ON THE UNSYMMETRICAL TOP.

By

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1. The problem of the top has an extensive literature but it is a literature of special cases, the specialization arising either in (a) the initial conditions, (b) the form of the momental ellipsoid, (c) the position of the centroid, or (d) a combination of the three. To this can be added a number of articles whose aim is to show that for general initial conditions there can not be an additional algebraic relation among the components of the angular velocity, excepting in the special cases that have been solved. References to the older papers are given by Stäckel[1] and by Klein and Sommerfeld[2]; references to a number of more recent papers are given by Whittaker.[3] This paper gives a solution of a special case which is so simple that it seems worthy of notice. The solution is of the type considered by N. Kowalewski.[4]

2. Let I_1, I_2, and I_3 be the principal moments of inertia of the body for the lines OX_1, OY_1, OZ_1, which are fixed in the body, let $(0, 0, h)$ be the coordinates of the centroid referred to the body axes, let θ, φ, ψ be Euler's angles, and let X, Y, Z be axes fixed in space. The point O is the fixed point, gravity is the only extraneous force, and the mass is m. For convenience mg will be represented by w in the equations of motion. If we call $\omega_1, \omega_2, \omega_3$ the components of the angular velocity along the body axes, Euler's equations are

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[1] Encyklopädie der Mathematischen Wissenschaften, Bd. 4, S. 581.

[2] Theorie des Kreisels.

[3] Analytical Dynamics, third Ed. p. 166.

[4] Mathematische Annalen, Bd. 65.

$$I_1 \dot{\omega}_1 + (I_3 - I_2)\, \omega_2 \omega_3 = hw \sin\theta \cos\varphi, \tag{1}$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3)\, \omega_3 \omega_1 = -\, hw \sin\theta \sin\varphi, \tag{2}$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1)\, \omega_1 \omega_2 = 0, \tag{3}$$

the values of $\omega_1 \omega_2$, and ω_3 are

$$\omega_1 = \dot{\theta} \cos\varphi + \dot{\psi} \sin\theta \sin\varphi, \tag{4}$$

$$\omega_2 = -\dot{\theta} \sin\varphi + \dot{\psi} \sin\theta \cos\varphi, \tag{5}$$

$$\omega_3 = \dot{\varphi} + \dot{\psi} \cos\theta. \tag{6}$$

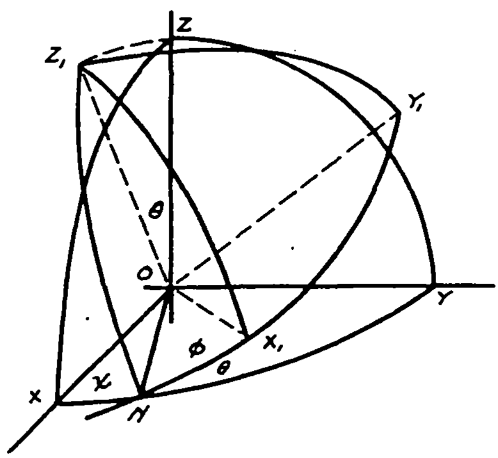

Fig. 1

In addition we have the two integrals, one of which states that the total energy is constant and the other that the projection of the angular momentum vector on the vertical line OZ is constant. They may be written

$$I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 - T = -\, 2\, wh \cos\theta, \tag{7}$$

$$I_1 \omega_1 \sin\theta \sin\varphi + I_2 \omega_2 \sin\theta \cos\varphi + I_3 \omega_3 \cos\theta = k, \tag{8}$$

k and T being the new constants.

3. From (1) and (3) we have

$$\sin\theta \cos\varphi = \frac{\omega_2}{h\, w\, I_3\, d\, \omega_3} [I_1 (I_1 - I_2)\, \omega_1\, d\omega_1 + I_3 (I_3 - I_2)\, \omega_3\, d\omega_3],$$

and from (2) and (3)

$$\sin\theta\sin\varphi = \frac{-\omega_1}{h\,w\,I_3\,d\,\omega_3}\left[I_2(I_1-I_2)\,\omega_2\,d\omega_2 + I_3(I_1-I_3)\,\omega_3\,d\omega_3\right].$$

Substituting these values and the value of $\cos\theta$ as given by equation (7) in equation (8) there results

$$\frac{-I_1\,\omega_1^2}{h\,w\,I_3\,d\,\omega_3}\left[I_2(I_1-I_2)\,\omega_2\,d\omega_2 + I_3(I_1-I_3)\,\omega_3\,d\omega_3\right] +$$

$$\frac{I_2\,\omega_2^2}{h\,w\,I_3\,d\,\omega_3}\left[I_1(I_1-I_2)\,\omega_1\,d\omega_1 + I_3(I_3-I_2)\,\omega_3\,d\omega_3\right] +$$

$$\frac{I_3\,\omega_3}{2\,w\,h}\left[T - I_1\,\omega_1^2 - I_2\,\omega_2^2 - I_3\,\omega_3^2\right] = k. \tag{9}$$

A second differential equation is obtained from the fact that

$$\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta = 1\text{; i.e.}$$

$$\frac{\omega_2^2}{h^2\,w^2\,I_3^2\,\overline{d\omega_3}^2}\left[I_1(I_1-I_2)\,\omega_1\,d\omega_1 + I_3(I_3-I_2)\,\omega_3\,d\omega_3\right]^2 +$$

$$\frac{\omega_1^2}{h^2\,w^2\,I_3^2\,\overline{d\omega_3}^2}\left[I_2(I_1-I_2)\,\omega_2\,d\omega_2 + I_3(I_1-I_3)\,\omega_3\,d\omega_3\right]^2 +$$

$$\frac{1}{4\,w^2\,h^2}\left[T - I_1\,\omega_1^2 - I_2\,\omega_2^2 - I_3\,\omega_3^2\right]^2 = 1. \tag{10}$$

Equations (9) and (10) may be simplified by the substitution

$$u = I_2(I_1-I_2)\,\omega_2^2 + I_3(I_1-I_3)\,\omega_3^2,$$

$$v = I_1(I_1-I_2)\,\omega_1^2 + I_3(I_3-I_2)\,\omega_3^2.$$

They then become

$$-\left[v - I_3(I_3-I_2)\,\omega_3^2\right]du + \left[u - I_3(I_1-I_3)\,\omega_3^2\right]dv +$$

$$I_3^2\,\omega_3\left[T(I_1-I_2) - (v+u)\right]d\omega_3 = 2\,k\,h\,w\,I_3(I_1-I_2)\,d\omega_3, \tag{11}$$

and

$$I_2\left[v - I_3(I_3-I_2)\,\omega_3^2\right]\overline{du}^2 + I_1\left[u - I_3(I_1-I_3)\,\omega_3^2\right]\overline{dv}^2 +$$

$$\frac{I_1\,I_2\,I_3^2}{I_1-I_2}\left[T(I_1-I_2) - (v+u)\right]^2\overline{d\omega_3}^2 = 4\,h^2\,w^2\,I_1\,I_2\,I_3^2\,(I_1-I_2)\,\overline{d\omega_3}^2, \tag{12}$$

Equations (11) and (12) define u and v in terms of ω_3 and our problem is to find such relations among the variables as will satisfy these equations.

4. It may be noted that $I_1 v + I_2 u$ is equal to $I_1 - I_2$ times the square of the angular momentum. Schiff[1] has mentioned the possibility of having a motion of such a nature that the angular momentum is preserved and the problem is also discussed by Stäckel[2] and others. It means, of course, that the body must move in such a way that the vector representing the torque is perpendicular to the vector representing the angular momentum.

As a first effort at obtaining a particular solution let us assume

$$I_1 v + I_2 u = I_1 I_2 C_0 + I_1 I_2 C_1 \omega_3^2 \qquad \text{or}$$

$$v = \frac{-I_2}{I_1} u + C_0 I_2 + C_1 I_2 \omega_3^2. \tag{13}$$

This value of v substituted in equation (11) gives

$$\frac{-I_2}{I_1} u\,du + 2\,C_1 I_2 \omega_3 u\,d\omega_3 + \frac{I_2}{I_1} u\,du - C_0 I_2\,du - C_1 I_2 \omega_3^2\,du +$$

$$I_3 \omega_3^2 \left[\frac{I_3}{I_1}(I_1 - I_2)\,du + (I_3 - I_1)\,2\,C_1 I_2 \omega_3\,d\omega_3\right] +$$

$$\left[T(I_1 - I_2) - \frac{I_1 - I_2}{I_1} u - C_0 I_2 - C_1 I_2 \omega_3^2\right] I_3^2 \omega_3\,d\omega_3 = 2\,k h w\, I_3 (I_1 - I_2)\,d\omega_3.$$

By taking

$$C_1 = \frac{I_3^2 (I_1 - I_2)}{2\,I_1 I_2}$$

the coefficient of $\omega_3 u\,d\omega_3$ becomes zero, and by taking[3]

$$k = 0 \text{ and } C_0 = \frac{T I_3 (I_1 - I_2)}{2\,I_1 I_2}$$

the equation reduces to

$$du = I_3 (2\,I_1 - I_3)\,\omega_3\,d\omega_3 \quad \text{or} \quad u = \frac{I_3 (2\,I_1 - I_3)}{2}\,\omega_3^2 + C'$$

C' being the constant of integration. The corresponding value of v is

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[1] Proc. Moscow Math. Soc. Vol. 24.

[2] Mathematische Annalen. Bd. 65, 67.

[3] Hess, Mathematische Annalen Bd. 37, studied a case where the projection of the angular momentum on a certain line was zero.

$$v = \frac{I_3(I_3 - 2I_2)}{2}\omega_3^2 - \frac{I_2}{I_1}C' + \frac{T I_3 (I_1 - I_2)}{2 I_1}.$$

These values of v and u substituted in equation (12) will satisfy the equation provided

$$I_3^2 = 2 I_1 I_2, \quad T = \frac{-2hw I_3 (I_1 + I_2 - I_3)}{(I_1 - I_3)(I_2 - I_3)},$$

$$C' = \frac{2hw I_1 I_2}{I_1 - I_3}.$$

There is an ambiguity in sign in solving for T and C', but if h is positive the sign must be as given for a real solution. The values of u and v which satisfy equations (11) and (12) are therefore

$$u = I_1 (I_3 - I_2)\omega_3^2 + \frac{2 I_1 I_2 h w}{I_1 - I_3}, \quad v = I_2 (I_1 - I_3)\omega_3^2 - \frac{2 I_1 I_2 h w}{I_2 - I_3}.$$

These values of u and v give

$$\omega_1^2 = \frac{-I_2}{I_1 - I_2}\left(\omega_3^2 + \frac{2hw}{I_2 - I_3}\right), \quad \omega_2^2 = \frac{I_1}{I_1 - I_2}\left(\omega_3^2 + \frac{2hw}{I_1 - I_3}\right).$$

5. As $\frac{d\omega_1}{d\omega_3}$ and $\frac{d\omega_2}{d\omega_3}$ are now determined, the values of $\sin\theta\cos\varphi$ and $\sin\theta\sin\varphi$ become

$$\sin\theta\cos\varphi = \frac{I_2(I_1 - I_3)}{hw I_3}\omega_3\omega_2, \quad \sin\theta\sin\varphi = \frac{I_1(I_2 - I_3)}{hw I_3}\omega_1\omega_3. \tag{14}$$

Hence

$$\tan\varphi = \frac{I_1(I_2 - I_3)\,\omega_1}{I_2(I_1 - I_3)\,\omega_2}. \tag{15}$$

From (7) $\cos\theta = 1 - \frac{I_3\omega_3^2}{2wh}$, and from (3)

$$t = \int \frac{I_3\, d\omega_3}{(I_1 - I_2)\,\omega_1\omega_2} = \pm \int \frac{I_3\, d\omega_3}{\sqrt{-I_1 I_2\left(\omega_3^2 + \frac{2hw}{I_1 - I_3}\right)\left(\omega_3^2 + \frac{2hw}{I_2 - I_3}\right)}} =$$

$$\pm \int \sqrt{\frac{2}{-\left(\omega_3^2 + \frac{2hw}{I_1 - I_3}\right)\left(\omega_3^2 + \frac{2hw}{I_2 - I_3}\right)}}\, d\omega_3 = \pm \int \sqrt{\frac{2}{f(\omega_3)}}\, d\omega_3.$$

The roots of $f(\omega_3) = 0$ are $\pm\sqrt{\frac{2hw}{I_3 - I_1}}$ and $\pm\sqrt{\frac{2hw}{I_3 - I_2}}$; the corresponding values of $\cos\theta$ are $\frac{I_1}{I_1 - I_3}$ and $\frac{I_2}{I_2 - I_3}$.

The absolute value of $\frac{I_1}{I_1 - I_3}$ is less than one only in case

$$I_3 > 2I_1, \text{ or } I_3^2 = 2I_1I_2 > 4I_1^2, \text{ i.e. } I_2 > 2I_1.$$

The second value of $\cos\theta$ is less than one in absolute value only in case $I_1 > 2I_2$. Hence the two conditions can not be satisfied simultaneously. We shall assume $I_2 > 2I_1$ and in that case $f(\omega_3) = 0$ has but two real roots viz.

$$\omega_3 = \pm\sqrt{\frac{2hw}{I_3 - I_1}}$$

Therefore ω_2 can become zero in the course of the motion but not ω_1; ω_3 oscillates between $+\sqrt{\frac{2hw}{I_3 - I_1}}$ and $-\sqrt{\frac{2hw}{I_3 - I_1}}$.

The square of the total angular velocity is

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = 2\omega_3^2 + \frac{2hwI_3}{(I_2 - I_3)(I_3 - I_1)},$$

and its maximum value is $\frac{2hw(2I_2 - I_3)}{(I_2 - I_3)(I_3 - I_1)}$.

6. As the moments of inertia for the principal axes are such that the sum of any two must be greater than the third, it is necessary to see that this condition can be satisfied. This is evident: for $I_2 > 2I_1$ and $I_3^2 = 2I_1I_2$. Hence if I_2 does not differ much from $2I_1$, the quantities I_1, I_2, I_3 will be nearly in the ratio 1 : 2 : 2 and can therefore form a triangle.

7. The polhodal cone. For the polhodal cone

$$\frac{x_1^2}{-I_2\left(\omega_3^2 + \frac{2hw}{I_2 - I_3}\right)} = \frac{y_1^2}{I_1\left(\omega_3^2 + \frac{2hw}{I_1 - I_3}\right)} = \frac{z_1^2}{\omega_3^2(I_1 - I_2)}.$$

Elimination of ω_3^2 gives the required equation which is

$$x_1^2 I_1 (I_2 - I_3) - y_1^2 I_2 (I_3 - I_1) - z_1^2 I_1 I_2 = 0,$$

a cone with its axis along OX_1.

8. The values of $\dot{\varphi}$ and $\dot{\psi}$. From equation (15)

$$\tan\varphi = \frac{I_1 (I_2 - I_3)\,\omega_1}{I_2 (I_1 - I_3)\,\omega_2}, \quad \sec^2\varphi\,\dot{\varphi} = \frac{I_1 (I_2 - I_3)}{I_2 (I_1 - I_3)}\,\frac{\omega_2\,\dot{\omega}_1 - \omega_1\,\dot{\omega}_2}{\omega_2^2} = \frac{hw\, I_1\, I_3\, \omega_3}{I_2 (I_1 - I_3)^2\, \omega_2^2},$$

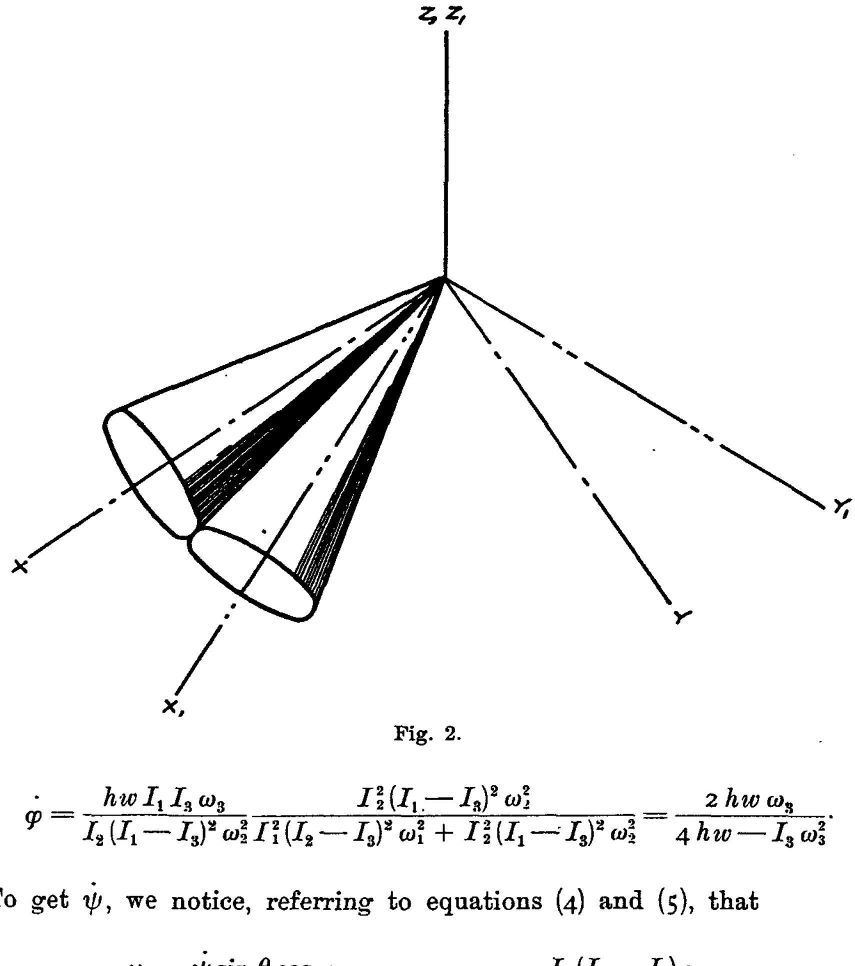

Fig. 2.

$$\dot{\varphi} = \frac{hw\, I_1\, I_3\, \omega_3}{I_2 (I_1 - I_3)^2\, \omega_2^2}\,\frac{I_2^2 (I_1 - I_3)^2\, \omega_2^2}{I_1^2 (I_2 - I_3)^2\, \omega_1^2 + I_2^2 (I_1 - I_3)^2\, \omega_2^2} = \frac{2\, hw\, \omega_3}{4\, hw - I_3\, \omega_3^2}.$$

To get $\dot{\psi}$, we notice, referring to equations (4) and (5), that

$$\frac{\omega_2 - \dot{\psi}\sin\theta\cos\varphi}{\omega_1 - \dot{\psi}\sin\theta\sin\varphi} = -\tan\varphi = -\frac{I_1 (I_2 - I_3)\,\omega_1}{I_2 (I_1 - I_3)\,\omega_2}.$$

If $\sin\theta\sin\varphi$ and $\sin\theta\cos\varphi$ are replaced by their values as given in equation (14), the value of $\dot{\psi}$ is

$$\dot{\psi} = \frac{I_1 I_2 I_3 h w \omega_3}{I_1^2 (I_2 - I_3)^2 \omega_1^2 + I_2^2 (I_1 - I_3)^2 \omega_2^2} = \frac{2 h w \omega_3}{4 h w - I_3 \omega_3^2} = \dot{\varphi},$$

$\psi = \varphi + \alpha$, α being a constant of integration which we shall take equal to zero.

9. The herpolhodal cone. Referring to Fig. 1, the vectors representing $\dot{\theta}$, $\dot{\psi}$ and $\dot{\varphi}$ have the directions ON, OZ and OZ_1 respectively. If $\omega_x, \omega_y, \omega_z$ are the components of ω along the fixed or space axes

$$\omega_x = \dot{\theta} \cos\psi + \dot{\varphi} \sin\theta \sin\psi = \dot{\theta} \cos\varphi + \dot{\psi} \sin\theta \sin\varphi = \omega_1,$$

$$\omega_y = \dot{\theta} \sin\psi - \dot{\varphi} \sin\theta \cos\psi = \dot{\theta} \sin\varphi - \dot{\psi} \sin\theta \cos\varphi = -\omega_2,$$

$$\omega_z = \dot{\psi} + \dot{\varphi} \cos\theta = \dot{\varphi} + \dot{\psi} \cos\theta = \omega_3.$$

Hence the equations of the polhodal and the herpolhodal cones are the same, only one is referred to the body and the other to the space axes. For the herpolhodal cone we have therefore the equation

$$x^2 I_1 (I_2 - I_3) - y^2 I_2 (I_3 - I_1) - z^2 I_1 I_2 = 0.$$

10. Geometrically the motion is represented by the rolling of a cone fixed in the body on an equal cone fixed in space and whose axis is horizontal. As $\cos\theta = 1$ when $\omega_3 = 0$ this is a convenient starting point. At that moment OZ and OZ_1 coincide and the angle between OX and OX_1 is $2 \arctan \sqrt{\frac{I_1 (I_2 - I_3)}{I_2 (I_3 - I_1)}}$ as sketched in Fig. 2. When the instantaneous axis is in the XY plane, θ is zero; when it is in the XZ plane, θ has its maximum value.

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