ADDITIONAL NOTE ON THE BOUNDARY BEHAVIOUR OF ELLIPTIC MODULAR FUNCTIONS.

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1. Let $\vartheta_3(0\,|\,\tau) = 1 + 2q + 2q^4 + 2q^9 + \cdots$, where $q = e^{i\pi\tau}$, and $\tau = x + iy$ is a complex variable. Let ξ be a quadratic surd, and let $\frac{p_n}{q_n}$ be the n-th convergent of the continued fraction for ξ. In a paper published recently in Vol. 52 of the Acta mathematica (p. 143—168) I gave an account of the asymptotic behaviour of $\vartheta_3(0\,|\,\tau)$ as $\tau \to \xi$ along the straight line $x = \xi$. The chief result consisted in proving the existence of an integer H for which the following theorem is true.

Theorem. *Suppose σ is a fixed number in the interval $0 \leqq \sigma < 1$ and R is a fixed integer in $1 \leqq R \leqq H$. Let*

$$\tau = \tau_n(\sigma) = \xi + iy = \xi + iy_n(\sigma),$$

where

$$y_n(\sigma) = \frac{\sigma}{q_n^2} + \frac{1-\sigma}{q_{n+1}^2}.$$

Let n tend to infinity by taking all values congruent to R (mod. H). Then $\sqrt[4]{y}\,\vartheta_3(0\,|\,\tau)$ tends to a finite limit.

Methods were also given for calculating the limits for all values concerned of σ and R.

2. The object of the present note is to point out that the above theorem is true (with a suitable H, whose existence can be proved) for some other classes of irrationals besides quadratic surds. The irrationals considered here are inte-

resting as they include some standard transcendental numbers like e, e^2 etc. As this note is essentially a sequel I shall assume (to save lengthy explanations) that this will be read along with the paper in the Acta mathematica cited above. I shall denote the latter shortly by an asterisk (*). The Lemmas 1, 2, 3, ... 8, the equations (7), (13), (15) and the congruences (17) referred to below are the ones that occur in (*).

Let us consider an irrational (not necessarily a quadratic surd) and let its expression as continued fraction be[1]

(i) $$[c_1, c_2, \dots c_n, \dots].$$

Let $\mathfrak{c}_n$ be the integer satisfying the conditions $1 \leqq \mathfrak{c}_n \leqq 8$, $\mathfrak{c}_n \equiv c_n \pmod{8}$. Let the continued fraction (i) be such that the sequence

$$\mathfrak{c}_1, \mathfrak{c}_2, \dots \mathfrak{c}_n, \dots$$

is periodic[2] with period r. We shall call such a continued fraction and the irrational represented by it, *residually periodic* (mod. 8) with period r. For these irrationals we can put in evidence the periodicity of the $\mathfrak{c}$-sequence and write it in the form

(ii) $$\mathfrak{b}_1, \mathfrak{b}_2, \dots \mathfrak{b}_m, \mathfrak{a}_1, \mathfrak{a}_2, \dots \mathfrak{a}_r, \mathfrak{a}_1, \mathfrak{a}_2, \dots \mathfrak{a}_r, \dots$$

the $\mathfrak{b}$'s forming the non-recurring part, and the $\mathfrak{a}$'s the recurring part. Now with reference to (ii) we can introduce the Γ-notation[3]; m and r being respectively the number of $\mathfrak{b}$'s and the number of $\mathfrak{a}$'s in (ii) and $\gamma_1, \gamma_2, \dots \gamma_n, \dots$ being any sequence we define

$$\Gamma_s^{(j)} = \gamma_{m+(j-1)r+s} \qquad \begin{pmatrix} s = 1, 2, \dots r \\ j = 1, 2, \dots \end{pmatrix}.$$

To indicate the relation between the γ's and Γ's we write as in (*) $\{\gamma_n\} = \{\Gamma_s^{(j)}\}$. With this notation let $\{c_n\} = \{C_s^{(j)}\}$.

3. An examination of the proofs of Lemmas 3 to 8 shows that these lemmas hold for all irrationals which are residually periodic (mod. 8). The numbers c_n, p_n, q_n which occur in the lemmas have now, of course, reference to the con-

[1] The notation is the same as that used in (*), p. 145, third foot-note.

[2] For the definition of a periodic sequence, see (*), p. 155.

[3] See (*), p. 154, 155.

tinued fraction of a residually periodic irrational[1]; the numbers m, r wherever they occur, should be taken to denote respectively the number of $\mathfrak{b}$'s and the number of $\mathfrak{a}$'s in (ii); and wherever the Γ-notation is used it should be considered to have reference to the sequence (ii). The proofs of Lemmas 3 to 8 will then be seen to hold verbally for residually periodic irrationals, with a few modifications of a minor character. The changes required are as follows.

On page 156 of (*), instead of the equations

$$\begin{cases} P_3^{(h)} = a_3 P_2^{(h)} + P_1^{(h)} \\ P_3^{(k)} = a_3 P_2^{(k)} + P_1^{(k)}, \end{cases}$$

we should write

$$\begin{cases} P_3^{(h)} = C_3^{(h)} P_2^{(h)} + P_1^{(h)} \\ P_3^{(k)} = C_3^{(k)} P_2^{(k)} + P_1^{(k)}, \end{cases}$$

and in addition to the congruences (17) use the fact that

$$C_3^{(h)} \equiv C_3^{(k)} \pmod{8},$$

and so deduce

$$P_3^{(h)} \equiv P_3^{(k)}.$$

The rest of the proof of Lemma 3 proceeds as before.

The second alteration is that the equation

$$c_{h+1} = c_{k+1}$$

on page 161 of (*) should be replaced by the congruence

$$c_{h+1} \equiv c_{k+1} \pmod{8}.$$

The rest of the proof of Lemma 7 remains unchanged.

Taking into account these alterations, it is easily seen that the arguments used in (*) establish the existence of the numbers H, Ω, $\mathfrak{m}$ (of Lemmas 7 and 8) for a residually periodic irrational.

4. Since Lemmas 3 to 8 hold for all residually periodic irrationals, it follows that the main theorem of (*) will be valid for such of these irrationals as satisfy the requirements of Lemmas 1 and 2 also (with a suitable g). The

[1] When we speak of residually periodic irrationals in this note the modulus concerned is always 8; so that we will drop the phrase »(mod. 8)» for simplicity.

irrationals ξ which we propose to consider in this note are subject to the following conditions:

1°. They are residually periodic (mod. 8) with period r.

2°. If g is the least common multiple of 2 and r, then for each fixed ϱ in $1 \leqq \varrho \leqq g$, the sequence

$$c_\varrho, c_{\varrho+g}, c_{\varrho+2g}, c_{\varrho+3g}, \ldots$$

behaves in one of two ways[1]: namely, either the terms of the sequence, from a certain point onwards, retain a constant value; or the sequence tends to infinity.

We shall describe these two conditions shortly as conditions ϖ.

With the g defined above we will prove presently the truth of Lemma 1 for irrationals satisfying conditions ϖ; that is to say, if p_n, q_n refer to the continued fraction of such an irrational ξ, and if ϱ is any fixed integer in $1 \leqq \varrho \leqq g$ and n tends to infinity through integers congruent to ϱ (mod. g), then $\frac{q_{n-1}}{q_n}$ and $p_n q_n - \xi q_n^2$ tend to finite limits. We shall denote these limits (as in (*)) respectively by[2] L_ϱ, $\varDelta_\varrho$.

5. Now Lemma 2 was substantially a deduction from Lemma 1; but in effecting this deduction in (*) we implicitly made certain assumptions, which were, no doubt, obviously true in the case of quadratic surds. The assumptions were, firstly, that $\varDelta_\varrho^2 + J_{\varrho+1}^2$ which occurs as the coefficient of $\mathfrak{X}_\varrho(\sigma)$, $\mathfrak{Y}_\varrho(\sigma)$ in equations (15) is never zero; and secondly, that L_ϱ which occurs as the denominator in the term $\frac{\varDelta_{\varrho-1}\varDelta_\varrho}{L_\varrho}$ in the first of the equations (15) is never zero. In the case of quadratic surds we had always $L_\varrho > 0$, $\varDelta_\varrho > 0$, and so also $\varDelta_\varrho^2 + J_{\varrho+1}^2 > 0$. Therefore the assumptions were then justified. For the irrationals we consider here it will be seen that some of the numbers L_ϱ, $\varDelta_\varrho$ are zero; and the assumptions mentioned above require consideration. The difficulty caused by the first assumption is easily disposed of by taking σ to lie in the interval $0 < \sigma \leqq 1$ instead of in the interval $0 \leqq \sigma < 1$. This change is of no significance (and could indeed have been made in the case of quadratic surds also); the main point being that in the interval for σ one of the end points 0, 1 should be included, and the other

[1] The behaviour need not be the same for two different values of ϱ.

[2] We shall also have by definition (as in (*)), $\varDelta_0 = \varDelta_g$, $L_{g+1} = L_1$.

excluded. When we take σ to be in $0 < \sigma \leqq 1$, we will have $J_{\varrho+1} \geqq \sigma > 0$, and so $\varDelta_\varrho^2 + J_{\varrho+1}^2 > 0$. As regards the second assumption, it will be seen from what follows that, in the case we are considering here, whenever L_ϱ vanishes, $\varDelta_{\varrho-1}$ will also simultaneously vanish; so that the term $\frac{\varDelta_{\varrho-1}\varDelta_\varrho}{L_\varrho}$ takes an indeterminate form. If now we refer to equation (13) from which the first of the equations (15) was derived, we see that the term $\frac{\varDelta_{\varrho-1}\varDelta_\varrho}{L_\varrho}$ is contributed by

$$\frac{q_n}{q_{n-1}}(p_{n-1}q_{n-1} - \xi q_{n-1}^2)(p_n q_n - \xi q_n^2);$$

so that, if we prove (in addition to Lemma 1) the existence of the limit of

$$\frac{q_n}{q_{n-1}}(p_{n-1}q_{n-1} - \xi q_{n-1}^2),$$

the difficulty caused by the second assumption would also be got over. We are thus led to consider the following lemma which we proceed to prove.

6. **Lemma 1-a.** *Suppose that ξ is an irrational satisfying the conditions ϖ, and that ϱ is a fixed integer in $1 \leqq \varrho \leqq g$. Let n tend to infinity through integers congruent to ϱ (mod. g). Then*

$$\frac{q_{n-1}}{q_n}, \qquad p_n q_n - \xi q_n^2, \qquad \frac{q_n}{q_{n-1}}(p_{n-1}q_{n-1} - \xi q_{n-1}^2)$$

tend to finite limits.

If the sequence formed by all the partial quotients c_n is bounded, it is seen from the conditions ϖ that ξ is a quadratic surd. We may therefore leave aside this case (as it has been already considered), and suppose that there is at least one integer s in $1 \leqq s \leqq g$, for which the sequence

$$c_s, \ c_{s+g}, \ c_{s+2g}, \ldots$$

tends to infinity. We shall denote the last written sequence by Σ_s.

Let us first consider the behaviour of $\frac{q_{n-1}}{q_n}$ and f_n, where as in (*)

(iii) $$f_n = [c_n, c_{n+1}, c_{n+2}, \ldots].$$

There are two cases to be examined according as[1] $c_n (n \equiv \varrho)$ tends to infinity or ultimately retains a constant value. In the first case, it is easily seen from the identity

$$\frac{q_{n-1}}{q_n} = [0, c_n, c_{n-1}, \ldots c_2]$$

that $\frac{q_{n-1}}{q_n} \to 0$; and also from (iii) it follows that $f_n \to \infty$. In the second case, the sequence of partial quotients in the continued fraction of $\frac{q_{n-1}}{q_n}$ begins (when n is sufficiently large) with a series of constants

$$0, \mathfrak{d}_1, \mathfrak{d}_2, \ldots \mathfrak{d}_{\mathfrak{h}},$$

and then follows a term of a sequence Σ_s which tends to infinity. The constants $\mathfrak{d}$ and their number $\mathfrak{h} \geqq 1$ depend only on ϱ and not on n. It therefore follows that

$$\frac{q_{n-1}}{q_n} \to [0, \mathfrak{d}_1, \mathfrak{d}_2, \ldots \mathfrak{d}_{\mathfrak{h}}],$$

the limit on the right being different from zero. Similarly when $c_n (n \equiv \varrho)$ ultimately retains a constant value, the sequence of partial quotients in the continued fraction of f_n begins (when n is sufficiently large) with a series of constants[2]

$$\mathfrak{e}_1, \mathfrak{e}_2, \ldots \mathfrak{e}_{\mathfrak{j}},$$

and then follows a term of a sequence[3] Σ_s which tends to infinity. The constants $\mathfrak{e}$ and their number $\mathfrak{j} \geqq 1$ depend only on ϱ and not on n. We therefore conclude as before that

$$f_n \to [\mathfrak{e}_1, \mathfrak{e}_2, \ldots \mathfrak{e}_{\mathfrak{j}}],$$

the limit on the right being different from zero.

We have thus proved that in any case $\frac{q_{n-1}}{q_n}$ tends to a finite limit which may be zero; and that f_n tends to infinity or to a finite limit different from zero. These results are true for every fixed ϱ in $1 \leqq \varrho \leqq g$.

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[1] To avoid constant repetition we shall understand that throughout the proof of the present Lemma the values through which n tends to infinity are all congruent to ϱ (mod. g).

[2] Clearly $\mathfrak{d}_1 = \mathfrak{e}_1$.

[3] This sequence need not be the same as the sequence Σ_s last mentioned.

Now by (7) we have

$$\frac{1}{p_n q_n - \xi q_n^2} = \eta_n f_{n+1} + \eta_n \frac{q_{n-1}}{q_n} = \eta_\varrho \left(f_{n+1} + \frac{q_{n-1}}{q_n} \right).$$

From what has been said above, it follows that there are only two alternatives to consider; if ϱ is such that $f_{n+1} \to \infty$, then $p_n q_n - \xi q_n^2 \to 0$; while, if ϱ is such that f_{n+1} tends to a finite limit, then $f_{n+1} + \frac{q_{n-1}}{q_n}$ tends to a limit different from zero, and so $p_n q_n - \xi q_n^2$ tends to a finite limit.

It now remains to consider

$$\text{(iv)} \qquad \frac{q_n}{q_{n-1}} (p_{n-1} q_{n-1} - \xi q_{n-1}^2).$$

If ϱ is such that $\frac{q_{n-1}}{q_n}$ tends to a limit different from zero, then from the results proved above we see that (iv) tends to a finite limit. We will now suppose that ϱ is such that

$$\text{(v)} \qquad \frac{q_{n-1}}{q_n} \to 0.$$

This happens (as indicated above) only when $c_n \to \infty$. In this case clearly

$$\frac{q_{n-1}}{q_n} \cdot c_n \to 1,$$

$$\frac{f_n}{c_n} \to 1,$$

and so

$$\text{(vi)} \qquad f_n \cdot \frac{q_{n-1}}{q_n} \to 1.$$

Now by (7) (with $n-1$ in place of n) we have

$$\frac{q_n}{q_{n-1}} (p_{n-1} q_{n-1} - \xi q_{n-1}^2) = \frac{q_n}{q_{n-1}} \cdot \frac{\eta_{\varrho-1} q_{n-1}}{f_n q_{n-1} + q_{n-2}} = \frac{\eta_{\varrho-1}}{f_n \cdot \frac{q_{n-1}}{q_n} + \frac{q_{n-2}}{q_{n-1}} \cdot \frac{q_{n-1}}{q_n}}.$$

Since $\frac{q_{n-2}}{q_{n-1}} < 1$, we obtain on using (v) and (vi) in the last equation,

(vii) $$\frac{q_n}{q_{n-1}}(p_{n-1}\,q_{n-1} - \xi\, q_{n-1}^2) \to \eta_{\varrho-1};$$

the lemma is therefore completely proved.

Denoting as in (*) the limits of $\frac{q_{n-1}}{q_n}$, $p_n\,q_n - \xi\, q_n^2$ by L_ϱ, $\varDelta_\varrho$ we see that a consequence of the existence of the limit (vii) is that, when $L_\varrho = 0$, we also have $\varDelta_{\varrho-1} = 0$.

7. After having proved Lemma 1-a, there is no difficulty in seeing that the arguments of paragraph 6 of (*), which constitute the proof of Lemma 2, hold substantially for irrationals ξ which satisfy the conditions ϖ. The only modifications are that instead of having $0 \leqq \sigma < 1$ we should have $0 < \sigma \leqq 1$; and in the special case when $L_\varrho = 0$, the first of the equations (15), which defines $\mathfrak{X}_\varrho(\sigma)$ should be replaced by

(viii) $$\mathfrak{X}_\varrho(\sigma)\,(\varDelta_\varrho^2 + J_{\varrho+1}^2) = \varDelta_\varrho.$$

It is also worth observing that when $L_{\varrho+1} = 0$, $L_\varrho \geqq 0$, the equations for $\mathfrak{X}_\varrho(\sigma)$, $\mathfrak{Y}_\varrho(\sigma)$ take the simple form[1]

$$\begin{cases} \mathfrak{X}_\varrho(\sigma) = \eta_{\varrho+1}\,L_\varrho \\ \mathfrak{Y}_\varrho(\sigma) = \dfrac{1}{\sigma}. \end{cases}$$

As all the lemmas of (*) have now been shown to be valid for irrationals satisfying conditions ϖ, it follows that the main theorem of (*) is also true for such irrationals, with the understanding that the interval for σ should be changed in the manner indicated, and that when $L_\varrho = 0$, the real part of $\mathfrak{T}_\varrho(\sigma) = \mathfrak{X}_\varrho(\sigma) + i\,\mathfrak{Y}_\varrho(\sigma)$ should be taken to be defined by (viii).

8. Among irrationals satisfying conditions ϖ, there are some standard transcendental numbers, for example[2]:

$$\begin{aligned} e &= [2, \overline{1, 2+2\nu, 1}]_{\nu=0}^{\infty} \\ &= [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \ldots] \end{aligned}$$

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[1] Use is made of the result (mentioned above) that when $L_{\varrho+1} = 0$, we also have $\varDelta_\varrho = 0$.

[2] See O. PERRON, Die Lehre von den Kettenbrüchen (1913), p. 134, 138. On pages 132—138 of this book will be found further examples of transcendental numbers related to e and satisfying conditions ϖ.

$$e^2 = [7, \overline{2+3\nu, 1, 1, 3+3\nu, 18+12\nu}]_{\nu=0}^{\infty}$$
$$= [7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, 42, \ldots].$$

It may be of interest to set down the numerical details when $\xi = e$. In this case $r = 12$, $g = 12$, and by calculation it is found that $H = 24$. The values of λ, ϱ, Ω, $\mathfrak{m}$ corresponding to $R = 1, 2, 3, \ldots 24$ are given in the first table below; and the values of L_ϱ, $\varDelta_\varrho$, $J_{\varrho+1}(\sigma)$, $\mathfrak{X}_\varrho(\sigma)$, $\mathfrak{Y}_\varrho(\sigma)$ corresponding to $\varrho = 1, 2, 3, \ldots 12$ are given in the second table. It is found on calculation that if $\varrho' - \varrho = 6$, then $L_\varrho = L_{\varrho'}$, $\varDelta_\varrho = \varDelta_{\varrho'}$ etc.; and so the pairs of values (1, 7), (2, 8), ... (6, 12) of ϱ are entered together in the first column of the second table. When y tends to zero in the manner described in the enunciation of the main theorem of (*) — with, of course, the modification in the interval of σ — the limit of $\sqrt[4]{y}\,\vartheta_3(0|\tau)$ for given values of R, σ can be easily read off from these two tables.[1]

R	ϱ	λ	$\mathfrak{m}$	$\Omega\sqrt{2}$	R	ϱ	λ	$\mathfrak{m}$	$\Omega\sqrt{2}$
1	1	$-i$	3	$1+i$	13	1	$-i$	3	$-1-i$
2	2	1	2	$\sqrt{2}$	14	2	1	2	$-\sqrt{2}$
3	3	$-i$	4	$-1+i$	15	3	$-i$	4	$1-i$
4	4	1	3	$i\sqrt{2}$	16	4	1	3	$-i\sqrt{2}$
5	5	$-i$	2	$-1+i$	17	5	$-i$	2	$1-i$
6	6	1	4	$-i\sqrt{2}$	18	6	1	4	$i\sqrt{2}$
7	7	$-i$	3	$1-i$	19	7	$-i$	3	$-1+i$
8	8	1	2	$-i\sqrt{2}$	20	8	1	2	$i\sqrt{2}$
9	9	$-i$	4	$-1-i$	21	9	$-i$	4	$1+i$
10	10	1	3	$-\sqrt{2}$	22	10	1	3	$\sqrt{2}$
11	11	$-i$	2	$-1-i$	23	11	$-i$	2	$1+i$
12	12	1	4	$-\sqrt{2}$	24	12	1	4	$\sqrt{2}$

[1] The $\sqrt{2}$ in the first table denotes the positive root.

ϱ	L_ϱ	Λ_ϱ	$J_{\varrho+1}(\sigma)$	$\mathfrak{X}_\varrho(\sigma)$	$\mathfrak{Y}_\varrho(\sigma)$
1, 7	1	$-\frac{1}{2}$	$\frac{1}{4}(1+3\sigma)$	$\frac{(1+3\sigma)^2-4}{(1+3\sigma)^2+4}$	$\frac{4(1+3\sigma)}{4+(1+3\sigma)^2}$
2, 8	$\frac{1}{2}$	0	σ	$-\frac{1}{2}$	$\frac{1}{\sigma}$
3, 9	0	$-\frac{1}{2}$	1	$-\frac{2}{5}$	$\frac{4}{5}$
4, 10	1	$\frac{1}{2}$	$\frac{1}{4}(1+3\sigma)$	$\frac{4-(1+3\sigma)^2}{4+(1+3\sigma)^2}$	$\frac{4(1+3\sigma)}{4+(1+3\sigma)^2}$
5, 11	$\frac{1}{2}$	0	σ	$\frac{1}{2}$	$\frac{1}{\sigma}$
6, 12	0	$\frac{1}{2}$	1	$\frac{2}{5}$	$\frac{4}{5}$