ON THE UNSYMMETRIC TOP

BY

JOHN J. CORLISS

of CHICAGO, ILL., U. S. A.

This paper is in the nature of a sequel to that published by the author in the Acta mathematica for August 1932, Vol. 59, page 423. In that paper the center of gravity was taken on one of the principal axes of the momental ellipsoid of the body corresponding to the fixed point. The kinetic energy and the angular momentum were assumed to be quadratic functions of ω_3, the projection of the angular velocity vector on the principal axis on which the center of gravity lies. This assumption led to the two cases given in that paper. The purpose of the paper is to find all similar cases[1] in which the kinetic energy and the angular momentum squared are expressible as polynomials in ω_3. It upholds the best traditions of workers on the top problem by giving one new case, but to the authors mind the most interesting part of the paper is its limiting character. It will be shown that there are no more cases of this particular type.

The equations of motion for the top with its center of gravity on the z-axis are:

$$(1) \qquad I_1 \dot{\omega}_1 + (I_3 - I_2)\,\omega_2\,\omega_3 = \quad W h \sin \Theta \cos \Phi$$

$$(2) \qquad I_2 \dot{\omega}_2 + (I_1 - I_3)\,\omega_1\,\omega_3 = - W h \sin \Theta \sin \Phi$$

$$(3) \qquad I_3 \dot{\omega}_3 + (I_2 - I_1)\,\omega_1\,\omega_2 = 0.$$

[1] N. Kowalevski: Math. Annalen 65, p. 528, 1908, tried to find all possible cases for which ω_1^2 and ω_2^2 can be expressed as polynominals of the third degree in ω_3. He found one new case, reference to which is made farther on in this paper.

The angular velocities $(\omega_1, \omega_2, \omega_3)$ are connected with Euler's angles (Θ, Φ, Ψ) by:

$$(4) \qquad \dot{\Theta} = \omega_1 \cos \Phi - \omega_2 \sin \Phi$$

$$(5) \qquad \dot{\Phi} = -\omega_1 \sin \Phi \cot \Theta - \omega_2 \cos \Phi \cot \Theta + \omega_3$$

$$(6) \qquad \dot{\Psi} = \omega_1 \sin \Phi \csc \Theta + \omega_2 \cos \Phi \csc \Theta$$

or by:

$$(7) \qquad \omega_1 = \dot{\Theta} \cos \Phi + \dot{\Psi} \sin \Theta \sin \Phi$$

$$(8) \qquad \omega_2 = -\dot{\Theta} \sin \Phi + \dot{\Psi} \sin \Theta \cos \Phi$$

$$(9) \qquad \omega_3 = \dot{\Phi} + \dot{\Psi} \cos \Theta$$

and the angular velocities $(\omega_x, \omega_y, \omega_z)$ are connected with Euler's angles by:

$$(10) \qquad \omega_x = \dot{\Theta} \cos \Psi + \dot{\Phi} \sin \Theta \sin \Psi$$

$$(11) \qquad \omega_y = \dot{\Theta} \sin \Psi - \dot{\Phi} \sin \Theta \cos \Psi$$

$$(12) \qquad \omega_z = \dot{\Psi} + \dot{\Phi} \cos \Theta$$

The origin, O, is taken at the fixed point and the following notation is used: (x, y, z) denote the fixed system of axes; (x_1, y_1, z_1) the moving system which is taken coincident with the principal axes of the momental ellipsoid at O; $(\omega_1, \omega_2, \omega_3)$ are the components of the instantaneous angular velocity vector, ω, along the moving axes; $(\omega_x, \omega_y, \omega_z)$ are its components along the fixed axes; (I_1, I_2, I_3) are the principal moments of inertia at O; W is the weight of the body; and, h is the distance of the center of gravity from the origin.

The classical integrals are:

$$(13) \qquad 2\, W h \cos \Theta = E - I_1 \omega_1^2 - I_2 \omega_2^2 - I_3 \omega_3^2$$

which states that the total energy is a constant, $\frac{E}{2}$;

$$(14) \qquad I_1 \omega_1 \sin \Theta \sin \Phi + I_2 \omega_2 \sin \Theta \cos \Phi + I_3 \omega_3 \cos \Theta = k$$

which states that the projection of the angular momentum on the vertical is a constant, k; and, the trigonometric identity

$$(15) \qquad (\sin \Theta \sin \Phi)^2 + (\sin \Theta \cos \Phi)^2 + \cos^2 \Theta = 1\,.$$

In order to simplify our problem we shall reduce it to the solution of two symmetric differential equations of the second order. Substituting for $\frac{1}{dt}$ its value from equation (3) in equations (1) and (2) we get:

$$I_1 (I_1 - I_2) \omega_1 \omega_2 d \omega_1 + I_3 (I_3 - I_2) \omega_2 \omega_3 d \omega_3 = W h I_3 d \omega_3 \sin \Theta \cos \Phi \tag{16}$$

$$I_2 (I_1 - I_2) \omega_1 \omega_2 d \omega_2 + I_3 (I_1 - I_3) \omega_1 \omega_3 d \omega_3 = - W h I_3 d \omega_3 \sin \Theta \sin \Phi . \tag{17}$$

Now let us introduce two new variables u and v:

$$u = I_2 (I_1 - I_2) \omega_2^2 + I_3 (I_1 - I_3) \omega_3^2 \tag{18}$$

$$v = I_1 (I_1 - I_2) \omega_1^2 + I_3 (I_3 - I_2) \omega_3^2 . \tag{19}$$

These variables are closely related to the variables used by Hess-Schiff[1] in the so called Hess-Schiff reduced differential equations since $I_2 u + I_1 v = (I_1 - I_2)(I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2)$, that is, equals a constant times angular momentum squared and $u + v = (I_1 - I_2)(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$, that is, equal a constant times kinetic energy.

In terms of the new variables equations (16), (17), and (13) become:

$$\omega_2 d v = 2 W h I_3 d \omega_3 \sin \Theta \cos \Phi \tag{20}$$

$$\omega_1 d u = - 2 W h I_3 d \omega_3 \sin \Theta \sin \Phi \tag{21}$$

$$2 W h \cos \Theta = \frac{E (I_1 - I_2) - (u + v)}{I_1 - I_2} = \eta . \tag{22}$$

Where η is a new variable defined by equation (22). If we now substitute for $\sin \Theta \sin \Phi$, $\sin \Theta \cos \Phi$, and $\cos \Phi$ in equations (14) and (15) their values in terms of u, v, and η we get:

$$- I_1 (I_1 - I_2) \omega_1^2 d u + I_2 (I_1 - I_2) \omega_2^2 d v + I_3^2 (I_1 - I_2) \eta \omega_3 d \omega_3 = 2 W h k I_3 (I_1 - I_2) d \omega_3 \tag{23}$$

$$I_1 I_2 (I_1 - I_2) \omega_1^2 \overline{d u}^2 + I_1 I_2 (I_1 - I_2) \omega_2^2 \overline{d v}^2 + I_1 I_2 (I_1 - I_2) I_3^2 \eta^2 \overline{d \omega_3}^2 = 4 W^2 h^2 I_1 I_2 I_3^2 (I_1 - I_2) \overline{d \omega_3}^2 . \tag{24}$$

Of course it would be possible to eliminate ω_1^2, ω_2^2, and η from equations (23) and (24) leaving two differential equations in three unknowns. However, for our future work this will not be necessary and the present form is easier and simpler to write. Throughout the paper the practice is followed of writing

[1] P. A. Schiff: Moskau Math. Samml. 24, p. 169, 1903. The Hess-Schiff variables are:

$$T = \frac{I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2}{2}, \quad U = \frac{I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2}{2}, \quad S = f I_1 \omega_1 + g I_2 \omega_2 + h I_3 \omega_3$$

where (f, g, h) are the coordinates of the center of gravity.

all equations in the simplest possible form, but of discussing said equations as though all replacements had been made. Equations (23) and (24) are necessary and sufficient conditions, that is, they are equivalent to the Euler equations as long as u, v, and ω_3 are all variable. It is obvious that they are necessary. The sufficiency part has been discussed at great length by both Stäckel and Lazzarino.[1] Stäckel and Lazzarino discussed the Hess-Schiff variables, however, the u and v of this paper are merely linear combinations of the Hess-Schiff variables.

Equations (23) and (24) are, as has been pointed out, necessary and sufficient, but on the other hand they are very complicated and not symmetric. Two symmetric equations which are much easier to work with will now be obtained. These new equations have the disadvantage that they are only neccessary conditions.

To obtain them we differentiate equations (23) and (24) with respect to ω_3. After substituting for $\frac{d\,\omega_1^2}{d\,\omega_3}$, $\frac{d\,\omega_2^2}{d\,\omega_3}$, and $\frac{d\,\eta}{d\,\omega_3}$ their values as obtained by differentiating equations (18), (19), and (22) with respect to ω_3 and collecting terms we get:

$$
(25)\quad \begin{aligned} & -I_1(I_1-I_2)\,\omega_1^2\frac{d^2u}{d\,\omega_3^2}+I_2(I_1-I_2)\,\omega_2^2\frac{d^2v}{d\,\omega_3^2}+I_3(I_3-2\,I_2)\,\omega_3\frac{du}{d\,\omega_3}+ \\ & \qquad + I_3(I_3-2\,I_1)\,\omega_3\frac{dv}{d\,\omega_3}+I_3^2(I_1-I_2)\,\eta=0 \end{aligned}
$$

$$
(26)\quad \begin{aligned} & I_1I_2(I_1-I_2)\,\omega_1^2\frac{du}{d\,\omega_3}\frac{d^2u}{d\,\omega_3^2}+I_1I_2(I_1-I_2)\,\omega_2^2\frac{dv}{d\,\omega_3}\frac{d^2v}{d\,\omega_3^2}+ \\ & +\frac{1}{2}\frac{du}{d\,\omega_3}\frac{dv}{d\,\omega_3}\left(I_2\frac{du}{d\,\omega_3}+I_1\frac{dv}{d\,\omega_3}\right)+I_2I_3(I_2-I_3)\,\omega_3\left(\frac{du}{d\,\omega_3}\right)^2+ \\ & \qquad +I_1I_3(I_3-I_1)\,\omega_3\left(\frac{dv}{d\,\omega_3}\right)^2-I_1I_2I_3^2\,\eta\left(\frac{du}{d\,\omega_3}+\frac{dv}{d\,\omega_3}\right)=0. \end{aligned}
$$

It is easily seen that equations (25) and (26) are linear combinations of the following:

$$
(27)\quad \begin{aligned} & I_1(I_1-I_2)\,\omega_1^2\frac{d^2u}{d\,\omega_3^2}+\frac{1}{2}\frac{du}{d\,\omega_3}\frac{dv}{d\,\omega_3}+I_1I_3\,\omega_3\frac{dv}{d\,\omega_3}+ \\ & \qquad +I_3(I_2-I_3)\,\omega_3\frac{du}{d\,\omega_3}-I_1I_3^2\,\eta=0 \end{aligned}
$$

[1] P. Stäckel: Math. Ann. 65, p. 538, 1908. Math. Ann. 67, p. 399, 1909. — O. Lazzarino: Rend. d. Soc. reale di Napoli (3^a) 17, p. 68—1911. R. Accademia dei Lincei atti 28_1, p. 266; p. 325; p. 341, 1919. R. Accademia dei Lincei atti 28_2, p. 9; p. 259; p. 329, 1919.

$$I_2(I_1 - I_2)\,\omega_2^2 \frac{d^2 v}{d\omega_3^2} + \frac{1}{2}\frac{du}{d\omega_3}\frac{dv}{d\omega_3} - I_2 I_3 \omega_3 \frac{du}{d\omega_3} + \tag{28}$$
$$+ I_3(I_3 - I_1)\,\omega_3 \frac{dv}{d\omega_3} - I_2 I_3^2 \eta = 0.$$

In actual computation it seems advisable to use both pair of equations, i. e., (23, 24) and (27, 28) although the sufficiency conditions might be determined by substituting directly back in Euler's equations and in the classical integrals. Certain obvious simplifications appear upon writing down the conditions that the values assumed for u and v shall satisfy both sets of equations.

Let us now assume that u and v may be expressed in the form:

$$\frac{du}{d\omega_3} = \sum_{i=1}^{n} B_i(b_i + 1)\,\omega_3^{b_i}, \text{ i. e., } \quad u = B_0 + \sum_{i=1}^{n} B_i\,\omega_3^{b_i+1} \tag{29}$$

$$\frac{dv}{d\omega_3} = \sum_{i=1}^{n} A_i(a_i + 1)\,\omega_3^{a_i}, \text{ i. e., } \quad v = A_0 + \sum_{i=1}^{n} A_i\,\omega_3^{a_i+1}. \tag{30}$$

For convenience we write down:

$$\left(\frac{du}{d\omega_3}\right)^2 = \sum_{i,j=1}^{n} B_i B_j (b_i + 1)(b_j + 1)\,\omega_3^{b_i+b_j} \tag{31}$$

$$\left(\frac{dv}{d\omega_3}\right)^2 = \sum_{i,j=1}^{n} A_i A_j (a_i + 1)(a_j + 1)\,\omega_3^{a_i+a_j} \tag{32}$$

$$\left(\frac{du}{d\omega_3}\right)\left(\frac{dv}{d\omega_3}\right) = \sum_{i,j=1}^{n} A_i B_j (a_i + 1)(b_j + 1)\,\omega_3^{a_i+b_j} \tag{33}$$

$$\frac{d^2 u}{d\omega_3^2} = \sum_{i=1}^{n} B_i b_i (b_i + 1)\,\omega_3^{b_i-1} \tag{34}$$

$$\frac{d^2 v}{d\omega_3^2} = \sum_{i=1}^{n} A_i a_i (a_i + 1)\,\omega_3^{a_i-1}. \tag{35}$$

From equations (18) and (19) we have

$$I_1(I_1 - I_2)\,\omega_1^2 = A_0 + I_3(I_2 - I_3)\,\omega_3^2 + \sum_{i=1}^{n} A_i\,\omega_3^{a_i+1} \tag{36}$$

$$I_2(I_1 - I_2)\,\omega_2^2 = B_0 + I_3(I_3 - I_1)\,\omega_3^2 + \sum_{i=1}^{n} B_i\,\omega_3^{b_i+1}. \tag{37}$$

From equation (22) we get for η

$$(I_1 - I_2)\,\eta = E(I_1 - I_2) - (A_0 + B_0) - \sum_{i=1}^{n} (A_i\,\omega_3^{a_i+1} + B_i\,\omega_3^{b_i+1}). \tag{38}$$

Upon substituting these values in equations (23), (24), (27), and (28) we obtain the following:

$$\begin{aligned}\sum_{i,j=1}^{n} A_i B_j (a_i - b_j)\,\omega_3^{a_i+b_j+1} + \sum_{i=1}^{n} [I_3 A_i \{(I_3 - I_1)\,a_i - I_1\}\,\omega_3^{a_i+2} + \\ + I_3 B_i \{(I_3 - I_2)\,b_i - I_2\}\,\omega_3^{b_i+2} + B_0 A_i (a_i + 1)\,\omega_3^{a_i} - A_0 B_i (b_i + 1)\,\omega_3^{b_i}] + \\ + I_3^2 [E(I_1 - I_2) - (A_0 + B_0)]\,\omega_3 = 2\,W h k I_3 (I_1 - I_2).\end{aligned} \tag{39}$$

$$\begin{aligned}\sum_{i,j,k=1}^{n} [I_1 A_i A_j B_k (a_i + 1)(a_j + 1)\,\omega_3^{a_i+a_j+b_k+1} + I_2 A_i B_j B_k (b_j + 1)(b_k + 1)\,\omega_3^{a_i+b_j+b_k+1}] + \\ + \sum_{i,j=1}^{n} [I_1 I_3 (I_3 - I_1) A_i A_j (a_i + 1)(a_j + 1)\,\omega_3^{a_i+a_j+2} + \\ + I_2 I_3 (I_2 - I_3) B_i B_j (b_i + 1)(b_j + 1)\,\omega_3^{b_i+b_j+2} + I_1 B_0 A_i A_j (a_i + 1)(a_j + 1)\,\omega_3^{a_i+a_j} + \\ + I_2 A_0 B_i B_j (b_i + 1)(b_j + 1)\,\omega_3^{b_i+b_j} + \frac{I_1 I_2 I_3^2}{I_1 - I_2} (A_i\,\omega_3^{a_i+1} + B_i\,\omega_3^{b_i+1})(A_j\,\omega_3^{a_j+1} + B_j\,\omega_3^{b_j+1})] - \\ - \frac{2\,I_1 I_2 I_3^2}{I_1 - I_2} [E(I_1 - I_2) - (A_0 + B_0)] \sum_{i=1}^{n} (A_i\,\omega_3^{a_i+1} + B_i\,\omega_3^{b_i+1}) + \\ + \frac{I_1 I_2 I_3^2}{I_1 - I_2} [E(I_1 - I_2) - (A_0 + B_0)]^2 = 4\,W^2 h^2 I_1 I_2 I_3^2 (I_1 - I_2).\end{aligned} \tag{40}$$

$$\begin{aligned}\sum_{i,j=1}^{n} \left(\frac{b_j + 1}{2}\right)(2\,b_j + a_i + 1)\,A_i B_j\,\omega_3^{a_i+b_j} + \sum_{i=1}^{n} \left[\left\{I_3 (I_2 - I_3)(b_i + 1)^2 + \frac{I_1 I_3^2}{I_1 - I_2}\right\} B_i\,\omega_3^{b_i+1} + \right. \\ \left. + \left\{I_1 I_3 (a_i + 1) + \frac{I_1 I_3^2}{I_1 - I_2}\right\} A_i\,\omega_3^{a_i+1} + A_0 B_i b_i (b_i + 1)\,\omega_3^{b_i-1}\right] - \\ - \frac{I_1 I_3^2}{I_1 - I_2} [E(I_1 - I_2) - (A_0 + B_0)] = 0.\end{aligned} \tag{41}$$

$$\sum_{i,j=1}^{n}\left(\frac{a_i+1}{2}\right)(2\,a_i+b_j+1)\,A_i\,B_j\,\omega_3^{a_i+b_j}+\sum_{i=1}^{n}\left\{I_3\,(I_3-I_1)\,(a_i+1)^2+\frac{I_2\,I_3^2}{I_1-I_2}\right\}A_i\,\omega_3^{a_i+1}+$$

$$(42)\qquad +\left\{-I_2\,I_3\,(b_i+1)+\frac{I_2\,I_3^2}{I_1-I_2}\right\}B_i\,\omega_3^{b_i+1}+B_0\,A_i\,a_i\,(a_i+1)\,\omega_3^{a_i-1}]-$$

$$-\frac{I_2\,I_3^2}{I_1-I_2}[E\,(I_1-I_2)-(A_0+B_0)]=0.$$

In order that the values assumed for u and v as given by equations (29) and (30) shall be solutions of equations (23) and (24) it is necessary that we shall be able to determine values for the arbitrary constants such that the expressions given in equations (39), (40), (41), and (42) become identities in ω_3.

Values of the constants which reduce equations (39) and (40) to identities in ω_3 lead to solutions of the Euler equations since equations (23) and (24) are necessary and sufficient conditions. Equations (41) and (42) are used in so far as possible to compute the constants since they are so much easier to work with.

There are several general conclusions which may be drawn:

I. To show that a_n and b_n cannot be taken as fractions:

The coefficients of $\omega_3^{a_n+b_n}$ in equations (41) and (42) vanish only if A_n or $B_n=0$ or if

$$2\,b_n+a_n+1=0$$

$$2\,a_n+b_n+1=0$$

that is if $a_n=b_n=-\frac{1}{3}$. But if we take $a_n=b_n=-\frac{1}{3}$, then in order to make the coefficients of $\omega_3^{a_n-1}$ and $\omega_3^{b_n-1}$ vanish we must take $A_0=B_0=0$. Furthermore, if $A_0=B_0=0$ then either $I_1=I_2$ or the energy constant, E, equals zero.

When $a_1=b_1=0$ we have:

$$\frac{A_1\,B_1}{2}-\frac{I_1\,I_3^2}{I_1-I_2}[E\,(I_1-I_2)-(A_0+B_0)]=0$$

$$\frac{A_1\,B_1}{2}-\frac{I_2\,I_3^2}{I_1-I_2}[E\,(I_1-I_2)-(A_0+B_0)]=0.$$

Subtracting shows $E=0$. If $I_1=I_2$ we have the Lagrange case.

For the top the energy constant, E, is an essentially positive quantity not zero. Therefore there is no point in studying the possibilities if $E=0$.

II. To prove that m must be taken equal to 2.

Let us now assume that

$$a_i = i - 1 \qquad i = 1, 2, \ldots, n$$

$$b_i = i - 1 \qquad i = 1, 2, \ldots, m.$$

For this part of the discussion we assume

$$n > m > 1.$$

(Assuming $n = m = 1$ leads to a case in which $\omega_3 =$ a constant and our equations do not apply.) In order to successfully equate coefficients in equations (41) and (42) we must have the exponents of the two highest degree terms equal, that is, the exponents of $\omega_3^{n-1+m-1}$ and ω_3^n must be equal which means that $m = 2$.

III. To show that we cannot hope to find a solution by taking $n \geq 7$. If we assume $n = 7$ and $m = 2$ then equations (41) and (42) are of the 7th degree in ω_3. Consequently upon equating coefficients of all powers of ω_3 to zero we obtain 16 equations to satisfy which we have at our disposal

3 B's + 8 A's + one condition on E + 2 relations among the I's = 14 arbitrary quantities at most.

Unless it should happen that the coefficients are not independent, then the equations cannot be satisfied. The relations which may be taken among the I's are somewhat limited since they are positive quantities and also since they must be chosen so as to form the sides of a triangle. For $n > 7$ the situation is still worse. A similar argument could be given to show that n cannot be taken equal 6. The case of $n = 6$ will be treated by computation also.

If we assume $m = 2$ and $n = 6$ and equate to the coefficients of all powers of ω_3 in equations (41) and (42) we obtain the following set of conditions to be satisfied:

$$(43) \qquad 2 A_0 B_2 + \frac{1}{2} A_1 B_1 - \frac{I_1 I_3^2}{I_1 - I_2} [E(I_1 - I_2) - (A_0 + B_0)] = 0$$

$$(44) \qquad 2 A_2 B_0 + \frac{1}{2} A_1 B_1 - \frac{I_2 I_3^2}{I_1 - I_2} [E(I_1 - I_2) - (A_0 + B_0)] = 0$$

$$(45) \qquad 3 A_1 B_2 + A_2 B_1 + I_1 I_3 A_1 + I_3 (I_2 - I_3) B_1 + \frac{I_1 I_3^2}{I_1 - I_2} (A_1 + B_1) = 0$$

$$(46) \quad 3A_2B_1 + 6A_3B_0 + A_1B_2 - I_2I_3B_1 + I_3(I_3 - I_1)A_1 + \frac{I_2I_3^2}{I_1 - I_2}(A_1 + B_1) = 0$$

$$(47) \quad 4A_2B_2 + 4I_3(I_2 - I_3)B_2 + \frac{3}{2}A_3B_1 + 2I_1I_3A_2 + \frac{I_1I_3^2}{I_1 - I_2}(A_2 + B_2) = 0$$

$$(48) \quad 4A_2B_2 + 4I_3(I_3 - I_1)A_2 + \frac{15}{2}A_3B_1 + 12A_4B_0 - 2I_2I_3B_2 + \\ + \frac{I_2I_3^2}{I_1 - I_2}(A_2 + B_2) = 0$$

$$(49) \quad 5A_3B_2 + 2A_4B_1 + 3I_1I_3A_3 + \frac{I_1I_3^2}{I_1 - I_2}A_3 = 0$$

$$(50) \quad 9A_3B_2 + 9I_3(I_3 - I_1)A_3 + 14A_4B_1 + 20A_5 - B_0 + \frac{I_2I_3^2}{I_1 - I_2}A_3 = 0$$

$$(51) \quad 6A_4B_2 + \frac{5}{2}A_5B_1 + 4I_1I_3A_4 + \frac{I_1I_3^2}{I_1 - I_2}A_4 = 0$$

$$(52) \quad 16A_4B_2 + 16I_3(I_3 - I_1)A_4 + \frac{45}{2}A_5B_1 + 30A_6B_0 + \frac{I_2I_3^2}{I_1 - I_2}A_4 = 0$$

$$(53) \quad 7A_5B_2 + 3A_6B_1 + 5I_1I_3A_5 + \frac{I_1I_3^2}{I_1 - I_2}A_5 = 0$$

$$(54) \quad 25A_5B_2 + 25I_3(I_3 - I_1)A_5 + 33A_6B_1 + \frac{I_2I_3^2}{I_1 - I_2}A_5 = 0$$

$$(55) \quad 8A_5B_2 + 6I_1I_3A_6 + \frac{I_1I_3^2}{I_1 - I_2}A_6 = 0$$

$$(56) \quad 36A_6B_2 + 36I_3(I_3 - I_1)A_6 + \frac{I_2I_3^2}{I_1 - I_2}A_6 = 0.$$

Equating the constant terms of equations (39) and (40) gives the following two equations which must also be satisfied:

$$(57) \quad A_1B_0 - A_0B_1 = 2Whk(I_1 - I_2)$$

$$(58) \quad I_1B_0A_1^2 + I_2A_0B_1^2 + \frac{I_1I_2I_3^2}{I_1 - I_2}[E(I_1 - I_2) - (A_0 + B_0)]^2 = \\ = 4W^2h^2I_1I_2I_3^2(I_1 - I_2).$$

IV. To show that A_6 and A_5 must be taken equal zero i. e., n must be taken ≤ 4. To satisfy equations (55) and (56) respectively B_2 must be chosen:

$$B_2 = \frac{I_3 (6 I_1 I_2 - 6 I_1^2 - I_1 I_3)}{8 (I_1 - I_2)}$$

$$B_2 = \frac{I_3 (36 I_1^2 - 36 I_1 I_2 - 36 I_1 I_3 + 35 I_2 I_3)}{36 (I_1 - I_2)}.$$

But if we eleminate $A_6 B_1$ between equations (53) and (54) we find that B_2 must also satisfy:

$$B_2 = \frac{I_3 (14 I_1 I_3 - 24 I_2 I_3 - 80 I_1^2 + 80 I_1 I_2)}{52 (I_1 - I_2)}.$$

For these three values of B_2 to be consistents the I's must satisfy:

$$63 I_1 I_3 - 70 I_2 I_3 - 126 I_1^2 + 126 I_1 I_2 = 0$$

$$41 I_1 I_3 - 48 I_2 I_3 - 82 I_1^2 + 82 I_1 I_2 = 0$$

which is not possible.

Therefore we must take A_6 or A_5 equal zero. Again if $A_6 = 0$, B_2 must be chosen so as to satisfy equations (53) and (54), that is,

$$B_2 = \frac{I_3 (5 I_1 I_2 - 5 I_1^2 - I_1 I_3}{7 (I_1 - I_2)}$$

$$B_2 = \frac{I_3 (25 I_1^2 - 25 I_1 I_2 - 25 I_1 I_3 + 24 I_1 I_2)}{25 (I_1 - I_2)}.$$

And also B_2 must be chosen so as to satisfy the equation obtained by eliminating $A_5 B_1$ between equations (51) and (52):

$$B_2 = \frac{I_3 (7 I_1 I_3 - 15 I_2 I_3 - 52 I_1^2 + 52 I_1 I_2)}{38 (I_1 - I_2)}.$$

For these three values of B_2 to be consistent the I's must satisfy:

$$50 I_1^2 - 50 I_1 I_2 - 25 I_1 I_3 + 28 I_2 I_3 = 0$$

$$174 I_1^2 - 174 I_1 I_2 - 87 I_1 I_3 + 105 I_2 I_3 = 0$$

which is not possible.

Finally if $A_5 = 0$ and $A_6 = 0$ we see at once from equations (53) and (54) that $B_1 = 0$. To satisfy equations (55) and (56) B_2 must be chosen:

$$B_2 = \frac{I_3 (6 I_1 I_2 - 6 I_1^2 - I_1 I_3)}{8 (I_1 - I_2)}$$

$$B_2 = \frac{I_3(36 I_1^2 - 36 I_1 I_2 - 36 I_1 I_3 + 35 I_2 I_3)}{36(I_1 - I_2)}.$$

And also B_2 must be chosen so as to satisfy the equation (51):

$$B_2 = \frac{I_3(4 I_1 I_2 - 4 I_1^2 - I_1 I_3)}{6(I_1 - I_2)}.$$

For these values of B_2 to be consistent the I's must satisfy:

$$63 I_1 I_3 - 70 I_2 I_3 - 126 I_1^2 + 126 I_1 I_2 = 0$$

$$I_1 I_3 \qquad - 2 I_1^2 + 2 I_1 I_2 = 0$$

which is not possible.

Therefore n cannot be taken greater than 4.

Taking $n = 1$ and $n = 2$ leads to the two cases given in my previous paper. The case of P. Field published in the Acta, vol. 56, is a special case of $n = 2$. Taking $n = 3$ corresponds to the case of N. Kowalevski.[1]

New Case.

If we take $n = 4$, that is if we assume

$$u = B_0 + B_1 \omega_3 + B_2 \omega_3^2$$

$$v = A_0 + \sum_{i=1}^{4} A_i \omega_3^i$$

we come to a new case.

The constants have the following values:

$$A_0 = \frac{I_1 I_3 (64 I_1^2 - 64 I_1 I_3 + 15 I_3^2) E}{2(I_3 - 2 I_1)(9 I_3^2 - 56 I_1 I_3 + 64 I_1^2)}$$

$$A_2 = \frac{I_3(4 I_1 - 3 I_3)(64 I_1^2 - 64 I_1 I_3 + 15 I_3^2)}{2(16 I_1 - 9 I_3)(I_3 - 2 I_1)}$$

$$A_4 = \frac{I_3(4 I_1 - 3 I_3)(16 I_1^2 - 16 I_1 I_3 + 3 I_3^2)(9 I_3^2 - 56 I_1 I_3 + 64 I_1^2)}{16(16 I_1 - 9 I_3)(I_3 - 2 I_1) I_1 E}$$

$$B_0 = -\frac{4 I_1 I_3 (I_3 - 2 I_1) E}{(9 I_3^2 - 56 I_1 I_3 + 64 I_1^2)}$$

$$B_2 = \frac{I_3(4 I_1 - 3 I_3)}{2}$$

and $B_1 = A_1 = A_3 = A_4 = 0$.

[1] N. Kowalevski: Math. Annalen 65, p. 528, 1908.

The I's must satisfy the relation:

$$I_2 = \frac{I_1(16 I_1 - 8 I_3)}{16 I_1 - 9 I_3}$$

and E must be taken:

$$E = \frac{4\, W h (I_3 - 2 I_1)(9 I_3^2 - 56 I_1 I_3 + 64 I_1^2)}{(4 I_1 - 3 I_3)(64 I_1^2 - 64 I_1 I_3 + 15 I_3^2)}.$$

The projection of the angular momentum on the vertical equal zero, that is

$$k = 0.$$

It is easy to show by numerical computation that the conditions cannot be satisfied by any other choice of constants.

The value of tan Φ is obtained from equations (20) and (21) and is:

$$\tan \Phi = -\frac{\omega_1 \dfrac{du}{d\omega_3}}{\omega_2 \dfrac{dv}{d\omega_3}}$$

To find Ψ and the time in terms of ω_3 elliptic functions must be introduced. Knowing Ψ the space cone could be found and the motion completely described. The equation of the body cone is obtained by eliminating the ω's from $\frac{x_1^2}{\omega_1^2} = \frac{y_1^2}{\omega_2^2} = \frac{z_1^2}{\omega_3^2}$ and is:

$$I_3(9 I_3 - 16 I_1)^3 (8 I_1 - 6 I_3)^2 z_1^4 - 256 I_1^4 (8 I_1 - 3 I_3)(8 I_1 - 5 I_3) y_1^4 -$$
$$- 16 I_1^2 (I_3 - 2 I_1)(9 I_3 - 16 I_1)^3 x_1^2 z_1^2 - 256 I_1^4 (I_3 - 2 I_1)(9 I_3 - 16 I_1) x_1^2 y_1^2 -$$
$$- 32 I_1^2 (27 I_3^3 - 144 I_3^2 I_1 + 320 I_3 I_1^2 - 256 I_1^3)(9 I_3 - 16 I_1) y_1^2 z_1^2 = 0.$$

The literature on the top problem is extensive but it is entirely a literature of special cases. Klein and Sommerfeld in their huge work: »Theorie des Kreisels» 1910, p. 391, have suggested the possibility of interpolating between the two movements. Thus if we can find enough special cases we may yet hope to know something of the motion of the unsymmetric top.