

ON A FUNICULAR  
SOLUTION OF BUFFON'S "PROBLEM OF THE NEEDLE"  
IN ITS MOST GENERAL FORM

BY

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in OXFORD.

Assisted by JAMES HAMMOND.

— «quaintly made of cords»  
(Two Gentlemen of Verona, act III, sc. 1.)

The founder of the theory of Local Probability appears to have been BUFFON (better known as a Naturalist, but who began his career as a Mathematician). Among a few other questions of a similar kind, which he proposed in his *Essai d'Arithmétique Morale*, the one which has obtained the greatest notoriety is the celebrated one which goes by the name of the *Problème de l'Aiguille*, the purport of which is as follows.

On an area of indefinite extent (say a planked floor) a number of parallel straight lines are ruled at equal distances, upon which a needle, not long enough to cross more than one of the parallels at the same time, is thrown down: the probability is required of its falling in such a position as to be intersected by one of the parallels.

An easier question of the same kind, which BUFFON treats before the other, is when a circle is used instead of the needle. This latter question he solves by simple geometrical considerations too obvious to need recapitulation; to obtain a solution of the former he, and after him LAPLACE, had recourse to a process of integration.

In a question given in the late M<sup>r</sup> TODHUNTER'S *Integral Calculus* (1<sup>st</sup> edition, 1857, p. 268) the solution of the problem is correctly stated for an ellipse, whose major axis is less than the distance between two conse-

cutive parallels, instead of for a circle or straight line: this important step in the development of the theory is, I am informed, currently attributed to the late M<sup>r</sup> LESLIE ELLIS, of the University of Cambridge.

In the year 1860, LAMÉ proposed to give a course of lectures on the subject at the Sorbonne, and, apparently without knowledge of the result contained in TODHUNTER'S treatise, reproduced the solution for the ellipse and for any equilateral polygon. In the same year M. EMILE BARBIER, whose lamented decease occurred in the course of the present year and who had attended LAMÉ'S lectures, discovered and published in the Journal of LIOUVILLE for that year a universal solution for an undivided plane contour of any form whatever.

The subsequent history I am not able to trace further than to state that in CZUBER'S *Geometrische Wahrscheinlichkeiten* (Leipzig, 1884) BARBIER'S solution is extended to the case of any two rigidly connected convex figures (in a plane).<sup>1</sup> I propose to give here the finishing stroke to the theory as regards plane figures by extending it to any number of them, rigidly connected and of any forms, in the same plane. It is always to be understood, in what precedes as in what follows, that the greatest diameter of the figure, or system of figures, is less than the distance between two consecutive parallels.

BARBIER'S principle (see CZUBER, pp. 117, 125) leads at once to the conclusion that the probability of any figure (subject to the restriction above stated) intersecting the system of parallels is to certainty as the length of a cord stretched round the figure is to the circumference of a circle touched by two adjoining parallels.<sup>2</sup> This circumference (with a view to simplicity of expression) we shall adopt as the unit of length in all subsequent formulae.

By the disjunctive probability of a set of figures I shall understand the probability of *one or more* of them intersecting one of the parallels: by the conjunctive probability of the same, the probability of *all* of them intersecting one of the parallels.

I start from BARBIER'S theorem that for a single figure the proba-

<sup>1</sup> See *Postscriptum*, p. 205.

<sup>2</sup> The case of a straight line (the original question of the *needle*) may be made to fall under this rule: for the line, as BARBIER has observed, may be regarded as an indefinitely narrow ellipse or other oval.

bility of intersection is measured by the length of a stretched string passing round it: this, it should be observed, is universally true whether the contour be curvilinear or rectilinear or mixtilinear, composed of a single line straight or curved or of any number of such — a theorem almost unexampled for its generality. The disjunctive probability for any number of figures  $A, B, C, \dots, H$  I shall for the present denote by  $A:B:C:\dots:H$ , the conjunctive by  $A.B.C\dots H$ .

Let there be  $n + 1$  figures given, let  $p_i$  be the sum of the conjunctive and  $\bar{w}_i$  of the disjunctive probabilities for these figures taken  $i$  and  $i$  together; so that  $\bar{w}_1$  and  $p_1$  are identical, and  $\bar{w}_{n+1}, p_{n+1}$  are monomial quantities. Then by a universal theorem of *logic* we have the reciprocal formulae

$$(1) \quad \bar{w}_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} p_i,$$

$$(2) \quad p_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} \bar{w}_i.$$

Let us now suppose that we have obtained expressions for the disjunctive and conjunctive probabilities of any number not exceeding  $n$  figures of any kind: we may extend these to the case of  $n + 1$  figures as follows.

1°. When the  $n + 1$  figures are so situated that it is impossible for all of them to be cut by the same straight line, we have  $p_{n+1} = 0$  so that  $\bar{w}_{n+1}$  can be found immediately in terms of  $p_1, p_2, \dots, p_n$  by using formula (1), or in terms of  $\bar{w}_1, \bar{w}_2, \dots, \bar{w}_n$  by using (2); i. e.  $\bar{w}_{n+1}$  can be found in terms of known quantities; for by hypothesis all the terms of  $p_i$  or of  $\bar{w}_i$  are known when  $i$  is any number not exceeding  $n$ .

2°. When all the  $n + 1$  figures are capable of being cut by the same straight line, let  $XY$  be some straight line which cuts them all and call the figures taken in the order in which they are cut by  $XY$

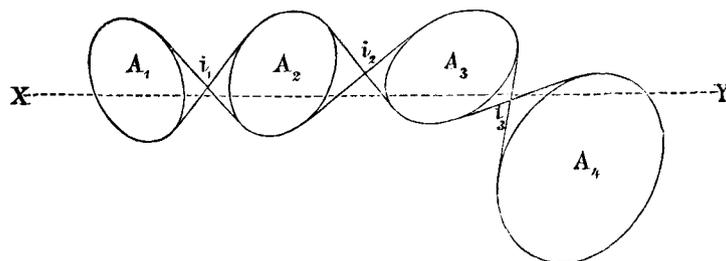
$$A_1, A_2, A_3, \dots, A_{n+1}.^1$$

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<sup>1</sup> It may be well to draw at once attention to the fact that different systems of straight lines do not necessarily cut the figures  $A_1, A_2, A_3, \dots$  in the same order; as ex. gr. if three circles touch, or so nearly touch one another that each blocks the channel between the other two, straight lines may be drawn whose intersections with *any* one of the three shall be intermediate to their intersections with the other two.

Let a stretched string be made to wind round these  $n + 1$  contours passing alternately from one side of  $XY$  to the other, as in Fig. 1, and crossing itself in the  $n$  points  $i_1, i_2, \dots, i_n$  lying between  $A_1, A_2; A_2,$

Fig. 1.



$A_3; \dots A_n, A_{n+1}$  respectively. Let us call the figures enclosed by the successive  $n + 1$  loops of the winding string

$$B_1, B_2, B_3, \dots, B_{n+1}.$$

It is obvious that any straight line which cuts all these loops will cut all the given figures, and *vice versa*.

Hence

$$A_1 \cdot A_2 \cdot A_3 \dots A_{n+1} = B_1 \cdot B_2 \cdot B_3 \dots B_{n+1}.$$

Let  $P_i, \Pi_i$  represent what  $p_i, \bar{w}_i$  become when for the figures  $A$  we substitute the loops  $B$ , so that

$$\Pi_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} P_i,$$

$$P_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} \Pi_i,$$

and

$$P_{n+1} = p_{n+1}.$$

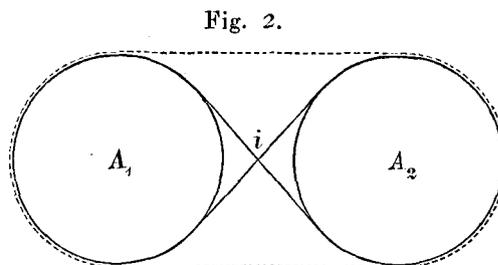
$\Pi_{n+1}$  is known by BARBIER'S rule, because the loops taken together form a single figure, in fact

$$\Pi_{n+1} = L,$$

where  $L$  is the length of the uncrossed string stretched round the system of figures  $B$ , which is no other than that stretched round the given

figures  $A$ . Also, by hypothesis,  $H_i$  is known for all values of  $i$  not exceeding  $n$ . We therefore know  $p_{n+1}$  which is the same as  $P_{n+1}$ . Hence  $\bar{\omega}_{n+1}$  is known from (1): thus then  $p_{n+1}$  and  $\bar{\omega}_{n+1}$  are both known, so that when the conjunctive and disjunctive probabilities are known in general for  $n$  figures they become known for  $n + 1$  figures; but when  $n = 1$ ,  $p_1$  and  $\bar{\omega}_1$  are equal to one another and to the length of a given stretched string. Hence, by the usual process of induction, we may conclude that the conjunctive and disjunctive probabilities for any number of figures can always be expressed as a linear function with positive and negative integer coefficients, or in a word as a Diophantine linear function, of a finite number of lengths of certain stretched strings.

When there are only *two* figures  $A_1, A_2$  we pass a stretched string between them crossing itself in  $i$  (see Fig. 2): then using  $(A_1 \times A_2)$  to denote the length of this string, and  $(A_1 A_2)$  to denote the length of the uncrossed string (indicated by dots in the figure) stretched round  $A_1, A_2$  we have



$$H_2 = (A_1 \times A_2) - P_2$$

and

$$\bar{\omega}_2 = (A_1) + (A_2) - p_2$$

(where  $(A_1), (A_2)$  denote the lengths of the separate bands round  $A_1, A_2$  respectively).

But

$$H_2 = (A_1 A_2),$$

and consequently

$$p_2 = P_2 = (A_1 \times A_2) - (A_1 A_2),$$

$$\bar{\omega}_2 = (A_1) + (A_2) + (A_1 A_2) - (A_1 \times A_2).$$

We will now proceed to consider in detail the application of the inductive method to the case of *three* figures for which, since each of these may be replaced by a convex band passing round it, we may if we please for greater graphical simplicity substitute three convexes (i. e. contours which any secant must intersect in exactly 2 points). Many cases requiring separate discussion will arise, but one important consequence, rising to the dignity of a principle, which holds good whatever may be the number of figures, governs them all; viz. that the final result for either probability is a linear homogeneous function of lengths of stretched bands drawn in various ways round the given figures and depending for their course on the forms and disposition of these figures exclusively, *wholly uninfluenced* by the presence of any points external to them. Lines drawn from the pointed ends, or apices, of the loops enclosing them do it is true make their appearance in the computations but either coalesce into portions of the bands referred to or else entering in pairs with opposite algebraical signs disappear from the final result. As a consequence, if for the sake of illustration we suppose the figures to be any closed *curves* without singular points, the probability, disjunctive or conjunctive, to be ascertained is a function exclusively of the complete system of lengths of double tangents that can be drawn between the curves and of the arcs into which they are severally divided by their points of contact with those tangents.

We have for all the cases of three figures

$$\bar{w}_3 = p_1 - p_2 + p_3$$

where

$$p_1 = (A_1) + (A_2) + (A_3)$$

and

$$p_2 = (A_2 \times A_3) - (A_2 A_3) + (A_3 \times A_1) - (A_3 A_1) + (A_1 \times A_2) - (A_1 A_2).$$

Thus

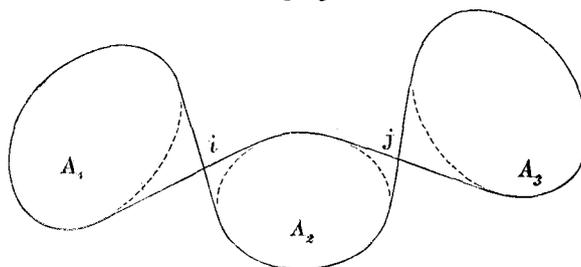
$$(3) \quad \bar{w}_3 - p_3 = (A_1) + (A_2) + (A_3) + (A_2 A_3) + (A_3 A_1) + (A_1 A_2) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2).$$

Similarly

$$\begin{aligned} \bar{w}_3 - P_3 &= (B_1) + (B_2) + (B_3) + (B_2 B_3) + (B_3 B_1) \\ &+ (B_1 B_2) - (B_2 \times B_3) - (B_3 \times B_1) - (B_1 \times B_2), \end{aligned}$$

where  $B_1, B_2, B_3$  are the loops of the string which passes round the figures  $A_1, A_2, A_3$  and crosses itself at  $i$  and  $j$ , as shown in Fig. 3. But  $P_3 = p_3$ , and  $H_3$  is the length of an uncrossed band stretched round the entire system of figures  $A_1, A_2, A_3$  (which will be expressed in symbols by writing  $H_3 = (A_1 A_2 A_3)$ ).

Fig. 3.



Hence

$$p_3 = (A_1 A_2 A_3) + (B_2 \times B_3) + (B_3 \times B_1) + (B_1 \times B_2) - (B_1) - (B_2) - (B_3) - (B_2 B_3) - (B_3 B_1) - (B_1 B_2).$$

Moreover

$$(B_1 \times B_2) = (B_1) + (B_2)$$

and

$$(B_2 \times B_3) = (B_2) + (B_3),$$

because  $B_1, B_2$  and  $B_2, B_3$  are pairs of consecutive loops. And whenever the three given figures are capable of being cut by a straight line in the order  $A_1, A_2, A_3$  (i. e. except in the case  $p_3 = 0$ , which is separately considered)

$$(B_3 B_1) = (A_3 A_1),$$

because both the crossing points,  $i$  and  $j$ , of the looped string necessarily fall inside the uncrossed band round  $A_1, A_3$ . Thus the value of  $p_3$  is given by the equation

$$(4) \quad p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (B_3 \times B_1) + (B_2) - (B_2 B_3) - (B_1 B_2)$$

which, for immediate purposes, we shall find convenient to write under the form

$$(5) \quad p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (B_2 \times B_3) - (B_2 B_3) + (B_3 \times B_1) - (B_1 B_2) - (B_3).$$

We shall apply the formula to the two classes which between them comprise all the cases of three figures, viz.

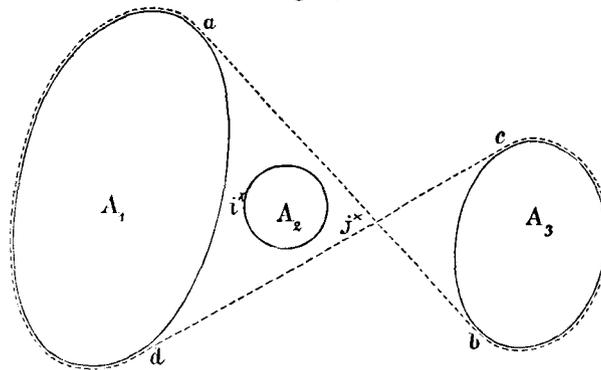
Class A. One of the figures, which we call  $A_2$ , lies either wholly or partially inside the crossed band round the other two.

Class B. Each figure lies entirely outside the crossed band round the other two.

In Class A we recognize 3 species, viz.

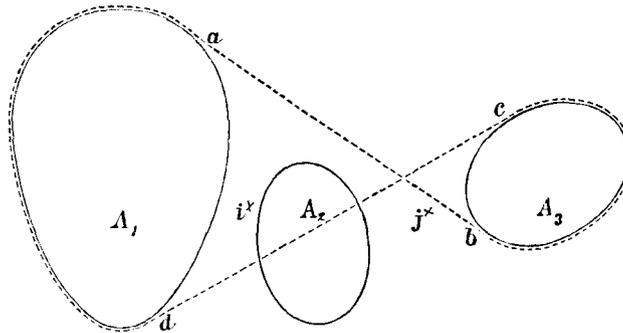
Aa. The figure  $A_2$  does not cut either of the crossed strings  $ab$ ,  $cd$  of the band looped round  $A_1$ ,  $A_2$  (Fig. 4), but lies wholly in the same loop as one of them, which we call  $A_1$ .

Fig. 4.



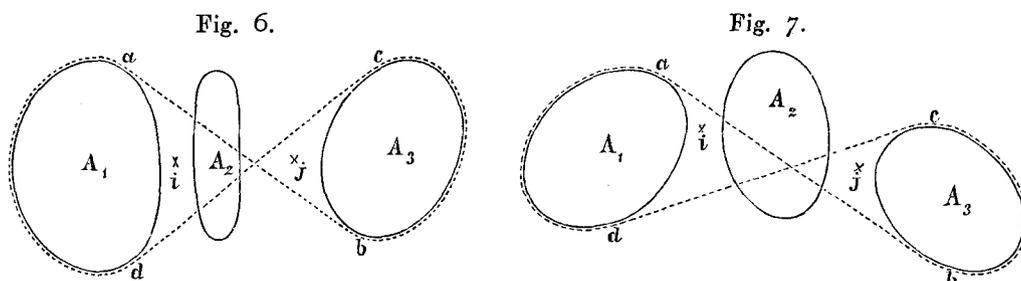
Ab. The figure  $A_2$  cuts one, but not both, of the crossed strings  $ab$ ,  $cd$  (Fig. 5), and part of it lies in the same loop as  $A_1$ .

Fig. 5.



Ac. The figure  $A_2$  cuts both the crossed strings  $ab$ ,  $cd$  (Fig. 6 and 7) and part of it lies in the same loop as  $A_1$ .

To avoid complicating these figures (4, 5, 6, 7) the band (looped round  $A_1, A_2, A_3$  as shown in Fig. 3) which crosses itself at  $i, j$  is not given, but the position of each crossing point is marked by a small cross. It should be observed that in Fig. 5 (species Ab)  $j$  lies outside the crossed band round  $A_1, A_3$ ; in Fig. 4 (species Aa)  $i$  and  $j$  lie in the same loop, and in Figs. 6, 7 (species Ac)  $i$  and  $j$  lie in opposite loops of the crossed band round  $A_1 A_3$ .



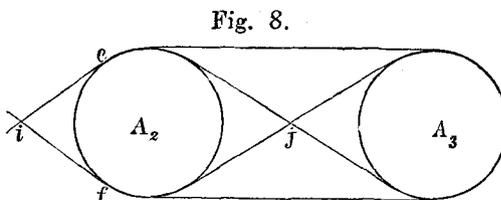
The discussion of species Aa is very simple; for it is clear that the conjunctive probability is

$$p_3 = (A_2 \times A_3) - (A_2 A_3)$$

since it is obviously impossible for a straight line to cut  $A_2$  and  $A_3$  without cutting  $A_1$ . Substituting this value for  $p_3$  in formula (3) we obtain the disjunctive probability

$$\bar{w}_3 = (A_1) + (A_2) + (A_3) + (A_1 A_2) + (A_1 A_3) - (A_1 \times A_2) - (A_1 \times A_3).$$

The remaining two species belonging to class A may be discussed simultaneously; for we have in all the cases (see Fig. 8), using  $e, f$  to denote the points of contact with the figure  $A_2$  of the strings which cross at the point  $i$  (between  $A_1$  and  $A_2$ ),



$$(B_2 \times B_3) = (A_2 \times A_3) + fi + ie - ef,$$

$$(B_2 B_3) = (A_2 A_3) + fi + ie - ef,$$



$$(B_1 \times B_3) = (A_1 \times A_3) + gj + jk + ka - ab - bg$$

and

$$(B_1 B_2) + (B_3) = D + gj + jk + ka - ab - bg,$$

where  $D$  is the band  $(abgchjlmna)$ , derived from the crossed band  $(abgcdna)$  round  $A_1, A_3$  by distending the loop which contains  $A_1$  until it also contains  $A_2$ .

Hence

$$(B_1 \times B_3) - (B_1 B_2) - (B_3) = (A_1 \times A_3) - D,$$

and the general formula for the conjunctive probability (for class A) becomes

$$(6) \quad p_3 = (A_1 A_2 A_3) + (A_1 \times A_3) + (A_2 \times A_3) - (A_1 A_3) - (A_2 A_3) - D.$$

Combining this with formula (3), which belongs to all cases of three figures we obtain

$$\bar{\omega}_3 = (A_1) + (A_2) + (A_3) + (A_1 A_2) + (A_1 A_2 A_3) - (A_1 \times A_2) - D.$$

The species Aa, Ab, Ac are distinguishable from one another by the difference in shape of the band  $D$  belonging to each. Thus in Aa the band  $D$  is not distended at all, but is simply  $(A_1 \times A_3)$ ; in Ab the loop containing  $A_1$  is distended on one side only; and in Ac is distended on both sides (see figures 10 and 11). This difference in shape will be denoted by writing  $D_1$  for  $D$  in the general formula when the species is Ab, and  $D_2$  for  $D$  when the species is Ac.

The dotted bands  
 $(pqjghjlmnp)$

Fig. 10.

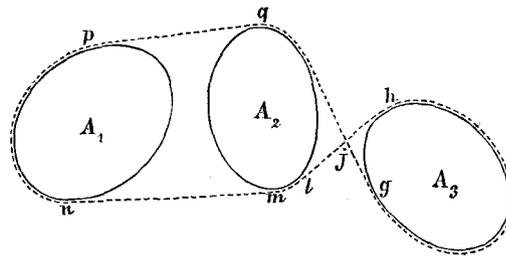
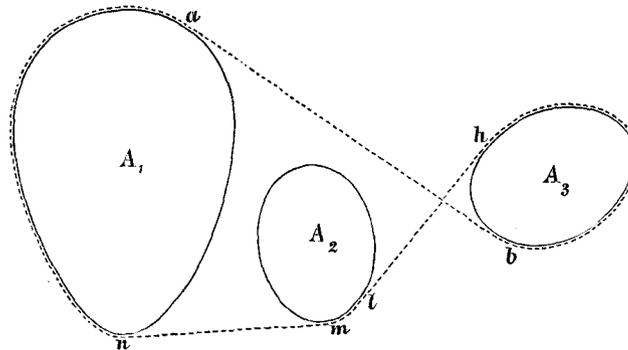


Fig. 11.



of Fig. 10, and  $(abhlmna)$  of Fig. 11 are what the dotted bands of Fig. 7 (species Ac) and Fig. 5 (species Ab) become, when the former is doubly and the latter singly distended.

Varieties of the species in class A (viz. one variety for Aa, two for Ab, and three for Ac, making 6 cases in all) occur when we consider the situation of the figure  $A_2$  with respect to the uncrossed band round  $A_1, A_3$ . In all cases where  $A_2$  lies wholly inside this band we have  $(A_1A_2A_3) = (A_1A_3)$ , so that in all such cases the general formula (6), which gives the conjunctive probability, becomes

$$p_3 = (A_1 \times A_3) + (A_2 \times A_3) - (A_2A_3) - D.$$

Aa. We have

$$D = (A_1 \times A_3)$$

so that

$$p_3 = (A_2 \times A_3) - (A_2A_3)$$

(the same as the result previously obtained from *à priori* considerations).

Ab. 1. The figure  $A_2$  lies wholly within the uncrossed band round  $A_1, A_3$

$$p_3 = (A_1 \times A_3) + (A_2 \times A_3) - (A_2A_3) - D_1.$$

Ab. 2. The figure  $A_2$  cuts the uncrossed band round  $A_1, A_3$

$$p_3 = (A_1A_2A_3) + (A_1 \times A_3) + (A_2 \times A_3) - (A_1A_3) - (A_2A_3) - D_1.$$

Ac. 1. The figure  $A_2$  lies wholly within the uncrossed band round  $A_1, A_3$ .

Ac. 2. The figure  $A_2$  cuts only one string of the uncrossed band round  $A_1, A_3$ . In these two cases the formulae which give  $p_3$  are the same as in the corresponding varieties of Ab, except that  $D_2$  takes the place of  $D_1$ .

Ac. 3. The figure  $A_2$  cuts both strings of the uncrossed band round  $A_1, A_3$ . In this case the formula for the conjunctive probability

$$p_3 = (A_1A_2A_3) + (A_1 \times A_3) + (A_2 \times A_3) - (A_1A_3) - (A_2A_3) - D_2$$

becomes greatly simplified; for (see Fig. 12)

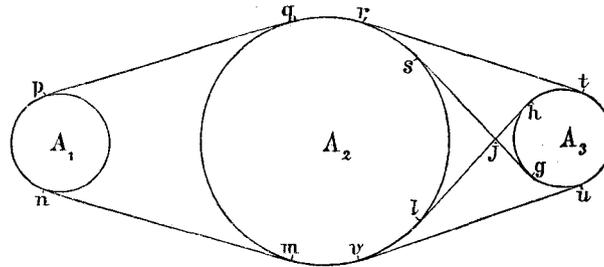
$$D_2 - (A_1A_2A_3) = rsjgu + vljht - rt - vu = (A_2 \times A_3) - (A_2A_3)$$

so that

$$p_3 = (A_1 \times A_3) - (A_1 A_3),$$

which is evidently true, since every straight line which cuts both  $A_1$  and  $A_3$  must also (in this case) cut  $A_2$ .

Fig. 12.

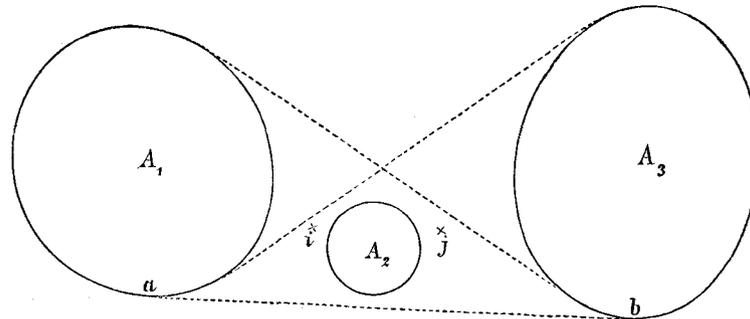


We have now enumerated all the 6 cases of Class A, and given in each case the formula for the conjunctive probability (from which, by means of formula (3), the disjunctive probability may be determined immediately). We proceed to the discussion of Class B.

In Class B (i. e. in the class where each figure lies entirely outside the crossed band round the other two) we recognize 4 species, and in one of them 2 varieties, making 5 cases in all. The enumeration is as follows.

Ba. There is one definite order of succession in which the three figures can be cut by a system of straight lines. There are two varieties of this species, viz.

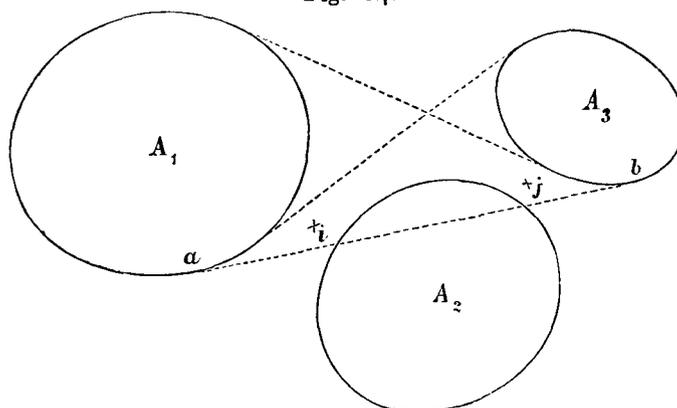
Fig. 13.



Ba. 1. The middle figure ( $A_2$ , see Fig. 13) lies wholly inside the

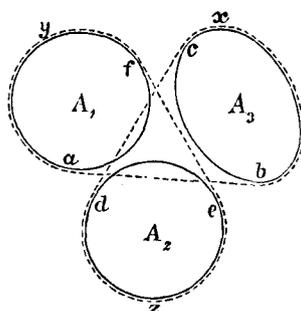
uncrossed band round the other two. The small crosses in this figure, as in others, indicate the positions of the points  $i, j$  where the string looped round  $A_1, A_2, A_3$  (see Fig. 3) crosses itself.

Fig. 14.



Ba. 2. The middle figure cuts the uncrossed band round the other two as shown in Fig. 14. In this, as in the preceding case, both  $i$  and  $j$  lie outside the crossed, but inside the uncrossed, band round  $A_1, A_3$ .<sup>1</sup>

Fig. 15.



Bb. The figures may be cut in two different orders by two distinct systems of straight lines (see Fig 15). One system of straight lines cuts the figures in the order  $A_1, A_2, A_3$ ; the other system cuts them in the order  $A_3, A_1, A_2$ .

<sup>1</sup> This circumstance enables us to discuss Ba. 1 and Ba. 2 simultaneously.

Bc. The figures may be cut by three distinct systems of straight lines (Fig. 16).

Bd. The three figures cannot all be cut by any straight line (Fig. 17).

Fig. 16.

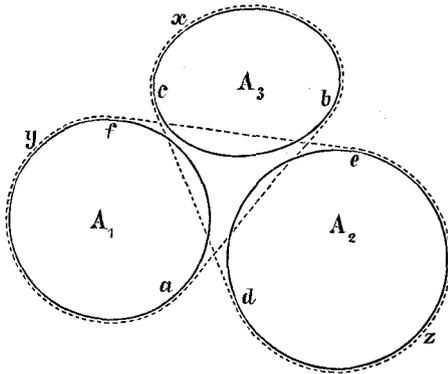
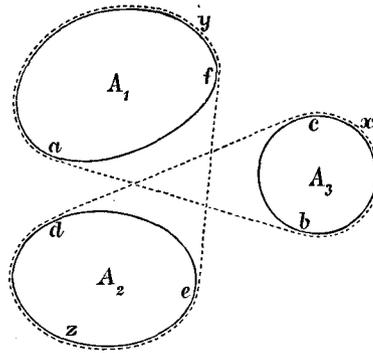


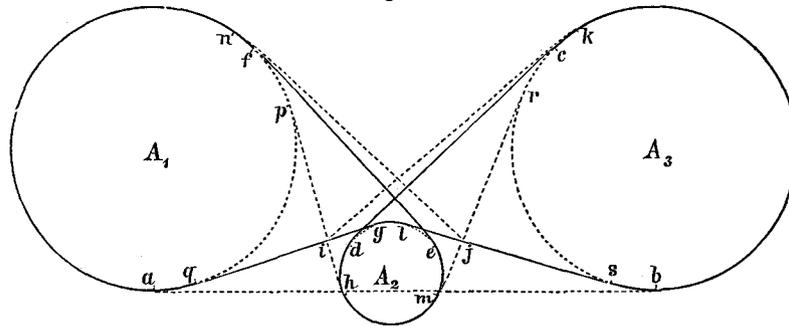
Fig. 17.



In all cases with the exception of Bd, which will be treated separately, we have (see formula 4 *ante*)

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (B_1 \times B_3) + (B_2) - (B_2 B_3) - (B_1 B_2).$$

Fig. 18.



In Ba (see Fig. 18) we have

$$(B_2 B_3) = (A_2 A_3) + hi + ik - kc - cd - dh,$$

$$(B_1 B_2) = (A_1 A_2) + mj + jn - nf - fe - em,$$

$$(B_1 \times B_3) = (B_1) + (B_3) + ik - kc - cr - rj + jn - nf - fp - pi.$$

Substituting these values in the general expression for  $p_3$ , we obtain

$$p_3 = (A_1A_2A_3) - (A_2A_3) - (A_3A_1) - (A_1A_2) + (B_1) + (B_2) + (B_3) - mr - rc + cd + dh - hp - pf + fe + em$$

where the term  $-mr$  comes from  $-mj - rj$ , and the term  $-hp$  comes from  $-hi - pi$ ; the other terms involving the points  $i, j$  or the points of contact  $k, n$  of tangents drawn from them to the original figures disappear in pairs. The terms

$$(B_1) + (B_2) + (B_3) - mr - rc + cd + dh - hp - pf + fe + em$$

will be seen to coalesce into a single band (whose course is marked in Fig. 18 by the continuous line  $aqiglj sbkcdhmefna$ , all other lines in the figure being dotted). This band we shall call  $\Delta_1$ .

Fig. 18 is drawn for the case Ba. 2, but the investigation of case Ba. 1 is precisely the same as that of Ba. 2. In both cases we find

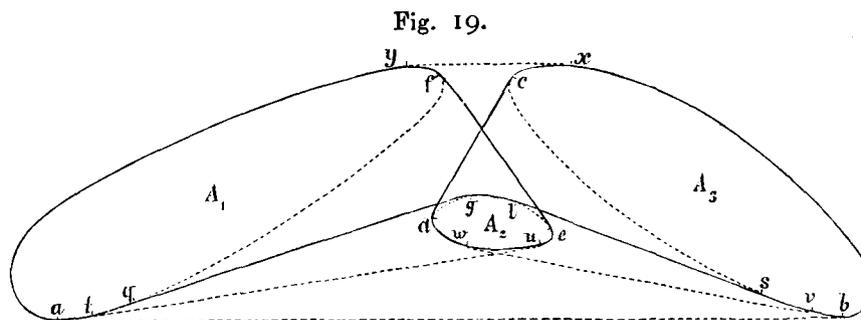
$$p_3 = (A_1A_2A_3) - (A_2A_3) - (A_3A_1) - (A_1A_2) + \Delta_1$$

for the conjunctive probability, and consequently

$$\bar{w}_3 = (A_1) + (A_2) + (A_3) + (A_1A_2A_3) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2) + \Delta_1$$

gives the disjunctive probability in both cases.

The band  $\Delta_1$  for the case Ba. 1 is shown by the continuous line



of Fig. 19, i. e.  $\Delta_1$  is the band  $atqgls vbxcdwuefy a$ : its course is precisely the same as that of the  $\Delta_1$  for the case Ba. 2.

The difference between the two cases is this: in Ba. 1 we have

$$(A_1 A_2 A_3) = (A_1 A_3)$$

so that

$$p_3 = \Delta_1 - (A_1 A_2) - (A_2 A_3),^1$$

whereas in Ba. 2 (and in all the cases to be subsequently considered) the terms  $(A_2 A_3) + (A_3 A_1) + (A_1 A_2) - (A_1 A_2 A_3)$  coalesce into a single band which we shall call  $\Delta$ , so that

$$p_3 = \Delta_1 - \Delta.$$

The course of the band  $\Delta$  is marked by the letters *abkcdhmfna* in Fig. 18. The band  $\Delta_1$  may be derived from  $\Delta$  by supposing its rectilinear portion *ab* to be pressed inwards by the figure  $A_2$  so as to occupy the position *aqglsb*.

The investigation of the case Bb proceeds on exactly the same lines as that of Ba. 2; we start from the same general formula and, by performing precisely similar work, obtain the result

$$p_3 = \Delta_2 - \Delta,$$

where (see Fig. 15)  $\Delta$  is the band *abxcdzefya* whose course is indicated by dots, and  $\Delta_2$  is the band derived from  $\Delta$  by supposing *two* of its rectilinear portions *ab*, *cd* to be pressed inwards by the figures  $A_1$  and  $A_2$ .

In the case Bc (Fig. 16) the work is simplified by observing that each of the figures  $A_1$ ,  $A_2$ ,  $A_3$  blocks the channel between the other two (i. e. no straight line can pass between any two of them without cutting the third). Hence every straight line which cuts the uncrossed band round all the figures must cut one or more of them; i. e.

$$\bar{\omega}_3 = (A_1 A_2 A_3)$$

<sup>1</sup> By an easy rearrangement of the bands the value of  $p_3$  for this case may be expressed as the difference of the two bands, *atuelgdwvbxya* and *atqgleuwdglsvbxya* (see Fig. 19), derived from the uncrossed band *abxya* round  $A_1$ ,  $A_3$  by *twisting* its rectilinear portion *ab* right round  $A_2$  in opposite directions.

and consequently formula (3) gives

$$p_3 = (A_1A_2A_3) - (A_2A_3) - (A_3A_1) - (A_1A_2) \\ + (A_2 \times A_3) + (A_3 \times A_1) + (A_1 \times A_2) - (A_1) - (A_2) - (A_3).$$

Now it is easily seen that

$$(A_2A_3) + (A_3A_1) + (A_1A_2) - (A_1A_2A_3) = \Delta$$

and

$$(A_2 \times A_3) + (A_3 \times A_1) + (A_1 \times A_2) - (A_1) - (A_2) - (A_3) = \Delta_3$$

where  $\Delta$  is the band *abxcdzefya* (shown by the dotted line in Fig. 16) and  $\Delta_3$  is what  $\Delta$  becomes when its rectilinear portions *ab*, *cd*, *ef* are pressed inwards by the figures  $A_1$ ,  $A_2$ ,  $A_3$ .

Thus

$$p_3 = \Delta_3 - \Delta.$$

The sole remaining case of three figures is Bd (Fig. 17), the case in which no straight line can possibly cut all three figures. In it we have obviously

$$p_3 = 0$$

and therefore

$$\bar{w}_3 = (A_1) + (A_2) + (A_3) + (A_2A_3) + (A_3A_1) \\ + (A_1A_2) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2).$$

This case forms no exception to the general rule for finding the conjunctive probability in cases belonging to class B.

We have

$$\Delta = abxcdzefya$$

(i. e.  $\Delta$  is the dotted band of Fig. 17), and since this band is not pressed inwards by any of the figures the conjunctive probability according to the rule would be  $\Delta - \Delta = 0$ , which is right.

Having thus pointed out the general method of procedure, and illustrated it by treating in detail the case of 3 figures, it does not seem

desirable to pursue the subject further in this direction for the present; but, before concluding, it may be worth while to notice that, in the general case of  $n$  limited right lines, the probabilities with which we have to do become Diophantine linear functions of the sides of the complete  $2n$ -gonal figure of which the  $n$  pairs of extremities of the lines are the angles. There will be a group of such linear functions depending on the mutual disposition of the  $n$  lines, but the number of formulae in any such group will be much greater than in the case of  $n$  general figures: for, when we pass from these to indefinitely narrow ovals, the portion of a definite band (appearing in any formula), partially surrounding any one of such ovals, may, according to the mutual disposition of their major axes, have in common with it an infinitesimal arc in some cases, in others an arc (to an infinitesimal *près*) equal to a circumference, and again in others to a semicircumference of the oval; which latter is ultimately the same as the length of the line whose double the complete circumference represents.

By way of illustration let us consider the question of two needles or limited straight lines rigidly connected. Neglecting the limiting cases, where one of the lines terminates in the other, there will remain 3 hypotheses:

- A. The lines intersect.
- B. The lines tend to intersect in a point external to each of them.
- C. One of the lines tends towards a point lying within the other.

Let  $p_2$  denote the chance of both the needles  $AB, CD$  being cut by one of the parallels,  $\bar{\omega}_2$  the chance of one or other of them being cut: then we have the general formulae

$$\bar{\omega}_2 = 2AB + 2CD - p_2$$

$p_2 =$  difference between the crossed and uncrossed bands round  $AB, CD$  applicable to all cases.

- A. When the lines intersect

$$\bar{\omega}_2 = AD + DB + BC + CA,$$

$$p_2 = 2AB + 2CD - AD - DB - BC - CA.$$

Fig. 20.

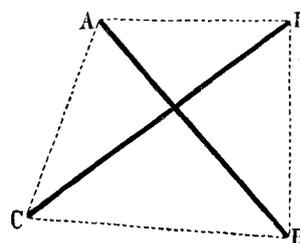
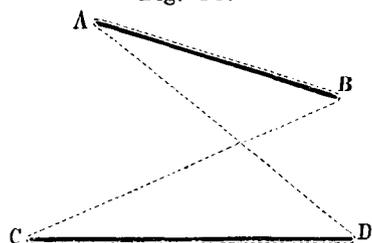


Fig. 21.

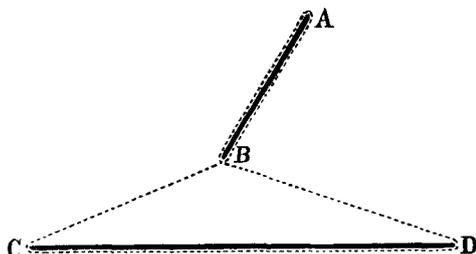


B. When the lines tend to intersect in a point external to each of them

$$\begin{aligned} p_2 &= (AB + BC + CD + DA) \\ &\quad - (AB + BD + DC + CA) \\ &= BC - CA + AD - DB,^1 \end{aligned}$$

$$\bar{\omega}_2 = 2AB + 2CD - BC + CA - AD + DB.$$

Fig. 22.



C. When one of the lines tends towards a point lying within the other

$$\begin{aligned} p_2 &= (2AB + BC + CD + DB) \\ &\quad - (AC + CD + DA) \\ &= 2AB + BC - CA - AD + DB, \end{aligned}$$

$$\bar{\omega}_2 = 2CD - BC + CA + AD - DB.$$

The complexity of cases for 3 right lines is such as would require a separate study even to obtain a perfect enumeration of them; consequently I shall leave it to others to pursue the subject further whether as regards principles or details. I will only add that the ascertainment of the general law that the formulae contain no other arguments than lengths of tight endless bands variously drawn round the given contours appears to me a distinct step achieved in the prosecution of this extensive Theory, and one that is far from being obvious à priori. Buffon's problem of the needle, it will be seen, has now expanded into a problem of  $n$  needles rigidly connected, which may be treated as a corollary to that of  $n$  entirely separate general contours, the mode of solution of which, it is believed, has been sufficiently indicated in the investigations which form the subject of this memoir.

New College, Oxford, Jan. 22<sup>nd</sup>, 1890.

<sup>1</sup> Imagine a string passing from  $B$  to  $C$ , from  $C$  to  $A$ , from  $A$  to  $D$ , and from  $D$  to  $B$ . This string cannot be kept tight unless fastened by pins at  $A, B, C, D$ . Inserting the necessary pins and tightening the string, we agree to consider the consecutive portions of the string as alternately positive and negative.

On these suppositions  $p_2$  is the algebraical length of the band  $BCADB$  stretched round the pins. The method of representation by means of pinned bands may be extended to the case of two (or any number of) general figures.

*Postscriptum.*

Since the above was set up in print my attention has been called to the fact that the extension of BARBIER'S theorem referred to on p. 186 is due to Prof<sup>r</sup> CROFTON and is given by him in his celebrated paper on the *Theory of Local Probability* contained in the Philosophical Transactions for 1868. Strange to say, no reference to this, so far as I can find, is made in CZUBER'S treatise. It is the more singular that I should have overlooked the fact inasmuch as it was an outcome of conversations with myself, when Prof<sup>r</sup> CROFTON was serving under me in the Royal Military Academy at Woolwich, that he was put upon the track of investigations in local probability in which he has since earned for himself so great and well merited celebrity. It may be added that Prof<sup>r</sup> CROFTON seems to have written in entire ignorance of BARBIER'S discovery as he makes no allusion to it in his paper.

It is indeed a romantic incident in mathematical history that BUFFON'S problem of the needle should have led up (as is undoubtedly the case) to CROFTON'S new and striking theorems in the integral calculus reproduced in BERTRAND'S *Calcul intégral*.

J. J. S.

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