ON THE INTEGRATION OF SERIES

BY

E. W. HOBSON
of CAMBRIDGE (England).

Since ABEL's researches in the theory of infinite series, some of the most important investigations on the subject have been concerned with the uniformity and non-uniformity of the convergence of such series. It was first pointed out by SEIDEL, and by STOKES independently, that a discontinuity in the sum of a convergent series, of which the terms are continuous functions of a real variable, is due to the non-uniform convergence of the series in the neighbourhood of points at which such discontinuity exists. It is further known that non-uniformity in the convergence of such a series does not necessarily involve discontinuity in the sum. The theory is of special importance in connection with the question regarding the conditions under which the series may be integrated term by term so that the series arising from such integration may have for its sum the integral of the sum of the original series.

If

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

is a series which converges everywhere in an interval (a, b) of the real variable x, and if $u_1(x), u_2(x), \ldots, u_n(x) \ldots$ are each continuous throughout the interval, it is well known that a sufficient condition that the sum of the integrals of the terms of the series taken through (a, b), or through an interval which is part of (a, b), may be represented by the integral of the sum-function $s(x)$ taken through the same interval, is that the series be uniformly convergent through the interval of integration. It has been, however, shewn by OSGOOD,[1] that in the case in which the sum-function $s(x)$ is continuous through the interval (x_0, x_1) of integration, a sufficient condition for term by term integration is that there should be in the interval (x_0, x_1) no point at which the measure of non-uniform convergence is indefinitely great.

It has been shewn by BAIRE[2] that the sum-function $s(x)$ is at most a point-wise discontinuous function. In the present communication the properties of the remainder-function $R_n(x) = s(x) - s_n(x)$, are considered on the lines of BAIRE's memoir, and the results are applied to prove that for the most general function $s(x)$ which is the sum of a series of the above type, the series may be integrated term by term and gives a series of which the sum is the integral of $s(x)$, provided (1) that $s(x)$ is integrable through the interval of integration, and (2) that in that interval there is no point at which the measure of non-uniform convergence is indefinitely great.

If $n = \frac{1}{y}$, we may consider $s_n(x)$, $R_n(x)$ as functions of x and y, defined for all values of x in the interval (a, b), and for values of y which are the reciprocals of any positive integer m. Following BAIRE's procedure, the functions may be defined for values of y intermediate between the values $y_m = \frac{1}{m}$, and $y_{m+1} = \frac{1}{m+1}$, so that writing $s(x, y)$, $R(x, y)$ for $s_n(x)$, $R_n(x)$,

$$s(x, y) = \frac{y - y_m}{y_{m+1} - y_m} s(x, y_{m+1}) + \frac{y_{m+1} - y}{y_{m+1} - y_m} s(x, y_m)$$

$$R(x, y) = \frac{y - y_m}{y_{m+1} - y_m} R(x, y_{m+1}) + \frac{y_{m+1} - y}{y_{m+1} - y_m} R(x, y_m).$$

If we further define $s(x, 0)$, $R(x, 0)$ to be $s(x)$, and zero respectively, the two functions $s(x, y)$, $R(x, y)$ are defined for every point inside and on the boundary of the rectangle contained by the four straight lines $x = a$, $x = b$, $y = 0$, $y = 1$.

The function $s(x, y)$ is everywhere continuous with regard to y, and is continuous with respect to x, everywhere except upon the bound-

[1] American Journal of Mathematics, Vol. XIX, 1897.

[2] See Annali di Math. (3) III, 1899.

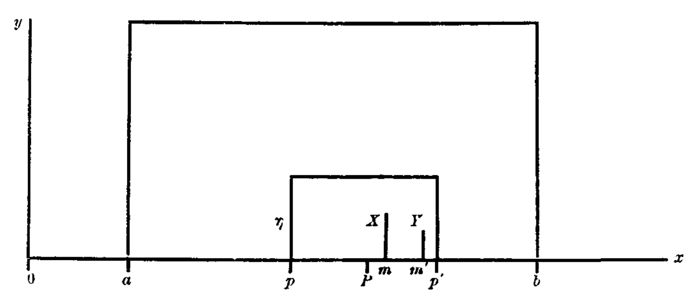
ary $y = 0$. BAIRE has shewn that this function is at most a point-wise discontinuous function with respect to (x, y), on any continuous curve within the rectangle, and in particular on the boundary $y = 0$. We shall here consider the function $R(x, y)$, which does not come under BAIRE's general case, as although it is everywhere continuous with regard to y, it is in general a point-wise discontinuous function of x, for any constant value of y between 0 and 1, the value $y = 0$ excepted, for which the function vanishes.

At any point $P(x, y)$, let a straight line of length 2ρ be drawn with P as middle point, and parallel to the y axis, and let $\omega(\rho)$ be the fluctuation (Schwankung) of the function $R(x, y)$ in the line 2ρ; the function $\omega(\rho)$ is a continuous function of ρ, and corresponding to an arbitrarily assigned positive number σ, let $\alpha_\sigma(x, y)$ be the upper limit of the values of ρ which are such that $\omega(\rho) \leqq \sigma$; if P is in the boundary $y = 0$, it will be sufficient to take the straight line of length ρ within the rectangle. The function $\alpha_\sigma(x, y)$ is thus defined for every point in the rectangle and is an essentially positive function. Moreover since $R(x, y) = s(x) - s(x, y)$, and since $s(x)$ is independent of y, the function $\alpha_\sigma(x, y)$ is the same as the corresponding function introduced by BAIRE for the function $s(x, y)$.

It has been shewn by BAIRE that $\alpha_\sigma(x, y)$ is a semi-continuous function, that is, that corresponding to an arbitrarily assigned positive number ε, a neighbourhood of the point P can be found such that for all points P' in this neighbourhood $\alpha_\sigma(P') < \alpha_\sigma(P) + \varepsilon$.

If P be a point $(x, 0)$ in the boundary $y = 0$, and a semi-circle of radius ρ, and centre P, be drawn within the rectangle, the lower limit of $|R(x, y)|$ in this semi-circle is zero, and the upper limit may be denoted by $\beta(\rho)$. The limit of $\beta(\rho)$ when ρ is indefinitely diminished may be called the measure of the non-uniform convergence of the given series at the point P; if this limit is zero, the convergence of the series at P is uniform. If we divided the semi-circle into quadrants by means of a radius, the limits when $\rho = 0$, of the upper limits of $|R(x, y)|$ in the two quadrants, may be called the measures of non-uniform convergence at P, on the right and on the left, respectively; these two measures are equivalent to OSGOOD's indices of the point P, of which he gives a different definition. The measure of non-uniform convergence of the given series is in accordance with the above definition, the saltus (Sprung) of the function $|R(x, y)|$ at the point $P(x, 0)$ with respect to the continuum (x, y).

The minimum of $\alpha_\sigma(x, y)$ at the point $P(x, 0)$, of the boundary $y = 0$, with respect to that boundary, is the limit when δ diminishes to zero, of the lower limit of α_σ in the neighbourhood $(x - \delta, x + \delta)$ of the point P. If this minimum at the point P is positive, a neighbourhood of P in the continuum (x, y) can be found, such that the fluctuation (Schwankung) of $R(x, y)$ in that neighbourhood is $\leqq 2\sigma$, and hence the saltus of $|R(x, y)|$ at P is $\leqq 2\sigma$. To prove this we observe that a neighbourhood pp' of P can be found such that β_σ at every point in pp'



is greater than a fixed number η which is less than the minimum of β_σ at P. Let X, Y be any two points in the rectangle whose base is pp' and height η, and let Xm, Ym' be perpendicular to the boundary. We have then

$$|R(X) - R(Y)| \leqq |R(X) - R(m)| + |R(Y) - R(m')|$$
$$\leqq 2\sigma$$

thus the required neighbourhood has been found.

It follows that if the saltus of $|R(x, y)|$ at P, is greater than 2σ, the minimum of $\alpha_\sigma(P)$ at P, must be zero.

Now it has been shewn by Baire that in every sub-interval of the boundary $y = 0$, points exist at which the minimum of $\alpha_\sigma(P)$ with respect to the straight line is positive, and this is the case however small σ may be.

It thus appears that in the interval (a, b) the points at which the given series is uniformly convergent are everywhere dense, and thus that the function $|R(x, y)|$ is on the boundary $y = 0$, a point-wise discontinuous function with respect to the continuum (x, y). It follows that the points of (a, b) at which the measure of non-uniform convergence of the given series exceeds an arbitrarily fixed positive number form a closed and non-dense aggregate.

Let it now be assumed that the point-wise discontinuous function $s(x)$ is an integrable function. The condition that the series $\Sigma\int_{x_0}^{x_1} u_n(x)dx$ converges to the value $\int_{x_0}^{x_1} s(x)dx$, is that a value y_0 of y, can be found corresponding to a given positive number ε, such that $\left|\int_{x_0}^{x_1} R(x, y)dx\right| < \varepsilon$, for any fixed value of y which is $\leqq y_0$.

It will be proved that this condition is satisfied, provided there is no point in the interval (x_0, x_1) at which the saltus of $|R(x, y)|$, the measure of non-uniform convergence, is indefinitely great. If the saltus of $|R(x, y)|$ is at every point finite, then $|R(x, y)|$ has a finite upper limit for every point within the fundamental rectangle; this follows from the fact proved above, that the points on $y = 0$, at which the saltus of $|R(x, y)|$ exceeds a fixed number, form a closed aggregate, and thus if at a converging series of points $x_1, x_2, \ldots, x_n, \ldots$ the values of this saltus formed a sequence of increasing numbers which had no finite upper limit, the saltus at the limiting point $\underset{n=\infty}{L} x_n$, would be indefinitely great.

Let A be a fixed positive number, then the aggregate G of points at which the saltus of $|R(x, y)|$ exceeds A, is closed and non-dense. It is well known that the aggregate G consists of the extremities of an enumerable aggregate of sub-intervals $\theta_1, \theta_2, \theta_3, \ldots,$ together with the limiting points of these extremities. Let I be the content of G, then if $l = x_1 - x_0$, $l - I$ is the limit of $\theta_1 + \theta_2 + \theta_3 + \ldots$

A number μ can be found corresponding to any fixed arbitrarily small number ε_1, such that $\theta_1 + \theta_2 + \ldots + \theta_\mu > l - I - \varepsilon_1$, and is $< l - I$. Inside each of the intervals θ, take an interval θ', this can be done so that $\sum_1^\mu \theta' = \sum_1^\mu \theta - \varepsilon_2$, where ε_2 is an arbitrarily assigned positive number. The sum $\Sigma\theta'$ lies between $l - I - \varepsilon_1 - \varepsilon_2$, and $l - I - \varepsilon_2$.

Let the interval l be devided into $\mu + s$ sub-intervals of which μ consist of the intervals θ', and the other s are $t_1, t_2, t_3, \ldots, t_s$; thus $l = \sum_1^s t + \sum_1^\mu \theta'$; all the points of G are in the intervals t.

We first consider the integral taken through the intervals θ'; on θ'_r as base a rectangle of height $\bar{y}_r$ can be drawn so that in that rectangle, $|R(x, y)| \leqq A + \eta$, when η is an arbitrarily small prescribed number. For if this is not the case, there would be points of the x-axis in θ'_r, such that the fluctuation of $|R(x, y)|$ in areas containing them are $> A$, however small y may be taken, contrary to the hypothesis that at every point of θ'_r the saltus of $|R(x, y)|$ is $\overline{\overline{<}} A$, hence $\bar{y}_r$ can be found corresponding to a given η; if $\bar{y}$ is the greatest of the μ numbers $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_\mu$, then if $y \overline{\overline{<}} \bar{y}$, for every x in the intervals θ', $|R(x, y)| \overline{\overline{<}} A + \eta$. It thus appears that $\left|\int_{x_0}^{x_1} R(x, y)dx\right|$, taken through the intervals θ', is $\leqq (A + \eta)\sum\theta'$ or $< (l - I - \varepsilon_2)(A + \eta)$, provided $y \leqq \bar{y}$. The numbers $\bar{y}, y$ converge to zero together.

Next consider the s intervals $t_1, t_2, \ldots, t_s$; for any point x of G, there is a value of y such that for it and all smaller values, $|R(x, y)| < \sigma$, where σ is a fixed positive number which we take $< A$; this arises from the continuity of $R(x, y)$ with respect to y, at the point (x, o). Take y_1 a value of y, and let G_{y_1} be the aggregate of points belonging to G, such that $|R(x, y)| < \sigma$, provided $y \overline{\overline{<}} y_1$. The points of G_{y_1} may be put into a finite number of intervals $\tau_1, \tau_2, \ldots, \tau_x$, where $\sum\tau < I_{y_1} + \delta$, I_{y_1} denoting the content of G_{y_1}, and δ an arbitrarily chosen positive number. The complementary intervals whose sum is $\sum t - \sum\tau$ contain only such points of G as do not belong to G_{y_1}. Since there are by hypothesis no points of G at which the upper limit of the fluctuation of $R(x, y)$ in (x, y) is not finite, and this upper limit is everywhere less than some fixed finite number, there exists a finite upper limit of $|R(x, y)|$ for all values of x which are in the intervals t but not in the intervals τ; let this be B. The integral taken through those parts of the intervals t which are not in the τ, is not greater than $B(\sum t - \sum\tau)$ or is $< B(I + \varepsilon_1 + \varepsilon_2 - I_{y_1})$; B cannot increase as y is diminished.

It now remains to consider the integral taken through the intervals τ; since $R(x, y)$ or $s(x) - s(x, y)$ is integrable in (x_0, x_1), these intervals τ

may be divided into a finite number of sub-intervals such that the sum of those sub-intervals in which the fluctuation of R is $\geqq$ an assigned number, is as small as we please. It thus appears that the intervals τ can be further sub-divided so that $\Sigma\tau = \Sigma\tau' + \Sigma\tau''$, where τ' are intervals in which the fluctuation of R for a fixed y, is $\geqq \alpha$, and the τ'' are intervals in which the fluctuation is $< \alpha$, where α is an arbitrarily chosen number; this can be done so that $\Sigma\tau'$ is arbitrarily small. Let $\alpha + \sigma < A$, then $\left|\int R(x, y)dx\right|$ through the intervals τ', is not greater than $B\Sigma\tau'$. Of the intervals τ'', some contain points of G_{y_1}, and others may not do so; let $\varkappa$ be the sum of the latter, then through these intervals the integral is not greater than $\varkappa B$. For any interval τ'' which contains a point of G_{y_1} $|R(x, y)|$ is everywhere less than $\sigma + \alpha$, where $y \overline{\overline{<}} y_1$; hence the integral through these intervals τ'' is $< (\sigma + \alpha)\Sigma\tau'' < A\Sigma\tau''$. It has now been shewn that

$$\left|\int_{x_0}^{x_1} R(x, y)dx\right| < (l - I - \varepsilon_2)(A + \eta) + B(I - I_{y_1} + \varepsilon_1 + \varepsilon_2) + B(\Sigma\tau' + \varkappa) + A\Sigma\tau''$$

where $A, y_1, \bar{y}$ are fixed, and ε_2 is arbitrarily small; y is $\leqq y_0$ where y_0 is the smaller of the numbers $y_1, \bar{y}$.

Thus the value of $\left|\int_{x_0}^{x_1} R(x, y)dx\right|$ is

$$< (A + \eta)(l - I + \Sigma\tau'') + B(I - I_{y_1} + \varepsilon_1) + B(\Sigma\tau' + \varkappa)$$

or, since $\Sigma\tau'$ is arbitrarily small,

$$< (A + \eta)(l - I + \Sigma\tau) + B(I - I_{y_1} + \varepsilon_1) + B\varkappa$$

$$< (A + \eta)(2l - I) + B(I - I_{y_1} + \varepsilon_1) + B\varkappa.$$

Now it has been shewn by Osgood, that y_1 may be chosen so small that $I - I_{y_1} < \lambda$, where λ is arbitrarily small; we have then also, $\varkappa < \lambda$. The integral is $< (A + \eta)2l + B(2\lambda + \varepsilon_1)$; let $A < \frac{p}{2l}$, and choose $\bar{y}$ so that $\eta < \frac{q}{2l}$, and y_1 so that $2B\lambda < r_1$, and let the θ' intervals be so chosen that $B\varepsilon_1 < s_1$, where p, q, r_1, s_1 are positive numbers such that $p + q + r_1 + s_1 = \varepsilon$. We now see that y_0 can be found such that $\left|\int_{x_0}^{x_1} R(x, y)dx\right| < \varepsilon$, if $y \overline{\overline{<}} y_0$; it has thus been established that the term by term integration of the series gives the same result as the integration of the sum $s(x)$ provided $s(x)$ is integrable through the interval of integration, and also the measure of non-uniform convergence is everywhere finite in that interval.