INVARIANTS ASSOCIATED WITH SINGULARITIES OF ALGEBRAIC CURVES.

BY

TEMPLE RICE HOLLCROFT

of AURORA, NEW YORK.

1. **Introduction.** Each singularity of an algebraic curve f, with the exception of distinct nodes, cusps, bitangents and stationary tangents, is associated with two distinct sets of invariants.[1] One set, in which the number of invariants is denoted by I_p, consists of the invariants among the coefficients of the equation in point coordinates of the curve f; the other set, in which the number is I_l, consists of the invariants among the coefficients in the line equation of f. The existence of both sets of invariants is necessary and sufficient for f to possess the designated singularity. Both I_p and I_l are independent of the order and class of f. The value of I_l for any given singularity is the same as the value of I_p for the reciprocal of this singularity.

An algebraic singularity, therefore, uniquely determines the two numbers I_p and I_l defined above. In this paper, the values of both I_p and I_l are found for a general algebraic singularity considered as defined by its constituent multiple points and their manner of combination. The chief problem is to find the value of I_l for a singularity so defined, that is, to determine the number of invariants among the coefficients of the equation of f in point coordinates associated with a general line singularity.

It has been proved by Lefschetz[2] that each node of f accounts for one invariant and his *Postulate of Singularities* states that a cusp of f always accounts

[1] The term »invariant» is used in this paper to mean an independent function of the coefficients of the equation of f whose vanishing is necessary in order that f possess a certain singularity.

[2] S. Lefschetz, On the existence of loci with given singularities, Transactions of the American Mathematical Society, Vol. 14 (1913), pp. 23—41.

for two invariants. The reciprocals of these statements will be used in studying the reciprocal singularity of a given singularity.

2. **Definitions.** Algebraic singularities may be divided into two general types, simple and compound. Simple point (line) singularities are built up of double points (lines) and contain no double lines (points). They are ordinarily called multiple points (lines). Compound singularities are those that contain both double points and double lines. Multiple points are formed by the coincidence of nodes and contain only those nodes which are created by the necessary crossings of the branches of the multiple point. Multiple lines are the reciprocals of multiple points.

Compound singularities result when certain multiple points of a curve become consecutive, that is, move into coincidence along given directions. The simplest example of this is the tacnode which consists of two consecutive nodes and two consecutive bitangents. Multiple points in becoming consecutive always involve the consecution of multiple lines, and conversely.

Another method of forming either simple or compound singularities, superposition, will be used in this paper. Two or more singular points are said to be superposed when they coincide so that the nature of each component singular point is not affected. The resulting singularity is the same as if the component singular points were lifted out of the plane successively and superposed without altering the relative positions of the branches within each singular point.

3. **Invariants associated with multiple points.** The postulation of a simple multiple point of order r is $r(r+1)/2$ and the number of invariants associated with it is $r(r+1)/2-2$. The multiple point contains $r-1$ loops each of which may vanish causing a node in the singularity to be replaced by a cusp.[1] For the most general multiple point of order r with $k \leq r-1$ cusps

$$I_p = r(r+1)/2 + k - 2.$$

[1] By a vanishing loop is meant a loop that disappears because and only because the tangents to its two branches become coincident. Some of the loops of the curve at a singularity may be imaginary corresponding to a certain number of imaginary nodes or acnodes among the nodes composing the singularity. Since the limiting form of an acnode as the two conjugate imaginary tangents approach coincidence is a real cusp and since the tangents of a pair of imaginary nodes may coincide respectively to form a pair of imaginary cusps, the *number* of invariants associated with a singularity is unaltered by the reality of its component double elements and therefore no distinction need be made as to the reality of these double elements.

Any number j of multiple points of orders r_i, $i=1, 2, \ldots, j$, may be superposed to form a multiple point of order Σr_i which will have as many distinct tangents as the sum of the distinct tangents of the component multiple points. A multiple point thus formed contains as many cusps as the sum of the numbers occurring in the component multiple points and no more, since only nodes are added by superposition. A multiple point formed by superposition of order $r=\Sigma r_i$ containing $k=\Sigma k_i$ cusps is the same as if it had been formed by the coincidence of $r(r-1)/2-k$ nodes and k cusps for which the value of I_p is given above.

If the j multiple points are superposed in such a way that any tangent of one coincides with any tangent of another, the resulting singularity is compound and will be discussed in the next section.

4. **Invariants associated with compound singularities.** A compound singularity on an algebraic curve f may be formed by the consecution of s multiple points of f each of order r_i. Assume that $r_1 \geq r_2 \geq r_3 \geq \cdots \geq r_s$. The component multiple point of highest order r_1 is called the principal point of the singularity and the order of the principal point is the order of the resulting singularity.

Enriques[1] has proved that the postulation of s consecutive points of orders r_i on f is the same as the total postulation of the s multiple points considered as distinct on f, that is

$$\frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1).$$

Assume that s points are consecutive on a curve. If one of the points P is fixed, one condition determines the direction of approach to P of each of the remaining $s-1$ points. The two parameters defining P and the $s-1$ parameters determining the $s-1$ directions through P total $s+1$ parameters. Therefore the number of independent parameters involved in the location of s consecutive points on any plane curve is $s+1$.

The postulation of a singularity is the total number of conditions necessary and sufficient to determine both the nature of the singularity and its position. The $\Sigma r_i(r_i+1)/2$ relations among the coefficients of f then involve the $s+1$

[1] F. Enriques, Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, Vol. 2, pp. 404—408.

parameters which determine the positions of the s consecutive points. From these $\Sigma r_i(r_i+1)/2$ relations the $s+1$ parameters may be eliminated in $\Sigma r_i(r_i+1)/2-s-1$ independent ways, each eliminant being an invariant associated with the singularity. Then for s consecutive multiple points of orders r_i, there results

$$I_p=\frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1)-s-1\,.$$

Since a compound singularity of order r at P involves r branches of f through P, these branches will form $r-1$ loops. When a loop vanishes, the node adjacent to the loop becomes a cusp. The introduction of each cusp increases by unity the value of I_p associated with the singularity.

The most general singularity that can occur on an algebraic plane curve consists of s principal r_1-fold point P to which j series of multiple points have become consecutive along distinct sets of branches of the principal point and, moreover, such that any point P_i of any series may have j_i series of multiple points consecutive along distinct sets of branches of P_i. For any singularity of this nature, the theorem of Enriques still holds.

In any such series of consecutive points, one condition is necessary and sufficient to determine the position of each point consecutive to P whether along the same or distinct sets of branches. The positions of the s consecutive points are, therefore, in the most general case, determined by $s+1$ conditions.

The most general algebraic singularity of order r_1 contains r_1 branches and therefore r_1-1 loops any of which may vanish and introduce a cusp which replaces a constituent node.

Finally, the most general algebraic singularity of order r_1 consisting of s consecutive r_i-fold points which have become consecutive in the general way described and containing $k\leq r_1-1$ cusps gives rise to the following number of invariants

$$I_p=\frac{1}{2}\Sigma r_i(r_i+1)+k-s-1\,.$$

Compound singularities are formed by superposition when multiple points are superposed so as to have a common tangent. Any number j of multiple points of orders r_i can be superposed so that all have a common tangent. The resulting singularity is of order Σr_i and consists of a principal point of order Σr_i and a consecutive j-fold point. The value of I_p is obtained by regarding the singularity as formed by the consecution of two multiple points of orders Σr_i and j.

Any number of compound singularities may be superposed so that two or more have a common tangent which is a multiple tangent for any or all of the points concerned. As above, the value of I_p for the resulting singularity is found by considering that singularity as formed by consecution.

A compound singularity may be a singular point some of whose tangents are multiple tangents. A tangent at a multiple point P may be a multiple tangent of any order q with simple contact at P or with any number of points of contact up to and including q located at P. In this case, the singularity accounts for one more invariant than the sum of the invariants belonging to the two singularities when distinct, because if either the point or line is given, the other has but one degree of freedom. The simplest example of such a singularity is the flecnode, for which $I_p = 2$.

5. **The determination of I_l for a given singularity.** The number I_l of invariants among the coefficients of the equation of f in line coordinates which express the condition that f possess a given singularity is identical with the number of invariants among the coefficients of the equation of f in point coordinates which express the condition that f possess the reciprocal of this given singularity. This follows from the principle of duality.

Nodes and cusps do not account for invariants among the coefficients of the line equation of a curve just as bitangents and stationary tangents account for no invariants among the coefficients of its point equation.

The following theorem will be useful in determining the value of I_l for a singularity whose point constituents are known:

Theorem I. *If among the coefficients of the equation in point coordinates of an algebraic curve f there exist I_p' invariants in addition to those accounted for by the nodes and cusps of f, then among the coefficients of the equation of f in line coordinates there must exist I_l' invariants in addition to those associated with the bitangents and stationary tangents of f, such that $I_p' = I_l'$.*

Consider and algebraic curve f of order n, class m, with δ nodes, $\varkappa$ cusps, τ bitangents and ι inflections. Let there be I_p' invariants among the coefficients of the point equation of f which vanish in addition to the $\delta + 2\varkappa$ invariants due to the double points and I_l' invariants among the coefficients of the line equation of f which vanish in addition to the $\tau + 2\iota$ invariants due to the double tangents. To prove $I_p' = I_l'$.

Since the curve f has a definite number of degrees of freedom, the equations of f in point and line coordinates must contain the same number of independent coefficients. This fact is expressed by the equality

$$\frac{1}{2}n(n+3) - \delta - 2\varkappa - I_p' = \frac{1}{2}m(m+3) - \tau - 2\iota - I_l'.$$

From Plücker's equations, there results for any algebraic curve

$$\frac{1}{2}n(n+3) - \delta - 2\varkappa = \frac{1}{2}m(m+3) - \tau - 2\iota.$$

Subtracting this relation from the one above, we obtain

$$I_p' = I_l'.$$

This theorem readily determines the value of I_l for a multiple point of order r containing $k \leq r-1$ cusps. We have seen that such a singularity contains $r(r-1)/2 - k$ nodes and k cusps and that for it $I_p = r(r+1)/2 + k - 2$. The nodes and cusps contained in this r-fold point account for $r(r-1)/2 + k$ invariants. Subtracting this number from I_p, we obtain $r-2$ as the number of invariants belonging to the singularity in addition to those accounted for by the nodes and cusps contained in it. Assuming that this singularity belongs to a curve that has no further point singularities or line singularities of order greater than two, apply the above theorem and there results:

For any multiple point of order r containing $k \leq r-1$ cusps, that is, with or without consecutive tangents,

$$I_l = r - 2.$$

This theorem may also be used to determine the value of I_l for a singularity formed by superposition. Superpose j singularities, simple or compound, each occurring at a single point so that no tangent of one coincides with a tangent of another. A certain number of nodes are added by the superposition. Consider that all the j singularities are possessed by the same curve and determine what part of the increase in I_p due to superposition is over and above the increase due to added nodes.

Let $I_{p,i}$ and $I_{l,i}$ respectively represent the numbers of invariants associated with the j distinct singularities of orders r_i. Some or all of these singularities may be compound. The value of $I_{l,i}$ for a compound singularity will be obtained in the next section. Its value does not affect the determination of the *additional* invariants due to superposition.

When singular points are superposed, the nodes added are created by the intersections of the branches of the component singular points, each branch of each singular point intersecting every branch of every other singular point. The number of additional nodes, therefore, depends only on the orders of the component singular points. The number d of additional nodes is therefore the same for j multiple points of orders r_i as for j compound singular points of the same orders r_i. Then in determining d for any set of superposed singular points, we need consider only the principal points of the singularities involved. The j principal points of orders r_i form a single principal point of order Σr_i which contains all of the d nodes added by superposition. The number d is, therefore,

$$d=\frac{1}{2}\sum_{i=1}^{j} r_i\left(\sum_{i=1}^{j} r_i-1\right)-\frac{1}{2}\sum_{i=1}^{j} r_i(r_i-1).$$

The various series of multiple points consecutive to the component principal points are, after superposition, consecutive to the resulting principal point along mutually exclusive branches. These consecutive multiple points occur in the same way as before superposition, since, by the definition of the process of superposition, the nature of each component singular point is not affected by it. Therefore the number of invariants associated with the consecutive multiple points is unchanged by superposition. The difference in the number of invariants associated with the principal point of the singularity formed by superposition and the sum of the numbers associated with each of the component principal points considered as distinct is, then, the total number of invariants added by superposition. This number is

$$\frac{1}{2}\Sigma r_i(\Sigma r_i+1)-2-\frac{1}{2}\Sigma r_i(r_i+1)+2j=d+2(j-1).$$

This reusult shows that in addition to the d invariants belonging to the additional nodes, $2(j-1)$ invariants are added by the superposition of j singular points of orders $r_i \geq 2$.

Therefore a singularity formed by the superposition of j singular points of orders $r_i \geqq 2$ has

$$I_p = \Sigma I_{p,i} + d + 2(j-1)$$
$$I_l = \Sigma I_{l,i} + 2(j-1).$$

The expression for I_l results directly from Theorem I.

6. **The determination of I_l for a compound singularity.** As stated in section four, the most general algebraic singularity at a single point can be formed by the consecution of multiple points. This singularity is defined by the number, orders and manner of combination of its component multiple points. For such a singularity, I_p has been found in section four. The problem now is to determine the value of I_l for a singularity so defined.

Let the singularity consist of a principal point P of order r to which j series each containing s_i multiple points are consecutive along j mutually exclusive sets of branches through P. Any series may subdivide as described in section four. Let the point of each series adjacent to the principal point P be denoted by P_i of order r_i, $i = 1, 2, \ldots, j$.

If a series of consecutive points should subdivide at any point, this will have no effect on the order of the multiple tangent formed by that series. For example, if two or more series of multiple points are consecutive to P_1 along mutually exclusive sets of branches of P_1, this does not alter the fact that all the r_1 branches of P_1 must have contact with each other at P because P_1 is consecutive to P along r_1 branches of P. Then, whether any series of consecutive points subdivides or not, the resulting singularity contains j multiple tangents of orders r_i.

Assume that between each two adjacent sets of the j sets of r_i branches there are a_i distinct simple branches of the principal point such that $a_i \geqq 0$. The entering and retiring branches of f through P may belong to sets of simple branches so there may be $j+1$ sets of branches of f through P each containing a_i branches with distinct simple tangents. The order of the resulting singularity is, therefore,

$$\sum_{i=1}^{j} r_i + \sum_{i=1}^{j+1} a_i = r.$$

The same singularity can also be formed by the superposition of $j+1$ multiple points of orders a_i and j compound singular points of orders r_i alternately. Let the value of I_l for each of these j component compound singular points be denoted by $I_{l,i}$. Then by the method of the preceding section, by which the value of I_l for a singularity formed by superposition was obtained, there results for the complete singularity at P

$$I_l = \sum_{i=1}^{j} (I_{l,i} - r_i) + r + 2(j-1).$$

In the above, the computation of I_l for the most general singularity at a single point has been reduced to the computation of the values of $I_{l,i}$ each of which is associated with a compound singularity at a single point with one and only one distinct tangent.

The following theorem will complete the process:

Theorem II. *If a compound singular point has but one distinct tangent and contains neither cusps nor inflections, then for it*

$$I_l = I_p.$$

It has been proved by C. A. Scott[1] that if a singularity at a point has but one distinct tangent, the number of its latent nodes is equal to the number of its latent bitangents. By latent nodes [bitangents] is meant all the nodes [bitangents] contained in the singularity in addition to the $r(r-1)/2$ nodes [$q(q-1)/2$ bitangents] necessary to form the principal point [line] of order r [q] of the singularity.

Consider a compound singularity of order r_1 consisting of s consecutive multiple points of orders r_i such that $r_1 = r_2 \geq r_3 \geq \cdots \geq r_s$. The singularity has but one tangent which has contact with r_1 branches at the singular point. This tangent is therefore of order r_1. The number of latent nodes d' and latent bitangents t' is

$$d' = t' = \frac{1}{2} \sum_{i=2}^{s} r_i (r_i - 1)$$

---

[1] C. A. Scott, The nature and effect of singularities of plane algebraic curves, American Journal of Mathematics, Vol. 15 (1893), p. 235.

and the total number of nodes d and bitangents t contained in the singularity is, therefore,

$$d = t = \frac{1}{2}\sum_{i=1}^{s} r_i(r_i - 1).$$

This singularity is assumed to contain no cusps. It will therefore contain no inflections, since only bitangents are involved in the consecution of multiple points with distinct branches. This singularity is also assumed to be built up by a single series of consecutive multiple points, but the series may subdivide at any point or points without altering the final result.

Let I_p and I_l be the numbers of invariants associated with this singularity. Let n be the order, m the class and τ and ι the total number of bitangents and inflections respectively of an algebraic curve possessing this singularity and no other except double lines. By Plücker's equations, since $d = t$,

$$\begin{aligned} m &= n(n-1) - 2t \\ \iota &= 3n(n-2) - 6t \\ 2\tau &= 2t + (n^2 - 2n - 2t)(n^2 - 2t - 9). \end{aligned}$$

For the reason given in Theorem I

$$\frac{1}{2}n(n+3) - I_p = \frac{1}{2}m(m+3) - I_l - (\tau - t) - 2\iota.$$

Substitute in this equality, the values of m, τ and ι in terms of n and t, simplify and there results

$$I_p = I_l$$

thereby proving the theorem.

The value of I_l for the complete singularity has been defined by a formula which involves the numbers $I_{l,i}$. These numbers $I_{l,i}$ are associated with compound singular points each of which has but one distinct tangent. Then by Theorem II each $I_{l,i}$ may be replaced by its corresponding $I_{p,i}$ and the final formula obtained defining the value of I_l for the complete singularity:

$$I_l = \sum_{i=1}^{j}(I_{p,i} - r_i) + r + 2(j-1)$$

wherein

$$I_{p,i} = \frac{1}{2}\sum_{h=1}^{s_i} r_{i,h}(r_{i,h}+1) - s_i - 1$$

is the number of invariants accounted for by s_i consecutive multiple points of orders $r_{i,h}$ where i is fixed for any one series and $r_{i,1} = r_{i,2}$.

The value of I_t for the most general singularity at a point P is therefore given in terms of the order r of the singularity, the number j of distinct sets of branches along which one or more multiple points are consecutive to the principal point, the number of branches r_i in each set and the numbers $I_{p,i}$ each of which is the number of invariants accounted for by one of the j sets of r_i branches considered as a distinct singularity.

Compound singularities formed by the superposition of singular points such that tangents coincide have been discussed in section four. In every case, the singularity could have been formed by the consecution of multiple points and the value of I_p was so obtained. Likewise in each case the value of I_t can be obtained by treating the singularity as if formed by consecutive multiple points.

At the end of section four, singularities that consist of singular points some of whose tangents are multiple lines were discussed. Since no double points or lines are added by the coincidence, by Theorem I the values of I_p and I_t are each increased by the same amount which is unity for each such coincidence. For example, the value of I_t for a flecnode is three.

7. **The effect of cusps on the value of I_t.** So far, in determining the value of I_t for a compound singularity, it has been assumed that the singularity contains no cusps. The occurrence of cusps in a simple multiple point does not affect the value of I_t, but this does not hold for compound singularities.

Any singularity of order r at P contains r branches of f through P and these branches form $r-1$ loops. The maximum number of cusps that may occur as components of any singularity, simple or compound, of order r is, therefore, $r-1$.

In a compound singularity, both branches of certain loops have the same tangent. Such a loop must, therefore, contain a point of inflection. This loop vanishes when the node adjacent to it becomes a cusp, but when this occurs, the stationary tangent at the inflection on the loop coincides with the tangent to the singularity, replacing one of the bitangents. This occurs whenever a loop formed by branches involved in the consecution of the principal point with one or more multiple points vanishes. Then, since replacing a bitangent by a stationary tangent increases the value of I_l for that singularity by unity and since this occurs when each cusp is introduced, it follows that replacing a node by a cusp in a compound singularity (except, as noted below, in that part of the principal point not involved in the consecution) increases the value of I_l as well as that of I_p by unity.

When a cusp is introduced in the principal point replacing a node whose branches are not involved in the consecution of the principal point with one or more multiple points, I_p is increased by unity, but I_l is not affected just as in the case of a simple multiple point.

Wells College, Aurora, New York, U. S. A.