ALGEBRAIC PROOFS OF THE RIEMANN-ROCH THEOREM AND OF THE INDEPENDENCE OF THE CONDITIONS OF ADJOINTNESS[1]

BY

J. C. FIELDS
of HAMILTON, CANADA.

§ 1.

The methods which I employ in the present paper are essentially those of which I have already made use in a paper *On the Reduction of the general Abelian Integral.*[2] They are purely algebraic in their character and furnish a very elementary introduction to the theory of the algebraic functions.

The substance of this paper was presented at the meeting of the Chicago Section of the American Mathematical Society held in Chicago in December 1900.

Let

$$F(z, u) = \sum_{r,s} e_{r,s} z^r u^s = 0 \tag{1}$$

be the equation to an irreducible algebraic curve of degree n. We shall assume that the multiple points of our curve are all double points with distinct tangents and separate from the branch points, that the asymptotes are all distinct from one another and none of them parallel to the axis of u and no two parallel to each other.

Any irreducible algebraic curve, as we know, can be transformed to this form by a birational transformation. The variable u we shall regard throughout as the dependent variable. The coefficient $e_{0,n}$ of u^n will on

[1] See note at end of paper.

[2] Transactions of the American Mathematical Society, Vol. 2.

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the above hypotheses have a value different from o and u will be an integral algebraic function of z.

Any rational function $R(z, u)$ of z, u we know may be expressed in the form

$$(2) \qquad R(z, u) = \sum_{\lambda} R_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right) + R_0(z, u)$$

where $R_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right)$ denotes a polynomial in $\left(\frac{1}{z - a_{\lambda}}, u\right)$ and $R_0(z, u)$ a polynomial in z, u the degrees of these polynomials in the variable u not being greater than $n - 1$.

In particular the most general rational function of z, u which becomes infinite only at ∞ may readily be shown to have the form

$$(3) \qquad \sum_{\lambda} \frac{y_{\lambda} F(a_{\lambda}, u)}{(z - a_{\lambda})(u - b_{\lambda})} + T(z, u)$$

where the summation with regard to λ is supposed to extend to the d double points $(a_{\lambda}, b_{\lambda})$ of our curve, the coefficients y_{λ} being arbitrary constants and the function $T(z, u)$ an arbitrary polynomial in z, u. On extending the above summation to a number of other points $(a_{\lambda}, b_{\lambda})$ the expression will evidently represent the most general rational function whose infinities other than those at ∞ are of the first order only and are included under the points here in question.

We shall now try to construct a function which is finite at ∞ and whose infinities, all of the first order, are to be found among those corresponding to the Q points $(a_{d+1}, b_{d+1}) \ldots (a_{d+Q}, b_{d+Q})$. These points further we shall assume to be all distinct from the d double points which we indicate by $(a_1, b_1) \ldots (a_d, b_d)$.

Our function if it exist must have the form

$$(4) \qquad \sum_{\lambda=1}^{d+Q} \frac{y_{\lambda} F(a_{\lambda}, u)}{(z - a_{\lambda})(u - b_{\lambda})} + T(z, u)$$

and our only task will evidently be to determine under what conditions a function of this form will not become infinite at ∞. Suppose the directions of the n points at ∞ on our curve to be given by the equations

$$u - k_1 z = 0 \,, \; \ldots, \; u - k_n z = 0$$

where, in accord with our original hypothesis in regard to the asymptotes of the curve, the n coefficients k all have different values. A homogeneous polynomial $t(z, u)$ of degree $n - 1$ in z, u will evidently become infinite to the order $n - 1$ for one at least of the n branches at ∞, as otherwise we should have $z^{n-1}t(1, k) = 0$ for $k = k_1 \ldots k_n$ and, as the degree of the equation $t(1, k) = 0$ in k is $n - 1$, this equation would be an identity and we should therefore also have $t(z, u) = 0$ identically.

We derive thence that $z^r t(z, u)$ must be infinite to the order $n + r - 1$ for one at least of the branches at ∞, and this of course whether r is positive or negative. It follows further also that a non-homogeneous polynomial $T(z, u)$ of degree $n + r - 1$ must become infinite to the order $n + r - 1$ for one at least of the branches at ∞, for the polynomial can evidently be written in the form

$$T(z, u) = z^r t(z, u) + T_{n+r-2}(z, u)$$

where the degree of $T_{n+r-2}(z, u)$ in z, u is not greater than $n + r - 2$.

Here and throughout the paper, where there is nothing to indicate the contrary, it is to be assumed as a matter of course that our functions are expressed in the reduced form in which the variable u does not appear to a power higher than the $(n - 1)^{\text{th}}$.

§ 2.

Returning to the consideration of the expression (4) which is to remain finite for $z = \infty$, we see that the individual elements of the summation therein appearing cannot become infinite to an order greater than $n - 2$ and that this therefore also must be the case for the polynomial $T(z, u)$.

The function which we have to consider then may be written in the form

$$\sum_{\lambda=1}^{d+Q} \frac{y_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + T_{n-2}(z, u) \tag{5}$$

where in $T_{n-2}(z, u)$ we employ the suffix $n - 2$ to indicate the degree of the polynomial.

We have to study the conditions under which the expression (5) does not become infinite at ∞.

From the equation to our curve we have

$$\frac{F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} = \frac{F(a_\lambda, u) - F(a_\lambda, b_\lambda)}{(z-a_\lambda)(u-b_\lambda)} = \frac{\sum_{r,s} e_{r,s} a_\lambda^r (u^s - b_\lambda^s)}{(z-a_\lambda)(u-b_\lambda)}.$$

This we may write in the form

$$(6) \qquad \frac{F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} = (z-a_\lambda)^{-1} \sum_{r,s} e_{r,s} a_\lambda^r (u^{s-1} + u^{s-2} b_\lambda + \ldots + b_\lambda^{s-1})$$

$$= (z-a_\lambda)^{-1} \sum_{s=1}^{n} \{a_\lambda, b_\lambda\}_s u^{n-s}$$

where

$$(7) \qquad \{a_\lambda, b_\lambda\}_s = \sum_{\sigma=0}^{s-1} \sum_{\rho=0}^{\sigma} e_{\rho, n-\sigma} a_\lambda^\rho b_\lambda^{s-\sigma-1}.$$

We then have

$$(8) \qquad \frac{F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} = \left\{\frac{1}{z} + \frac{a_\lambda}{z^2} + \frac{a_\lambda^2}{z^3} + \ldots + \frac{a_\lambda^{n-3}}{z^{n-2}} + \frac{a_\lambda^{n-2}}{z^{n-2}(z-a_\lambda)}\right\} \sum_{s=1}^{n} \{a_\lambda, b_\lambda\}_s u^{n-s}$$

$$= \left\{\frac{1}{z} + \frac{a_\lambda}{z^2} + \frac{a_\lambda^2}{z^3} + \ldots + \frac{a_\lambda^{n-3}}{z^{n-2}}\right\} \sum_{s=1}^{n} \{a_\lambda, b_\lambda\}_s u^{n-s} + P_\lambda(z, u)$$

where $P_\lambda(z, u)$ is a rational function of z, u which is finite for $z = \infty$.

This we may further reduce to the form

$$(9) \qquad \frac{F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} = \sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} a_\lambda^{r-1} \{a_\lambda, b_\lambda\}_s z^{-r} u^{n-s} + \bar{P}_\lambda(z, u)$$

where $\bar{P}_\lambda(z, u)$ is a rational function of z, u which is finite for $z = \infty$.

For the summation in (5) we shall then have

$$\sum_{\lambda=1}^{d+Q} \frac{y_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} = \sum_{\lambda=1}^{d+Q} \sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} y_\lambda a_\lambda^{r-1} \{a_\lambda, b_\lambda\}_s z^{-r} u^{n-s} + P(z, u)$$

where $P(z, u)$ is a rational function of z, u which is finite for $z = \infty$.

This we may write in the form

$$(10) \qquad \sum_{\lambda=1}^{d+Q} \frac{y_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} = \sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} c_{n-s-r, n-s} z^{-r} u^{n-s} + P(z, u)$$

where

$$(11) \qquad c_{n-s-r,\,n-s} = \sum_{\lambda=1}^{d+Q} y_\lambda a_\lambda^{r-1} \{a_\lambda\,,\, b_\lambda\}_s.$$

For the suffixes of the coefficients c we here select for reasons of convenience the sum of the exponents of z and u and the exponent of u respectively.

On writing

$$(12) \qquad y_{i,k} = \sum_{\lambda=1}^{d+Q} y_\lambda a_\lambda^i b_\lambda^k$$

it is readily seen that we have

$$(13) \qquad c_{n-s-r,\,n-s} = \sum_{\sigma=0}^{s-1} \sum_{\rho=0}^{\sigma} e_{\rho,\,n-\sigma} y_{r-1+\rho,\,s-1-\sigma}.$$

Since the exponents of z which appear in the summation on the right of (10) are all negative whereas in the polynomial $T_{n-2}(z\,,\,u)$ in (5) no such exponents make their appearance, it will not be inconsistent with the notation employed above if we write

$$(14) \qquad T_{n-2}(z\,,\,u) = \sum_{s=2}^{n} \sum_{r=0}^{s-2} c_{n-s+r,\,n-s} z^r u^{n-s}.$$

Combining (10) and (14) we have

$$(15) \qquad T_{n-2}(z\,,\,u) + \sum_{\lambda=1}^{d+Q} \frac{y_\lambda F(a_\lambda\,,\,u)}{(z-a_\lambda)(u-b_\lambda)}$$

$$= \sum_{s=2}^{n} \sum_{r=0}^{s-2} c_{n-s+r,\,n-s} z^r u^{n-s} + \sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} c_{n-s-r,\,n-s} z^{-r} u^{n-s} + P(z\,,\,u)$$

$$= \sum_{s=1}^{n} \sum_{r=-(n-s-1)}^{s-2} c_{n-s+r,\,n-s} z^r u^{n-s} + c_{0,0} + P(z\,,\,u).$$

This again we may write in the form

$$(16) \qquad \sum_{\lambda=1}^{d+Q} \frac{y_\lambda F(a_\lambda\,,\,u)}{(z-a_\lambda)(u-b_\lambda)} + T_{n-2}(z\,,\,u) = \sum_{\mu=1}^{n-2} z^{-\mu} p_\mu(z\,,\,u) + c_{0,0} + P(z\,,\,u)$$

where the functions $p_\mu(z, u)$ are homogeneous polynomials in z, u of degree $n-1$ which are given by the formula

$$p_\mu(z, u) = \sum_{s=1}^{n} c_{n-1-\mu,\, n-s} z^{s-1} u^{n-s}.$$

Now if the expression (16) is not to become infinite for any of the n branches at ∞ the polynomials $p_1(z, u) \ldots p_{n-2}(z, u)$ must all vanish identically, as otherwise the expression would become infinite to the order $n-r-1$ for some one at least of these branches in case $p_r(z, u)$ be the first of these polynomials which does not vanish identically. Apart from $c_{0,0}$ then the coefficients c on the right of (15) must all have the value 0. These coefficients however include among them the coefficients of the polynomial $T_{n-2}(z, u)$ and this polynomial therefore must reduce to the constant $c_{0,0}$.

Our expression on the left of (16) will then take the form

$$(17) \qquad \sum_{\lambda=1}^{d+Q} \frac{y_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} + c_{0,0}.$$

Since 0 is the value also of the remaining coefficients

$$c_{n-s-r,\, n-s} \qquad r=1,2,\ldots,n-s-1, \quad s=1,2,\ldots,n-2$$

on the right of (15), the equations (13) become

$$(18) \qquad 0 = \sum_{\sigma=0}^{s-1} \sum_{\rho=1}^{\sigma} e_{\rho,\, n-\sigma} y_{r-1+\rho,\, s-1-\sigma}. \qquad r=1,2,\ldots,n-s-1, \quad s=1,2,\ldots,n-2$$

This system of equations we shall attempt to satisfy by a proper choice of the quantites $y_{i,k}$ on the right. The number of these equations is evidently $\frac{1}{2}(n-1)(n-2)$ and this also is the number of the quantities $y_{i,k}$ which present themselves in the equations, as may readily be shown. Namely the sum of the suffixes in a quantity $y_{r-1+\rho,\, s-1-\sigma}$ is $r+s-2+\rho-\sigma$ and for given values of r and s the greatest value which this sum can have is $r+s-2$ corresponding to the values $\rho=0$, $\sigma=0$. Also from (18) we see that $r+s-2$ cannot exceed $n-3$ in value. It follows therefore that the sum of the suffixes in a quantity $y_{i,k}$ which appears in our equations, is in no case greater than $n-3$. Now the number of possible pairs of integers (i, k) whose sum is not greater than $n-3$ is $\frac{1}{2}(n-1)(n-2)$, and that all the $\frac{1}{2}(n-1)(n-2)$ corresponding quantities $y_{i,k}$ actually present themselves in our system of equations is immediately apparent, for each one of them in fact appears in some one of the equations with $e_{0,n}$ as coefficient, as we see on substituting for r, s in $y_{r-1,s-1}$ the several pairs of values given in (18).

§ 3.

The number of the quantities $y_{i,k}$ appearing in the equations (18) we have just seen to be equal to the number of the equations and we shall now seek to determine these quantities so as to satisfy the equations. For this purpose we shall write our equations in the form

$$(19)\qquad 0 = e_{0,n}y_{r-1,s-1} + e_{1,n-1}y_{r,s-2} + e_{2,n-2}y_{r+1,s-3} + \cdots$$
$$+ e_{s-2,n-s+2}y_{r+s-3,1} + e_{s-1,n-s+1}y_{r+s-2,0} + \Gamma_{r-1,s-1}$$

where evidently $\Gamma_{r-1,s-1}$ involves only quantities $y_{i,k}$ in which the sum of the suffixes is less than $r+s-2$.

The number of equations of the type (19) in which the sum of the suffixes of the quantities $y_{i,k}$ in evidence on the right has a given value $r+s-2=\tau$ is $\tau+1$, and these $\tau+1$ equations we shall group together and consider in the form

$$(20)\left\{\begin{aligned}
e_{0,n}y_{0,\tau} + e_{1,n-1}y_{1,\tau-1} + e_{2,n-2}y_{2,\tau-2} + \cdots + e_{\tau-1,n-\tau+1}y_{\tau-1,1} + e_{\tau,n-\tau}\quad y_{\tau,0} &= -\Gamma_{0,\tau}\\
e_{0,n}\quad y_{1,\tau-1} + e_{1,n-1}y_{2,\tau-2} + \cdots + e_{\tau-2,n-\tau+1}y_{\tau-1,1} + e_{\tau-1,n-\tau+1}y_{\tau,0} &= -\Gamma_{1,\tau-1}\\
e_{0,n}\quad y_{2,\tau-2} + \cdots + e_{\tau-3,n-\tau+1}y_{\tau-1,1} + e_{\tau-2,n-\tau+2}y_{\tau,0} &= -\Gamma_{2,\tau-}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
e_{0,n}\qquad y_{\tau-1,1} + e_{1,n-1}\qquad y_{\tau,0} &= -\Gamma_{\tau-1,1}\\
e_{0,n}\qquad y_{\tau,0} &= -\Gamma_{\tau,0}.
\end{aligned}\right.$$

Suppose now that the quantities $y_{i,k}$ in which the sum of the suffixes is less than τ are all equal to 0. The same will then evidently also be true of the quantities l' on the right of the equations (20), for they are linear and homogeneous in the quantities $y_{i,k}$ the sum of whose suffixes is less than τ.

The equations (20) then become

$$(21)\begin{cases} e_{0,n}y_{0,\tau}+e_{1,n-1}y_{1,\tau-1}+e_{2,n-2}y_{2,\tau-2}+\dots+e_{\tau-1,n-\tau+1}y_{\tau-1,1}+e_{\tau,n-\tau}\,y_{\tau,0}=0 \\ e_{0,n}\,y_{1,\tau-1}+e_{1,n-1}y_{2,\tau-2}+\dots+e_{\tau-2,n-\tau+2}y_{\tau-1,1}+e_{\tau-1,n-\tau+1}y_{\tau,0}=0 \\ e_{0,n}\,y_{2,\tau-2}+\dots+e_{\tau-3,n-\tau+3}y_{\tau-1,1}+e_{\tau-2,n-3+2}y_{\tau,0}=0 \\ \dots\dots\dots\dots\dots\dots \\ e_{0,n}\,y_{\tau-1,1}+e_{1,n-1}\,y_{\tau,0}=0 \\ e_{0,n}\,y_{\tau,0}=0. \end{cases}$$

Here the determinant of the coefficients is $e_{0,n}^{\tau+1}$ and is therefore different from 0, since by our hypothesis with regard to the curve $F(z,u)=0$ the coefficient $e_{0,n}$ has a value other than 0. It follows hence that the quantities $y_{0,\tau}, y_{1,\tau-1}, \dots, y_{\tau,0}$ must all have the value 0. If then all the quantities $y_{i,k}$ in which the sum of the suffixes is less than τ have the value 0, this also will be so in the case of those quantities in which the sum of the suffixes is equal to τ.

Now in the case $\tau=0$ the system of equations (20) reduces to the single equation

$$e_{0,n}y_{0,0}=0.$$

We therefore have $y_{0,0}=0$ and it follows thence by our inductive reasoning above that the quantities $y_{0,1}$ and $y_{1,0}$ corresponding to the case $\tau=1$ must also have the value 0, and by successive induction we see that the quantities $y_{i,k}$ for which $\tau=i+k$ has one of the values $2,3,\dots,$ $n-3$ must all be equal to 0. Substituting for these $\frac{1}{2}(n-1)(n-2)$ quantities the value 0 in the corresponding equations of the type (12), we obtain the system of equations

$$(22)\qquad \sum_{\lambda=1}^{d+Q} y_\lambda a_\lambda^i b_\lambda^k=0. \qquad {\scriptstyle i+k=0,1,2,\dots,n-3}$$

Any rational function then whose infinities are of the first order only and correspond to ones among the Q points $(a_{d+1}, b_{d+1}) \dots (a_{d+Q}, b_{d+Q})$ must be of the form (17), and its coefficients y_λ must satisfy the conditions implied in the system of equations (22).

Conversely if the coefficients y_λ in a function of the form (17) satisfy the system of equations (22) the infinities of this function will be included under the Q infinities here in question. For a function of the type (17), constructed with the values of the quantites y_λ in question and reduced to the form on the right of (16), will evidently, by virtue of the equations (12) and (13), have all its coefficients c with the exception of $c_{0,0}$ equal to 0 and will therefore be finite for $z = \infty$.

The form (17) then in which the coefficients y_λ are subject to the conditions (22) give all those and only those rational functions whose infinities are included under the infinities of the first order corresponding to the Q points $(a_{d+1}, b_{d+1}) \dots a_{d+Q}, b_{d+Q})$.

§ 4.

We shall now study the equations (22) more in detail and in the first place we shall consider the case in which $Q = 0$.

In this case our equations become

$$(23) \qquad \sum_{\lambda=1}^{d} y_\lambda a_\lambda^i b_\lambda^k = 0. \qquad {\scriptstyle i+k=0,1,2,\dots,n-3}$$

The satisfaction of this system of equations by values of the quantities y_λ which are not all 0 would imply the existence of a function

$$\sum_{\lambda=1}^{d} \frac{y_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + c_{0,0}$$

which is nowhere infinite and which at the same time is not a constant. This however as we know is impossible, for an algebraic function must either be a constant or must somewhere become infinite. It follows therefore that the system of equations (23) can only be satisfied when all the quantities y_λ have the value 0 and this proves that the conditions for adjointness are independent of one another. The points $(a_1, b_1) \dots (a_d, b_d)$ namely are the d double points of the curve and the conditions for adjointness are those conditions which must be satisfied by the coefficients $\partial_{i,k}$ in the equation to a curve

$$\sum_{i,k} \partial_{i,k} z^i u^k = 0 \tag{24}$$

in order that it may pass through the double points. These conditions in the case of a curve of degree $n-3$ will be given by the d equations

$$\sum_{i+k \overline{\overline{<}} n-3} \partial_{i,k} a_\lambda^i b_\lambda^k = 0 \qquad \lambda=1,2,\ldots,d \tag{25}$$

and these equations in the quantities $\partial_{i,k}$ must evidently be linearly independent of one another since, as we have seen, the system of equations (23) can only be satisfied when the quantities y_λ all have the value 0.

Since the d conditions for adjointness are independent of one another in the case of a curve of degree $n-3$ they will evidently also be independent of one another in the case of a curve whose degree is greater than $n-3$.

On defining the genus of a curve as the number of linearly independent adjoint polynomials of degree $n-3$ and on indicating the same by the letter p we shall evidently have

$$p = \frac{1}{2}(n-1)(n-2) - d.$$

§ 5.

We shall now consider the system of equations (22) in the case where we have $Q > 0$.

For $Q \overline{\overline{<}} p$ we see that equations (22) in general can only be satisfied on putting 0 for each of the quantities y_λ as otherwise the $d+Q$ equations

$$\sum_{i+k \overline{\overline{<}} n-3} \partial_{i,k} a_\lambda^i b_\lambda^k = 0 \qquad \lambda=1,2,\ldots,d+Q \tag{26}$$

would not be independent of one another, whereas it is evident that these $d+Q$ equations are independent of one another in case the Q points $(a_{d+1}, b_{d+1}) \ldots (a_{d+Q}, b_{d+Q})$ are arbitrary, for the number $d+Q$ in this case is $\overline{\overline{<}} \frac{1}{2}(n-1)(n-2)$ the number of coefficients in a polynomial of degree $n-3$.

It follows then that it is impossible to construct a function of the form (17) other than a constant whose infinities, all of the first order, correspond to ones among the Q arbitrary points $(a_{d+1}, b_{d+1}) \ldots (a_{d+Q}, b_{d+Q})$ in the case where we have $Q \overline{\overline{<}} p$.

It might however happen that we could construct such a function for a special set of Q points. If we can construct such a function in the case where $Q = p$ the p points $(a_{d+1}, b_{d+1}) \ldots (b_{d+Q}, b_{d+Q})$ will evidently all lie on an adjoint curve of degree $n - 3$, for in this case the $d + Q$ equations (26) will not all be independent of one another.

In the case where we have $Q \overline{\overline{>}} p + 1$ it is always possible to satisfy the equations (22) by quantities y_λ not all of which have the value 0, for in this case these quantities are in excess of the number of the equations. It is therefore always possible to construct a rational function of z, u whose infinities are included under an arbitrary set of $p + 1$ infinities of the first order.

The strength of a system of Q points $(a_{d+1}, b_{d+1}) \ldots (a_{d+Q}, b_{d+Q})$ in determining an adjoint curve of degree $n - 3$ is defined as the number q of conditions to which the coefficients of the general adjoint curve of degree $n - 3$ must be subjected in order that it may pass through these Q points. The strength of any system of points can evidently not be greater than p for this is the number of arbitrary coefficients involved in the expression of the general adjoint curve of degree $n - 3$.

Returning to our system of equations (22) we see, that the number of arbitrary quantities y_λ presenting themselves in the most general system of solutions of these equations is equal to $Q - q$ the number namely of the $d + Q$ equations (26) taken in any given order, which are dependent on the equations preceding them in this order.

In other words the number of arbitrary coefficients, including the constant term, involved in the expression of the most general rational function of z, u whose infinities are included under a certain set of Q infinities, is $Q - q + 1$ where q is the strength of the set of infinities in question. This is the RIEMANN-ROCH Theorem. The infinities which cannot actually present themselves in the function are those for which the corresponding quantities y_λ in the equations (22) must have the value 0, and to each of these we see corresponds an equation in the system of equations (26) which is independent of the remaining equations of the system. We may then evidently say that in the general function here in question infinity corresponding to a point (a_λ, b_λ) does or does not actually present itself, according as the omission of this point from the system of Q points $(a_{d+1}, b_{d+1}) \dots (a_{d+Q}, b_{d+Q})$ does not or does diminish the strength of the system.

The number of points of intersection of an adjoint curve of degree $n - 3$ with our original curve, over and above the double points, is $n(n-3) - 2d = 2p - 2$. These $2p - 2$ points determine our adjoint curve completely and their strength is therefore $p - 1$.

The strength of $2p - 1$ or more points is evidently p and from any set of $2p$ or more points we may therefore omit any point without diminishing the strength of the set. It will follow that in the case where we have $Q \overline{\overline{>}} 2p$ all Q infinities will actually present themselves in the most general function whose infinities are included under the Q infinities in question.

§ 6.

In proving the Riemann-Roch Theorem we have excluded infinities corresponding to the double points. Such infinities however may also be taken account of by a slight extension of our reasoning.

Namely on indicating by (ξ, η) a pair of parameters connected by the equation to our curve $F(\xi, \eta) = 0$, we may readily shew that the function

$$\left[\frac{d}{d\xi}\frac{F(\xi, u)}{(z-\xi)(u-\eta)}\right]_{\xi=a, b=\eta}$$

becomes infinite to the first order for the double point (a, b) and for that branch only through the double point with regard to which the differentiation is effected, while it remains finite for all other finite points of the curve.

To indicate this function we shall employ the less accurate but more concise notation

$$\frac{d}{da}\frac{F(a, u)}{(z-a)(u-b)}.$$

We shall write

$$y = g + g' \frac{d}{da}$$

where g and g' are arbitrary constants and y is therefore an operator involving two arbitrary constants, so that we have

$$y \frac{F(a , u)}{(z - a)(u - b)} = \frac{gF(a , u)}{(z - a)(u - b)} + g' \frac{d}{da} \frac{F(a , u)}{(z - a)(u - b)}.$$

If we would determine the possibility of constructing a function whose infinities are included under a certain set of Q infinities some of which, say r in number, correspond respectively to r of the double points we should have, instead of the expression (5), to consider an expression

$$\sum_{\lambda=1}^{d+Q-r} y_\lambda \cdot \frac{F(a_\lambda , u)}{(z - a_\lambda)(u - b_\lambda)} + \tau_{n-2}(z , u)$$

where r of the symbols y_λ indicate symbolic operators of the form

$$g_\lambda + g'_\lambda \frac{d}{da_\lambda}.$$

With regard to this expression our reasoning would be the same as in the case of the expression (5), with some self-evident modifications due to replacing certain of the factors y_λ by operators which operate on whatever functions of (a_λ , b_λ) they happen to precede.

In case infinities corresponding to each of the branches through a double point (a , b) present themselves under the Q infinities in question, we should have to introduce a symbolic factor of the form

$$y = g + g' \left(\frac{d}{da}\right)_1 + g'' \left(\frac{d}{da}\right)_2$$

where by the suffixes 1 and 2 we distinguish between operations having reference to the separate branches.

In any case our conclusion is that $Q - q + 1$ is the number of arbitrary constants involved in the expression of the most general rational function whose infinities are included under a certain set of Q infinities, where q is the strength of the corresponding set of Q zeros in determining an adjoint polynomial of degree $n-3$ and further that a given one of the infinities in question will or will not actually present itself in such most general function, according as the omission of the corresponding zero from the set of Q zeros does not or does diminish the strength of the set.

Note. In his lectures for the year 1869 WEIERSTRASS in constructing a function possessing Q infinities of the first order makes use of the representation by partial fractions in the case where the fundamental algebraic curve possesses only double points.[1] In the present paper we are concerned with the same special case and employ the like representation though instead of treating our function with WEIERSTRASS under the form of a constant plus a linear expression in Q functions, each one of which possesses one of the Q infinities in question and a certain set of p other infinities, we handle the function directly in the form (5). What however more particularly characterizes the paper is its method of dealing with the equations of condition, a method which is applicable also in the case where our algebraic equation possesses any singularities whatever and where we would prove the RIEMANN-ROCH Theorem for any arbitrary combination of infinities. This will appear in a later paper where I shall present the theory of the algebraic functions as I have developed it for an arbitrary algebraic equation on employing the representation by means of partial fractions. The special theory here given then may be regarded as a preparation for the more general theory though in the specialization the important apparatus necessary to the handling of the higher singularities falls away.

Hamilton, Canada, June 11, 1901.

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[1] See BRILL and NOETHER's *Bericht über die Entwicklung der Theorie der algebraischen Functionen.*