SUR LE MULTIPLICATEUR
DES
FONCTIONS HYPERELLIPTIQUES DE PREMIER ORDRE

PAR

MARTIN KRAUSE
à ROSTOCK.

Si on pose:

$$N\tau'_{11} = (cd)_{02} + (ac)_{02}\tau_{11} + 2(bc)_{02}\tau_{12} + (db)_{02}\tau_{22} + (ab)_{02}(\tau_{12}^2 - \tau_{11}\tau_{22})$$

$$N\tau'_{12} = (cd)_{12} + (ac)_{12}\tau_{11} + [2(bc)_{12} - n]\tau_{12} + (db)_{12}\tau_{22} + (ab)_{12}(\tau_{12}^2 - \tau_{11}\tau_{22})$$

$$N\tau'_{22} = (cd)_{31} + (ac)_{31}\tau_{11} + 2(bc)_{31}\tau_{12} + (db)_{31}\tau_{22} + (ab)_{31}(\tau_{12}^2 - \tau_{11}\tau_{22})$$

$$N = (cd)_{23} + (ac)_{23}\tau_{11} + 2(bc)_{23}\tau_{12} + (db)_{23}\tau_{22} + (ab)_{23}(\tau_{12}^2 - \tau_{11}\tau_{22})$$

$$C_2 = c_2 - a_2\tau_{21} - b_2\tau_{22}, \qquad C_3 = - c_3 + a_3\tau_{21} + b_3\tau_{22}$$

$$D_2 = d_2 - a_2\tau_{11} - b_2\tau_{12}, \qquad D_3 = - d_3 + a_3\tau_{11} + b_3\tau_{12},$$

où n désigne un degré de transformation complètement arbitraire, on aura:

$$\frac{\partial\tau'_{11}}{\partial\tau_{11}} = n\frac{C_2^2}{N^2}, \qquad \frac{\partial\tau'_{11}}{\partial\tau_{12}} = -\frac{2nC_2D_2}{N^2}, \qquad \frac{\partial\tau'_{11}}{\partial\tau_{22}} = n\frac{D_2^2}{N^2}$$

$$\frac{\partial\tau'_{12}}{\partial\tau_{11}} = n\frac{C_2C_3}{N^2}, \qquad \frac{\partial\tau'_{12}}{\partial\tau_{12}} = -\frac{n(C_2D_3 + C_3D_2)}{N^2}, \qquad \frac{\partial\tau'_{12}}{\partial\tau_{22}} = n\frac{D_2D_3}{N^2}$$

$$\frac{\partial\tau'_{22}}{\partial\tau_{11}} = n\frac{C_3^2}{N^2}, \qquad \frac{\partial\tau'_{22}}{\partial\tau_{12}} = -\frac{2nC_3D_3}{N^2}, \qquad \frac{\partial\tau'_{22}}{\partial\tau_{22}} = n\frac{D_3^2}{N^2}.$$

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Il suit de là que le déterminant fonctionnel est:

$$\begin{vmatrix} \frac{\partial \tau'_{11}}{\partial \tau_{11}} & \frac{\partial \tau'_{11}}{\partial \tau_{12}} & \frac{\partial \tau'_{11}}{\partial \tau_{22}} \\ \frac{\partial \tau'_{12}}{\partial \tau_{11}} & \frac{\partial \tau'_{12}}{\partial \tau_{12}} & \frac{\partial \tau'_{12}}{\partial \tau_{22}} \\ \frac{\partial \tau'_{22}}{\partial \tau_{11}} & \frac{\partial \tau'_{22}}{\partial \tau_{12}} & \frac{\partial \tau'_{22}}{\partial \tau_{22}} \end{vmatrix} = -\frac{n^3(C_2D_3 - C_3D_2)^3}{N^6} = -n^3(C_2D_3 - C_3D_2)^{-3}.$$

Si l'on pose ensuite:

$$A_0 = a_0 + a_3\tau'_{11} + a_2\tau'_{12}, \qquad A_1 = a_1 + a_3\tau'_{21} + a_2\tau'_{22}$$

$$B_0 = b_0 + b_3\tau'_{11} + b_2\tau'_{12}, \qquad B_1 = b_1 + b_3\tau'_{21} + b_2\tau'_{22}$$

on a, d'après une remarque de Brioschi (Comptes rendus des séances de l'Académie des sciences t. 47, p. 311)

$$-(A_0B_1 - B_0A_1)(C_2D_3 - C_3D_2) = n^2.$$

Il suit de là que le déterminant fonctionnel prend la valeur:

$$\frac{(A_0B_1 - B_0A_1)^3}{n^3}$$

En maintenant les désignations que j'ai employées dans mon travail sur les équations qui donnent le multiplicateur des fonctions hyperelliptiques de premier ordre (Mathematische Annalen T. 20), on a:

$$A_0B_1 - B_0A_1 = \frac{M(K_{11}K_{22} - K_{12}K_{21})}{(C_{11}C_{22} - C_{12}C_{21})}.$$

Par conséquent:

$$(1) \qquad \begin{vmatrix} \frac{\partial \tau'_{11}}{\partial \tau_{11}} & \frac{\partial \tau'_{12}}{\partial \tau_{11}} & \frac{\partial \tau'_{22}}{\partial \tau_{11}} \\ \frac{\partial \tau'_{11}}{\partial \tau_{12}} & \frac{\partial \tau'_{12}}{\partial \tau_{12}} & \frac{\partial \tau'_{22}}{\partial \tau_{12}} \\ \frac{\partial \tau'_{11}}{\partial \tau_{22}} & \frac{\partial \tau'_{12}}{\partial \tau_{22}} & \frac{\partial \tau'_{22}}{\partial \tau_{22}} \end{vmatrix} = \frac{M^3}{n^3}\,\frac{(K_{11}K_{22} - K_{12}K_{21})^3}{(C_{11}C_{22} - C_{12}C_{21})^3}.$$

Posons maintenant d'après ROSENHAIN:

$$k^2 = \frac{\vartheta_{23}^2 \cdot \vartheta_{01}^2}{\vartheta_4^2 \cdot \vartheta_5^2}, \qquad \lambda^2 = \frac{\vartheta_2^2 \cdot \vartheta_{23}^2}{\vartheta_{34}^2 \cdot \vartheta_4^2}, \qquad \mu^2 = \frac{\vartheta_2^2 \cdot \vartheta_{01}^2}{\vartheta_{34}^2 \cdot \vartheta_5^2}$$

$$k_1^2 = \frac{\vartheta_{03}^2 \cdot \vartheta_{12}^2}{\vartheta_4^2 \cdot \vartheta_5^2}, \qquad \lambda_1^2 = \frac{\vartheta_0^2 \cdot \vartheta_{03}^2}{\vartheta_{34}^2 \cdot \vartheta_4^2}, \qquad \mu_1^2 = \frac{\vartheta_{12}^2 \cdot \vartheta_0^2}{\vartheta_5^2 \cdot \vartheta_{34}^2}$$

$$\lambda_k^2 = \frac{\vartheta_{14}^2 \cdot \vartheta_{03}^2 \cdot \vartheta_{23}^2}{\vartheta_{34}^2 \cdot \vartheta_4^2 \cdot \vartheta_5^2}, \qquad \mu_\lambda^2 = \frac{\vartheta_{14}^2 \cdot \vartheta_0^2 \cdot \vartheta_2^2}{\vartheta_{34}^2 \cdot \vartheta_4^2 \cdot \vartheta_5^2}, \qquad \mu_k^2 = \frac{\vartheta_{14}^2 \cdot \vartheta_{12}^2 \cdot \vartheta_{01}^2}{\vartheta_{34}^2 \cdot \vartheta_4^2 \cdot \vartheta_5^2}$$

Alors on a:

$$\frac{\partial(k^2)}{\partial\tau_{ik}} = 2k^2 \left[\frac{\frac{\partial\vartheta_{01}}{\partial\tau_{ik}}}{\vartheta_{01}} + \frac{\frac{\partial\vartheta_{23}}{\partial\tau_{ik}}}{\vartheta_{23}} - \frac{\frac{\partial\vartheta_5}{\partial\tau_{ik}}}{\vartheta_5} - \frac{\frac{\partial\vartheta_4}{\partial\tau_{ik}}}{\vartheta_4} \right]$$

$$\frac{\partial(\lambda^2)}{\partial\tau_{ik}} = 2\lambda^2 \left[\frac{\frac{\partial\vartheta_2}{\partial\tau_{ik}}}{\vartheta_2} + \frac{\frac{\partial\vartheta_{23}}{\partial\tau_{ik}}}{\vartheta_{23}} - \frac{\frac{\partial\vartheta_{34}}{\partial\tau_{ik}}}{\vartheta_{34}} - \frac{\frac{\partial\vartheta_4}{\partial\tau_{ik}}}{\vartheta_4} \right]$$

$$\frac{\partial(\mu^2)}{\partial\tau_{ik}} = 2\mu^2 \left[\frac{\frac{\partial\vartheta_2}{\partial\tau_{ik}}}{\vartheta_2} + \frac{\frac{\partial\vartheta_{01}}{\partial\tau_{ik}}}{\vartheta_{01}} - \frac{\frac{\partial\vartheta_{34}}{\partial\tau_{ik}}}{\vartheta_{34}} - \frac{\frac{\partial\vartheta_5}{\partial\tau_{ik}}}{\vartheta_5} \right]$$

Si l'on considère les relations:

$$4\pi i \frac{\partial\vartheta_a}{\partial\tau_{ii}} = \left[\frac{\partial^2\vartheta_a(v_1 v_2)}{\partial v_i^2} \right]_0 = \vartheta_a''(v_i)_0$$

$$2\pi i \frac{\partial\vartheta_a}{\partial\tau_{12}} = \left[\frac{\partial^2\vartheta_a(v_1 v_2)}{\partial v_1 \partial v_2} \right]_0,$$

qu'on pose ensuite:

$$\frac{\vartheta_{01}''(v_i)_0}{\vartheta_{01}} - \frac{\vartheta_5''(v_i)_0}{\vartheta_5} = a_{ii}, \qquad \frac{\vartheta_{23}''(v_i)_0}{\vartheta_{23}} - \frac{\vartheta_4''(v_i)_0}{\vartheta_4} = b_{ii}, \qquad \frac{\vartheta_2''(v_i)_0}{\vartheta_2} - \frac{\vartheta_{34}''(v_i)_0}{\vartheta_{34}} = c_{ii}$$

et qu'on définisse d'une manière analogue les grandeurs a_{12}, b_{12}, c_{12}, on aura:

$$\frac{\partial(k^2)}{\partial\tau_{ii}} = \frac{k^2}{2\pi i}(a_{ii} + b_{ii}), \qquad \frac{\partial(\lambda^2)}{\partial\tau_{ii}} = \frac{\lambda^2}{2\pi i}(b_{ii} + c_{ii}), \qquad \frac{\partial(\mu^2)}{\partial\tau_{ii}} = \frac{\mu^2}{2\pi i}(c_{ii} + a_{ii})$$

$$\frac{\partial(k^2)}{\partial\tau_{12}} = \frac{k^2}{\pi i}(a_{12} + b_{12}), \qquad \frac{\partial(\lambda^2)}{\partial\tau_{12}} = \frac{\lambda^2}{\pi i}(b_{12} + c_{12}), \qquad \frac{\partial(\mu^2)}{\partial\tau_{12}} = \frac{\mu^2}{\pi i}(c_{12} + a_{12}).$$

De là résulte l'équation:

$$\begin{vmatrix} \frac{\partial(k^2)}{\partial\tau_{11}} & \frac{\partial(\lambda^2)}{\partial\tau_{11}} & \frac{\partial(\mu^2)}{\partial\tau_{11}} \\ \frac{\partial(k^2)}{\partial\tau_{12}} & \frac{\partial(\lambda^2)}{\partial\tau_{12}} & \frac{\partial(\mu^2)}{\partial\tau_{12}} \\ \frac{\partial(k^2)}{\partial\tau_{22}} & \frac{\partial(\lambda^2)}{\partial\tau_{22}} & \frac{\partial(\mu^2)}{\partial\tau_{22}} \end{vmatrix} = \frac{1}{2}\left(\frac{1}{\pi i}\right)^3 k^2\lambda^2\mu^2 \begin{vmatrix} a_{11} & b_{11} & c_{11} \\ a_{12} & b_{12} & c_{12} \\ a_{22} & b_{22} & c_{22} \end{vmatrix}.$$

Mais maintenant on a les équations:

$$\frac{\vartheta_2''(v_i)_0}{\vartheta_2} = \frac{\vartheta_5''(v_i)_0}{\vartheta_5} - \frac{\vartheta_1'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_{12}^2}{\vartheta_2^2} + \frac{\vartheta_3'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_{23}^2}{\vartheta_2^2}$$

$$\frac{\vartheta_4''(v_i)_0}{\vartheta_4} = \frac{\vartheta_5''(v_i)_0}{\vartheta_5} - \frac{\vartheta_1'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_{14}^2}{\vartheta_4^2} - \frac{\vartheta_3'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_{34}^2}{\vartheta_4^2}$$

$$\frac{\vartheta_{01}''(v_i)_0}{\vartheta_{01}} = \frac{\vartheta_5''(v_i)_0}{\vartheta_5} - \frac{\vartheta_1'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_0^2}{\vartheta_{01}^2} - \frac{\vartheta_{13}'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_{03}^2}{\vartheta_{01}^2}$$

$$\frac{\vartheta_{23}''(v_i)_0}{\vartheta_{23}} = \frac{\vartheta_5''(v_i)_0}{\vartheta_5} - \frac{\vartheta_3'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_2^2}{\vartheta_{23}^2} - \frac{\vartheta_{13}'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_{12}^2}{\vartheta_{23}^2}$$

$$\frac{\vartheta_{34}''(v_i)_0}{\vartheta_{34}} = \frac{\vartheta_5''(v_i)_0}{\vartheta_5} + \frac{\vartheta_3'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_4^2}{\vartheta_{34}^2} + \frac{\vartheta_{13}'(v_i)_0^2}{\vartheta_5^2}\frac{\vartheta_{14}^2}{\vartheta_{34}^2}$$

et des équations semblables pour les quantités:

$$\left[\frac{\partial^2\vartheta_\alpha(v_1 v_2)}{\partial v_1 \partial v_2}\right]_0$$

Il suit de là que le déterminant des quantités a prend la forme:

$$-\frac{2\,.\,\vartheta_{12}^2\,.\,\vartheta_{14}^2\,.\,\vartheta_0^2\,.\,\vartheta_{03}^2}{\vartheta_{01}^2\vartheta_5^4\,.\,\vartheta_4^2\,.\,\vartheta_{23}^2\,.\,\vartheta_2^2\,.\,\vartheta_{34}^2}\left\{\vartheta_1'(v_1)_0\vartheta_{13}'(v_2)_0-\vartheta_1'(v_2)_0\vartheta_{13}'(v_1)_0\right\}$$

$$.\left\{\vartheta_{13}'(v_1)_0\vartheta_3'(v_2)_0-\vartheta_{13}'(v_2)_0\vartheta_3'(v_1)_0\right\}\left\{\vartheta_3'(v_1)_0\vartheta_1'(v_2)_0-\vartheta_3'(v_2)_0\vartheta_1'(v_1)_0\right\}$$

ou encore:

$$-2\pi^6\,.\,k_1^2\lambda_1^2\mu_1^2\,.\,\lambda_k^2\mu_\lambda^2\,.\,\mu_k^2\,4^3(K_{11}K_{22}-K_{12}\,.\,K_{21})^3$$

puisqu'on a les équations:

$$\begin{aligned}
\vartheta_1'(v_1)_0\vartheta_{13}'(v_2)_0-\vartheta_1'(v_2)_0\vartheta_{13}'(v_1)_0 &= -\pi^2\vartheta_5\vartheta_{12}\vartheta_{01}\vartheta_{14}\\
\vartheta_{13}'(v_1)_0\vartheta_3'(v_2)_0-\vartheta_{13}'(v_2)_0\vartheta_3'(v_1)_0 &= \pi^2\vartheta_5\vartheta_{34}\vartheta_{03}\vartheta_{23}\\
\vartheta_3'(v_1)_0\vartheta_1'(v_2)_0-\vartheta_3(v_2)_0\vartheta_1'(v_1)_0 &= -\pi^2\vartheta_5\vartheta_0\vartheta_4\vartheta_2\,.
\end{aligned}$$

Par conséquent:

$$(2)\qquad \begin{vmatrix}
\frac{\partial(k^2)}{\partial\tau_{11}} & \frac{\partial(\lambda^2)}{\partial\tau_{11}} & \frac{\partial(\mu^2)}{\partial\tau_{11}}\\[2ex]
\frac{\partial(k^2)}{\partial\tau_{12}} & \frac{\partial(\lambda^2)}{\partial\tau_{12}} & \frac{\partial(\mu^2)}{\partial\tau_{12}}\\[2ex]
\frac{\partial(k^2)}{\partial\tau_{22}} & \frac{\partial(\lambda^2)}{\partial\tau_{22}} & \frac{\partial(\mu^2)}{\partial\tau_{22}}
\end{vmatrix} = -\pi^3k^2\lambda^2\mu^2.\,k_1^2\lambda_1^2\mu_1^2\lambda_k^2\mu_\lambda^2\,.\,\mu_k^2\,4^3(K_{11}K_{22}-K_{12}K_{21})^3.$$

Si nous désignons les quantités transformées correspondant aux quantités k, λ, μ, K, par c, l, m, C, on a d'une manière anlogue:

$$(3)\qquad \begin{vmatrix}
\frac{\partial(c^2)}{\partial\tau_{11}'} & \frac{\partial(l^2)}{\partial\tau_{11}'} & \frac{\partial(m^2)}{\partial\tau_{11}'}\\[2ex]
\frac{\partial(c^2)}{\partial\tau_{12}'} & \frac{\partial(l^2)}{\partial\tau_{12}'} & \frac{\partial(m^2)}{\partial\tau_{12}'}\\[2ex]
\frac{\partial(c^2)}{\partial\tau_{22}'} & \frac{\partial(l^2)}{\partial\tau_{22}'} & \frac{\partial(m^2)}{\partial\tau_{22}'}
\end{vmatrix} = -\pi^3c^2l^2m^2c_1^2l_1^2m_1^2l_c^2m_l^2m_c^2\,4^3(C_{11}C_{22}-C_{12}C_{21})^3.$$

Si donc nous posons:

$$F = \begin{vmatrix} \frac{\partial(c^2)}{\partial(k^2)} & \frac{\partial(l^2)}{\partial(k^2)} & \frac{\partial(m^2)}{\partial(k^2)} \\ \frac{\partial(c^2)}{\partial(\lambda^2)} & \frac{\partial(l^2)}{\partial(\lambda^2)} & \frac{\partial(m^2)}{\partial(\lambda^2)} \\ \frac{\partial(c^2)}{\partial(\mu^2)} & \frac{\partial(l^2)}{\partial(\mu^2)} & \frac{\partial(m^2)}{\partial(\mu^2)} \end{vmatrix}$$

nous obtiendrons la relation:

$$M^3 = \frac{n^3 F \cdot k^2 \cdot \lambda^2 \cdot \mu^2 \cdot k_1^2 \lambda_1^2 \mu_1^2 \lambda_k^2 \mu_\lambda^2 \cdot \mu_k^2}{c^2 \cdot l^2 \cdot m^2 \cdot c_1^2 l_1^2 m_1^2 \cdot l_c^2 m_l^2 m_c^2}. \tag{4}$$