COMPLETELY INTEGRABLE DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES.[1]

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Introduction. The primary object of this paper is to obtain existence theorems for the abstract completely integrable[2] differential equation

$$d_{\xi}^{x} f(x) = F(x, f(x), \xi),$$

where the left member is the Fréchet differential[3] of $f(x)$ with increment ξ, and the ranges and domains of the functions involved are in Banach spaces.[4] By a generalization of the well-known method of successive approximations and the use of most of the known and several new properties of abstract differentials and integrals, we prove two main theorems, one local in character and the other 'in the large', by means of which we obtain new existence theorems for Pfaffian differential equations in Hilbert space and the well-known space of continuous functions, and also a new existence theorem for abstract implicit functions. Kerner's recent theorem[5], in which $F(x, f(x), \xi)$ is independent of $f(x)$, is an immediate corollary of the first main theorem. By specializing the Banach spaces, we obtain several of the recent improvements in the theory of the classical

[1] Presented to the Amer. Math. Soc. (1934). Cf. Bull. Amer. Math. Soc., 40, 530 (1934).

[2] The condition of complete integrability (the premise in (ii) of Theorem I) is suggested by a theorem of Kerner on the symmetry in the increments of a repeated Fréchet differential; it is definitely a necessary condition for the existence of the solution in Theorem II.

[3] M. FRÉCHET, Annales Sc. Ec. Norm. Sup., t. 42, 293—323 (1925). See also T. H. HILDEBRANT and L. M. GRAVES, Trans. Amer. Math. Soc., 29 (1927).

[4] S. BANACH, Fund. Math., 3, 133—181 (1922). See also his book, Théorie des opérations linéaires, (1932). A Banach space is briefly a complete normed vector space closed under multiplication by real numbers.

[5] M. KERNER, Annals of Math. (1933).

Pfaffian systems due to Nikliborc, Nikodym, and others, as well as the theorem of Kerner for abstractly valued functions of a real variable. Finally, an aplication of the main theorems to differential equations of the second order is outlined, and a few indications are given of related outstanding problems in abstract analysis.

Most of the necessary notations and known results in the abstract differential and integral calculus are collected in section 1. The first main theorem is proved in section 2, and its special cases are contained in section 3. Section 4 contains the second main theorem. The existence theorems for Pfaffian differential equations in Hilbert space, function space, and certain normed rings, including a Volterra ring of permutable functions, are given in section 5. In section 6 the theorem on abstract implicit functions is proved by means of the first main theorem and a lemma on the differentiability with respect to a parameter of the inverse of a solvable linear function.

1. Definitions and Known Results.

Let R be the real number system,
in which ε, δ are positive numbers,
n is a non-negative integer,
and i is any non-negative integer $\leq n$.
Let E_i be a Banach space, that is, a complete normed linear space closed under multiplication by elements in R; let
x_i, ξ_i be elements in E_i,
and let $\|x_i\|$ be the norm of x_i.

In each of the immediately following definitions, $F(x_1 \ldots x_n)$ is in E_0 for the values of the arguments considered.

Def. 1.1 $\Delta^{x_1 \ldots x_n}_{\xi_1 \ldots \xi_n} F(x_1 \ldots x_n)$ *is* $F(x_1 + \xi_1 \ldots x_n + \xi_n) - F(x_1 \ldots x_n)$.

Def. 1.2 $F(x_1 \ldots x_n)$ *is additive in* $x_1 \ldots x_n$ if and only if for any x_i, ξ_i

$$\Delta^{x_1 \ldots x_n}_{\xi_1 \ldots \xi_n} F(x_1 \ldots x_n) = F(\xi_1 \ldots \xi_n).$$

Def. 1.3 $F(x_1 \ldots x_n)$ *is continuous in* $x_1 \ldots x_n$ if and only if for any ε some δ exists such that for any ξ_i

$$\max \|\xi_i\| < \delta \text{ implies } \|\Delta^{x_1 \ldots x_n}_{\xi_1 \ldots \xi_n} F(x_1 \ldots x_n)\| < \varepsilon.$$

Def. 1.4 $F(x_1 \dots x_n)$ *is linear in* $x_1 \dots x_n$ if and only if it is additive in $x_1 \dots x_n$, and continuous in $x_1 \dots x_n$ for $x_i = 0$. Here, as in all the following, 0 is the zero of R or E_i according to the context.

Def. 1.5 $d^{x_1 \dots x_n}_{\xi_1 \dots \xi_n} F(x_1 \dots x_n)$, *the differential of* $F(x_1 \dots x_n)$ *in* $x_1 \dots x_n$ *with increments* $\xi_1 \dots \xi_n$, is in E_0 for any ξ_i, linear in $\xi_1 \dots \xi_n$, and for any ε some δ exists such that for any ξ_i, $\max \|\xi_i\| < \delta$ implies

$$\| \Delta^{x_1 \dots x_n}_{\xi_1 \dots \xi_n} F(x_1 \dots x_n) - d^{x_1 \dots x_n}_{\xi_1 \dots \xi_n} F(x_1 \dots x_n) \| \leq \varepsilon \max \|\xi_i\|.$$

It is easily shown that the differential is unique if it exists.

Def. 1.6 $F(x_1 \dots x_n)$ *is differentiable in* $x_1 \dots x_n$ if and only if $d^{x_1 \dots x_n}_{\xi_1 \dots \xi_n} F(x_1 \dots x_n)$ exists.

Definitions of continuity and differentiability equivalent to those given above are obtained when $\max \|\xi_i\|$ is replaced by any function $\varrho(\xi_i)$ on $E_1 \dots E_n$ to non-negative R, with the property that for some positive a, b, c in R

$$a \max \|\xi_i\| \leq \varrho(\xi_i) \leq b \max \|\xi_i\|$$

for any ξ_i such that $\max \|\xi_i\| < c$. This will be evident from the two simple propositions which follow, in which $\sigma(\xi_1 \dots \xi_n)$ and $\tau(\varepsilon)$ are functions to non-negative R, on $E_1 \dots E_n$ and on positive R respectively.

A. $\sigma(\xi_1 \dots \xi_n) < \varepsilon$ *if* $\max \|\xi_i\| < \tau(\varepsilon)$ *implies*

$\sigma(\xi_1 \dots \xi_n) < \varepsilon$ *if* $\varrho(\xi_i) < a\tau(\varepsilon)$,

and $\sigma(\xi_1 \dots \xi_n) < \varepsilon$ *if* $\varrho(\xi_i) < \tau(\varepsilon)$ *implies*

$\sigma(\xi_1 \dots \xi_n) < \varepsilon$ *if* $\max \|\xi_i\| < \frac{1}{b}\tau(\varepsilon)$.

B. $\sigma(\xi_1 \dots \xi_n) < \varepsilon \max \|\xi_i\|$ *if* $\max \|\xi_i\| < \tau(\varepsilon)$ *implies*

$\sigma(\xi_1 \dots \xi_n) < \varepsilon \varrho(\xi_i)$ *if* $\varrho(\xi_i) < a\tau\left(\frac{\varepsilon}{a}\right)$,

and $\sigma(\xi_1 \dots \xi_n) < \varepsilon \varrho(\xi_i)$ *if* $\varrho(\xi_i) < \tau(\varepsilon)$ *implies*

$\sigma(\xi_1 \dots \xi_n) < \varepsilon \max \|\xi_i\|$ *if* $\max \|\xi_i\| < \frac{1}{b}\tau(b\,\varepsilon)$.

Evidently $\max \|\xi_i\|$ is an instance of $\varrho(\xi_i)$, with $a \leq 1 \leq b$, and for any positive t in R

$$\varrho_t(\xi_i) = \left\{ \sum_{i=1}^{n} \|\xi_i\|^t \right\}^{\frac{1}{t}}$$

is an instance of $\varrho(\xi_i)$, with $a \leq 1$ and $n^{\frac{1}{i}} \leq b$. More complicated instances are easily constructed. $\varrho_1(\xi_i)$ and $\varrho_2(\xi_i)$ have sometimes been used in defining continuity and differentiability, but, since the results do not depend on the choice of $\varrho(\xi_i)$, we have preferred the simpler function $\max \|\xi_i\|$.

The derivative and Riemann integral of a function on R to E_0 are familiar notions, but for completeness we give their definitions here. t, τ, t_k, τ_k are in R.

Def. 1.7 For some δ, let $p(\tau)$ be in E_0 for $|\tau - t| < \delta$. Then $\frac{d}{dt}p(t)$, *the derivative of* $p(t)$ *with respect* to t, is in E_0, and for any ε some δ exists such that for any τ

$$|\tau| < \delta \text{ implies } \left\| \Delta_\tau^t p(t) - \tau \frac{d}{dt} p(t) \right\| \leq \varepsilon |\tau|.$$

We shall write $\gamma_\sigma p(\sigma)$ for $\left\{ \frac{dp(t)}{dt} \right\}_{t=0}$, the σ being a mere mark, like the σ in Def. 1.8. Comparing definitions 1.7 and 1.5, we see that if either one of the differential of $p(t)$ and the derivative of $p(t)$ exists, so does the other, and

$$d_\tau^t p(t) = \tau \frac{d}{dt} p(t).$$

Def. 1.8 Let a, b be in R. If $a < b$, let t_k, τ_k satisfy

$$a = t_1 < t_2 < \cdots < t_m = b \text{ and } t_k \leq \tau_k \leq t_{k+1} \text{ for } k = 1, 2, \ldots, m-1,$$

where m is a positive integer, evidently uniquely determined by the function t_k. Let $p(t)$ be in E_0 for $a \leq t \leq b$ or $a = t = b$ or $a \geq t \geq b$, according as $a < b$, $a = b$, $a > b$. Then

$$\int_a^b p(\sigma)\, d\sigma,$$

the Riemann integral of $p(\sigma)$ *in* σ *from* a *to* b, is in E_0 and satisfies

1) if $a < b$, then for any ε some δ exists such that for any t_k, τ_k

$$\max_{1 \leq k \leq m-1} (t_{k+1} - t_k) < \delta \text{ implies}$$

$$\left\| \sum_{k=1}^{m-1} (t_{k+1} - t_k) p(\tau_k) - \int_a^b p(\sigma)\, d\sigma \right\| < \varepsilon;$$

2) if $a = b$, then

$$\int_a^b p(\sigma)\,d\sigma = 0;$$

3) if $a > b$, then

$$\int_a^b p(\sigma)\,d\sigma = -\int_b^a p(\sigma)\,d\sigma.$$

It is easily shown that the integral is unique if it exists.

We shall frequently require the following theorems, proofs of which will be found in the papers indicated. In each theorem involving $F(x_1 \ldots x_n)$ the function is in E_0 for the values of the arguments considered.

Theorem 1.1 If *$F(x_1 \ldots x_n)$ is differentiable in $x_1 \ldots x_n$, then for any ξ_i $\gamma_\sigma F(x_1 + \sigma\xi_1 \ \ldots \ x_n + \sigma\xi_n)$ exists equal to*[1]

$$d^{x_1 \ldots x_n}_{\xi_1 \ldots \xi_n} F(x_1 \ldots x_n).$$

Theorem 1.2 *Let $\eta_1 \ldots \eta_n$ be any permutation of $x_1 \ldots x_n$, and $v_1 \ldots v_n$ the same permutation of $\xi_1 \ldots \xi_n$, so that $\eta_i = x_{k_i}$ and $v_i = \xi_{k_i}$. Let ζ_i be the zero in E_{k_i}. If $F(x_1 \ldots x_n)$ is differentiable in $x_1 \ldots x_n$, then it is differentiable in $\eta_1 \ldots \eta_i$ and*[1]

$$d^{\eta_1 \ldots \eta_i}_{v_1 \ldots v_i} F(x_1 \ldots x_n) = d^{\eta_1 \ldots \eta_i \eta_{i+1} \ldots \eta_n}_{v_1 \ldots v_i \zeta_{i+1} \ldots \zeta_n} F(x_1 \ldots x_n).$$

Theorem 1.3 *Let $G_i = G_i(x_1 \ldots x_n)$ be on $E_1 \ldots E_n$ to E_i. If $F(G_1 \ldots G_n)$ is differentiable in $G_1 \ldots G_n$, and for any i $G_i(x_1 \ldots x_n)$ is differentiable in $x_1 \ldots x_n$, then $F(G_1(x_1 \ldots x_n) \ldots G_n(x_1 \ldots x_n))$ is differentiable in $x_1 \ldots x_n$, and*

$$d^{x_1 \ldots x_n}_{\xi_1 \ldots \xi_n} F(G_1(x_1 \ldots x_n) \ldots G_n(x_1 \ldots x_n)) = d^{G_1 \ldots G_n}_{H_1 \ldots H_n} F(G_1 \ldots G_n),$$

where[1]

$$H_i = d^{x_1 \ldots x_n}_{\xi_1 \ldots \xi_n} G_i(x_1 \ldots x_n).$$

Theorem 1.4 *Let η_i be in E_i. If for some δ and any η_i, ξ_i such that $\max\limits_{1 \le j \le i} \|\eta_j - x_j\| < \delta$, $F(\eta_1 \ldots \eta_n)$ is differentiable in $\eta_1 \ldots \eta_i$, linear in $\eta_{i+1} \ldots \eta_n$, and*

[1] M. Fréchet, Ann. Sc. Ec. Normale, loc. cit.

$$d_{\xi_1 \ldots \xi_i}^{\eta_1 \ldots \eta_i} F(\eta_1 \ldots \eta_n)$$

is continuous in $\eta_1 \ldots \eta_i$, then $F(x_1 \ldots x_n)$ *is differentiable*[1] in $x_1 \ldots x_n$.

Now let a, b, r, s, t be in R, and suppose $a < b$.

Theorem 1.5 *Let* $p(t)$ *be in* E_0 *for* $a \leq t \leq b$. *If* $p(t)$ *is continuous in* t *for* $a \leq t \leq b$, *then*

$$\int_a^b p(\sigma)\, d\sigma$$

exists.[2]

Theorem 1.6 *For some* δ, *let* $p(r, t)$ *be in* E_0 *for* $|r - s| < \delta$ *and* $a \leq t \leq b$. *If* $p(s, t)$ *is continuous in* s, t *for* $a \leq t \leq b$, *then*

$$\int_a^b p(s, \sigma)\, d\sigma$$

is continuous in[3] s.

Theorem 1.7 *Let* $p(t)$ *be in* E_0 *for* $a \leq t \leq b$. *If* $\frac{d}{dt}p(t)$ *exists continuous in* t *for* $a \leq t \leq b$, *then*[4]

$$p(b) - p(a) = \int_a^b \frac{d}{d\sigma} p(\sigma)\, d\sigma.$$

Theorem 1.8 *Suppose that for some* δ, $p(r, t)$ *is in* E_0 *and* $\frac{d}{dr}p(r, t)$ *exists continuous in* r, t *for* $|r - s| < \delta$ *and* $a \leq t \leq b$. *Then the integrals*

$$\frac{d}{ds}\int_a^b p(s, \sigma)\, d\sigma, \quad \int_a^b \frac{d}{ds} p(s, \sigma)\, d\sigma$$

exist, are continuous in s, *and are equal.*[3]

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[1] A. D. Michal, Annali di Matematica (1936).
V. Elconin, Thesis (Calif. Inst. of Tech., 1937).

[2] L. M. Graves, Trans. of Amer. Math. Soc., vol. 29 (1927), pp. 163—177. Cf. also M. Kerner, Prace Matematyczno-Fizyczne, vol. XL (1932), pp. 47—67.

[3] V. Elconin, loc. cit.

[4] M. Kerner, Prace Mat.-Fiz., loc. cit. See also Graves, loc. cit.

In this section no use has been made of the completeness of $E_1, E_2, \ldots, E_n$, and that of E_0 is needed only in the proofs of theorems 1.5, 1.6, 1.7 and 1.8.

Special definitions and notations will be given when necessary, but if the contrary is not explicitly indicated, the symbolism of this section will apply throughout the paper.

2. The Differential Equation $d_\xi^x f(x) = F(x, f, \xi)$.

In this section E, Σ are Banach spaces,

$$x, u, \xi, z, \zeta \text{ are in } E,$$

$$y, v, \eta \text{ are in } \Sigma,$$

$$a, b, c, g \text{ are positive in } R,$$

$$m \text{ is a non-negative integer in } \mathrm{R}$$

and $\Phi(x)$ is any function on E to Σ. The main results of the section are assembled in

Theorem I. *For any x, y, z, ξ, η such that $\|x-u\| < a$ and $\|y-v\| \leq b$, let $F(x, y, z)$ be in Σ, linear in z, and such that*

$$G(x, y, z, \xi, \eta) \equiv d_{\xi\eta}^{xy} F(x, y, z)$$

exists, continuous in x, y; and for $\|x-u\| < a$ and any m, let

$$f_0(x) \text{ be } v$$

and let $f_{m+1}(x)$ *be* $v + \int_0^1 F(u+\sigma(x-u), f_m(u+\sigma(x-u)), x-u)\, d\sigma$. *Then*

1) *For some $c \leq a$ and any x such that $\|x-u\| < c$,*

$$f_m(x) \text{ exists and } \|f_m(x) - v\| < b.$$

2) *For any such c: if*

$$\|G(x, y, z, 0, \eta)\| \leq g\|z\|\|\eta\|$$

for some g and any x, y, z, η such that $\|x-u\| < c$ and $\|x-v\| \leq b$, then

(i) $$f(x) \equiv \lim_{m\to\infty} f_m(x) \text{ exists for } \|x-u\| < c;$$

(ii) $$G(x, y, z, \xi, F(x, y, \xi)) = G(x, y, \xi, z, F(x, y, z))$$

for any x, y, z, ξ *such that* $\|x-u\| < c$ *and* $\|y-v\| \leq b$ *implies*[1]

$$f(u) = v$$
$$\|f(x) - v\| \leq b$$
$$d_\xi^x f(x) = F(x, f, \xi)$$
$$d_z^x d_\xi^x f(x) = G(x, f, \xi, z, F(x, f, z))$$

for any x, ξ, z *such that* $\|x-u\| < c$;

(iii)
$$\text{if} \quad \Phi(u) = v,$$
$$\|\Phi(x) - v\| \leq b,$$
$$d_\xi^x \Phi(x) = F(x, \Phi, \xi),$$

for any x, ξ *such that* $\|x-u\| < c$, *then for* $\|x-u\| < c$

$$\Phi(x) = f(x).$$

Proof. To simplify the notations we assume $u = 0$, $v = 0$, but the argument will be valid for any u, v; or it may be shown that the theorem is true if true for $u = 0$, $v = 0$.

It is well known that a function is continuous if it is differentiable. Hence for $\|x\| < a$ and $\|y\| < b$, $F(x, y, z)$ is continuous in x, y for any z; and since it is linear in z by hypothesis, we conclude, by theorems due to Kerner[2], that

(2. 1) $F(x, y, z)$ is continuous in x, y, z if $\|x\| < a$ and $\|y\| \leq b$,

and for some positive p, q, r in R such that $p \leq a$ and $q \leq b$

(2. 2) $\|F(x, y, z)\| < r\|z\|$ if $\|x\| < p$ and $\|y\| \leq q$.

Let p, q, r be any such numbers, and suppose $c = \min.\left(p, \dfrac{q}{r}\right)$. Then $\|x\| < c$ implies: if $f_m(sx)$ exists continuous in s and

$$\|f_m(sx\| < b \text{ for } 0 \leq s \leq 1,$$

then by (2. 1) and the continuity of a composite function formed of continuous functions, $F(s\sigma x, f_m(s\sigma x), sx)$ is continuous in s, σ for $0 \leq s$, $\sigma \leq 1$;

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[1] The preceding identity is the condition of complete integrability for the differential equation $d_\xi^x f(x) = F(x, f, \xi)$.

[2] M. Kerner, Studia Mathematica, vol. III (1931) pp. 156—162; Annals of Math., loc. cit.

$$f_{m+1}(sx) = \int_0^1 F(s\sigma x,\ f_m(s\sigma x),\ sx)\,d\sigma$$

exists continuous in s, by theorems 1.5, 1.6; and

$$\| f_{m+1}(sx) \| < sr \| x \| < sq < b.$$

But

$$f_0(x) = 0 \text{ if } \| x \| < c \leq a.$$

Hence by induction the first conclusion of the theorem is established, with $c = \min\left(p,\ \frac{q}{r}\right)$.

Now let c have any value $\leq a$ for which conclusion 1) holds; let s, t satisfy $0 \leq s$, $t \leq 1$, and suppose x, ξ such that $\| sx + t\xi \| < c$ for any s, t. By an induction similar to the one just completed, it follows that for any m

(2.3) $f_m(sx + t\xi)$ is continuous in s, t, and $\| f_m(sx + t\xi) \| < b$.

Hence writing $\nu = sx + t\xi$ and using theorems 1.1, 1.2, 1.7,

$$f_{m+1}(\nu) - f_m(\nu)$$
$$= \int_0^1 \int_0^1 G(\sigma\nu,\ (1-\tau) f_{m-1}(\sigma\nu) + \tau f_m(\sigma\nu),\ \nu,\ 0,\ f_m(\sigma\nu) - f_{m-1}(\sigma\nu))\,d\sigma\,d\tau,$$

and, by the premise in conclusion 2),

$$\| f_{m+1}(\nu) - f_m(\nu) \| \leq g \| \nu \| \int_0^1 \| f_m(\sigma\nu) - f_{m-1}(\sigma\nu) \| \,d\sigma. \tag{2.4}$$

But by (2.1) the function $\| F(\sigma\nu,\ 0,\ \nu) \|$ is continuous in s, t, σ for $0 \leq \sigma \leq 1$. Hence for some positive p in R

$$\| F(\sigma\nu,\ 0,\ \nu) \| < p \text{ for } 0 \leq s,\ t,\ \sigma \leq 1,$$

and

$$\| f_1(\nu) - f_0(\nu) \| = \| f_1(\nu) \| \leq \int_0^1 \| F(\sigma\nu,\ 0,\ \nu) \| \,d\sigma < p.$$

By induction we obtain for any m

$$\| f_{m+1}(\nu) - f_m(\nu) \| < p \frac{\| g\nu \|^m}{m!} < cp \frac{g^n \| \nu \|^{m-1}}{m!} < p \frac{(gc)^m}{m!}; \tag{2.5}$$

from which

(2.6) $$f(\nu) = \lim_{m \to \infty} f_m(\nu) \quad \text{exists,}$$

and

(2.7) $f_m(sx + t\xi)$ converges to $f(sx + t\xi)$ uniformly in s, t.

By (2.3), (2.7), and the continuity of the limit of a uniformly convergent sequence of continuous functions,

(2.8) $f(sx + t\xi)$ is continuous in s, t,

and

$$\| f(sx + t\xi \| \leq b.$$

Therefore by theorem 1.5

(2.9) $$\int_0^1 F(\sigma\nu, f(\sigma\nu), \nu) d\sigma \quad \text{exists.}$$

The argument leading to (2.4) shows that

$$\| F(\nu, f_m(\nu), z) - F(\nu, f(\nu), z) \| \leq g \, |z| \, \| f_m(\nu) - f(\nu) \|,$$

and by (2.7)

(2.10) $F(\nu, f_m(\nu), \nu)$ converges to $F(\nu, f(\nu), \nu)$ uniformly in s, t.

In particular $F(\sigma\nu, f_m(\sigma\nu), \nu)$ converges to $F(\sigma\nu, f(\sigma\nu), \nu)$ uniformly in σ for $0 \leq \sigma \leq 1$; from which, by (2.9) and a well known theorem on the integral of a uniformly convergent sequence of integrable functions[1]

$$\lim_{m \to \infty} \int_0^1 F(\sigma\nu, f_m(\sigma\nu), \nu) d\sigma = \int_0^1 F(\sigma\nu, f(\sigma\nu), \nu) \, d\sigma.$$

This with (2.6) gives

(2.11) $$f(x) = \int_0^1 F(\sigma x, f(\sigma x), x) d\sigma$$

for any x such that $\|x\| < c$.

Let $\Theta(x)$ be any solution of this integral equation such that $\|\Theta(x)\| \leq b$ for $\|x\| < c$. Then by an easy induction

[1] M. Kerner, Prace Mat.-Fiz., loc. cit.

$$\| \Theta(x) - f_m(x) \| < \| \Theta(x) \| \frac{(g c)^m}{m!} \leq b \frac{(g c)^m}{m!},$$

and

(2. 12) $$\Theta(x) = f(x) \text{ for } \|x\| < c.$$

From (2. 6) conclusion (i) of the theorem follows immediately.

From (2. 8) and (2. 11),

(2. 13) $$f(0) = 0 \text{ and } \|f(x)\| \leq b \text{ for } \|x\| < c,$$

which contains the first two parts of conclusion (ii), the remainder of which is obtained as follows. Suppose that $\|x\| < a$ and $\|y\| \leq b$. Then $G(x, y, z, 0, \eta)$ is linear in z, since it is evidently additive in z, and it is continuous in z at $z = 0$ by

$$\| G(x, y, z, 0, \eta) \| \leq g \|z\| \|\eta\|.$$

Moreover $G(x, y, z, \xi, 0)$ is linear in z, since by the premise in conclusion (ii), it is equal to

$$G(x, y, \xi, z, 0) + G(x, y, \xi, 0, F(x, y, z)) - G(x, y, z, 0, F(x, y, \xi)),$$

each term of which is linear in z. Hence

(2. 14) $$G(x, y, z, \xi, \eta) \text{ is linear in } z;$$

and since it is by definition linear in ξ, η, a double application of one of Kerner's theorems[1] gives

(2. 15) $$G(x, y, z, \xi, \eta) \text{ is continuous in } x, y, z, \xi, \eta.$$

Also from (2. 14), and theorems 1. 2, 1. 4 $F(x, y, z)$ is differentiable in x, y, z, and

(2. 16) $$d^{xyz}_{\xi\eta\zeta} F(x, y, z) = F(x, y, \zeta) + G(x, y, z, \xi, \eta).$$

Now suppose x, ξ such that $\|sx + t\xi\| < c$ for $0 \leq s$, $t \leq 1$, and write $\nu = sx + t\xi$. If for $0 \leq s$, $t \leq 1$, $\frac{d}{dt} f_m(\nu)$ exists continuous in s, t, then, by (2. 15), (2. 16), and theorems 1. 1, 1. 2, 1. 3, $0 \leq s$, t, $\sigma \leq 1$ implies

$$\frac{d}{dt} F(\sigma\nu, f_m(\sigma\nu), \nu)$$

exists continuous in s, t, σ, and equal to

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[1] M. Kerner, Annals of Math., loc. cit.

$$F(\sigma\nu,\ f_m(\sigma\nu),\ \xi) + G\left(\sigma\nu,\ f_m(\sigma\nu),\ \nu,\ \sigma\xi,\ \frac{d}{dt} f_m(\sigma\nu)\right);$$

and by theorem 1.8

$$\frac{d}{dt} f_{m+1}(\nu) \quad \text{exists equal to}$$

$$\int_0^1 \frac{d}{dt} F(\sigma\nu,\ f_m(\sigma\nu),\ \nu)\, d\sigma.$$

But $0 \leq s,\ t \leq 1$ implies $\frac{d}{dt} f_0(\nu) = 0$. Hence by induction

$$(2.17)\quad \begin{cases} \text{for any } m,\ 0 \leq s,\ t \leq 1 \text{ implies} \\ \frac{d}{dt} f_{m+1}(\nu) \text{ exists, continuous in } s,\ t, \text{ and equal to} \\ \int_0^1 F(\sigma\nu,\ f_m(\sigma\nu),\ \xi)\, d\sigma + \int_0^1 G\left(\sigma\nu,\ f_m(\sigma\nu),\ \nu,\ \sigma\xi,\ \frac{d}{dt} f_m(\sigma\nu)\right) d\sigma. \end{cases}$$

Now suppose $\|x\| < c$. By (2.17)

$$\gamma_\sigma f_{m+1}(x + \sigma\xi) = \int_0^1 F(sx,\ f_m(sx),\ \xi)\, ds + \int_0^1 s\gamma_\sigma F(sx + \sigma\xi,\ f_m(sx + s\xi),\ x)\, ds.$$

The last integral equals

$$\int_0^1 s \frac{d}{dt} F(sx,\ f_m(sx),\ \xi)\, ds + \int_0^1 s\gamma_\sigma \{F(sx + \sigma\xi,\ f_m(sx + \sigma\xi),\ x) - F(sx + \sigma x,\ f_m(sx + \sigma x),\ \xi)\}\, ds.$$

Integrating the first of these by parts:

$$\gamma_\sigma f_{m+1}(x + \sigma\xi) = F(x, f_m(x),\ \xi) + \int_0^1 s\gamma_\sigma \{F(sx + \sigma\xi,\ f_m(sx + \sigma\xi),\ x) - F(sx + \sigma x,\ f_m(sx + \sigma x),\ \xi)\}\, ds.$$

Hence

$$\gamma_\sigma f_{m+1}(x+\sigma x) = F(x, f_m(x), x),$$

and

$$\gamma_\sigma f_{m+1}(x+\sigma\xi) = F(x, f_m(x), \xi) + \int_0^1 s\{G(sx, f_m(sx), x, \xi, \gamma_\sigma f_m(sx+\sigma\xi))$$

$$- G(sx, f_m(sx), \xi, x, \gamma_\sigma f_m(sx+\sigma x))\}\, ds = F(x, f_m(x), \xi)$$

$$+ \int_0^1 s\{G(sx, f_m(sx), x, 0, \gamma_\sigma f_m(sx+\sigma\xi) - F(sx, f_m(sx), \xi))$$

$$- G(sx, f_m(sx), \xi, 0, F(sx, f_{m-1}(sx), x) - F(sx, f_m(sx), x))\}\, ds,$$

since

$$G(sx, f_m(sx), x, \xi, F(sx, f_m(sx), \xi))$$

$$- G(sx, f_m(sx), \xi, x, F(sx, f_m(sx), x)) = 0.$$

For $m > 0$, let

$$A_m(x, \xi) = \gamma_\sigma f_m(x+\sigma\xi) - F(x, f_m(x), \xi),$$

and

$$B_m(x, \xi) = F(x, f_{m-1}(x), \xi) - F(x, f_m(x), \xi).$$

Then

$$A_{m+1}(x, \xi) = B_{m+1}(x, \xi) + \int_0^1 \{G(sx, f_m(sx), x, 0, A_m(sx, s\xi))$$

$$- G(sx, f_m(sx), \xi, 0, B_m(sx, sx))\}\, ds.$$

Let ξ be such that $\|x+t\xi\| < c$ for $0 \le t \le 1$, and write $v = x + t\xi$. The argument leading to (2.4) shows that for $0 \le t \le 1$:

$$\|B_m(v, \xi)\| \le g\|\xi\|\,\|f_m(v) - f_{m-1}(v)\|,$$

and as in (2.5), for some positive p in R,

$$\|B_m(v, \xi)\| < cp\frac{g^m\|v\|^{m-2}\|\xi\|}{(m-1)!} < q\frac{(gr)^{m-1}}{(m-1)!},$$

where $g = cpq$ and $r = c + \|\xi\|$; and

$$\|A_{m+1}(\nu,\ \xi)\| < q\frac{(gr)^m}{m!} + gr\int_0^1 \{\|A_m(s\nu,\ s\xi)\| + \|B_m(s\nu,\ s\nu)\|\}\,ds$$

$$< 2q\frac{(gr)^m}{m!} + gr\int_0^1 \|A_m(s\nu,\ s\xi)\|\,ds.$$

By (2. 1), for some positive h in R, $0 \leq t \leq 1$ implies

$$\|A_1(\nu,\ \xi)\| < 2q + gr\int_0^1 \|F(s\nu,\ 0,\ s\xi)\|\,ds < 2q + hgr;$$

and by induction: $0 \leq t \leq 1$ implies

$$\|A_m(\nu,\ \xi)\| < 2q\frac{(2gr)^{m-1}}{(m-1)!} + h\frac{(gr)^m}{m!} \quad \text{for } m > 0;$$

hence $A_m(\nu, \xi)$ converges to zero uniformly in t; and by (2. 10)

$$(2.18) \qquad \begin{cases} \dfrac{d}{dt} f_m(x + t\xi) \text{ converges to } F(x + t\xi,\ f(x + t\xi),\ \xi) \\ \text{uniformly in } t \text{ for } 0 \leq t \leq 1. \end{cases}$$

From (2. 17) and theorem 1. 7

$$f_m(x + \xi) - f_m(x) = \int_0^1 \frac{d}{dt} f_m(x + t\xi)\,dt;$$

and

$$f(x + \xi) - f(x) = \lim_{m\to\infty} \int_0^1 \frac{d}{dt} f_m(x + t\xi)\,dt = \int_0^1 F(x + t\xi,\ f(x + t\xi),\ \xi)\,dt$$

from (2. 18) and the theorem on integrals of uniformly convergent sequences used to prove (2. 11). Hence

$$\Delta_\xi^x f(x) - F(x, f, \xi)$$

$$= \int_0^1 \int_0^\sigma G(x + \tau\xi,\ f(x + \tau\xi),\ \xi,\ \xi,\ F(x + \tau\xi,\ f(x + \tau\xi),\ \xi))\,d\sigma\,d\tau,$$

and by one of Kerner's theorems[1], it can be shown that the norm of the integrand in the right member is less than som multiple of $\|\xi\|^2$ for all sufficiently small $\|\xi\|$. Since $F(x, f, \xi)$ is linear in ξ, it follows that

$$d_\xi^x f(x) = F(x, f, \xi), \tag{2. 19}$$

and, differentiating,

$$d_z^x d_\xi^x f(x) = G(x, f, \xi, z, F(x, f, z))$$

for $\|x\| < c$. This establishes conclusion (ii).

If the premise of conclusion (iii) is satisfied, then

$$\|\Phi(x)\| \leq b$$

and by theorem 1. 7,

$$\Phi(x) = \int_0^1 F(\sigma x, \Phi(\sigma x), x)\, d\sigma$$

for $\|x\| < c$. By (2. 12) conclusion (iii) follows, and the proof of the theorem is complete.

3. Special Cases of Theorem I.

The definitions in section 2 preceding Theorem I are retained in this section.

If $F(x, y, z)$ satisfies the premise of Theorem I for *any* $y\,(b = \infty)$, then we can strengthen the conclusion of the theorem and obtain

Theorem 3.1. *For any x, y, z, ξ, η such that $\|x - u\| < a$, let*

$F(x, y, z)$ be in Σ, linear in z, and such that

$d_{\xi\eta}^{xy} F(x, y, z) = G(x, y, z, \xi, \eta)$ exists, continuous in x, y;

and for $\|x - u\| < a$ and any m, let

$f_0(x)$ be v

and let $f_{m+1}(x)$ be $v + \int_0^1 F(u + \sigma(x - u), f_m(u + \sigma(x + u)), x - u)\, d\sigma.$

[1] M. Kerner, Annals of Math., loc. cit.

Then

1) *For any* x *such that* $\|x-u\|<a$, $f_m(x)$ *exists.*

2) *If*

$$\|G(x, y, z, 0, \eta)\| \leq g\|z\|\|\eta\|$$

for some g *and any* x, y, z, η *such that* $\|x-u\|<a$, *then*

$$\text{(i)} \qquad f(x)=\lim_{m \to \infty} f_m(x) \quad \textit{exists for} \quad \|x-u\|<a;$$

$$\text{(ii)} \qquad G(x, y, z, \xi, F(x, y, \xi)) = G(x, y, \xi, z, F(x, y, z))$$

for any x, y, z, ξ *such that* $\|x-u\|<a$ *implies*

$$f(u)=v, \ d_\xi^x f(x) = F(x, f, \xi),$$

$$d_z^x d_\xi^x f(x) = G(x, f, \xi, z, F(x, y, z))$$

for any x, ξ, z *such that* $\|x-u\|<a$;

$$\text{(iii)} \qquad \textit{if} \quad \Phi(u)=v \quad \textit{and}$$

$$d_\xi^x \Phi(x) = F(x, \Phi, \xi)$$

for any x, ξ *such that* $\|x-u\|<a$, *then for* $\|x-u\|<a$

$$\Phi(x)=f(x).$$

Proof. The proof of Theorem I applies here, with the following simplifications: replace $c \leq a$ by $c=a$; omit the proof that

$$\|f_m(x)\|<b, \ \|f(x)\| \leq b;$$

and in the statement preceding (2. 12), omit the phrase »such that $\|\Theta(x)\| \leq b$ for $\|x\|<c$» and the inequality

$$\|\Theta(x)\| \frac{(gc)^m}{m!} \leq b \frac{(gc)^m}{m!}.$$

If moreover $F(x, y, z)$ is *constant* in y, the first condition in conclusion 2) above becomes redundant. Removing it, we obtain the following theorem, which has been proved in a different way by Kerner.[1]

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[1] Annals of Math., loc. cit.

Theorem 3.2. *If for any x, ξ, z such that $\|x-u\|<a$*

$$F(x, z) \text{ is in } \Sigma, \text{ linear in } z,$$

$$d_{\xi}^{x} F(x, z) \text{ exists, continuous in } x, \text{ then}$$

(i)
$$f(x) = v + \int_{0}^{1} F(u+\sigma(x-u),\ x-u)\, d\sigma$$

exists for $\|x-u\|<a$;

(ii) *if*

$$d_{\xi}^{x} F(x, z) = d_{z}^{x} F(x, \xi)$$

for any x, ξ, z such that $\|x-u\|<a$, then

$$f(u)=v,\ d_{\xi}^{x} f(x) = F(x, \xi),\ d_{z}^{x}\, d_{\xi}^{x} f(x) = d_{z}^{x} F(x, \xi)$$

for any x, ξ, z such that $\|x-u\|<a$;

(iii) *if $\Phi(u)=v$ and*

$$d_{\xi}^{x}\, \Phi(x) = F(x, \xi)$$

for any x, ξ such that $\|x-u\|<a$, then for $\|x-u\|<a$

$$\Phi(x) = f(x).$$

Proof. Regarding $F(x, z)$ as a function of x, y, z constant in y, we have by theorem 1.2,

$$d_{0\,\eta}^{x\,y} F(x, z) = d_{\eta}^{y} F(x, z) = 0,$$

and by hypothesis

$$d_{\xi}^{x} F(x, z) - d_{z}^{x} F(x, \xi) = d_{\xi\,\eta}^{x\,y} F(x, z) - d_{z\,\eta}^{x\,y} F(x, \xi) = 0$$

identically in η. The conditions of conclusion 2) in Theorem 3.1 are therefore satisfied, and it is easy to verify that the others are also. The present theorem then follows from Theorem 3.1. Note that here $f_1(x) = f_m(x) = f(x)$.

Now suppose E is R. If $F(x, y, z)$ satisfied the conditions of Theorem I, then evidently

$$F(x, y, z) = z\, F(x, y, 1)$$

and

$$G(x, y, z, \xi, F(x, y, z)) = z\xi\ G(x, y, 1, 1, F(x, y, 1)).$$

Hence the symmetry condition in conclusion 2) of Theorem I is redundant. Removing it, we obtain

Theorem 3.3. *Suppose that for any* x, y, ξ, η *such that* $\|x-u\| < a$ *and* $\|y-v\| \leq b$

$F(x, y)$ *is in* Σ,

$d_{\xi\eta}^{xy} F(x, y) = G(x, y, \xi, \eta)$ *exists, continuous in* x, y;

and for $\|x-u\| < a$ *and any* m, *let* $f_0(x)$ *be* v *and*

$$f_{m+1}(x) \text{ be } v + \int_u^x F(\sigma, f_m(\sigma))\, d\sigma.$$

Then

1) *For some* $c \leq a$ *and any* x *such that* $\|x-u\| < c$

$$f_m(x) \text{ exists and } \|f_m(x) - v\| < b.$$

2) *For any such* c: *if*

$$\|G(x, y, 0, \eta)\| \leq g\|\eta\|$$

for some g *and any* x, y, η *such that* $\|x-u\| < c$ *and* $\|y-v\| \leq b$, *then*

(i) $$f(x) = \lim_{m\to\infty} f_m(x) \text{ exists for } \|x-u\| < c;$$

(ii) *for any* x *such that* $\|x-u\| < c$

$$f(u) = v,\ \|f(x) - v\| \leq b,\ \frac{d}{dx} f(x) = F(x, f),$$

$$\frac{d^2 f(x)}{dx^2} = G(x, f, 1, 1, F(x, f));$$

(iii) *if*

$$\Phi(u) = v,\ \|\Phi(x) - v\| \leq b,\ \frac{d\Phi(x)}{dx} = F(x, \Phi)$$

for any x *such that* $\|x-u\| < c$, *then for* $\|x-u\| < c$

$$\Phi(x) = f(x).$$

Proof. The theorem follows from Theorem I and the integral identity

$$(3.1)\qquad \int_0^1 F(u+\sigma(x-u),\ \Phi(u+\sigma(x-u)))(x-u)\,d\sigma = \int_u^x F(\sigma,\ \Phi(\sigma))\,d\sigma.$$

Moreover, using this identity we can replace the 'differentiation under the integral sign' in the proof of Theorem I by differentiation with respect to the upper limit, and so prove the following theorem due to Kerner[1], in which the hypotheses are much weaker than those of Theorem 3.3 — from these we can still conclude the existence of $\frac{d}{dx}f(x)$, but not that of $\frac{d^2}{dx^2}f(x)$.

Theorem 3.4. *Suppose that for any x, y, η such that $|x-u|<a$ and $|y-v|\le b$*

$$F(x,\ y) \textit{ is in } \Sigma \textit{ continuous in } x,\ y$$

$$d_\eta^y F(x,\ y) \textit{ exists, continuous in } x,\ y;$$

and for $|x-u|<a$ and any m, let $f_0(x)$ be v and

$$f_{m+1}(x) \textit{ be } v + \int_u^x F(\sigma,\ f_m(\sigma))\,d\sigma.$$

Then

1) *For some $c\le a$ and any x such that $|x-u|<c$*

$$f_m(x) \textit{ exists and } |f_m(x)-v|<b;$$

2) *For any such c: if*

$$|d_\eta^y F(x,\ y)| \le g\,|\eta|$$

for some g and any x, y, η such that $|x-u|<c$ and $|y-v|\le b$, then

(i) $$f(x)=\lim_{m\to\infty} f_m(x) \textit{ exists for } |x-u|<c;$$

(ii) *for any x such that $|x-u|<c$*

$$f(u)=v,\quad |f(x)-v|\le b,\quad \frac{d}{dx}f(x)=F(x,\ f);$$

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[1] Prace Mat.-Fiz., loc. cit.

(iii) *if*

$$\Phi(u) = v, \ \|\Phi(x) - v\| \leq b, \ \frac{d\Phi(x)}{dx} = F(x, \Phi)$$

for any x *such that* $\|x - u\| < c$, *then* $\Phi(x) = f(x)$ *for* $\|x - u\| < c$.

4. A Theorem Related to Theorem I.

Def. 4. 1. A set L of elements in a Banach space E form a »*domain*» if and only if for any x in L, some δ exists such that $\|x - \xi\| < \delta$ implies ξ is in L for any ξ in E.

Def. 4. 2. A set L of elements in a Banach space E is *convex* if and only if $0 \leq t \leq 1$ implies $tx + (1 - t)\xi$ is in L for any x, ξ in L.

In this section we shall understand that E, Σ are Banach spaces; x, u, ξ, z, ζ are in E; y, v, η are in Σ; L, $\varDelta$ are domains in E, Σ; λ is a domain in Σ whose closure lies in $\varDelta$; g is positlve in R; m is a non-negative integer in R; and $\Phi(x, v, u)$ is any function on L, $\varDelta$, L to $\varDelta$.

Let $\varDelta$ be a domain in λ such that for some δ, $\|y - \eta\| > \delta$ for any y, η in $\varDelta$, $\Sigma - \lambda$; and let μ be any such δ. Let D be a convex domain in L such that for some δ, $\|x\| < \delta$ for any x in D. And, for a considerable gain in brevity, let x, y be restricted to L, $\varDelta$ in the remainder of this section. With these notations we can state the following extension of Theorem I:

Theorem II. *Suppose that for any* x, y, z, ξ, η

$$F(x, y, z) \text{ is in } \Sigma, \text{ linear in } z,$$

$$d^{xy}_{\xi\eta} F(x, y, z) = G(x, y, z, \xi, \eta) \text{ exists, continuous in } x, y;$$

and for any m, x, v, u, *let* $f_0(x, v, u)$ *be* v *and* $f_{m+1}(x, v, u)$ *be*

$$v + \int_0^1 F(u + \sigma(x - u), f_m(u + \sigma(x - u), v, u), x - u)\, d\sigma.$$

Then for any $\varDelta$:

1) *Some* D *exists such that* $f_m(x, v, u)$ *exists and* $f_m(x, v, u)$ *is in* λ *whenever* x, v, u *are in* D, $\varDelta$, D.

2. *For any such D: If g exists such that*

$$\|G(x, y, z, 0, \eta)\| \leq g\|z\| \, \|\eta\|$$

whenever x, y are in D, Δ, then

(i)
$$f(x, v, u) = \lim_{m \to \infty} f_m(x, v, u)$$

exists for x, v, u in D, Δ, D;

(ii) *if*

$$G(x, y, z, \xi, F(x, y, \xi)) = G(x, y, \xi, z, F(x, y, z))$$

whenever x is in D, then

$$f(u, v, u) = v, \; f(x, v, u) \text{ is in } \Delta,$$

$$d_\xi^x f(x, v, u) = F(x, f, \xi),$$

$$d_z^x \, d_\xi^x f(x, v, u) = G(x, f, \xi, z, F(x, f, z)),$$

for x, v, u in D, Δ, D;

(iii) *if* $\Phi(u, v, u) = v$, $\Phi(x, v, u)$ *is in* Δ,

$$d_\xi^x \Phi(x, v, u) = F(x, \Phi, \xi)$$

whenever x, v, u are in D, Δ, D, then

$$\Phi(x, v, u) = f(x, v, u)$$

for x, v, u in D, Δ, D.

Proof. As in the proof of Theorem I, we can show that

(4.1) $F(x, y, z)$ is continuous in x, y, z.

Now suppose that

(4.2)
$$\|F(x, y, z)\| < \delta \|z\|$$

whenever x, y are in D, Δ. This will be true for some D, Δ, δ, by a theorem of Kerner[1]; and, as in the proof of Theorem I, we find by induction that

$$\|f_m(x, v, u) - v\| \leq \int_0^1 \|F(\ldots)\| \, d\sigma < \delta \|x - u\| < \mu$$

[1] Annals of Math., loc. cit.

for any x, v, u in d, $\varDelta$, d, where d is a domain in D such that

$$\|x-u\|<\frac{\mu}{\delta}$$

whenever x, u are in d. But it is evident that

$$\|f_m(x,\ v,\ u)-v\|<\mu$$

implies $f_m(x, v, u)$ is in λ whenever x, v, u are in L, $\varDelta$, L. Hence conclusion 1) of the theorem follows, with $D=d$. The remainder of the proof is a direct extension of that given for Theorem I.

The theorems of section 3 can evidently be modified to give special cases of Theorem II. For example, we have

Theorem 4. 1. *Assume the hypothesis of Theorem II, and suppose that the domain λ, and hence $\varDelta$, is the Banach space Σ. Then $f_m(x, v, u)$ exists if x, v, u are in D, Σ, D, and the remaining conclusions of Theorem II hold, with $\varDelta$ replaced by Σ.*

5. Applications of the Preceding Theorems.

Several known results may be obtained as instances of the preceding theorems.[1] Moreover, the abstract theory is useful in the proof of new results. For example, if E and Σ are each the classical real Hilbert space[2] H, Theorem I may be used to prove

Theorem 5. 1. *Let a, b, g be positive real numbers; let $u=(u^1, u^2, \ldots)$, $v=(v^1, v^2, \ldots)$ be in H, and suppose that for any $x=(x^1, x^2, \ldots)$ and $y=(y^1, y^2, \ldots)$ in H such that $\|x-u\|<a$ and $\|y-v\|\leq b$*

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[1] NIKLIBORC, W., Studia Math., 1 (1929), pp. 41—49;
THOMAS, T. Y., Annals of Math., 35 (1934), p. 734;
KERNER, M., Prace Matematyczno-Fizyczne, 40 (1932), pp. 47—67.
Also see our note, Proc. of Nat. Acad. of Sciences, 21 (1935), pp. 534—536, in which some of our results are summarized.

[2] STONE, Linear Transformations in Hilbert Space (1932).
HILBERT, Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. The *norm* $\left\{\sum_{i=1}^{\infty}(x^i)^2\right\}^{\frac{1}{2}}$ of an element $x=(x^1, x^2, \ldots)$ in H will be denoted by $\|x\|$; x^i is the ith *coordinate* of x.

(1) $$\Phi^i_j(x^1, x^2, \ldots; y^1, y^2, \ldots)$$

is in R;

(2) $$\sum_{i,j} (\Phi^i_j)^2 \text{ exists;}$$

(3) $$\frac{\partial \Phi^i_j}{\partial x^k}, \frac{\partial \Phi^i_j}{\partial y^k}, \qquad i, j, k = 1, 2, \ldots \text{ exist;}$$

(4) $$\sum_{i,j,k} \left(\frac{\partial \Phi^i_j}{\partial y^k}\right)^2 \text{ exists.}$$

Then

(i) $$\sum_l \frac{\partial \Phi^i_j}{\partial y^l} \Phi^l_k, \qquad i, j, k = 1, 2, \ldots \text{ exist.}$$

Moreover, if

(5) $$\sum_{i,j,k} \left(\frac{\partial \Phi^i_j}{\partial x^k}\right)^2 \text{ exists;}$$

(6) $$\sum_{i,j,k} \left(\frac{\partial \Phi^i_j}{\partial y^k}\right)^2 \leq g^2;$$

(7) *for any* $\varepsilon > 0$ *some* $\delta > 0$ *exists such that*

$$\sum_{i,j,k} \left(\Delta^{x^1 x^2 \ldots;\, y^1 y^2 \ldots}_{\xi^1 \xi^2 \ldots;\, \eta^1 \eta^2 \ldots} \frac{\partial \Phi^i_j}{\partial x^k}\right)^2 < \varepsilon \text{ and } \sum_{i,j,k} \left(\Delta^{x^1 x^2 \ldots;\, y^1 y^2 \ldots}_{\xi^1 \xi^2 \ldots;\, \eta^1 \eta^2 \ldots} \frac{\partial \Phi^i_j}{\partial y^k}\right)^2 < \varepsilon$$

if $\|\xi\|, \|\eta\| < \delta$;

(8) $$\frac{\partial \Phi^i_j}{\partial x^k} + \sum_l \frac{\partial \Phi^i_j}{\partial y^l} \Phi^l_k = \frac{\partial \Phi^i_k}{\partial x^j} = \sum_l \frac{\partial \Phi^i_k}{\partial y^l} \Phi^l_j;$$

then

(ii) *for some positive number* $c \leq a$, *the system of differential equations*[1]

(5. 1) $$df^i = \sum_j \Phi^i_j(x^1, x^2, \ldots; f^1, f^2, \ldots)\, dx^j, \quad i = 1, 2, \ldots,$$

and the conditions

(5. 2) $$\begin{cases} \|f - v\| \leq b, \text{ where } f = (f^1, f^2, \ldots), \\ f^i(u^1, u^2, \ldots) = v^i, \quad i = 1, 2, \ldots, \end{cases}$$

have a unique solution $f^i(x^1, x^2, \ldots)$, $i = 1, 2, \ldots$, *for* $\|x - u\| < c$.

---

[1] $df^i(x^1, x^2, \ldots) \equiv d^x_\omega \Psi^i(x)$, $i = 1, 2, \ldots$; *where* $x = (x^1, x^2, \ldots)$ *and* $\omega = (dx^1, dx^2, \ldots)$ *are in* H, *and* $\Psi^i(x) = f^i(x^1, x^2, \ldots)$, $i = 1, 2, \ldots$.

Proof. (i) follows from (1)—(4) and the Schwarzian inequality. For $\|x-u\|<a$, $\|y-v\|\leq b$, and any elements z, ξ, η in H, let $F(x, y, z)$, $G(x, y, z, \xi, \eta)$ be functions with values in H whose ith coordinates are $\sum_j \Phi^i_j z^j$ and

$$\sum_{j,k}\left(\frac{\partial \Phi^i_j}{\partial x^k} z^j \xi^k + \frac{\partial \Phi^i_j}{\partial y^k} z^j \eta^k\right) \tag{5.3}$$

respectively. The existence of (5.3) follows from (1)—(5) and the Schwarzian inequality. $F(x, y, z)$ is evidently linear in z. The continuity of $G(x, y, z, \xi, \eta)$ in x, y now follows from (7) and repeated use of the Schwarzian inequality. Moreover, by the triangular inequality,

$$\begin{aligned}
&\|\Delta^{xy}_{\xi\eta} F(x, y, z) - G(x, y, z, \xi, \eta)\| \\
&\quad\leq \left(\sum_i \left\{\sum_{j,k}\left(\frac{1}{\xi^k}\Delta^{x^k}_{\xi^k} E^{x^{k+1}\,x^{k+2}\cdots}_{\xi^{k+1}\,\xi^{k+2}\cdots}\Phi^i_j - \frac{\partial \Phi^i_j}{\partial x^k}\right) z^j \xi^k\right\}^2\right)^{\frac{1}{2}}, \\
&\quad+ \left(\sum_i \left\{\sum_{j,k}\left(\frac{1}{\eta^k}\Delta^{y^k}_{\eta^k} E^{y^{k+1}\,y^{k+2}\cdots}_{\eta^{k+1}\,\eta^{k+2}\cdots}\Phi^i_j - \frac{\partial \Phi^i_j}{\partial y^k}\right) z^j \eta^k\right\}^2\right)^{\frac{1}{2}}
\end{aligned}$$

where $E^{x^{k+1}\,x^{k+2}\cdots}_{\xi^{k+1}\,\xi^{k+2}\cdots}\Phi^i_j$ denotes $\Phi^i_j(x^1, x^2, \ldots, x^k, x^{k+1}+\xi^{k+1}, \ldots;\ y^1+\eta^1, \ldots)$. Hence, using the Schwarzian inequality and the mean-value theorem for real functions of a real variable, we have for some real numbers $\lambda^k = \Theta^k\xi^k$, $\mu^k = \varrho^k\eta^k$, where Θ^k, ϱ^k are positive numbers ≤ 1:

$$\begin{aligned}
&\|\Delta^{xy}_{\xi\eta} F(x, y, z) - G(x, y, z, \xi, \eta)\| \\
&\quad\leq \left(\sum_i \left\{\sum_{j,k}\left(E^{x^k\,x^{k+1}\cdots}_{\xi^k\,\xi^{k+1}\cdots} - 1\right)\frac{\partial \Phi^i_j}{\partial x^k} z^j \xi^k\right\}^2\right)^{\frac{1}{2}} \\
&\quad+ \left(\sum_j \left\{\sum_{j,k}\left(E^{y^k\,y^{k+1}\cdots}_{\mu^k\,\eta^{k+1}\cdots} - 1\right)\frac{\partial \Phi^i_j}{\partial y^k} z^j \eta^k\right\}^2\right)^{\frac{1}{2}} \\
&\quad\leq \left(\sum_{i,j,k}\left\{\left(E^{x^k\,x^{k+1}\cdots}_{\lambda^k\,\xi^{k+1}\cdots} - 1\right)\frac{\partial \Phi^i_j}{\partial x^k}\right\}^2\right)^{\frac{1}{2}} \|z\|\,\|\xi\| \\
&\quad+ \left(\sum_{i,j,k}\left\{\left(E^{y^k\,y^{k+1}\cdots}_{\mu^k\,\eta^{k+1}\cdots} - 1\right)\frac{\partial \Phi^i_j}{\partial y^k}\right\}^2\right)^{\frac{1}{2}} \|z\|\,\|\eta\| \\
&\quad\leq \varepsilon\|z\| \max(\|\xi\|, \|\eta\|) \text{ if } \max(\|\xi\|, \|\eta\|) < \delta
\end{aligned}$$

for some δ, by (7). Therefore

$$d_{\xi\,\eta}^{x\,y} F(x, y, z) = G(x, y, z, \xi, \eta).$$

From (8), the condition of complete integrability in (ii) of Theorem I is clearly satisfied. Since also, by (6) and Schwarz's inequality,

$$\|G(x, y, z, 0, \eta)\| \leq \left\{ \sum_{i,j,k} \left(\frac{\partial \Phi_j^i}{\partial y^k} \right)^2 \right\}^{\frac{1}{2}} \|z\| \|\eta\| \leq g \|z\| \|\eta\|,$$

the present theorem follows by an application of Theorem I. In fact, the above proof establishes the stronger result, that the differential equation

$$d_\xi^x f(x) = F(x, f, \xi)$$

and the conditions

$$\|f - v\| \leq b, \; f(u) = v$$

have a unique solution $f(x)$ for $\|x - u\| < c$ where

$$f(x) \equiv (\Psi^1, \Psi^2, \ldots) \text{ and } \Psi^i(x) \equiv f^i(x^1, x^2, \ldots), \; i = 1, 2, \ldots.$$

The existence of $d_\xi^x f(x)$ implies that of $d f^i(x^1, x^2, \ldots)$, which in turn implies the existence of $\frac{\partial f^i}{\partial x^j}$; so that under the hypothesis of theorem 5.1 the system of partial differential equations

$$\frac{\partial f^i}{\partial x^j} = \Phi_j^i(x^1, x^2, \ldots; f^1, f^2, \ldots), \quad i, j = 1, 2, \ldots,$$

and the conditions (5.2) have a unique solution for $\|x - u\| < c$.

In the same way, Theorem II can be used to show the existence under hypotheses similar to those of Theorem 5.1, of a unique solution $f^i(x^1, x^2, \ldots; u^1, u^2, \ldots; v^1, v^2, \ldots)$ of the equation (5.1) and the conditions (5.2) for all x, u, v in certain domains of H.

Theorems I and II can also be used to prove existence theorems for functional Pfaffian equations. For example, let E and Σ be the Banach spaces of real continuous functions x^s, y^t, with norms $\|x^s\|$ and $\|y^t\|'$, on the real intervals $d \leq s \leq e$ and $d' \leq t \leq e'$ respectively; where $d \leq d'$, $e' \leq e$. Then with the aid of Theorem I we can prove

Theorem 5. 2. *Suppose a, b, g_1, g_2 are positive real numbers; α, β are in the intervals (d', e'), (d, e); and u^s, v^t are in E, Σ respectively. Suppose further that for $\|x^s - u^s\| < a$ and $\|y^t - v^t\|' \leq b$:*

(1) *For each α, β $\Phi^\alpha(x^s, y^t)$, $\Phi^\alpha_\beta(x^s, y^t)$ are on E, Σ to R, continuous in α and in α, β respectively*[1];

(2) *$\Phi^\alpha(x^s, y^t)$ and $\Phi^\alpha_\beta(x^s, y^t)$ are differentiable in x^s, y^t uniformly with respect to α and β;*

(3) *$d^{x\,y}_{\xi\,\eta}\Phi^\alpha(x^s, y^t)$ and $d^{x\,y}_{\xi\,\eta}\Phi^\alpha_\beta(x^s, y^t)$ are continuous in α and in α, β respectively; and are continuous in x^s, y^t uniformly with respect to α and to α, β respectively;*

(4) $$\|d^y_\eta \Phi^\alpha(x^s, y^t)\| \leq g_1\|\eta\|'$$

and

$$\|d^y_\eta \Phi^\alpha_\beta(x^s, y^t)\| \leq g_2\|\eta\|'.$$

Then, the complete integrability of the functional Pfaffian equation

$$d^x_z f^\alpha(x^s) = \Phi^\alpha(x^s, f^t)z^\alpha + \Phi^\alpha_\beta(x^s, y^t)z^\beta, \tag{5.4}$$

where the repetition of the index β indicates integration over the interval (d, e), implies that for some $c \leq a$ equation (5.4) and the conditions $f^\alpha(u^s) = v^\alpha$, $\|f^\alpha - v^\alpha\|' \leq b$ have a unique solution $f^\alpha(x^s)$ for $\|x^s - u^s\| < c$.

If in theorem 5.2, $\Phi^\alpha(x^s, y^t) = 0$ and $\Phi^\alpha_\beta(x^s, y^t)$ is constant with respect to α, then we obtain as a corollary an existence theorem for the functional Pfaffian equation

$$d^x_z f(x^s) = \Phi_\beta(x^s, f)z^\beta,$$

where $\Phi_\beta(x^s, y)$ is on E, R to R for each β.

If to the postulated addition and scalar multiplication in Banach space new functions are postulated, then instances of the right member of the completely integrable differential equation

$$d^x_z f(x) = F(x, f, z)$$

can be written explicitly, and the equation defines functions $f(x)$ whose existence

[1] Clearly $\Phi^\alpha(x^s, y^t)$ and $\Phi^\alpha_\beta(x^s, y^t)$, for each β, can be regarded as elements of Σ.

and structure in terms of only the postulated functions are demonstrated by Theorems I or II. For example, if E is Σ and there is a bilinear function $x \cdot y$, in E for all x, y in E, then the simplest non-trivial instance of $F(x, y, z)$ is $y \cdot z$; if moreover

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y \tag{5.5}$$

for all x, y, z in E, then the equation

$$d_z^x f(x) = f \cdot z \tag{5.6}$$

is completely integrable, and its solution is

$$f(x) = v + \sum_{n=1}^{\infty} v \cdot \frac{(x-u)^n}{n!}$$

where $f(u) = v$ and x^n is defined by $x^1 = x$, $x^n = x^{n-1} \cdot x$ for $n = 2, 3, \ldots$.

Examples of spaces E and functions $x \cdot y$ for which equation (5.5) is satisfied will be given below. That the equation is not always satisfied may be shown by taking for E the set of real quaternions[1] and for $x \cdot y$ the quaternion product; in fact (5.5) cannot be satisfied if E has a unit element and $x \cdot y$ is not commutative.

Example I. Let E be the set of elements $x = \sum_{i=1}^{3} x_i e_i$ of real normed algebra, and let $x \cdot y = \sum_{i,j} x_i y_j e_i e_j$, where the $e_i e_j$ are given by

$e_i e_j$	e_1	e_2	e_3
e_1	0	S	$-S$
e_2	S	0	$-S$
e_3	S	$-S$	0

$$S = e_1 + e_2 + e_3.$$

[1] A real linear algebra of elements $x = \sum_{i=1}^{n} x_i e_i$ can be normed in many ways to form a Banach space. For example, $\|x\|$ may be defined to be $\left\{\sum_{i=1}^{n} x_i^2\right\}^{\frac{1}{2}}$.

Then (5. 5) is satisfied, E has no unit, $x \cdot y$ is not associative and not commutative, and $x^n = 0$ if $n > 2$.

Example II. If the multiplication table in Example I is replaced by

$e_i\, e_j$	e_1	e_2	e_3
e_1	$\frac{1}{2} S$	$\frac{1}{2} S$	$-S$
e_2	$-S$	$\frac{1}{2} S$	$\frac{1}{2} S$
e_3	$\frac{1}{2} S$	$-S$	$\frac{1}{2} S$

then E and $x \cdot y$ have the properties stated in Example I, except that $x \cdot y$ is associative.

Example III. Let E be the set of elements $x = \sum_{i=1}^{n} x_i e_i$ of a real normed algebra; let

$$x\,y = \sum_{i,j} x_i\, y_j\, e_i\, e_j = \sum_{i,j,k} x_i\, y_j\, \gamma_{i,j}^k\, e_k,$$

where $\gamma_{ij}^k = a_i\, a_j\, \Phi^k + \Psi_{ij}^k$, $\Psi_{ij}^k = -\Psi_{ji}^k$, $\sum_{i=1}^{n} a_i \Psi^i \neq 0$, and not all the $\Psi_{ij}^k = 0$.

If $x \cdot y = \frac{1}{2}(xy + yx)$, then (5. 5) is satisfied, $x \cdot y$ is associative and commutative, $x^n \not\equiv 0$ for all n; but xy is neither symmetric nor skew-symmetric — so that $x \cdot y \not\equiv x\,y$ and $x \cdot y \not\equiv 0$.

Example IV. A special case, in which xy is associative, of Example III is obtained if $n = 4$; $a_1 = a_4 = 0$, $a_2 = a_3 = 1$; $\Phi^1 = \Phi^4 = 0$, $\Phi^2 = \Phi^3 = 1$; $\Psi_{ij}^k = 0$ if i, j is not a permutation of 1, 4, $\Psi_{14}^k = -\Psi_{41}^k = \Psi^k$, where $\Psi^1 = \Psi^4 = 0$, $\Psi^2 = -\Psi^3 = 1$. The table for the $e_i\, e_j$ is then

$e_i e_j$	e_1	e_2	e_3	e_4
e_1	0	0	0	S_2
e_2	0	S_1	S_1	0
e_3	0	S_1	S_1	0
e_4	$-S_2$	0	0	0

$S_1 = e_2 + e_3$

$S_2 = e_2 - e_3$

There exists no unit-element.

More generally, if E is any Banach space in which a symmetric bilinear function $x \cdot y$ is defined, the condition (5.5) for the complete integrability of the equation (5.6) is satisfied if and only if $x \cdot y$ is associative, so that E is a commutative abstract ring. Suppose $x \cdot y$ is associative, and $\Phi_i(\nu, \xi) = \nu \sum_{n=0}^{\infty} a_{in} \xi^n$, $f_i(v, x) = v \cdot \sum_{n=0}^{\infty} a_{in} x^n$, $x^n = x^{n-1} \cdot x$ for $n = 2, 3, \ldots$, $x^0 = I$, where ν, ξ and the a_{in} are real numbers, v, x are elements in E, and the mark I, introduced to simplify the notation for abstract series, has the properties $x \cdot I = I \cdot x$, $(x \cdot I) \cdot y = x \cdot (I \cdot y)$, for any x, y in E; $x + I$ is not defined. Then to the identity $F(\Phi_1, \Phi_2, \ldots, \Phi_n) = 0$ in ν, ξ corresponds the identity $F(f_1, f_2, \ldots, f_n) = 0$ in v, x, where $F(\Phi_1, \ldots, \Phi_n)$ is a power series in $\Phi_1, \ldots, \Phi_n$. This is simply because the two corresponding identities are reducible to the same infinite set of simultaneous equations in the a_{in}. Thus to the identity $\nu e^{\xi} \nu e^{\eta} = \nu (\nu e^{\xi + \eta})$ corresponds

$$e(v, x) \cdot e(v, y) = e(v, x + y),$$

where $e(v, x) = v \cdot \sum_{n=0}^{\infty} \frac{x^n}{n!}$; and to $(\nu \sin \xi)^2 + (\nu \cos \xi)^2 = \nu^2$ corresponds

$$(\sin(v, x))^2 + (\cos(v, x))^2 = v^2,$$

where $\sin(v, x) = v \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ and $\cos(v, x) = v \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$. Term-by-term differentiation of these abstract series (justified below) gives the identities

$$d_z^x e(v, x) = e(v, x) \cdot z,$$

$$d_z^x \sin(v, x) = \cos(v, x) \cdot z,$$

$$d_z^x \cos(v, x) = -\sin(v, x) \cdot z,$$

where the functions differentiated in the left members are respectively equal to v, o, v when $x = 0$. These equations are completely integrable. The are, in fact, rather simple instances of the equation considered in the following theorem.

Theorem 5. 3. *Let a_{mn} be in the abstract ring E for $m, n = 0, 1, 2, \ldots,$ and such that for some positive numbers ϱ, σ, τ*

$$\|a_{mn}\| < \varrho \sigma^{-m} \tau^{-n}$$

for all sufficiently large m, n. Let μ be the modulus of the bilinear commutative ring product $x \cdot y$ and let $\lambda = \min\left(1, \frac{1}{\mu}\right)$. Then

1) *if $\|x\| > \frac{\sigma}{\mu}$, $\|y\| < \frac{\tau}{\mu}$, the double series $D = \sum_{m,n} a_{mn} \cdot x^m \cdot y^n$ converges absolutely and uniformly;*

2) *if $\|x\| < \lambda\sigma$, $\|y\| < \lambda\tau$, the double series D is term-by-term Fréchet differentiable in x, y, the derived series converges absolutely and uniformly, and the equation*

$$d_z^x f(x) = D \cdot z$$

is completely integrable.

Proof.

$$\|a_{mn} \cdot x^m \cdot y^n\| \leq \mu^{m+n} \|a_{mn}\| \|x\|^m \|y\|^n < \varrho \left\{\frac{\|x\|}{\frac{\sigma}{\mu}}\right\}^m \left\{\frac{\|y\|}{\frac{\tau}{\mu}}\right\}^n \text{ for all sufficiently large } m, n.$$

Hence if $\|x\| < \frac{\sigma}{\mu}$, $\|y\| < \frac{\tau}{\mu}$, the double series for $\varrho\left(1 - \frac{\|x\|}{\frac{\sigma}{\mu}}\right)^{-1}\left(1 - \frac{\|y\|}{\frac{\tau}{\mu}}\right)^{-1}$ ultimately dominates D, and 1) follows. Moreover, if $\|x\| < \sigma$, $\|y\| < \tau$, the double series for $\varrho\left(1 - \frac{\|x\|}{\sigma}\right)^{-1}\left(1 - \frac{\|y\|}{\tau}\right)^{-1}$ ultimately dominates $\sum_{m,n} \|a_{mn}\| \|x\|^m \|y\|^n$; hence for sufficiently small $\|\xi\|$, $\|\eta\|$

$$\sum_{m,n} \|a_{mn}\| \{\|x\| + \|y\|\}^m \{\|y\| + \|\eta\|\}^n = \sum_{m,n} \|a_{mn}\| \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \|x\|^{m-r} \|y\|^{n-s} \|\xi\|^r \|\eta\|^s$$

$$= \sum_{r,s} \left\{ \sum_{m,n} \|a_{m+r,\, n+s}\| \binom{m+r}{r} \binom{n+s}{s} \|x\|^m \|y\|^n \right\} \|\xi\|^r \|\eta\|^s,$$

by a known re-arrangement theorem for series of non-negative numbers. Hence if $\|x\| < \lambda\sigma$, $\|y\| < \lambda\tau$, and sufficiently small $\|\xi\|$, $\|\eta\|$, the series

$$D_{mn} = \sum_{r,s} \binom{r+m}{m} \binom{s+n}{n} a_{r+m,\, s+n} \cdot x^r \cdot y^s,$$

$$\sum_{m,n} D_{mn} \cdot \xi^m \cdot \eta^n$$

connerge absolutely and uniformly,

$$\sum_{m,n} a_{mn} \cdot (x + \xi)^m \cdot (y + \eta)^n = \sum_{m,n} D_{mn} \cdot \xi^m \cdot \eta^n,$$

$$\|\Delta^{x\,y}_{\xi\,\eta} D - D_{10} \cdot \xi - D_{01} \cdot \eta\| \leq \mu \|\xi\| \|\eta\| \left\| \sum_{m,n} D_{m+1,\, n+1} \cdot \xi^m \cdot \eta^n \right\|,$$

$$\text{so that } d^{x\,y}_{\xi\,\eta} D = D_{10} \cdot \xi + D_{01} \cdot \eta,$$

and the condition for complete integrability becomes

$$(D_{10} + D \cdot D_{01}) \cdot (\xi \cdot z - z \cdot \xi) = 0,$$

an evident identity. This completes the proof.

By an argument similar to that used above for double series, term-by-term differentiation of the single series $\sum_m a_m \cdot x^m$ is valid for $\|x\| < \lambda\sigma$, if $\|a_n\| < \varrho\sigma^{-n}$ for some ϱ, σ and all sufficiently large m. Since $\lim_{m\to\infty} \frac{\sigma^m}{m!} = 0$ for any number σ, $\left\|\frac{v}{m!}\right\| < \varrho\sigma^{-m}$ for any ϱ, σ and all sufficiently large m. Hence the term-by-term differentiation of $e(v, x)$, $\sin(v, x)$, $\cos(v, x)$ is valid for any x.

If E is the set of real functions $x(\alpha, \beta)$, continuous in α, β for $a \leq \alpha \leq b$, $a \leq \beta \leq b$, and $x \cdot y \equiv \int_\alpha^\beta x(\alpha, \sigma)\, y(\sigma, \beta)\, d\sigma$, then E is the ring with respect to $x \cdot y$

studied by Volterra[1]. If moreover $\|x\| = \max\limits_{a \leq \alpha, \beta \leq b} |x(\alpha, \beta)|$, then $x \cdot y$ is bilinear and E is a normed vector ring. If, with Volterra, we suppose that a function $\xi(\alpha, \beta)$ exists such that $\xi(\alpha, \alpha) \neq 0$ for $a \leq \alpha \leq b$ and $x \cdot \xi = \xi \cdot x$ for any x in E, then, as Volterra[1] and others have shown, $x \cdot y$ is commutative in E, and theorem 5.3 expresses new properties of Volterra's 'permutable' functions. The ring E contains null-factors, since non-zero elements x, y exist in E, such that

$$x(\alpha, \beta) = 0 \quad \text{if} \quad \alpha \leq \beta \leq b,$$
$$y(\alpha, \beta) = 0 \quad \text{if} \quad \beta \leq \alpha \leq b,$$

$$\text{and hence } x \cdot y = \int_{\alpha}^{\beta} x(\alpha, \sigma) y(\sigma, \beta) \, d\sigma = 0.$$

Altho E contains no unit-element, a normed ring S with a unit element exists in which E is an ideal[2]. For if S is defined to be the set of all ordered pairs (x, k) where x is in E and k is a real number, and if for any $X = (x, k)$, $Y = (y, l)$ in S,

$$X \cdot Y = (x \cdot y + ky + lx,\ kl), \quad \|X\| = \|x\| + |k|,$$

then S is a normed ring, $(0, 1)$ is a unit-element in S, the set T of elements $(x, 0)$ in S is isomorphic and isometric to E, and for any X, Y in T and Z in S, the elements $X \cdot Z$, $Z \cdot Y$, $X - Y$ are in T. Hence if in S and the definitions of addition and multiplication, T is replaced by E, S becomes a ring S^1 which contains E as an ideal. In fact, E is a prime ideal, since its residue class ring $\frac{S^1}{E}$ contains no null-factors.

6. Existence Theorems for the Equation $K(x, f(x)) = 0$.

The notations used for differences and differentials in the preceding sections are occasionally ambiguous. For example, the value of $d_\xi^y f(y, x)$ when $y = x$ is not always $d_\xi^x f(x, x)$, and similarly for the differences $\Delta_\xi^y f(y, x)$ and $\Delta_\xi^x f(x, x)$; that is, the notations are not completely substitutive. Whenever completely substitutive notations are necessary, we shall write[3]

[1] Theory of Functionals, London, 1930.

[2] B. L. Van Der Waerden, Moderne Algebra, Vol. 1, Berlin, 1930.

[3] V. Elconin, loc. cit.

$$\mathop{\Delta}\limits^{x_1 \dots x_n}_{\xi_1 \dots \xi_n}{}_{\sigma_1 \dots \sigma_n} F(\sigma_1 \dots \sigma_n) \text{ for } \Delta^{x_1 \dots x_n}_{\xi_1 \dots \xi_n} F(x_1 \dots x_n),$$

and

$$\mathop{d}\limits^{x_1 \dots x_n}_{\xi_1 \dots \xi_n}{}_{\sigma_1 \dots \sigma_n} F(\sigma_1 \dots \sigma_n) \text{ for } d^{x_1 \dots x_n}_{\xi_1 \dots \xi_n} F(x_1 \dots x_n).$$

With these new forms, $d^y_\xi f(y, x) = \mathop{d}\limits^{x}_{\xi}{}_\sigma f(\sigma, x)$ if $y = x$, whereas $d^x_\xi f(x, x) = \mathop{d}\limits^{x}_{\xi}{}_\sigma f(\sigma, \sigma)$.

Before proceeding to the theorem of this section, we shall prove an important lemma. Let n be a positive integer; $b, a_1, a_2, \dots, a_n$ be positive real numbers; α, β be in Banach E, Σ respectively; and u_i, x_i, ξ_i be in a Banach space E_i, for $i = 1, 2, \dots, n$.

Lemma. *If for any $x_1 \dots x_n, \xi_1 \dots \xi_n, \alpha, \beta$ such that $\|x_i - u_i\| < a_i$, $H(x_1 \dots x_n, \beta)$, $P(x_1 \dots x_n, \alpha)$ are in E, Σ respectively,*

$$H(x_1 \dots x_n, P(x_1 \dots x_n, \alpha)) = \alpha, \quad P(x_1 \dots x_n, H(x_1 \dots x_n, \beta)) = \beta,$$

$H(x_1 \dots x_n, \beta)$ is linear in β, and $d^{x_1 \dots x_n}_{\xi_1 \dots \xi_n} H(x_1 \dots x_n, \beta)$ is continuous in $x_1 \dots x_n$. Then for any $x_1 \dots x_n, \xi_1 \dots \xi_n, \alpha$ such that $\|x_i - u_i\| < a_i$, $P(x_1 \dots x_n, \alpha)$ is linear in α, and

$$d^{x_1 \dots x_n}_{\xi_1 \dots \xi_n} P(x_1 \dots x_n, \alpha) + P\left(x_1 \dots x_n, \mathop{d}\limits^{x_1 \dots x_n}_{\xi_1 \dots \xi_n}{}_{\sigma_1 \dots \sigma_n} H(\sigma_1 \dots \sigma_n, P(x_1 \dots x_n, \alpha))\right) = 0.$$

Proof. Assume the premise for $n = 1$; the argument will be valid for any n. Let $a = a_1$, $u = u_1$, $x = x_1$, $\xi = \xi_1$, and suppose $\|x - u\| < a$. By a theorem of Schauder-Banach[1], $P(x, \alpha)$ is linear in α; hence for some positive number A, depending on x,

$$\|P(x, \alpha)\| < A \|\alpha\|,$$

so that

$$\|P(x, H(x, \beta))\| = \|\beta\| < A \|H(x, \beta)\|,$$

and, by a theorem of Kerner, for some B, b

$$\left| \Delta^x_\xi \|H(x, \beta)\| \right| \leq \|\Delta^x_\xi H(x, \beta)\| \leq \int_0^1 \|d^x_\xi H(x + \sigma\xi, \beta)\| d\sigma < B \|\beta\| \|\xi\|$$

[1] Studia Mathematica, 1929, 1930.

if $\|\xi\| < b$, since $d_\xi^x H(x, \beta)$ is easily shown to be linear in β by a theorem of Banach.[1] Hence for some A, b and any ξ such that $\|\xi\| < b$

$$(6.1) \qquad \|H(x+\xi, \beta)\| > A\|\beta\|,$$

$$(6.2) \qquad \Delta_\xi^x H(x, P(x, \alpha)) = \Delta_{\sigma\atop\xi}^{x} H(\sigma, P(x+\xi, \alpha)) + H(x, \Delta_\xi^x P(x, \alpha)) = 0,$$

$$(6.3) \qquad \Delta_\xi^x H(x, P(x, \alpha)) = \Delta_{\sigma\atop\xi}^{x} H(\sigma, P(x, \alpha)) + H(x+\xi, \Delta_\xi^x P(x, \alpha)) = 0;$$

using (6.1), (6.3):

$$A\|\Delta_\xi^x P(x, \alpha)\| < \|H(x+\xi, \Delta_\xi^x P(x, \alpha))\| = \|\Delta_{\sigma\atop\xi}^{x} H(\sigma, P(x, \alpha))\|,$$

and since $H(x, \beta)$ is continuous in x

$$(6.4) \qquad P(x, \alpha) \text{ is continuous in } x.$$

Using (6.2) and (6.4): for some A, B, b and any ξ such that $\|\xi\| < b$

$$\Delta_\xi^x P(x, \alpha) + P\Big(x, \Delta_{\sigma\atop\xi}^{x} H(\sigma, P(x+\xi, \alpha))\Big) = 0,$$

$$\begin{aligned}
&\Big\|\Delta_\xi^x P(x, \alpha) + P\Big(x, d_{\sigma\atop\xi}^{x} H(\sigma, P(x, \alpha))\Big)\Big\| \\
&\leq \Big\|P\Big(x, \Big(\Delta_{\sigma\atop\xi}^{x} - d_{\sigma\atop\xi}^{x}\Big) H(\sigma, P(x, \alpha))\Big)\Big\| + \Big\|P\Big(x, \Delta_{\sigma\atop\xi}^{x} \Delta_{\tau\atop\xi}^{x} H(\sigma, P(\tau, \alpha))\Big)\Big\| \\
&< A\Big\{\Big\|\Big(\Delta_{\sigma\atop\xi}^{x} - d_{\sigma\atop\xi}^{x}\Big) H(\sigma, P(x, \alpha))\Big\| + \Big\|\Delta_{\sigma\atop\xi}^{x} H\Big(\sigma, \Delta_{\tau\atop\xi}^{x} P(\tau, \alpha)\Big)\Big\|\Big\} \\
&\leq A\Big\{\Big\|\Big(\Delta_{\sigma\atop\xi}^{x} - d_{\sigma\atop\xi}^{x}\Big) H(\sigma, P(x, \alpha))\Big\| + \int_0^1 \Big\|d_{\sigma\atop\xi}^{x+s\xi} H\Big(\sigma, \Delta_{\tau\atop\xi}^{x} P(\tau, \alpha)\Big)\Big\|\Big\} \\
&< A\Big\{\Big\|\Big(\Delta_{\sigma\atop\xi}^{x} - d_{\sigma\atop\xi}^{x}\Big) H(\sigma, P(x, \alpha))\Big\| + B\|\xi\|\Big\|\Delta_{\sigma\atop\xi}^{x} P(\tau, \alpha)\Big\|\Big\},
\end{aligned}$$

and by (6.4) again and the definition of differential

$$(6.5) \qquad d_\xi^x P(x, \alpha) + P\Big(x, d_{\sigma\atop\xi}^{x} H(\sigma, P(x, \alpha))\Big) = 0.$$

This completes the proof.

Now let n be a non-negative integer; a, b, c, e, g, h, A be positive numbers; E_1, E_2, E_3 be Banach spaces; u, x, α, γ be in E_1; v, y, β, δ be in E_2; z be in E_3; and let $\Phi(x)$ be any function on E_1 to E_2.

[1] Fundamenta Math., loc. cit.

Theorem 5.1. *For any* $x, y, z, \alpha, \beta, \gamma, \delta$ *such that* $\|x-u\| < a$, $\|y-v\| < b$, *suppose that* $P(x, y, z)$ *is in* E_2; $F(x, y)$, $G(x, y, \alpha)$, $H(x, y, \beta)$ *are in* E_3; $F(u, v) = 0$; $d^{xy}_{\alpha\beta}F(x, y)$ *exists equal to* $G(x, y, \alpha) + H(x, y, \beta)$; $d^{xy}_{\gamma\delta} d^{xy}_{\alpha\beta} F(x, y) \equiv K(x, y; \alpha, \beta; \gamma, \delta)$ *exists, continuous in* x, y;

$$H(x, y, P(x, y, z)) = z, \quad P(x, y, H(x, y, \beta)) = \beta;$$

let $Q(x, y, \alpha) = -P(x, y, G(x, y, \alpha))$; *let* $f_0(x) = v$; *and for any* n, *let* $f_{n+1}(x)$ *be*

$$v + \int_0^1 Q(u + \sigma(x-u), \ f_n(u + \sigma(x-u)), \ x-u) d\sigma.$$

Then

(1) *for some* A, $g \leq a$, $h \leq b$

$$\|P(x, y, K(x, y; \alpha, Q(x, y, \alpha); 0, \beta))\| < A \|\alpha\| \|\beta\|$$

if $\|x-u\| < g$, $\|y-v\| < h$;

(2) *for any such* A, g, h *and any* $c < h$, *some* $e \leq g$ *exists such that*

$$f(x) \equiv \lim_{n \to \infty} f_n(x) \ \textit{exists}, \quad \|f_n(x) - v\| < c,$$

if $\|x-u\| < e$; *and for any such* e:

(i) $$\Phi(u) = v, \ F(x, \Phi(x)) \equiv 0, \quad \|\Phi(x) - v\| < b$$

for $\|x-u\| < e$, *if and only if* $\Phi(x) = f(x)$ *for* $\|x-u\| < e$;

(ii) $$d^x_\alpha f(x) = Q(x, f(x), \alpha),$$

and

$$d^x_\gamma d^x_\alpha f(x) = -P(x, f(x), K(x, f(x); \alpha, Q(x, f(x), \alpha); \gamma, Q(x, f(x), \gamma))).$$

Proof. Assume the premise. The proof will be reduced to an application of Theorem I by taking for the functions F, G of that theorem the functions $Q(x, y, \alpha)$, $-P(x, y, K(x, y; \alpha, Q(x, y, \alpha); \gamma, \delta))$ respectively, and showing that condition (iii) of Theorem I is then equivalent to condition (ii) of the present theorem.

If $\|x-u\| < a$, $\|y-v\| < b$, then $G(x, y, \alpha)$, $H(x, y, \beta)$, evidently linear in α and in β respectively, are continuous in x, y since they each have a differential in x, y; these differentials are themselves continuous, hence $P(x, y, z)$ is continuous in x, y, linear in z, by the preceding lemma;

(6. 6) $Q(x, y, \alpha)$ is continuous in x, y, linear in α; and $P(x, y, K(x, y; \alpha, Q(x, y, \alpha); 0, \beta))$ is continuous in x, y, linear in α, linear in β, which implies that for some A, $g \leq a$, $h \leq b$

(6. 7) $$\| P(x, y, K(x, y; \alpha, Q(x, y, \alpha); 0, \beta)) \| < A \|\alpha\| \|\beta\|$$

if $\|x - u\| < g$, $\|y - v\| < h$. Let A, g, h be any such numbers. Then $\|x - u\| < g$, $\|y - v\| < h$ implies

$$d^{xy}_{\gamma\delta} Q(x, y, \alpha) + P(x, y, d^{xy}_{\gamma\delta}(G(x, y, \alpha)) + d^{xy}_{\sigma\tau\,\gamma\delta} P(\sigma, \tau, G(x, y, \alpha)) = 0,$$

and by the preceding lemma

$$d^{xy}_{\sigma\tau\,\gamma\delta} P(\sigma, \tau, G(x, y, \alpha)) = P\Big(x, y, d^{xy}_{\sigma\tau\,\gamma\delta} H(\sigma, \tau, Q(x, y, \alpha))\Big),$$

so that

(6. 8) $$d^{xy}_{\gamma\delta} Q(x, y, \alpha) = - P(x, y, K(x, y; \alpha, Q(x, y, \alpha); \gamma, \delta));$$

and

(6. 9) $$P(x, y, K(x, y; \alpha, Q(x, y, \alpha); \gamma, Q(x, y, \gamma))$$

is symmetric in α, γ, since

$$K(x, y; \alpha, \beta; \gamma, \delta) = K(x, y; \gamma, \delta; \alpha, \beta).$$

Moreover, for any $e \leq g$, if $\|x - u\| < e$ implies

$$\Phi(u) = v, \ F(x, \Phi(x)) = 0, \ \|\Phi(x) - v\| < h, \text{ and } d^x_\alpha \Phi(x) \text{ exists,}$$

then $\|x - u\| < e$ implies

$$\Phi(u) = v, \ \|\Phi(x) - v\| < h, \text{ and } d^x_\alpha(\Phi(x)) = Q(x, \Phi(x), \alpha),$$

since

$$d^x_\alpha F(x, \Phi(x)) = G(x, \Phi(x), \alpha) + H(x, \Phi(x), d^x_\alpha \Phi(x)) = 0;$$

and the converse is true. In view of this and the results in (6. 6), (6. 7), (6. 8), (6. 9), Theorem I can be applied to complete the proof.

Concluding Remarks.

Altho we have restricted ourselves in this paper to first order differential equations, the method of successive approximations may be applied to obtain existence theorems for certain differential equations of higher order, and for certain systems of differential equations. It is sometimes possible to reduce the proof of such existence theorems to an application of Theorems I or II. For example, the second order differential equation

$$d_\xi^x d_\eta^x g(x) = H(x, g(x), d_\eta^x g(x), \xi, \eta)$$

may be replaced by the equivalent system

$$d_\eta^x g(x) = P(x, \eta)$$

$$d_\xi^x P(x, \eta) = H(x, g(x), P(x, \eta), \xi, \eta)$$

of first order equations, which in the product space[1] Σ^2 is equivalent to the single equation

$$d_\xi^x f(x, \eta) = F(x, f(x, \eta), \xi, \eta),$$

where

$$f = \{g(x), P(x, \eta)\}, \; F = \{P(x, \eta), H(x, g(x), P(x, \eta), \xi, \eta)\}.$$

Hence if the initial conditions of the second order equation are on $g(x)$ and $d_\eta^x g(x)$, the existence of $f(x, \eta)$ now follows from Theorem I, if the premise of the latter is satisfied. In general, however, such a reduction is not possible; if in the preceding example the one-point initial condition is replaced by a two-point boundary condition, Theorem I cannot be applied. We have already begun the study of higher order equations with many-point boundary conditions. We also intend to study the most general first-order differential equation

$$d_{\xi}^{\overset{x}{\sigma}} f(\sigma) = F_{\xi}^{\overset{x}{\sigma}} f(\sigma),$$

which cannot always be reduced to the form considered in this paper, as is evident from the example

$$d_{\xi}^{\overset{x}{\sigma}} f(\sigma) = f(g(x, \xi)),$$

where $g(x, \xi)$ is a given function.

Since our analysis of the dependence on the parameters v, u of the solution $f(x, v, u)$ in Theorem II is still incomplete, we have preferred to reserve it for a future paper rather than delay unduly the present publication.

[1] Σ^2 is the Banach space of ordered pairs $\{x, y\}$ of elements x, y in Σ, with $\|\{x, y\}\| = \max\{\|x\|, \|y\|\}$. However, many other definitions of equivalent norms are possible. See section I.