\delta-1)}{-4(w-b)^2} + 2\frac{b\delta}{4(w-b)^3}\right)\right.\\ &\left.\times \frac{(w^2-1)(1+2\delta w^2+w^4)}{(w^2-b^2)(1-w^2b^2)2\sqrt{(w^2+\delta)(1+w^2\delta)}}\right] dw\\ &+\frac{1}{2\pi i}\int_{\{|w|=1\}}\left[\left(\frac{1+b^2(-8\delta+4)}{-4(w-b)^2}+2\frac{b}{4(w-b)^3}\right)\right.\\ &\left.\times\frac{(w^2-1)\sqrt{(w^2+\delta)(1+w^2\delta)}}{(w^2-b^2)(1-w^2b^2)}\right] dw + O(b^2). \qquad (21)\end{aligned}$$

Letting $\delta=0$ and recalling that $\frac{\partial G}{\partial\delta}(0,0)=-\frac{1}{4}$, we see that

$$\begin{aligned}
&\frac{\partial G}{\partial\delta}(0,b)-\frac{\partial G}{\partial\delta}(0,0)\\
&\quad=\frac{1}{2\pi i}\int_{\{|w|=1\}}\left[\left(\frac{-1}{4(w-b)^2}+\frac{2b}{4(w-b)^3}\right)\frac{(w^2-1)w}{(w^2-b^2)(1-w^2b^2)}\right]dw+\frac{1}{4}+O(b^2)\\
&\quad=\frac{1}{2\pi i}\int_{\{|w|=1\}}\left[\left(\frac{2b}{4(w-b)^3}\right)\frac{(w^2-1)w}{(w^2-b^2)(1-w^2b^2)}\right]dw\\
&\qquad+\frac{1}{2\pi i}\int_{\{|w|=1\}}\frac{1}{4}\left[(w^2-1)w\left(\frac{1}{w^4}-\frac{1}{(w-b)^2(w^2-b^2)(1-w^2b^2)}\right)\right]dw+O(b^2).
\end{aligned}$$

In the last equality we used the fact that

$$\frac{1}{2\pi i}\int_{\{|w|=1\}}\frac{(w^2-1)w}{w^4}\,dw=1. \tag{22}$$

By Taylor expansion,

$$\frac{1}{w^4}-\frac{1}{(w-b)^2(w^2-b^2)(1-w^2b^2)}=\frac{1}{w^4}\left(-\frac{2b}{w}+O(b^2)\right),$$

and hence

$$\begin{aligned}
\frac{\partial^2 G}{\partial b\partial\delta}(0,0)&=\lim_{b\to 0}\frac{1}{b}\left(\frac{\partial G}{\partial\delta}(0,b)-\frac{\partial G}{\partial\delta}(0,0)\right)\\
&=\frac{1}{2\pi i}\int_{\{|w|=1\}}\left[\left(\frac{2(w^2-1)w}{4w^5}\right)\right]dw-\frac{1}{2\pi i}\int_{\{|w|=1\}}\left[\left(\frac{2(w^2-1)w}{4w^5}\right)\right]dw\\
&=0.
\end{aligned}$$

We turn now to the computation of $\frac{\partial^2 G}{\partial\delta^2}(0,0)$. Using (21) and (22), we have that

$$\begin{aligned}
&\frac{\partial G}{\partial\delta}(\delta,0)-\frac{\partial G}{\partial\delta}(0,0)\\
&\qquad=\frac{1}{2\pi i}\int_{\{|w|=1\}}-\frac{1}{8}\left[\frac{\delta(w^2-1)(1+2\delta w^2+w^4)}{w^4\sqrt{(w^2+\delta)(1+w^2\delta)}}\right]dw\\
&\qquad\quad+\frac{1}{2\pi i}\int_{\{|w|=1\}}-\frac{1}{4}\left[\frac{(w^2-1)\sqrt{(w^2+\delta)(1+w^2\delta)}}{w^4}\right]dw+\frac{1}{4}\\
&\qquad=\frac{1}{2\pi i}\int_{\{|w|=1\}}-\frac{1}{8}\left[\frac{\delta(w^2-1)(1+2\delta w^2+w^4)}{w^4\sqrt{(w^2+\delta)(1+w^2\delta)}}\right]dw\\
&\qquad\quad+\frac{1}{2\pi i}\int_{\{|w|=1\}}\frac{1}{4}\left[\frac{(w^2-1)w}{w^4}\left(1-\sqrt{1+\delta\left(\frac{1}{w^2}+w^2\right)+\delta^2}\right)\right]dw.
\end{aligned}$$

Taylor expansion

$$\sqrt{1+\delta\left(\frac{1}{w^2}+w^2\right)+\delta^2}=1+\frac{\delta}{2}\left(\frac{1}{w^2}+w^2\right)+O(\delta^2)$$

yields that

$$\begin{aligned}\frac{\partial^2 G}{\partial\delta^2}G(0,0)&=\lim_{\delta\to 0}\frac{1}{\delta}\left(\frac{\partial G}{\partial\delta}G(\delta,0)-\frac{\partial G}{\partial\delta}G(0,0)\right)\\&=\frac{1}{2\pi i}\int_{\{|w|=1\}}-\frac{1}{8}\left[\frac{(w^2-1)(1+w^4)}{w^5}\right]dw\\&\quad+\frac{1}{2\pi i}\int_{\{|w|=1\}}\frac{1}{8}\left[\frac{(w^2-1)w}{w^4}\left(\frac{1}{w^2}+w^2\right)\right]dw\\&=\frac{1}{8}-\frac{1}{8}=0.\end{aligned}$$

References

1. Davis, P., *The Schwarz Function and Its Application*, Carus Mathematical Monographs **14**, The Mathematical Association of America, 1974.
2. Eremenko, A. and Lundberg, E., Non-algebraic quadrature domains, *Potential Anal.* **38** (2013), 787–804.
3. Gustafsson, B., *Applications of variational inequalities to a moving boundary problem for Hele-Shaw flows*, research report TRITA-MAT-1981-9, Royal Inst. of Technology, Stockholm.
4. Gustafsson, B., On quadrature domains and on an inverse problem in potential theory, *J. Anal. Math.* **44** (1990), 172–215.
5. Gustafsson, B. and Sakai, M., Properties of some balayage operators with applications to quadrature domains and moving boundary problems, *Nonlinear Anal.* **22** (1994), 1221–1245.
6. Karp, L., Construction of quadrature domains in $\mathbb{R}^n$ from quadrature domains in $\mathbb{R}^2$, *Complex Var. Elliptic Equ.* **17** (1992), 179–188.
7. Khavinson, D. and Shapiro, H. S., *The Schwarz potential in* $\mathbb{R}^n$ *and Cauchy's problem for the Laplace equation*, TRITA-MAT-1989-36, Royal Institute of Technology, Stockholm.
8. Lundberg, E., Laplacian growth, elliptic growth, and singularities of the Schwarz potential, *J. Phys. A: Math. Theor.* **44**, 135202 (2011).
9. Neumann, C., Über das logarithmische Potential einer gewissen Ovalfläche, *Abh. der math.–phys. Klasse der Königl. Sächs. Gesellsch. der Wiss. zu Leibzig* **60** (1908), 53–56. Dritte Mitteilung, *Abh. der math.–phys. Klasse der Königl. Sächs. Gesellsch. der Wiss. zu Leibzig* **60** (1908), 240–247.
10. Sakai, M., *Quadrature Domains*, Lecture Notes in Mathematics **934**, Springer-Verlag, Berlin–Heidelberg–New York, 1982.

11. Shapiro, H. S., *The Schwarz Function and Its Generalization to Higher Dimensions*, Arkansas Lecture Notes in the Mathematical Sciences **9**, John Wily & Sons, Inc., New York, 1992.

Lavi Karp
Department of Mathematics
ORT Braude College
P.O. Box 78
IL-21982 Karmiel
Israel
karp@braude.ac.il

Erik Lundberg
Department of Mathematical Sciences
Florida Atlantic University
Boca Raton FL 33431
U.S.A
elundber@fau.edu

Received October 16, 2016