THE EXISTENCE OF MINIMAL SURFACES OF GIVEN TOPOLOGICAL STRUCTURE UNDER PRESCRIBED BOUNDARY CONDITIONS.

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The purpose of the present paper is the solution of the boundary value problems for minimal surfaces when the boundaries are not, or not entirely-fixed Jordan curves but are free to move on prescribed manifolds. At the same time I shall present modifications and simplifications of my previous solution of the Plateau' and Douglas' problem for fixed boundary curves and prescribed topological structure and incidentally discuss certain features of the problem in order to clarify its relation to the theory of conformal mapping. Though based on previous publications[1], the paper may, except for some references, be read independently.

Introduction.

A minimal surface S in the m-dimensional Euclidean space with the rectangular coordinates $x_1, \ldots, x_m$ — combined as a vector $\mathfrak{x}$ — is defined by means of two parameters u, v as follows: In a domain B of the u, v-plane, $\mathfrak{x} = \mathfrak{x}(u, v)$ is harmonic in the parameters u, v; which means that for $\mathfrak{x}$ or all its components the Laplace equation

$$(1) \qquad \Delta \mathfrak{x} = \mathfrak{x}_{uu} + \mathfrak{x}_{vv} = 0$$

[1] Nos. [9], [10], [11], [12], [13], [14] of the bibliography at the end of the paper. References to this list are made in square brackets throughout this paper.

or

$$\triangle x_\mu = 0$$

holds; moreover $\mathfrak{x}$ satisfies the two non-linear additional conditions

$$(2) \qquad \begin{aligned} E - G &= \mathfrak{x}_u^2 - \mathfrak{x}_v^2 = \Sigma x_{\mu_u}^2 - \Sigma x_{\mu_v}^2 = 0, \\ F &= \mathfrak{x}_u \mathfrak{x}_v = \Sigma x_{\mu_u} x_{\mu_v} = 0 \end{aligned}$$

which characterize u, v as isometric parameters on S and the mapping of B on S as conformal.

If as usual $\Re$ means »real part» and $\Im$ »imaginary part» we have

$$(3) \qquad x_\mu = \Re f_\mu(w),$$

where $f_\mu(w)$ is an analytic function of the complex variable

$$(4) \qquad w = u + iv$$

in B. Then, by introducing the analytic function

$$(5) \qquad \varphi(w) = \Sigma f'_\mu(w)^2 = \Sigma (x_{\mu_u} - i x_{\mu_v})^2 = (E - G) - 2iF$$

the condition (2) reduces to

$$(2\text{ a}) \qquad \varphi(w) = 0,$$

which, incidentally, makes it evident that the conditions (2) do not overdetermine the problem but essentially amount to only one boundary condition for the linear system (1) of differential equations.[1]

The classical »*Problem of Plateau*» is to determine a simply connected minimal surface bounded by a given Jordan curve Γ. To solve it, one may suppose that the parameters u, v range over the unit circle B in the w-plane, with the boundary C; then one has to find the vector $\mathfrak{x}$, harmonic in B and continuous in $B + C$, so that C is mapped in a continuous and monotonic way[2] on Γ and that (2 a) is satisfied.

It is in this formulation that the Plateau problem was first solved completely (1930) with independent methods by T. Radó and J. Douglas[3], both

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[1] See also Courant-Hilbert, Meth. der math. Phys. Vol. II, p. 130 ff. for a more general analysis of this fact.

[2] It is remarkable that the one-to-one correspondence between C and Γ follows as a consequence. Moreover, as I shall show elsewhere, the solution remains unchanged if we modify the problem by dropping even the requirement of monotonicity.

[3] See [1], [2], and the papers quoted there.

methods being based on variational problems. Radó makes use of the theory of conformal mapping, while Douglas avoids this theory as much as possible and rather includes Riemann's mapping theorem for simply connected Jordan domains as a by-product for the case of a plane curve Γ. Douglas, as early as 1931, formulated a much more general problem: To find a minimal surface S of prescribed topological structure — i. e. prescribed genus, or, in case of non-orientability, characteristic number — with a prescribed boundary Γ consisting of k separated Jordan curves $\Gamma_1, \ldots, \Gamma_k$ (oriented if S is to be orientable). This »*Douglas' problem*» presents essentially greater difficulties and new interesting aspects. Douglas has treated first the cases of minimal surfaces topologically equivalent to an annular ring and to a Möbius strip respectively. In 1936 he communicated a general result and gave details and proofs to supplement his previous reasonings so as to make them cover the general problem. In 1938 he amplified these communications and announced more detailed publications one of which, [8], appeared in 1939.[1]

In a note (June 1936)[2] I published another method for the solution of Plateau's and Douglas' problem with two alternatives, one using the other avoiding the theory of conformal mapping. This method also permits, for the first time, the solution of the corresponding *problems with »free boundaries»*, when parts of the boundary or the whole boundary are free on prescribed continuous manifolds. It seems that these »free» problems — as previously envisaged in special cases by Gergonne, Riemann, Schwarz — are not accessible to the other methods mentioned above.[3] Moreover, as was first observed by M. Shiffman[4], my method also permits the proof of the existence of minimal surfaces with prescribed boundaries which do not give an absolute but only a relative minimum for the variational problem. The method was presented in detail for the case of genus zero and k boundary curves in a paper which also elaborates sufficiently the necessary additional steps for arbitrary topological structure.[5]

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[1] See [3], [4], [5], [7], [8], in particular the detailed last paper. In [10] I referred to [3], [4] as preliminary announcements. Prof. Douglas called my attention to the fact that these papers were intended to give his proof in sufficient detail.

[2] See [9].

[3] See [13]; the case when the free boundaries are planes is treated in a paper by J. Ritter [23], not yet published.

[4] See [17].

[5] [10]. See also Shiffman [16], where the case of a relative minimum under a certain condition is treated for higher topological structure.

In [12] and [13] modifications and extensions are indicated which form the basis of parts of the present paper.

For the Plateau problem our point of departure is the following variational problem involving the *Dirichlet integral*

$$D(\mathfrak{x}) = \frac{1}{2} \iint_B (\mathfrak{x}_u^2 + \mathfrak{x}_v^2)\, du\, dv: \tag{6}$$

We consider this integral for vectors $\mathfrak{x}(u, v)$ in a domain B with the boundary C. These vectors are supposed to be continuous in $B + C$ and to have piecewise continuous[1] first derivatives in B. B may be the unit circle and $\mathfrak{x}$ is supposed to map C in a continuous and monotonic way on the prescribed Jordan curve Γ. Then we seek among all these admissible vectors one, $\mathfrak{x}$, which minimizes the Dirichlet integral $D(\mathfrak{x})$.

We expressly suppose that $D(\mathfrak{x})$ admits of vectors $\mathfrak{x}$ with finite $D(\mathfrak{x})$. This is certainly true if Γ is rectifiable[2], an assumption which we shall henceforth make for all boundary curves.

In the cases of the Douglas problem we have to consider a similar variational problem for domains B of the prescribed topological character. These domains B however cannot be fixed in advance but must be free within a class of domains depending on a certain number of arbitrary parameters which together with $\mathfrak{x}$ have to be determined by the variational problem.

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[1] A function is called piecewise continuous in a domain if in every closed subdomain the continuity is interrupted only in a finite number of points and smooth arcs, i. e., arcs with a continuously turning tangent.

[2] For, Γ permits with the arc length s as parameter, the total length of Γ being 2π, the representation

$$x_\nu = \frac{a_0^\nu}{2} + \Sigma(a_m^\nu \cos m s + b_m^\nu \sin m s),$$

and corresponding harmonic vectors $\mathfrak{x}$ with these boundary values have, for a concentric circle with radius $r < 1$ the Dirichlet integral

$$D_r(\mathfrak{x}) = \pi \sum_{m,\nu} r^{2m} (a_m^{\nu\,2} + b_m^{\nu\,2})\, m.$$

Since the existence of the arc length implies the convergence of

$$\Sigma(a_m^{\nu\,2} + b_m^{\nu\,2})\, m^2,$$

the existence of $D(\mathfrak{x}) = \lim_{r \to 1} D_r(\mathfrak{x})$ is obvious.

The Euler equation of these problems is (1), and it will be seen that the degree of freedom in the boundary representation, together with the degree of freedom in the choice of the domain B, leads to the relations (2 a) as »natural boundary conditions».

Douglas, starting from the same Dirichlet integral, restricts the admissible vectors to harmonic vectors and then considers $D(\mathfrak{x})$ explicitly as a functional of the boundary values which depends on functions of only one variable, whereas the systematic exploitation of the greater degree of freedom in the two dimensional integral (6) is essential for our method. This accounts for the possibility of an extension of our method to the problems with free boundaries for which such a boundary representation would not be feasible. The viewpoint of the two-dimensional problem also permits an intrinsic consideration without explicit calculations.

The interconnection between the *theory of conformal mapping* and that of *the Plateau-Douglas problem* may be illuminated by the following remarks: Originally the conditions (1) and (2) characterizing a minimal surface are local conditions, invariant under conformal mapping. It is, accordingly, not required that the whole minimal surface be represented in a continuous way by the same uniform parameter $w = u + iv$; instead any abstract Riemann domain B of suitable connectivity with different local variables w might be chosen. However, for solving the Plateau-Douglas problem by convergent processes involving sequences of domains B, we shall have to restrict the admissible domains B to certain compact classes in which such passages to a limit can be performed. Therefore our solution appears dependent on the underlying choice of domains B. To free our results from the reference to the class of parameter domains B, we have to know that more general Riemann domains can be *»uniformized»*, that is conformally represented by *»normal domains»* B of the type under consideration. A certain knowledge of this kind is likewise necessary to establish the equivalence of our problem with that of the *surfaces of least area.*[1]

In the case of genus zero it is possible to obtain satisfactory results without preliminary use of mapping theorems because sufficiently general mapping theorems can be obtained as a consequence of the solution of the general Douglas problem by verifying that certain sufficient conditions for the solvability are

[1] See e. g. [10] p. 717 ff.

satisfied[1]. For higher topological structure, however, the results obtainable without use of mapping theorems are decidedly less complete as a critical examination will show. The mapping theory seems therefore preferable as a basis in these higher cases, all the more as thereby also the variational part of the investigation is greatly simplified.

In the first part of the following paper first, assuming our variational problem solved, we shall prove that the solution is a minimal surface. Secondly we shall establish the existence of the solution under suitable sufficient conditions. To make these conditions more easily verifiable we shall transform them into another form. It is at this point where in case of higher topological structure the theory of conformal mapping becomes indispensable. — In the second part, we shall discuss the case of free boundaries.

Part I. The Plateau-Douglas problem.

§ 1. Proof by Conformal Mapping that the Solution is a Minimal Surface.

We first show how simply the proof in the most general case can be given (provided the existence of a solution of the variational problem is assumed), if the theory of conformal mapping of Riemann domains is used. Suppose the variational problem be solved by an admissible vector $\mathfrak{x}$ and a domain B of our class, so that

$$D(\mathfrak{x}) = d$$

is the minimum value. The vector $\mathfrak{x}$, according to the Dirichlet Principle, must then be a harmonic vector.[2] In regard to the class of admissible domains B we assume the following mapping theorem to the true: Every Riemann domain of the prescribed topological structure with k piecewise smooth boundary lines can be mapped conformally onto a domain B of the class.

The type of such »normal domains» for which the proof of this mapping theorem can be given most easily is that of the »parallel slit domains». These domains consist of the whole w-plane or the upper half w-plane except for a

[1] See e. g. [10] p. 707 ff. or [12].

[2] The Dirichlet principle states that, with given continuous boundary values, the minimum of the Dirichlet integral over B of a function with piecewise continuous first derivatives is given by the harmonic function and only by it. This principle, which here is needed only for our special class of domains B, is equivalent to the boundary value problem of the Laplace equation. For its proof see e. g. Courant-Hilbert, Meth. der math. Phys. vol. II, (1937). Chap. VII.

finite number of slits parallel to the u-axis. In case of domains of genus zero the parallel boundary slits are of finite length. In case of domains not of genus zero and of characteristic number $\varkappa$ there are $\varkappa$ pairs of unilaterally infinite slits whoses edges are coordinated in a simple manner[1]. — To fix the ideas we may suppose that B is a slit domain.

To prove that our solution $\mathfrak{x}$, B represents a minimal surface we first show that $\mathfrak{x}$ furnishes a minimum of the Dirichlet integral also in comparison with certain discontinuous vectors $\mathfrak{z}$.[2] We consider in B a small straight segment L with the end points $A_1 A_2$ through an arbitrary point P, e.g. the segment $|u| \leq a$, $v = 0$ through the origin, and in L the function $\lambda(u) = (u^2 - a^2)$; ε may be a small parameter and Q a rectangle in B adjacent to L, e.g. $|u| < a$ and $0 \leq v \leq b$.

The domain B is now cut along the segment L and the minimizing harmonic vector $\mathfrak{x}(u, v)$ is replaced by a vector $\mathfrak{z}(u, v)$ which is identical with $\mathfrak{x}$ outside of Q and which, in Q, is defined by

$$\mathfrak{z}(u, v) = \mathfrak{x}(u + \varepsilon \varrho, v)$$

with

$$\varrho = \lambda(u) \frac{b - v}{b} = \frac{(u^2 - a^2)(b - v)}{b}.$$

The varied vector $\mathfrak{z}$, therefore, is discontinuous along the cut L, but analytic along the interior of either edge of L. Our statement now is

$$D_B(\mathfrak{z}) \geq D_B(\mathfrak{x}) = d,$$

or, which is equivalent

$$D_Q(\mathfrak{z}) \geq D_Q(\mathfrak{x}). \tag{7}$$

In other words: The vector $\mathfrak{x}$ gives a minimum of the Dirichlet integral with respect to the vector $\mathfrak{z}$ for the rectangle Q.

To prove this we consider the Riemann domain G which we obtain by cutting the domain B along L and by coordinating the lower and upper edge of the cut in such a way that to a point with the coordinate u on the lower edge the point with the coordinate $u + \varepsilon \lambda(u)$ corresponds on the upper edge. By assuming $|\varepsilon 2 a| < 1$ we ensure that $u + \varepsilon \lambda$ is monotonic in u and that there-

[1] See e.g. Hurwitz-Courant, Funktionentheorie, 3d edition p. 472 ff. For higher genus, Courant, Math. Zeitschrift vol. 3. (1919) p. 1 ff., and [10], pp. 721 ff. and [15].

[2] See [9], [10].

fore we have a one-one correspondence. The boundary of G consists of the boundary b of B plus the end points A_1 and A_2 of L. Hence according to the mapping theorem assumed above, we can map G conformally on a domain G' of the type B (e. g. slit domain), so that the boundary slits b of B are transformed in a one-one way into new boundary slits b' and the points A_1, A_2 into points (or slits) A'_1, A'_2.

Corresponding interior points of the different edges of the cut L, forming together an interior point of the Riemann domain G, will be transformed into an interior point of G' and thus the vector $\mathfrak{z}$ will be transformed into a vector $\mathfrak{z}'$, in G', which is continuous not only in G' but also in the domain $B' = G' + A'_1 + A'_2$. The domain B' is an admissible domain in our variational problem and $\mathfrak{z}'$ is there an admissible vector since there is a continuous and one-one correspondence between the boundaries b and b'. Therefore, because B and $\mathfrak{x}$ were supposed to solve the minimum problem, we have

$$D_{B'}(\mathfrak{z}') \geq D_B(\mathfrak{x}) = d.$$

On the other hand, because of the invariance of the Dirichlet integral under our conformal mapping, we have

$$D_{B'}(\mathfrak{z}') = D_B(\mathfrak{z}) = D_B(\mathfrak{x}(u + \varepsilon\varrho, v)),$$

and because of $\varrho = 0$, except in the rectangle Q, we finally obtain

$$D_Q(\mathfrak{z}) = D_Q(\mathfrak{x}(u + \varepsilon\varrho, v)) \geq D_Q(\mathfrak{x}),$$

as stated.

To show that the solution is a minimal surface becomes now a matter of the classical formalism of the variational calculus, since $\mathfrak{x}$ and therefore $\mathfrak{z}$ is analytical in u, v and ε in the rectangle Q and on its boundary. The Dirichlet integral of $\mathfrak{z}$ over Q must have a minimum for $\varepsilon = 0$, which by differentiation under the integral sign can by expressed by

$$\text{(8)} \qquad \iint_Q (\mathfrak{z}_u \mathfrak{z}_{u\varepsilon} + \mathfrak{z}_v \mathfrak{z}_{v\varepsilon})\, du\, dv = 0, \quad \text{for} \quad \varepsilon = 0.$$

By transforming the left side by Green's formula, and observing that $\varrho(u, v)$ and hence $\mathfrak{z}_\varepsilon$ vanish on the boundary of Q, except on L, that $\mathfrak{x}$ is harmonic, and that for $\varepsilon = 0$ we have on L

$$\mathfrak{z}_\varepsilon = \lambda(u)\, \mathfrak{x}_u(u, 0),$$

we conclude

(9) $$\int_L \lambda\, \mathfrak{x}_u \mathfrak{x}_v \, du = 0.$$

Since λ is positive in L and since a can be chosen arbitrarily small it follows by the classical reasoning that in the point P

$$F = \mathfrak{x}_u \mathfrak{x}_v = 0.$$

In the same way, by choosing as our cut a segment $u - v = \text{const.}$, we obtain

$$E - G = (\mathfrak{x}_u - \mathfrak{x}_v)(\mathfrak{x}_u + \mathfrak{x}_v) = 0.$$

Therefore the equations (2) characterizing S as a minimal surface are proved for every point P in B.

It should be mentioned that the same mapping theorem which permits the »*sewing together*» different analytically coordinated edges of a cut, also serves to furnish the proof for the minimum area property of the minimal surface, as shown in [10].

§ 2. Proof without Use of Conformal Mapping.

In this section, again assuming that the domain B and the harmonic vector $\mathfrak{x}$ solve the variational problem, we shall prove the relation $\varphi(w) = 0$ without using any theorems on conformal mapping. We shall do this not only for the case of surfaces of genus zero but also for the case of higher topological structure.

By performing suitable variations, we first establish variational conditions in a rather general form from which then we shall obtain the condition $\varphi(w) = 0$ for different types of normal domains B.

1. General Variational Formula.

To express analytically the fact that $\mathfrak{x}$ and B furnish a minimum with respect to variations of the boundary values of $\mathfrak{x}$ and of the domain B we can proceed as follows:[1] We transform the domain B of the variables u, v or the complex variable $w = u + iv$ into another admissible domain B' of the complex variable $w' = u' + iv'$ by a one-one transformation of the form

[1] See also Shiffman [16].

$$
(10) \qquad
\begin{aligned}
u &= u' + \varepsilon \Lambda \\
v &= v' + \varepsilon M \\
w &= w' + \varepsilon (\Lambda + i M),
\end{aligned}
$$

where ε is a small parameter and the quantities Λ and M are continuous functions of u, v, ε or u', v', ε in B or B' with piecewise continuous first derivatives[1]. The derivatives with respect to all three variables are supposed to be absolutely bounded in the domain B.

We shall use the symbol

$$A \equiv 0,$$

if in the whole domain concerned we have

$$|A| < a \varepsilon^2,$$

where a is a constant in this domain. Then we have, if for $\varepsilon = 0$ the notation

$$
(11) \qquad
\begin{aligned}
\Lambda (u, v, 0) &= \lambda (u, v) \\
M (u, v, 0) &= \mu (u, v),
\end{aligned}
$$

is introduced,

$$(12) \qquad \varepsilon \Lambda_u \equiv \varepsilon \Lambda_{u'} \equiv \varepsilon \lambda_u \equiv \varepsilon \lambda_{u'}, \quad \text{etc.};$$

$$
(13) \qquad
\begin{aligned}
\frac{\partial (u', v')}{\partial (u, v)} &\equiv 1 - \varepsilon (\lambda_u + \mu_v); \\
\frac{\partial (u, v)}{\partial (u', v')} &\equiv 1 + \varepsilon (\lambda_u + \mu_v).
\end{aligned}
$$

Now we introduce a variation of the vector $\mathfrak{x}$, replacing $\mathfrak{x}$ in B by a vector $\mathfrak{z}(u', v')$ in B', by the definition

$$\mathfrak{z}(u', v') = \mathfrak{x}(u, v).$$

The functions Λ, M are chosen so that B' again is an admissible domain; therefore we certainly have, because of the minimum property of $\mathfrak{x}$ and B,

$$D_{B'}(\mathfrak{z}) \geq D_B(\mathfrak{x}).$$

[1] If B is a Riemann surface then our functions are supposed to be univalued there, but not necessarily in the simple plane.

This gives:

$$\frac{1}{2}\iint\limits_B \{[\mathfrak{x}_u(1+\varepsilon A_{u'}) + \mathfrak{x}_v \varepsilon M_{u'}]^2 + [\mathfrak{x}_u \varepsilon A_{v'} + \mathfrak{x}_v(1+\varepsilon M_{v'})]^2\} \frac{\delta(u', v')}{\delta(u, v)} du\, dv$$

$$\geq D_B(\mathfrak{x})$$

Taking into account the fact that the Dirichlet integral $D(\mathfrak{x})$ and hence also

$$\iint\limits_B |\mathfrak{x}_u \mathfrak{x}_v|\, du\, dv$$

is finite, we obtain, because of (11), (12), and (13)

$$D_B(\mathfrak{x}) \leq D_{B'}(\mathfrak{z}) \equiv D(\mathfrak{x}) + \varepsilon \frac{1}{2}\iint\limits_B [p(\lambda_u - \mu_v) - q(\lambda_v + \mu_u)]\, du\, dv, \tag{14}$$

where

$$p + iq = \varphi(w) = (\mathfrak{x}_u^2 - \mathfrak{x}_v^2) - 2i\,\mathfrak{x}_u \mathfrak{x}_v. \tag{15}$$

Hence the minimum property is expressed by the equation

$$\iint\limits_B [p(\lambda_u - \mu_v) - q(\lambda_v + \mu_u)]\, du\, dv = 0. \tag{16}$$

This fundamental equation can be modified as follows: We make the assumption that B is divided into two parts $B = B_1 + B_2$, having the common piecewise smooth boundary line L in the interior of B, and we suppose that $A + iM$ and $\lambda + i\mu$ are analytic functions of w in B_2. Then by (10) the domain B_2 is mapped conformally onto a domain B_2'; therefore the corresponding part of the Dirichlet integral remains invariant, and we may apply our whole reasoning only to B_1. Hence under the assumption of the boundedness of the derivatives of λ, μ for B_1, not necessarily for B_2, we have instead of (16) the relation

$$\iint\limits_{B_1} \{p(\lambda_u - \mu_v) - q(\lambda_v + \mu_u)\}\, du\, dv = 0. \tag{16 a}$$

Under the further assumption that λ and μ vanish in a neighborhood of the boundary lines of B_2, except L, the condition (16 a) is, by integration by parts equivalent with

$$(17) \qquad \int_{L\uparrow} \lambda (p\,dv + q\,du) + \mu (p\,du - q\,dv) = 0.$$

For, the expressions p and q satisfy the Cauchy-Riemann equations, so that the resulting domain integral over B_1 vanishes and only the contour integral remains.

If $\Im$ denotes the imaginary part of a complex quantity, we may write the variational condition (17) in the convenient form

$$(18) \qquad \Im \int_L (\lambda + i\mu)\,\varphi(w)\,dw = 0.$$

Our general conditions (16) and (18) will now be applied to definite types of domains B, which requires a suitable choice of the functions $\varLambda$, M.

2. Variation of the Boundary Representation.

We suppose that B is bounded by circles, one of which e, g. C_1, may be the unit circle with corresponding polar coordinates r, θ. Keeping B fixed, we establish the variational condition referring to the representation of Γ_1 on the corresponding boundary curve C_1. For this purpose we choose $\varLambda + iM = 0$ except in a small annular ring R_{r_0} adjacent to C_1. With R_r we denote such an annular ring between the circles with the radii r and 1, with B_r the domain $B - R_r$, with C_r the circle with the radius r. From (16) we infer, since the existence of the Dirichlet integral $D(\mathfrak{x})$ implies

$$\iint_{R_r} (\mathfrak{x}_u^2 + \mathfrak{x}_v^2)\,du\,dv \to 0, \quad \text{for} \quad r \to 1$$

and therefore for $r \to 1$

$$\iint_{R_r} (|p| + |q|)\,du\,dv \to 0,$$

that

$$(19) \qquad \iint_{B_r} \{p(\lambda_u - \mu_v) - q(\lambda_v + \mu_u)\}\,du\,dv \to 0,$$

for $r \to 1$. Since $\lambda + i\mu$ was assumed equal to zero except in a small ring R_{r_0} adjacent to C_1 and since p and q satisfy the Cauchy-Riemann equations we obtain by integration by parts

$$\Im \int_{C_r \uparrow} (\lambda + i\mu)\, \varphi(w)\, dw \to 0, \quad \text{for} \quad r \to 0. \tag{20}$$

Now, with an arbitrary real function $\alpha(r,\theta)$ having continuous first derivatives, we choose in the neighborhood of C_1 the variation

$$\Lambda + i M = - w \frac{e^{i\varepsilon\alpha(r,\theta)} - 1}{\varepsilon}.$$

Our transformation (10) then becomes $w' = w e^{i\varepsilon\alpha}$ and transforms the circle C_1 into itself in a one-one way if ε is sufficiently small. Since near C_1

$$\lambda + i\mu = i w \alpha(r,\theta),$$

our condition (20) becomes

$$\Re \int_{C_r} \alpha(r,\theta)\, w\, \varphi(w)\, dw \to 0, \tag{21}$$

where $\Re$ means »real part». By $dw = i w\, d\theta$, this is equivalent to

$$\Im \int_{C_r} \alpha(r,\theta)\, w^2 \varphi(w)\, d\theta \to 0. \tag{22}$$

Since α is arbitrary, we can, for any boundary cirle C, easily deduce from this formula[1]:

The analytic function:

$$\psi(w) = w^2 \varphi(w) \tag{23}$$

has real boundary values on the boundary circle C. This incidentally implies that the function $\psi(w)$ and hence $\varphi(w)$ is regular on the boundary.

In the simplest case of the Plateau problem we conclude immediately that in the unit circle $\psi(w) = 0$ is a real constant, which must be zero since $\varphi(w)$ is regular for $w = 0$. Thus, in this case, the solution is recognized as a minimal surface.

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[1] See [10] p. 712 or [12].

3. Variation of Circular Boundaries.

We now consider variations of the domain B. In particular, again supposing B to be bounded by circles, we vary B with respect to a boundary circle C_1 by displacing it, or by expanding or contracting it around its center.

A translation of the circle in direction of the u-axis is effected by putting

$$\Lambda + iM = \lambda + i\mu = 1,$$

in a neighborhood of C_1, bounded by a line L which can, in B, be deformed into C_1, and $\Lambda + iM = 0$ in the neighborhood of all the other boundary components. Since in the ring B_2 between L and C_1 the expression $\lambda + i\mu$ is constant, hence analytic in w, we can apply (18) and obtain immediately

$$\Im \int_L \varphi(w)\, dw = 0.$$

In the same way we obtain, by choosing $\lambda + i\mu = i$ in B_2, the equation

$$\Re \int_L \varphi(w)\, dw = 0,$$

which combined gives

$$\int_L \varphi(w)\, dw = 0. \tag{24}$$

A dilatation of the unit circle C_1 can be represented by putting in B_2

$$w' = (1 + \varepsilon) w \qquad \text{or} \qquad w' = (1 + i\varepsilon) w,$$

which gives

$$\lambda + i\mu = w \qquad \text{or} \qquad \lambda + i\mu = iw,$$

and therefore, as above

$$\int_L w\, \varphi(w)\, dw = 0, \tag{25}$$

as an expression of the variability of the radius.

Because of the regularity of $\varphi(w)$ on the boundary (24) and (25) are equivalent to

$$\int_C \varphi(w)\,dw = 0, \tag{24 a}$$

$$\int_C w\,\varphi(w)\,dw = 0 \tag{25 a}$$

for each boundary circle which may be arbitrarily varied within the class of admissible domains.

The relation $\varphi(w) = 0$ is a consequence of the conditions (23), (24), (25), if the domains B are plane domains bounded by k circles, which correspond to the case of genus zero and k contours. For the proof we refer to [10].

As pointed out there (p. 721 f.) the same reasoning yields $\varphi(w) = 0$ also in case of higher topological structure, if e. g. the class B consists of fundamental domains of Schottky-groups from which k circular discs are removed.[1]

In this paper I want to carry out the variational analysis for another class of representing domains B namely Riemann surfaces, all of whose boundary lines are unit circles.[2] For the analysis of such domains we must study the effect of a variation of branch points which here is the only admissible form of a variation of the domain.

4. Variation of Branch Points.

We now suppose that the domain B is a Riemann domain over the w-plane containing a branch point P, e. g. the point $w = 0$, which is not fixed for the class of admissible domains. Then we perform variations of the domain B by only deforming a neighborhood N of the branch point enclosing a smaller neighborhood B_2 which is bounded by a closed curve L on the Riemann domain B. We again choose λ and μ as zero outside of the larger neighborhood N, and $\lambda + i\mu$ as an analytic function of w in B_2; then our formula (18) is applicable. If, in particular, we choose $\lambda + i\mu = 1$ or $\lambda + i\mu = i$ in B_2, we obtain immediately as before the variational condition

$$\int_L \varphi(w)\,dw = 0 \tag{26}$$

[1] In [10] it was supposed that such fundamental domains are bounded by circles. However, this restriction is not essential and should be dropped.

[2] These domains were introduced in [14]. See also [12].

where L is any closed curve on the Riemann domain enclosing the branch point P and only this branch point, where φ, as we shall see, may have a singularity.

In the case of a simple branch point this is the only variational condition. However, if P is a branch point of higher order, say of order r, we have to supplement the condition by others corresponding to a resolution of P into branch points of lower order. This resolution is effected in a simply connected neighborhood B_2 of P, bounded by L, by an analytic transformation of the form

$$w' - w = \varepsilon\, w^{\frac{\nu}{r+1}} \tag{27}$$

$$w' - w = i\,\varepsilon\, w^{\frac{\nu}{r+1}} \tag{28}$$

where ν may range from 0 to $r-1$.[1] Accordingly we choose $w' - w = (\lambda + i\mu)\varepsilon$ in B_2 and $\lambda + i\mu = 0$ outside of a wider simply connected neighborhood N around P.

Then our conditions (18) immediately yield as before

$$\Re \int_L w^{\frac{\nu}{r+1}} \varphi(w)\, dw = 0, \qquad \Im \int_L w^{\frac{\nu}{r+1}} \varphi(w)\, dw = 0,$$

or

$$\int_L w^{\frac{\nu}{r+1}} \varphi(w)\, dw = 0, \tag{29}$$

for $\nu = 0, 1, \ldots, r-1$.

For these conditions the following interpretation can be given: An analytic function of the form

$$\varphi(w) = \Sigma f'_\mu(w)^2$$

has in general at the branch point P a polar singularity of the order $2r$.

Indeed, by

$$\sigma = w^{\frac{1}{r+1}}$$

the neighborhood of the branch point $w = 0$ is transformed into the simple neighborhood of $\sigma = 0$; since for $\sigma \to 0$, the functions $f_\mu(w)$ have their real parts bounded, they must be regular in σ; hence

$$\chi(\sigma) = \varphi(w) = \left(\frac{d\sigma}{dw}\right)^2 \Sigma \left(\frac{df_\mu}{d\sigma}\right)^2 = \frac{1}{(r+1)^2}\,\frac{1}{\sigma^{2r}}\,A(\sigma)$$

[1] As easily seen by first mapping the vicinity of P on the simple neighborhood of $\sigma = 0$ by $w = \sigma^{r+1}$, (27) and (28) produce for B' one $(\nu - 1)$-fold and $r + 1 - \nu$ simple branch points.

where $A(\sigma)$ is regular for $\sigma = 0$, which exhibits the character of the singularity at $\sigma = 0$.

Now the conditions (29) can be written as conditions in the σ-plane:

$$\int_{L'} \sigma^{\nu + r} \varphi \, d\sigma = \int_{L'} \sigma^{\nu + r} \chi(\sigma) \, d\sigma = 0 \qquad (\nu = 0, 1, \ldots, r - 1)$$

where L' is a simple closed curve around the pole $\sigma = 0$.

Therefore, in the expansion of $\chi(\sigma)$ in powers of σ

$$\varphi(w) = \chi(\sigma) = \cdots + \frac{a_1}{\sigma} + \cdots + \frac{a_r}{\sigma^r} + \frac{b_1}{\sigma^{r+1}} + \cdots + \frac{b_r}{\sigma^{2r}},$$

all the coefficients b_ν must vanish and we have as a final expression of the variational conditions for a branch point: *The function $\varphi(w)$ has at a branch point of order r a pole of an order at most r.*

5. Evaluation of the Variational Condition for Riemann Domains B Bounded by Unit Circles.

On the basis of the previous results, the proof of the characteristic relation $\varphi(w) = 0$ for the solution of the variational problem becomes very simple, if the underlying class of domains B is chosen not as a domain in the plane but as a Riemann surface all of whose boundary lines are unit circles.[1] This class is defined as follows:

We consider for the case of genus zero a k-fold connected domain B formed by the discs of k unit circles which are connected in branch points of the total multiplicity $2k - 2$. For higher genus p, we obtain domains B by affixing to the k-fold circular disc p full planes each in 4 branch points[2]. Branch points connecting two circular discs are supposed to be interior points, while branch points connecting full planes with circular discs may lie on boundary circles.

We make the assumption — for the proof under suitable conditions see § 4 — that our variational problem is solved by a vector $\mathfrak{x}$ in a domain B of this class.

By reflecting our domains on all boundary circles, we could consider instead a closed symmetric Riemann surface with all these boundary circles as symmetry

[1] See also [12] and [14].

[2] Each such full plane represents a »handle» and increases the genus by 1.

lines and then require that the functions $\mathfrak{x}$ have the same values on the Riemann surface at points corresponding by this symmetry. Such closed symmetric surfaces which remain connected after being cut along the unit circles, also take care of non-orientable minimal surfaces.[1]

Since under a linear transformation of the unit circle into itself the Dirichlet integral is invariant, we may conveniently assume for the solution $\mathfrak{x}$, B that $w = 0$ is no branch point.

In considering first the case of genus zero, we count the zeros and poles of

$$\psi(w) = w^2 \varphi(w)$$

in B. If N is the total multiplicity of the former, P that of the latter, and if we assume that $\psi(w)$ is not identically zero, we have

$$N - P = \frac{1}{2\pi i} \sum_{\mu} \int_{C_\mu^* \uparrow} d \log \psi(w), \tag{30}$$

the integrals being extended in the positive sense along C_μ^*, where C_μ^* is the unit circle C_μ except for small halfcircles circumventing, in the negative sense, zeros of $\psi(w)$ which may lie on C_μ.

Since $\psi(w)$, according to the variational condition established in No. 2, is real on C_μ, the arcs of C_μ do not contribute to the imaginary parts of the integrals; while each circumventing halfcircle contributes $-\pi i$ for a simple zero, and $-s\pi i$ for a zero of order s. Hence,

$$N - P \leq 0,$$

or

$$N \leq P.$$

From the result of No. 4 we know that $P \leq 2k - 2$; on the other hand the factor w^2 in $\psi(w)$ provides a double zero at each of the k origins of the discs forming B and hence ensures $N \geq 2k$. Thus

$$2k \leq N \leq 2k - 2$$

would result. This being absurd, $\psi(w) = 0$, hence, $\varphi(w) = 0$ is proved.[2]

For higher topological structure the same reasoning holds. The equality (30) again leads to a contradiction. For in addition to the $2k$ zeros at the

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[1] See p. 78. Closed, symmetric Riemann surfaces as domains of representation are used in a general way by Douglas. Cf. [8].

[2] The following variant of the reasoning, due to M. Shiffman, may be indicated: We can replace the variational condition (29) by the equivalent: The function

k origins of the unit circles, we have $2p$ more zeros at the origins of the p full planes and $2p$ more at their points of infinity. The latter follows because $f_\mu(w)$ is bounded, hence regular there; therefore $f'_\mu(w)$ has a zero there of order at least 2 and likewise $w^2 \Sigma f'_\mu(w)^2$ has a zero of at least the order 2. Thus $N \geq 2k + 4p$, while $P \leq 2k - 2 + 4p$, so that a contradiction $N \leq P$ again results if the right hand side in (30) is non positive which was proved if all the branch points are interior points.

However, in the case of higher topological structure, if branch points lie on the boundary[1], the following supplementary argument is necessary, because such branch points make a positive contribution to the right hand side of (30). As appears immediately from considering the corresponding symmetric surface B^*, such a branch point connecting $r + 1$ sheets in B is a $2r$-fold branch point on the symmetric surface B^*. Hence $\varphi(w)$ may have at this point R of C_μ a pole of order not higher than $2r$.[2] On the other hand $\psi(w) = w^2 \varphi(w)$ is real on C_μ as before. By the same reasoning as above, circumventing R by a small circular arc, we find now as contribution to the right hand side of (30) at most the positive value $\frac{1}{2} 2r = r$, while our branch point on the boundary reduces the total multiplicity of interior branch points, and therefore the number P, by r. Thus the conclusion above, leading to a contradictory inequality, subsists.

In a similar way the reasoning for non-orientable surfaces can be carried out.

$$\int_{w_0}^{w} w\varphi(w)\,dw = \chi(w) = \varrho + i\sigma$$

is regular and univalued in B. (See also [10]).

Now the condition (23) shows that $\sigma = \text{const.} = \sigma_\nu$ on each boundary C_ν. Hence not only σ but also ϱ can be extended beyond C_ν and has equal values in points near C_ν and symmetric to C_ν. But ϱ must attain its maximum in $B + C$ in a point R on a circle C_ν. This maximum would thus be a maximum of ϱ in a whole neighborhood of R, which is impossible for a regular not constant harmonic function.

[1] This occurence can, as the construction of the solution in § 4 shows, not be excluded, unless the genus is zero.

[2] This follows by the method of Nr. 4. We first transform B by a linear transformation so that the unit circle, i. e. the symmetryline, becomes the real axis and that the branch point falls into the origin. Then we apply in the vicinity of the origin the variation $w' - w = \varepsilon w^{\frac{\nu}{2r+1}}$ for odd ν and for $\nu = 0$, and the variation $w' - w = \varepsilon \left(w^{\frac{\nu}{2r+1}} + w^{\frac{\nu-1}{2r+1}}\right)$ for even positive ν. Thereby the symmetry of the image of B^* and the one-one correspondence of the boundaries of B and B' is preserved so that the reasoning of Nr. 4 remains applicable. Because of the symmetry of $\varphi(w)$ the condition (29) is again a consequence, L now being a closed curve on B^*.

§ 3. Solution of the Variational Problem in the Simplest Case.

What remains to be shown is the existence of the solution of the variational problem. The discussion of this cardinal point is based on a simple Lemma.

1. Fundamental Lemma:

In a domain $G = G_n$ of the u, v plane, which may vary with the index n, we consider a class of continuous vectors $\mathfrak{x} = \mathfrak{x}_n$ with piecewise continuous first derivatives, so that their Dirichlet integrals are equally bounded by a constant M:

$$D_{G_n}(\mathfrak{x}_n) \leq M.$$

Around a fixed arbitrary point Q, we draw circles of the radius r; C_r may denote an arc or a set of arcs of such a circle contained in G and s may be the arc length on C_r. Then there exists for each positive $\delta < 1$ a value ϱ with

$$\delta \leq \varrho \leq \sqrt{\delta},$$

so that

$$\int_{C_\varrho} \mathfrak{x}_s^2 \, ds \leq \frac{\varepsilon(\delta)}{\varrho} \tag{31}$$

with

$$\varepsilon(\delta) = \frac{2M}{\log \frac{1}{\delta}} \to 0 \tag{32}$$

for $\delta \to 0$.

Consequently for the length L_ϱ of the image C'_ϱ of C_ϱ in the $\mathfrak{x}$-space, we have

$$L_\varrho^2 \leq 2\pi\varepsilon(\delta). \tag{33}$$

The proof of (31) follows immediately by introducing in $D(\mathfrak{x})$ polar coordinates r, ϑ around Q and reducing $D(\mathfrak{x})$ to a double integral[1]; then (33) follows by Schwarz' inequality because of

$$L_\varrho = \int_{C_\varrho} \sqrt{\mathfrak{x}_s^2} \, ds.$$

[1] See [10] p. 688 f. The fact that G is an open domain presents no difficulty since the integrand is positive in G.

2. Minimizing Sequences. Equicontinuity.

We consider our variational problem for the simple Plateau problem, B being the unit circle with the circumference C mapped by the admissible vectors $\mathfrak{x}$ monotonically on the (rectifiable) Jordan curve Γ. A sequence $\mathfrak{x}_n$ of admissible vectors for which

$$D(\mathfrak{x}_n) \to d$$

is called a minimizing sequence if, as before, d denotes the lower limit of $D(\mathfrak{x})$. Since a linear transformation of the unit circle B into itself leaves the Dirichlet integral unchanged, we may in advance assume that by such a *transformation* the vectors $\mathfrak{x}_n$ coordinate three given points P_1, P_2, P_3 on C to three fixed points P_1', P_2', P_3' on Γ (three point condition), so that the mutual distance between the latter points is greater than a positive quantity a.

We state: The boundary values of a set of admissible vectors $\mathfrak{x}_n$ are equicontinuous if the $\mathfrak{x}_n$ satisfy the three point condition and if $D(\mathfrak{x}_n) \leq M$ with a fixed M.

Proof: Any Jordan curve Γ has the following property. There exists for $\tau > 0$ a $\sigma(\tau)$ with $\sigma(\tau) \to 0$ for $\tau \to 0$, so that for any pair of two points A', B' on Γ whose distance is not greater than τ, one of the two arcs $A'B'$ on Γ has a diameter not exceeding $\sigma(\tau)$. Now let Q be any point on C. We choose for a given small ε the quantity δ according to (32). Then, by our fundamental Lemma, there exists a ϱ with $\delta \leq \varrho \leq \sqrt{\delta}$, so that the inner arc AB of the circle with the radius ϱ around Q, i. e. the closed intersection of this circle with $B + C$, is mapped on a curve with the length not exceeding $\tau = \sqrt{2\pi\varepsilon}$.

The endpoints A, B of this arc, which are on C, are mapped on two points A', B' on Γ whose distance does not exceed τ; and hence, one of the arcs $A'B'$ of Γ has a diameter not exceeding $\sigma(\tau)$. For sufficiently small ε, or δ, or τ, this arc must correspond to the small arc AQB of C, because the larger of the two arcs on Γ must contain at least two of the three fixed points P_1', P_2', P_3' and because the three point condition prevents the small are AQB, for sufficiently small δ, from containing two of the fixed points P_1, P_2, P_3 on C. Hence it is proved that the oscillation of $\mathfrak{x}_n$ on any arc of C with a chord not exceeding δ does not exceed σ, with $\sigma \to 0$ for $\delta \to 0$. Since all these quantities depend on M only, the statement concerning equicontinuity is proved. The vectors $\mathfrak{x}_n$ need not be harmonic. But in this case, by replacing $\mathfrak{x}_n$ by a harmonic vector with the same boundary values, we obtain a vector with a smaller Dirichlet integral according to the classical Dirichlet Principle.[1] Therefore, we may assume that the minimizing sequence under consideration consists of harmonic vectors.

To solve our variational problem, we now choose among the uniformly bounded and equicontinuous boundary value functions of the vectors $\mathfrak{x}_n$ a uniformly converging subsequence. The corresponding harmonic vectors $\mathfrak{x}_n$ then converge uniformly also in B and

$$\mathfrak{x} = \lim \mathfrak{x}_n$$

is an admissible harmonic vector. Since for each closed subdomain $\bar{B}$ the derivatives of the harmonic vectors converge uniformly also, we have

$$D_{\bar{B}}(\mathfrak{x}) = \lim D_{\bar{B}}(\mathfrak{x}_n) \leq \lim D(\mathfrak{x}_n) = d.$$

Hence

$$D_B(\mathfrak{x}) \leq d$$

and therefore, since d was the lower bound,

$$D_B(\mathfrak{x}) = d.$$

Thus $\mathfrak{x}$ represents a solution of the variational problem and therefore, because of § 2 and § 3, of the Plateau problem.

3. Remarks — Semicontinuity.

The preceding reasoning holds if $\mathfrak{x}_n$ does not map C exactly on Γ but on a continuous (rectifiable) curve $\Gamma^{(n)}$ which tends to Γ in the strong sense,[2] i. e. so that together with two points $P^{(n)}$, $Q^{(n)}$ on $\Gamma^{(n)}$ tending to P, Q on Γ the whole arc $\widehat{P^n Q^n}$ on $\Gamma^{(n)}$ tends to the arc $\widehat{PQ}$ on Γ. We need not suppose that $\Gamma^{(n)}$ is a Jordan curve, but we permit $\Gamma^{(n)}$ to have multiple points and corresponding small loops which disappear in the limit. If $\mathfrak{x}_n$ satisfies a three point condition, by mapping three fixed points P_1, P_2, P_3 of C on the points $P_1'^{(n)}$, $P_2'^{(n)}$, $P_3'^{(n)}$ which tend to three points P_1', P_2', P_3' on Γ, then equicontinuity of the boundary values of $\mathfrak{x}_n$ is proved exactly as above. Hence, the concept of a minimizing sequence may be generalized by permitting for $\mathfrak{x}_n$ a mapping on $\Gamma^{(n)}$, without changing our reasoning. At the same time we draw the conclusion:

If $d^{(n)}$ is the lower limit of $D(\mathfrak{x})$ under the condition that $\mathfrak{x}$ maps the boundary C on $\Gamma^{(n)}$, then

$$d \leq \lim\ \inf\ d^{(n)}.$$

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[1] See [11].

[2] Or in the »Frechet sense».

In other words: *The lower limit depends on the boundary in a semi-continuous way.*

A second remark is: The solution $\mathfrak{x}$ furnishes eo ipso a one-one correspondence between the boundaries C and Γ. For the simple proof we refer to [10].

Thirdly: The three point condition serves to ensure the equicontinuity of the boundary values. It is possible to attain the same objective in other ways. For example, we consider in the $\mathfrak{x}$-space a Jordan curve H which interlocks with Γ. Then each surface $\mathfrak{x}$ bounded by Γ must have a point in common with H. Hence, if α is a lower bound for the distances between points of H and points of Γ, there must be a point A in B, so that the corresponding $\mathfrak{x}$ has a distance greater than α from Γ. By a linear transformation of B into itself we may throw the point A into the origin. We assume that the vectors $\mathfrak{x}$ of a minimal sequence be subject to this transformation which, because of the invariance of the Dirichlet integral under conformal mapping, does not change the character of a minimizing sequence. Then we can prove equicontinuity of this new sequence $\mathfrak{x}$ by slightly modifying an argument from No. 2: If not the arc $c: AQB$, but the complementary arc c^* were mapped on an arc γ of Γ with a diameter less than σ, then we consider the subdomain B' of B bounded by c^* and the circular arc c_ϱ with the radius ϱ. On the boundary of B', the oscillation of $\mathfrak{x}_n$ is less than $\sigma + \sqrt{2\pi\varepsilon}$. Hence $\mathfrak{x}$ at the origin, according to the maximum and minimum principle of potential theory, cannot differ by more than $\sigma + \sqrt{2\pi\varepsilon}$ from a boundary value for B', e. g. from the value in A, which is on Γ. If ε is sufficiently small, we have $\sigma + \sqrt{2\pi\varepsilon} < \alpha$ and this would contradict our assumption. Hence, again the equicontinuity is established.

Thus we may impose the following condition in our variational problem, instead of the three point condition: *The origin of B shall have an image $\mathfrak{x}$ at a distance not less than α from Γ.*

Fourth: As was first emphasized by Douglas, the solution of the Plateau problem contains — for the special case of a plane curve Γ — a proof of the Riemann mapping theorem stating that the unit circle can be mapped conformally on the interior of a plane curve Γ. In addition the one-one correspondence of the boundaries follows.

4. Critical Analysis of the Method.

The method requires no more knowledge of potential theory than the Poisson solution of the Dirichlet problem for the circle and its minimum property.

The conformal equivalence of the unit circle B with other plane domains bounded by a Jordan curve being a consequence, we might have chosen for B, instead of a circle, such a more general domain. This remark apparently removes the objection that the method refers to a special type of parameter domains B. However, we could have chosen as parameter domain quite different simply connected Riemann domains B for which the previous method does not establish the conformal equivalence to a circle. For example, B may be a parallel strip $0 \leq u \leq 1$; $-\infty < v < \infty$ except the circle $u^2 + v^2 < \frac{1}{4}$. Simple connectivity is established by coordinating and »identifying» the boundary points of the strip on $u = 0$ which those on $u = 1$. E. g., we may identify the point $u = 0$, $v = a$ with the point $u = 1$, $v = ta$ with fixed t. In such corresponding points the values of $\mathfrak{x}$ shall be required to be equal. Or we might choose for B any simply connected Riemann surface bounded by a Jordan curve. A priori it is conceivable that such domains would provide a different lower limit d and hence different solutions. This is not the case because Riemann's mapping theorem can be generalized to any such simply connected domain. However, the proof of this fact is not obtained by our method and therefore an a priori knowledge of some of the theory of conformal mapping of Riemann domains[1] seems unavoidable, if one wants to free our solution from reference to special classes of domains B.

It is also on the basis of such mapping theorems that the equivalence of our solution with that of the least area problem follows: d is the lower limit of the areas of all surfaces, images of B, which are bounded by Γ.[2]

§ 4. Solution of the Variational Problem in the General Case.

1. Condition of Cohesion.

In the general problem of Douglas the boundary Γ consists of k separated[3] Jordan curves $\Gamma_1, \ldots, \Gamma_k$, and the minimal surface under consideration may

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[1] Or, what for simply-connected domains — and only for these — is equivalent, of Green's function.

[2] See [10] p. 721.

[3] It may be pointed out that curves Γ_ν may even be permitted to have points in common. Our methods can then easily be applied, and the result contains an alternative. E. g., for two Jordan curves with a point P in common, we obtain either a regular minimal surface or two different surfaces bounded by Γ_1 and Γ_2 respectively and having P in common; of these cases the one occurs in which the lower limit of the Dirichlet integral — or the area — is smaller. It is easy to verify this result on the basis of the subsequent reasonings.

have the genus zero or any prescribed genus p or, if non-orientable, characteristic number $\varkappa$. Accordingly the domains B of representation must have the same topological structure.

As simple examples show, it may be that the general problem has no proper solution. For example, there is no doubly connected minimal surface of revolution to be spanned between two parallel circles, if these circles are far apart. Or for a single plane boundary curve there certainly does not exist a minimal surface of genus one. Therefore we have to specify the problem by additional conditions, sufficient for the solvability. In this section the existence of the solution will be shown under the condition that for minimizing sequences a certain tendency to degeneration is excluded a priori. In the next section this condition will be replaced by another in the form of an inequality, first introduced by Douglas which is more easily verified in concrete cases. It is in connection with this form of sufficient conditions that recourse to the mapping theory for higher topological structure seems inevitable.

We define: A sequence of surfaces $\mathfrak{x}_n$ in the $\mathfrak{x}$-space satisfies the *condition of cohesion* or condition $\mathfrak{C}$, if there is a positive α so that every simple closed curve on $\mathfrak{x}_n$[1] of diameter less than α can, on the surface, be continuously contracted to a point (or is homotopic to zero).[2] Otherwise the sequence is said to tend to degeneration, which means essentially that the surfaces tend to degenerate either into separated surfaces connected only in single points, or for higher topological structure, e.g. genus p, possibly to degenerate into a surface of lower structure, e.g. of lower p.

If for our variational problem, formulated for a certain class of domains B of representation, there exists a minimizing sequence satisfying the condition $\mathfrak{C}$ of cohesion, then also the problem is said to satisfy the condition $\mathfrak{C}$.

It may be emphasized that in important cases the condition can be verified.[3]

2. Solution of the Variational Problem for Genus Zero and Plane Circular Domains B.

Now the main theorem is: *If the condition $\mathfrak{C}$ is satisfied, the variational problem can be solved. Then, either by § 1 or by § 2, the solution of the Douglas-Plateau problem is established.*

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[1] By this is meant a curve corresponding to a closed Jordan curve in the parameter domain B.

[2] The process of deformation is always defined with respect to the parameter domain.

[3] See e.g. for the case of plane boundaries [12], where mapping theorems appear as a consequence.

The proof is essentially the same for the different types of normal domains B considered in § 2.

To construct the solution under the assumption that the condition $\mathfrak{C}$ is satisfied, we consider the minimizing sequence $\mathfrak{x}_n$ in corresponding domains B_n such that

$$D_{B_n}(\mathfrak{x}_n) \to d.$$

We have to show that we can select a subsequence of the domains B_n tending to a limiting domain B of the prescribed topological type and that on the boundaries of B the vectors $\mathfrak{x}_n$ are equicontinuous functions; whereafter, the reasoning proceeds exactly as in § 3.

We carry out the proof for the case of genus zero, assuming the domains B to be plane regions bounded by k circles. (The reasoning is typical of that for other suitable classes B). By a linear transformation we may transform such a domain into the whole plane exterior to k circles, or into a domain bounded by two concentric circles, C_1, C_2, one being the unit circle, and $k-2$ circles lying in the ring between C_1 and C_2. This latter normalization — which replaces the three point condition of § 2 — shall now be assumed for every B_n.

We prove that the B_n define a limiting k-fold connected domain B. This is evident, if B_n cannot degenerate in one of the following ways:

1) Two circles, e. g. C_1 and C_μ, come arbitrarily near at a point P, while their radii remain above a positive bound α.

2) The same happens, but the radius of one of them, say C_μ, shrinks to zero.

3) One or more circles, e. g. C_2, C_3, shrink to the same point P while P remains bounded away from the non-shrinking circles.

The first type of degeneration is excluded as follows: By our Lemma, § 3, there is for a fixed arbitrarily small δ a circle around P with radius ϱ between δ and $\sqrt{\delta}$ so that the image of any arc of this circle by $\mathfrak{x}_n$ has a length L_n with $L_n^2 \leq 2\pi\,\varepsilon(\delta)$ where $\varepsilon(\delta) = \dfrac{2M}{\log \frac{1}{\delta}}$ and M is a bound for $D(\mathfrak{x}_n)$. But an arc of this circle joins, for sufficiently small δ and sufficiently large n, a point of C_1 with one of C_μ; the image therefore joins a point on Γ_1 with one on Γ_μ. The distance between points on these curves is bounded away from zero. Since δ and hence ε and thus L_n can be made arbitrarily small, we therefore have a contradiction.

The third type of degeneration is impossible, because here we can, with fixed sufficiently large n, include the circles shrinking to P in a circle K_ϱ with a radius ϱ around P, so that the image of K_ϱ by $\mathfrak{x}_n$ has a length L_ϱ not exceeding $\sqrt{2\pi\varepsilon(\delta)}$. But this shows that $\mathfrak{x}_n$ tends to degeneration in contradiction to our assumption $\mathfrak{C}$.

To exclude the second type of degeneration, we consider the typical case that a circle C_3 shrinks to a point P on C_1, while C_2, concentric with C_1, stays away from C_1. Again, by our lemma we can, for suitably small fixed δ and n sufficiently large, draw a circular arc $c = c_n$ around P joining two points A und B on C_1, so that the length L_n of the image $\gamma = \gamma_n$ of c by $\mathfrak{x}_n$ is less than $\sqrt{2\pi\varepsilon(\delta)} = \eta(\delta)$. The arc APB and the complementary arc c^* of C_1 are mapped on two complementary arcs of Γ, whose endpoints, the images of A und B, have a distance less than $\eta(\delta)$ so that one of them has a diameter arbitrary small for sufficiently small δ. This arc together with the arc γ_n then defines on the surface $\mathfrak{x}_n$ a closed curve with a total diameter arbitrarily small if δ is chosen suitably small. This curve is, on B_n, not homotopic to a point. For it separates on the surface Γ_2 from Γ_3, because in B_n the corresponding curve separates C_2 from C_3. But this expresses the fact that $\mathfrak{x}_n$ tends to separation, in contradiction to our assumption.

Hence degeneration of B_n is excluded, and we may assume that the sequence B_n or a subsequence tends to a domain B of the same type.

Now equicontinuity of $\mathfrak{x}_n$ on each boundary, e. g. C_1, is proved as follows: As above, there exists an inner circular arc $c = c_n$ with radius $\varrho = \varrho_n$ around P on C_1 joining two points A, B of C_1 with $\delta \leq \varrho \leq \sqrt{\delta}$, so that for the length L_n of the image γ_n of c_n we have $L_n^2 \leq 2\pi\varepsilon(\delta) = \eta(\delta)^2$. Equicontinuity of $\mathfrak{x}_n$ on C_1 means that the oscillation of $\mathfrak{x}_n$ on the arc APB remains arbitrarily small with δ. If this is not so, then c_n together with the arc c_n^* of C_1 complementary to the arc APB has as image a closed curve t_n on $\mathfrak{x}_n$ whose diameter can be made arbitrarily small with δ and which cannot be contracted to a point because it is homotopic to the curve Γ_1. In other words, non equicontinuity would mean tendency to separation in contradiction to our assumption. Hence the equicontinuity is proved.

The existence proof then is completed exactly as in § 3 after we replace the $\mathfrak{x}_n$ by harmonic vectors with the same equicontinuous boundary values, so that a suitable subsequence of them converges to an admissible harmonic vector $\mathfrak{x}$ in B for which $D(\mathfrak{x}) = d$.

3. Solution for Other Cases and Other Normal Domains.

If B, instead of being a circular domain, is of another type, e. g. a parallel slit domain (see § 2), the reasoning remains the same. However, if we prescribe a higher topological structure for the minimal surface bounded by Γ, then B cannot any longer be chosen as a simple plane domain. We have the choice between plane domains B with »*inner edges*» coordinated by analytic correspondences, as slit domains, or fundamental domains of groups of linear transformations (see [10]), or between Riemann surfaces B with several sheets and branch points. In either case the proof is very similar to that above. For parallel slit domains B we refer to [10] and [16]. For the Riemann surface type, we consider as particularly simple the surfaces introduced in § 2, bounded by k unit circles and, in the case of genus p, having $k + p$ sheets. By a linear transformation, they always can be normalized so that one branch point has a fixed position, e. g. at $w = 0$ or at $w = \frac{1}{2}$, bounded away from the boundary circles.

The slight modification necessary may be explained for two-fold connected domains B, consisting of two unit circles with two connecting branch points one of which, Q, may be fixed, e. g. at $w = \frac{1}{2}$. The only possible variability of B is the position of the free branch point P, and we have to show that this branch point P in B_n tends neither to Q nor to the boundary. The first degeneration is excluded as above by our Lemma[1]; the second as follows: If P tends to a boundary point R on the unit circle C_1, then we draw around R for sufficiently small δ a circle C_ϱ with radius ϱ between δ and $\sqrt{\delta}$ as in our lemma. P will, for large n, be separated from Q by C_ϱ, and thus C_ϱ will join a point A on C_1 with a point B on C_2. Since on C_ϱ the oscillation of $\mathfrak{x}_n$ becomes arbitrarily small with δ, while the images of A and B must have a distance at least equal to the minimum distance between the curves Γ_1 and Γ_2, this type of degeneration is excluded.

Higher topological structure does not affect our previous reasoning to exclude degeneration of B_n. — In the case of *non-orientable surfaces* it is preferable to use the closed Riemann surfaces symmetric by reflection on k unit circles. (See § 2). For example, for the type of the Moebius strip we may use a

[1] It is obvious that the Lemma is valid also for Riemann surfaces B.

surface consisting of 3 sheets, symmetric with respect to the unit circle in one sheet, this sheet being connected by two branch points to each of the other two and those latter by two branch points connected together.

§ 5. Further Discussion of the Solution.

1. Lemmas. — Theorems.

The *sufficient condition* $\mathfrak{C}$ *of* § 4 *can be replaced by another* which appears more explicit. A surface $\mathfrak{x}$ having the boundary Γ is, with respect to the required topological structure, said to be degenerate, if it consists of two separate surfaces (possibly meeting each other in points) having together as boundary Γ and a total characteristic number not exceeding the prescribed $\varkappa$; or if, without being decomposed, it has a lower genus or characteristic number than prescribed. The lower limit for such a type of degenerate surfaces with the boundary Γ may be called d^*. If $\mathfrak{x}$ is decomposed into two surfaces $\mathfrak{x}'$, $\mathfrak{x}''$ with Γ' and Γ'' as boundaries respectively and with prescribed characteristic numbers $\varkappa'$, $\varkappa''$ as well as prescribed character of orientability, then we define the Dirichlet integrals $D'(\mathfrak{x}')$, $D''(\mathfrak{x}'')$ and the lower limits d' and d'' correspondingly and, for this type of degeneration

$$d^* = d' + d''.$$

We shall show that for every type of degeneration

$$d \leq d^*$$

and, in the next section, we shall prove the *main theorem:*

The variational problem and with it the Douglas-Plateau problem has a solution, if the strict inequality condition

$$d < d^*$$

is satisfied for every possible degeneration.[1]

To prove these statements without making use of the fact that the lower limits d, d^*, d', d'' are really lower limits of the area, we shall formulate two properties of the Dirichlet integral in two Lemmas and then prove the lower semicontinuous character of the lower limit d in its dependence on Γ.

Lemma 2. *If* $\mathfrak{z}(u, v)$ *is an arbitrary continuous vector with piecewise continuous first derivatives in* B *with finite* $D(\mathfrak{z})$ *and with* $|\mathfrak{z}| < M$, *then to every point* P

[1] An equivalent statement was first formulated by Douglas [3].

of B, e. g. the origin, and to any prescribed σ there exists an arbitrarily small η and a vector $\mathfrak{y}(u, v)$ such that $\mathfrak{y}$ is equal to $\mathfrak{z}$ outside the neighborhood $u^2 + v^2 < \eta^2$ of P and has a prescribed fixed value, e. g. $\mathfrak{y} = 0$ in the smaller neighborhood $u^2 + v^2 < \eta^4$ and that

(34) $$D(\mathfrak{y}) < D(\mathfrak{z}) + \sigma.$$

In other words, without essentially increasing the Dirichlet integral, we can locally pull out a spine from the surface $\mathfrak{z}$ reaching to a given point. For the proof[1] we define with $r^2 = u^2 + v^2$ and with given $\eta < 1$ the function $p(r) = p(u, v)$ by

$$p = 1 \text{ for } r > \eta,\ p = 0 \text{ for } r < \eta^2,\ p = 1 + \frac{1}{\log \eta} \log \frac{\eta}{r} \text{ for } \eta^2 \leq r \leq \eta,$$

and $\mathfrak{y}$ by

(35) $$\mathfrak{y}(u, v) = p(u, v)\, \mathfrak{z}(u, v).$$

We find $D(p) = \frac{1}{2} \int\int (p_u^2 + p_v^2)\, du\, dv = -\pi \frac{1}{\log \eta} = \varepsilon$ and, using $|p| < 1$, $|\mathfrak{z}| < M$ and Schwarz' inequality

$$D(\mathfrak{y}) \leq D(\mathfrak{z}) + M^2 D(p) + \int\int_{\eta^2 < r < \eta} p\, \mathfrak{z}\, (p_u \mathfrak{z}_u + p_v \mathfrak{z}_v)\, du\, dv \leq D(\mathfrak{z}) + M^2 \varepsilon^2 + 2 M \sqrt{\varepsilon D(\mathfrak{z})}$$

or

(36) $$D(\mathfrak{y}) \leq (\sqrt{D(\mathfrak{z})} + M \sqrt{\varepsilon})^2.$$

Since we can satisfy (34) by making η and hence ε sufficiently small, our Lemma is proved.

An immediate consequence of Lemma 2 is the *theorem: There is always*

(37) $$d \leq d^*$$

in particular

(38) $$d \leq d' + d''$$

for every type of degeneration of the surfaces under consideration.[2] We prove this, e. g. for Riemann domains bounded by unit circles, assuming the degenerate surfaces to have k' and k'' boundary curves forming the boundaries Γ' and Γ'', respectively. Then we consider two such corresponding domains B' and B'' with vectors $\mathfrak{x}'$, $\mathfrak{x}''$ for which, with fixed arbitrarily small ε,

[1] See [10] p. 685 f.

[2] See [10] p. 699, where the proof is given in detail for plane circular domains.

$$D_{B'}(\mathfrak{x}') < d' + \varepsilon, \qquad D_{B''}(\mathfrak{x}'') < d'' + \varepsilon.$$

In two congruent sufficiently small circles K', K'' in B' and B'' respectively, we replace according to Lemma 2 these vectors by vectors $\mathfrak{y}'$ and $\mathfrak{y}''$ respectively, which vanish identically in these circles and for which

$$D_{B'}(\mathfrak{y}') < d' + 2\varepsilon, \qquad D_{B''}(\mathfrak{y}'') < d'' + 2\varepsilon.$$

Finally we join B' and B'' by two branch points situated within K' and K''. Thus we obtain a domain B in which $\mathfrak{y}'$ and $\mathfrak{y}''$ together define a vector $\mathfrak{y}$ admissible in the problem for Γ and with the Dirichlet integral

$$D_B(\mathfrak{y}) = D_{B'}(\mathfrak{y}') + D_{B''}(\mathfrak{y}'') < d' + d'' + 4\varepsilon.$$

Hence, whatever ε may be, we have

$$d \leq D_B(\mathfrak{y}) < d' + d'' + 4\varepsilon,$$

which proves our theorem.

The semicontinuity of d and the main theorem will appear as a consequence of a further lemma, by means of which the treatment given in previous papers is essentially simplified:

Lemma 3: *If the sequence $\mathfrak{z}_n$ of admissible vectors with $D(\mathfrak{z}_n) \leq M$ tends to degeneration as described in § 4, 1, we can replace $\mathfrak{z}_n$ by a vector $\mathfrak{y}_n$ in B_n so that $\mathfrak{y}_n$ is actually degenerated and so that*

$$D(\mathfrak{y}_n) \leq D(\mathfrak{z}_n) + \sigma_n, \tag{39}$$

with $\sigma_n \to 0$ and with the boundary $\Gamma^{(n)}$ of $\mathfrak{y}_n$ tending to Γ in the strong sense.

In other words, without noticeably interfering with the Dirichlet integral or the boundaries one can slightly deform a sequence of surfaces tending to degeneration into another sequence whose members are actually degenerated, the different parts touching in a single point.

Proof: We may suppose that our sequence degenerates around the origin $\mathfrak{x} = 0$, i. e. that there is on $\mathfrak{z}_n$ a curve τ_n, on $\mathfrak{z}_n$ not homotopic to zero, whose largest distance δ_n from the origin tends to zero with increasing n. Then we subject the whole m-dimensional z-space to a deformation which contracts the interior of a small sphere around the origin into the origin and leaves all the points of the space outside a larger, but still small, sphere unchanged: The point z is taken into the point y by

$$y_\mu = p(r)\, z_\mu \quad \text{or} \quad \mathfrak{y} = p(r)\, \mathfrak{z}$$

where $p(r)$ is the following[1] function of the distance $r = \sqrt{z_1^2 + \cdots + z_m^2}$ and a parameter η:

$$p = 1 \text{ for } r > \eta;\ p = 0 \text{ for } r < \eta^2;\ p = 1 + \frac{1}{\log \eta} \log \frac{\eta}{r} \text{ for } \eta^2 \leq r \leq \eta.$$

Now we substitute for z_μ the values given by the vector $\mathfrak{z}(u, v)$ and for η the value $\sqrt{\delta_n}$. Then the vector

(40) $$\mathfrak{y}_n = p(r)\, \mathfrak{z}_n$$

represents a degenerated surface as described in the Lemma. We have to establish the inequality (39).

Omitting in the following the index n we consider in B the open point set B^* where $\eta^2 < |\mathfrak{z}| = r < \eta$. We have

(41) $$D(\mathfrak{y}) = \frac{1}{2} \iint\limits_B \{(p\, \mathfrak{z}_u + p_u\, \mathfrak{z})^2 + (p\, \mathfrak{z}_v + p_v\, \mathfrak{z})^2\}\, du\, dv = a + b + c,$$

where because of $p \leq 1$

(42) $$a = \frac{1}{2} \iint\limits_B p^2 (\mathfrak{z}_u^2 + \mathfrak{z}_v^2)\, du\, dv \leq D(\mathfrak{z}),$$

furthermore

(43) $$b = \frac{1}{2} \iint\limits_B \mathfrak{z}^2 (p_u^2 + p_v^2)\, du\, dv = \frac{1}{2} \iint\limits_{B^*} \mathfrak{z}^2 (p_u^2 + p_v^2)\, du\, dv,$$

(44) $$c = \iint\limits_B (p p_u\, \mathfrak{z}\mathfrak{z}_u + p p_v\, \mathfrak{z}\mathfrak{z}_v)\, du\, dv = \iint\limits_{B^*} (p p_u\, \mathfrak{z}\mathfrak{z}_u + p p_v\, \mathfrak{z}\mathfrak{z}_v)\, du\, dv.$$

Now we have in B^*, with

$$\varepsilon = \frac{1}{\log \frac{1}{\eta}},$$

$$|p_u| = \varepsilon \frac{1}{r} |\mathfrak{z}_u|, \quad |p_v| = \varepsilon \frac{1}{r} |\mathfrak{z}_v|, \quad |p p_u\, \mathfrak{z}\mathfrak{z}_u| \leq \varepsilon\, \mathfrak{z}_u^2, \quad |p p_v\, \mathfrak{z}\mathfrak{z}_v| \leq \varepsilon\, \mathfrak{z}_v^2.$$

Hence, because of $|\mathfrak{z}| = r$,

$$\mathfrak{z}^2 (p_u^2 + p_v^2) = \varepsilon^2 (\mathfrak{z}_u^2 + \mathfrak{z}_v^2)$$

[1] This function is essentially different from that used for Lemma 2, because it refers to the vector space, not to the parameter domain B.

and

$$b \leq \varepsilon^2 D_{B^*}(\mathfrak{z}) \leq \varepsilon^2 D_{B'}(\mathfrak{z}),$$

$$c \leq 2\,\varepsilon\, D_{B^*}(\mathfrak{z}) \leq 2\,\varepsilon\, D_{B'}(\mathfrak{z}).$$

Therefore

$$D(\mathfrak{y}) \leq D_B(\mathfrak{z})\,(1 + 2\,\varepsilon + \varepsilon^2) = (1 + \varepsilon)^2 D(\mathfrak{z}),$$

which proves the lemma since ε tends to zero with η.

2. Semicontinuity of the Lower Limit d. — Sufficient Condition.

We want to establish the following theorem: *If $\Gamma^{(n)}$ is a system of k continuous contours — not necessarily without multiple points — which converge to Γ in the strong sense, and if $\mathfrak{x}_n$ in the domain B_n of the prescribed structure is a vector of the admissible type mapping the boundary of B_n monotonically on $\Gamma^{(n)}$, then we have the relation*

$$d \leq \lim\ \inf\ D_{B_n}(\mathfrak{x}_n), \tag{45}$$

which expresses the semicontinuous dependence of d on Γ. For the proof we may use induction assuming the theorem to be true for lower values of k or for lower topological structure.

There are two possibilities: First: the $\mathfrak{x}_n$ satisfy the condition $\mathfrak{C}$ of cohesion. Then our reasoning of § 4 subsists literally; the domains B_n form a compact set and define a limiting domain B; we have equicontinuity of the $\mathfrak{x}_n$ on the boundaries, and the corresponding harmonic vectors yield a limiting vector $\mathfrak{x}$ with

$$D(\mathfrak{x}) \leq \lim\ \inf\ D(\mathfrak{x}_n)$$

which maps the boundary of B on Γ and has the prescribed topological properties. All the more (45) holds.

Second: the $\mathfrak{x}_n$ tend to degeneration. Then we may assume — if necessary after choosing a suitable subsequence — that there is on the surface $\mathfrak{x}_n$ a closed curve τ_n not homotopic to a point, so that on τ_n we have $|\mathfrak{x}_n| < \delta = \delta_n \to 0$. To the curve τ_n there belongs in B_n a closed simple curve t_n which in case of genus zero separates B_n into the domains B_n', B_n'' bounded by t_n and by the systems C' and C'' respectively, consisting of $k' > 0$ and $k'' > 0$ boundary curves of B_n with $k' + k'' = k$, and which in case of higher topological structure may dissect B_n into a domain of lower topological structure[1].

[1] Or separates into different surfaces.

We first discuss the case of genus zero[1] supposing that the representing domains are plane domains, e. g. domains bounded by k circles, two of them concentric.

According to Lemma 3, we replace the surface $\mathfrak{x}_n$ defined in B_n by an actually degenerated surface $\mathfrak{y}_n$ so that $D_{B_n}(\mathfrak{y}_n) < D_{B_n}(\mathfrak{x}_n) + \sigma_n$ with $\sigma_n \to 0$, and so that $\mathfrak{y}_n$ takes the boundary C of B into a system $\Gamma^{*(n)}$ of curves which tends to Γ as well as $\Gamma^{(n)}$ does. — As a matter of fact, $\Gamma^{*(n)}$ is identical with $\Gamma^{(n)}$ unless the origin is on Γ.

Certainly we have

$$\liminf D_{B_n}(\mathfrak{y}_n) \leq \liminf D_{B_n}(\mathfrak{x}_n). \tag{46}$$

We may assume that the curve t_n in B_n, on which $\mathfrak{y}$ vanishes, contains B'_n in its interior and B''_n in its exterior. Then we define $B^*_n = B'_n$ plus the whole exterior of t_n, and $B^{**}_n = B''_n$ plus the whole interior of t_n; and define $\mathfrak{y}^*_n = \mathfrak{y}_n$ in B'_n and $\mathfrak{y}^*_n = 0$ outside of t_n, $\mathfrak{y}^{**}_n = \mathfrak{y}_n$ in B''_n and $\mathfrak{y}^{**}_n = 0$ inside of t_n.

Then we have

$$D_{B^*_n}(\mathfrak{y}^*) + D_{B^{**}_n}(\mathfrak{y}^{**}) = D_{B_n}(\mathfrak{y}), \tag{47}$$

and $\mathfrak{y}^*_n$, $\mathfrak{y}^{**}_n$ are continuous and have piecewise continuous first derivatives in B^*_n, B^{**}_n respectively. They furthermore take the boundaries into $\Gamma^{*(n)}$ and $\Gamma^{**(n)}$ where $\Gamma^{*(n)} \to \Gamma'$, $\Gamma^{**(n)} \to \Gamma''$. Thus they correspond to variational problems relating to lower numbers k' and k'' of boundary curves. For such lower numbers the semicontinuity may be assumed already proved. Then we have

$$\liminf D_{B^*_n}(\mathfrak{y}^*) \geq d'$$

$$\liminf D_{B^{**}_n}(\mathfrak{y}^{**}) \geq d''$$

where d' and d'' refer to the partition of the boundary Γ into Γ' and Γ''. Consequently by (46) and (47) we have

$$d' + d'' \leq \liminf D_{B_n}(\mathfrak{x}_n). \tag{48}$$

Since by (38)

$$d \leq d' + d'',$$

our theorem is proved; for it was proved for $k = 1$ so that induction is possible.

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[1] See [10] p. 683.

Incidentally, if a degenerating minimizing sequence $\mathfrak{x}_n$ exists, the right hand side of (45) becomes d, and our reasoning yields the equality

$$d = d' + d''.$$

A consequence of the preceding analysis is our main theorem of No. 1, which for genus zero states:

A sufficient condition in the case of genus zero for the solvability of the variational problem is the inequality

$$d < d' + d''$$

for all partitions of the boundary Γ into Γ' and Γ''.

For this condition, according to the reasoning above, excludes tendency to degeneration.

Our proof can without difficulty be modified to cover types of domains B which are Riemann surfaces of the kind considered in § 3. It made no use of conformal mapping. However, if the genus of the domain is not zero, the proof requires a modification using mapping theorems. The domains of lower topological structure which we obtain in this case by a construction as above using the curve t_n are in general not of the same type as our domains B. Hence to complete the proof we must know that such domains, or rather all domains, can be mapped conformally to domains of the type B. (See the detailed discussion in No. 3.)

The sufficient condition stated by the main theorem is in a general way expressed by

$$d < d^*$$

where d^* refers to any type of degenerated surfaces with the same total boundary, degeneration including the possibility of a topological structure lower than the prescribed[1].

The sufficient conditions of this section are easily verified in every case in which the degenerate solutions or such of lower topological type are self intersecting. Then the inequality condition for the higher type becomes evident if we identify the lower limit of the Dirichlet integral with that of the area. For along the lines of self-intersection we can pull the different parts of the surface of lower

[1] That the condition is not necessary is apparent if we realize that it is violated e. g., for a minimal surface of revolution, if the area or the Dirichlet integral is exactly equal to the sum of the areas of the two boundary circles.

type apart in such a way that the self-intersection disappears and the area decreases. Thereby surfaces of higher type originate or degenerations disappear.

For example, for $p = 0$ and $k = 2$, if Γ_1 and Γ_2 are interlocking, it is seen immediately that for the degenerate surface consisting of the two simply connected minimal surfaces through Γ_1 and Γ_2 the area is larger than that of other doubly connected surfaces which we obtain from an intersecting pair of surfaces by a deformation eliminating the self-intersection.

Similarly we can see that *through a knotted curve Γ we always have besides the self-intersecting simply connected minimal surface one of higher structure.*[1]

3. Remarks. Critical Analysis of the Method.[2]

As already mentioned our sufficient conditions can be verified directly for the genus zero if the boundary curves Γ_k are in a plane. (See [10] and [12].) Then mapping theorems result as a consequence by means of a general continuity theorem. By this latter it can be shown that the sufficient conditions, if satisfied for a system Γ of contours, remain satisfied if Γ is deformed in a suitable neighborhood. Consequently the solvability of the problem is assured for boundaries Γ sufficiently near to a plane and sufficiently smooth.

The inclusion of mapping theorems in our theory removes the objection to the specific reference to certain classes of domains B of representation. But it must be stated that there exist Riemann domains B of connectivity k for which our theory does not immediately yield the conformal equivalence to domains of our type B, so that they might possibly yield a smaller value of d.

The difficulty is enhanced for higher topological structure and may be explained in the sufficiently general case of one contour and genus one: $k = 1$ and $p = 1$. Then we choose for B the unit circle plus another full plane connected to the unit circle by 4 branch points. If a minimizing sequence B_n degenerates in such a way that two branch points tend to the same point, cancelling each other, so that in the limit only two branch points remain, we would have degeneration into a simply connected domain with $p = 0$. But this domain, consisting of a unit circle with a full plane affixed in two branch points, is of a different type from the domains B used originally for defining the lower limit d for simply connected domains of genus zero. Conformal mapping must be

[1] The result that two interlocking curves always define a doubly connected minimal surface, was first obtained by Douglas.

[2] See also § 3, 3.

applied to establish the equivalence of such domains obtained by processes of degeneration with domains of the originally admitted type B.

The same situation arises with other types of domains B, e. g. for plane domains defined by fundamental domains of Schottky groups of linear substitutions with p generating transformations. Since the group and the boundary circles depend on only a finite number of parameters, the reasoning concerning the solution of the variational problem proceeds exactly as that in §§ 3, 4. Also the variational part of the theory, the proof of $\varphi(w) = 0$, is similar to that given above. But again a degeneration of B_n may occur so that in the limit B becomes of lower genus but still is defined by a group with p generating transformations; namely, if two corresponding boundaries of the fundamental domains touch each other in corresponding points or, as one says, if the fundamental domain of the limiting group has a »parabolic vertex». Then the genus of the limiting domain will be lowered and therefore this domain will no longer belong to the admitted type for the lower genus, so that such an equivalence must be established by some mapping theorems. — For slit domains corresponding considerations hold.[1]

Part II. Free Boundaries.

§ 1. Preliminaries.

1. Position of the Problem.

In the second part of this paper we give the solution to the Plateau problem with free boundaries. This means, we prove the existence of minimal surfaces of least area d or least Dirichlet integral[2] whose boundaries, or parts of whose boundaries, are free to move on prescribed continuous manifolds of less than m dimensions. These »free problems» present a much greater variety than those with fixed Jordan curves as boundaries. For, not only may the topological structure of the minimal surface be prescribed in the problem, but so also may topological properties relative to the given manifolds. All such questions, in particular the proof of sufficient conditions in topologically higher cases, can

[1] The preceding remarks, which indicate that the theory of conformal mapping is the preferable basis for the treatment of Douglas' problem for higher topological structure, seem to apply also to the presentation of Douglas' theory in [8].

[2] In our proof we shall not make use of their equivalence but refer to it for convenience.

be treated in a manner similar to the corresponding theory of part I. I shall leave an analysis of the general possibilities for another occasion, and treat in detail solely the typical *case of a doubly-connected minimal surface, one of whose boundaries is free on a closed manifold M,*[1] *while the other is a Jordan curve Γ monotonically described.*

The free boundaries, under the very general assumptions concerning M, are not necessarily continuous curves. The methods of part I, therefore, inasmuch as consideration of vectors $\mathfrak{x}$ as functions on the boundary of the domain B of representation is concerned, must be replaced by a reasoning referring to the interior of the domain B. Correspondingly, the behavior of the solution is analyzed by means of a theorem on harmonic — or more generally, monotonic — vectors, given in No. 2.

To formulate our problem precisely we suppose the surfaces under consideration to be represented by continuous vectors $\mathfrak{x}(u, v)$ with piecewise-continuous first derivatives in a concentric annular ring B of the u, v-plane between the unit circle C_1 and a concentric circle C_2 of radius a, so that $\mathfrak{x}$ has continuous boundary values on C_2, mapping C_2 monotonically on Γ; and that the boundary of $\mathfrak{x}$ corresponding to C_1 is on M. This latter property is defined as follows: Denote by $g(\mathfrak{x})$ the distance of the point $\mathfrak{x}$ from M. If $\mathfrak{x} = \mathfrak{x}(u, v)$ is a surface defined in B, $g(\mathfrak{x})$ becomes a function $g(u, v)$ of u, v in B; if $g(u, v)$ tends to zero as the point u, v in B tends to C_1, then we say that the boundary of $\mathfrak{x}(u, v)$ corresponding to C_1 is on M. It is immediately clear that, in polar coordinates r, ϑ, the distance $g(u, v) = g(r, \vartheta)$ tends to zero uniformly in ϑ as r tends to one. — Note that our definition does not imply existence of continuous boundary values of $\mathfrak{x}$ on C_1.

Our problem now is to find a doubly-connected minimal surface of least area d bounded by Γ and M. *We shall prove in § 2 that such a minimal surface exists, provided that the lower limit d is smaller than the lower limit d^* belonging to the Plateau problem for Γ as the sole boundary.* The solution is obtained as solution of the variational problem: To find a domain B and an admissible vector $\mathfrak{x}$ as above, for which $D(\mathfrak{x}) = d$ is a minimum.

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[1] In [13] I have discussed the case when one part of the boundary is a prescribed Jordan arc, another free on a manifold M. — In [23] Mr. Ritter has treated the case of a »Schwarz Chain», where the boundary consists of k fixed Jordan arcs alternating with k parts free on manifolds M_i which in his paper are planes, but with the methods of the present paper or [13] could be chosen as general continuous manifolds.

2. Theorem on Boundary Values.

The proof will be based on a theorem which, for convenience, we first state for a half-plane B: $v > 0$. *Let $\mathfrak{x} = \mathfrak{x}_\nu(u, v)$ be a sequence of harmonic vectors in B: $v > 0$, having the boundary on a closed manifold M_ν, for which*

$$D(\mathfrak{x}) = \frac{1}{2}\iint_B (\mathfrak{x}_u^2 + \mathfrak{x}_v^2)\, du\, dv \leq A^2$$

is bounded by a constant A^2. In each closed subdomain of the halfplane B the $\mathfrak{x}_\nu$ may converge uniformly to a harmonic vector $\mathfrak{x}$. Furthermore, we assume that the manifolds M_ν tend to a continuous manifold M so that the longest distance of points of M_ν from M tends to zero. Then the boundary of the vector $\mathfrak{x}$ is on M. — Note that no assumptions are made concerning the dimensions of M_ν and M. In our application M_ν will be a curve, M a surface.

Proof: We observe that the uniform convergence of the $\mathfrak{x}_\nu$ implies that of the derivatives in closed subdomains of B, and hence the inequality

$$D(\mathfrak{x}) \leq A^2.$$

By the existence of $D(\mathfrak{x})$ we have for the small strip B_h: $0 < v < 2h$

$$\varepsilon(h)^4 = \iint_{B_h} (\mathfrak{x}_u^2 + \mathfrak{x}_v^2)\, du\, dv \to 0 \qquad \text{for } h \to 0. \tag{49}$$

We now appraise the oscillation of $\mathfrak{x}$ on the line $v = h$, in particular on a segment $L = L_h$:

$$|u - u_0| < th,\ v = h, \text{ with } t = \frac{1}{\varepsilon(h)}$$

whose length relative to h tends to ∞ as $h \to 0$. Along $L = L_h$ we have, by the mean value theorem of potential theory,

$$\mathfrak{x}_u = \frac{1}{\pi h^2}\iint_{K_h} \mathfrak{x}_u\, du\, dv$$

where the integral is extended over a circle of radius h around the point u, h of L; hence by Schwarz' inequality and (49)

$$\mathfrak{x}_u^2 < \frac{1}{\pi h^2}\varepsilon(h)^4 < \frac{1}{h^2}\varepsilon(h)^4. \tag{50}$$

From

$$\mathfrak{x}(u, h) - \mathfrak{x}(u_0, h) = \int_{u_0}^{u} \mathfrak{x}_u(u, h)\, d\, u$$

we obtain immediately by (50) along L

$$(51) \qquad |\mathfrak{x}(u, h) - \mathfrak{x}(u_0, h)| < \frac{1}{h}\, \varepsilon(h)^2 |u - u_0| < \varepsilon(h)$$

which shows that the oscillation of $\mathfrak{x}$ on L_h is small with h, uniformly in u_0.

To link the boundary of $\mathfrak{x}$ with that of $\mathfrak{x}_\nu$ we choose ν sufficiently large so that on L_h, because of the convergence of $\mathfrak{x}_\nu$ to $\mathfrak{x}$,

$$|\mathfrak{x}_\nu(u, h) - \mathfrak{x}(u, h)| < \varepsilon(h);$$

then by (51) we have on L_h

$$|\mathfrak{x}_\nu(u, h) - \mathfrak{x}(u_0, h)| \leq 2\, \varepsilon(h).$$

For every positive δ with $\delta < h$ we have now, with u, h in L,

$$(52) \qquad |\mathfrak{x}(u_0, h) - \mathfrak{x}_\nu(u, \delta)| \leq 2\, \varepsilon(h) + |\mathfrak{x}_\nu(u, h) - \mathfrak{x}_\nu(u, \delta)| \leq$$

$$\leq 2\, \varepsilon(h) + \int_{\delta}^{h} \left| \frac{\partial\, \mathfrak{x}_\nu(u, v)}{\partial\, v} \right| d\, v.$$

The distance of the point $\mathfrak{x}_\nu(u, \delta)$ from M_ν is, for sufficiently small δ, less than a quantity $\eta(\delta)$ tending to zero, if δ and ν are fixed. Since the distance to M_ν satisfies the triangular inequality, we have, from (52), for the distance $g_\nu[\mathfrak{x}(u_0, h)] = g_\nu(u_0, h)$ of the point $\mathfrak{x}(u_0, h)$ from M_ν

$$g_\nu(u_0, h) - \eta(\delta) \leq 2\, \varepsilon(h) + \int_{\delta}^{h} \left| \frac{\partial\, \mathfrak{x}_\nu}{\partial\, v} \right| d\, v;$$

for, the distance g_ν is not greater than the distance from $\mathfrak{x}(u_0, h)$ to $\mathfrak{x}_\nu(u, \delta)$ plus the distance $\mathfrak{x}_\nu(u, \delta)$ from M_ν.

By integration with respect to u over the interval

$$|u - u_0| < t\, h = h\, \varepsilon(h)^{-1}$$

we find

$$t\, h\, [g_\nu(u_0, h) - \eta(\delta)] \leq 2\, h + \int\!\!\int \left| \frac{\partial\, \mathfrak{x}_\nu}{\partial\, v} \right| d\, u\, d\, v,$$

where the integral on the right hand side is extended over the rectangle $|u-u_0| \leq th$, $\delta < v \leq h$. Hence we have by $\eta(\delta) \to 0$ for $\delta \to 0$ and by Schwarz' inequality

$$h\,t \liminf g_\nu(u_0, h) \leq 2h + A\sqrt{h\,t\,h} = h(2 + A\sqrt{t}),$$

and, since for the distance g from M by the triangular inequality $g(u_0, h) \leq \liminf g_\nu(u_0, h)$ holds, we obtain finally

$$g(u_0, h) \leq 2/t + A/\sqrt{t}.$$

For $h \to 0$ we have $t \to \infty$ uniformly in u_0; hence we have for the distance $g(u_0, h)$ the relation $g \to 0$ for $h \to 0$, which expresses our theorem.[1]

It is obvious that the theorem subsists if from the half-plane one or more domains are cut out, so that B becomes a multiply-connected domain. Furthermore, by conformal mapping, the theorem is extended literally to the interior B of a circle or to an annular ring B.

§ 2. Solution of the Problem.

1. Construction of the Solution.

We suppose

$$d < d^*,$$

where d^* is the minimum for the Plateau problem referring to the single contour Γ, and we consider a minimizing sequence of ring domains B_n together with admissible vectors $\mathfrak{x}_n$ in B_n with $D_{B_n}(\mathfrak{x}_n) \to d$.

First we show, in a manner similar to § 3 and § 4 of part I, that B_n cannot tend to degeneration. If the radius $a = a_n$ of the inner circle C_2 of B_n were to come arbitrarily near to 1, we would have exactly the same contradiction as in § 4 of part I, because M and Γ have a positive distance α. If $a = a_n$ should tend to zero for a subsequence B_n, then, according to the fundamental Lemma in part I, we would have, for an arbitrarily small δ and sufficiently large n, a concentric circle C of radius $\varrho = \varrho_n$ such that

$$\delta \leq \varrho \leq \sqrt{\delta}$$

and that on C the oscillation of $\mathfrak{x}_n$ is less than the square root of

[1] In [13] it is pointed out that the theorem can immediatly be generalized in different ways, one generalization — useful in connection with the classical problem of Riemann — permitting the manifolds to depend continuously on the boundary point of B.

$$2\pi\varepsilon = \frac{2A\,2\pi}{\log\frac{1}{\delta}},$$

where A is a common upper bound for $D_{B_n}(\mathfrak{x}_n)$.

Then, for sufficiently large n the circle C_2 is inside C and defines with C an annular ring B'_n. We may suppose that with increasing n the quantity δ and hence ε tends to zero, while always $a_n < \delta$. We certainly have

$$D_{B_n}(\mathfrak{x}_n) \geq D_{B'_n}(\mathfrak{x}_n).$$

According to Lemma 3 of part I, § 5, we replace $\mathfrak{x}_n$ by a vector $\mathfrak{y}_n$ which has a constant value on C, say zero, for which

$$D_{B'_n}(\mathfrak{y}_n) < D_{B'_n}(\mathfrak{x}_n) + \sigma(\varepsilon),$$

with $\sigma(\varepsilon) \to 0$ for $\varepsilon \to 0$, and which maps C_2 on a continuous curve $\Gamma^{(\varepsilon)}$ tending in the strong sense to Γ for $\varepsilon \to 0$. If we extend $\mathfrak{y}_n$ as identically zero into the whole plane outside C, we have for the domain B^* outside of C_2, for sufficiently large n,

$$D_{B^*_n}(\mathfrak{y}_n) \leq D_{B'}(\mathfrak{x}_n) + \sigma(\varepsilon) \leq D_{B_n}(\mathfrak{x}_n) + \sigma(\varepsilon).$$

On the other hand we have $d^*_n \leq D_{B^*}(\mathfrak{y}_n)$, where d^*_n is the lower limit of the Dirichlet integral in the Plateau problem for the single contour $\Gamma^{(\varepsilon)}$.

If we now let n tend to infinity and δ, ε at the same time to zero, we have, because of the semi-continuity of the lower limit of $D(\mathfrak{x})$ in the Plateau problem and because of $\sigma(\varepsilon) \to 0$,

$$d^* \leq \liminf d^*_n \leq \lim D_{B_n}(\mathfrak{x}_n) + \sigma(\varepsilon) = d,$$

which contradicts our assumption $d < d^*$. Hence there is no degeneration of B_n possible, and we can choose a subsequence of the domains B_n which tend to an annular ring B with radii 1 and a, where $0 < a < 1$.

In the same way as in part I, § 4 we see that the boundary values of $\mathfrak{x}$ on C_2 are equicontinuous, so that, at least for a suitable subsequence, they converge uniformly to a monotonic, continuous representation of Γ. Next, the vectors $\mathfrak{x}_n$ are replaced by harmonic vectors $\mathfrak{y}_n$ in B having on the inner circle the boundary values $\mathfrak{x}_n(a_n, \vartheta)$ and on the unit circle the boundary values $\mathfrak{x}_n(r_n, \vartheta)$, where r_n is a sequence tending to 1. These values $\mathfrak{x}_n(r_n, \vartheta)$ represent a continuous curve M_n, and we suppose r_n so near to C_1 that M_n tends to M.[1]

For the harmonic vectors $\mathfrak{y}_n$ we certainly have

$$\liminf D_B(\mathfrak{y}_n) \leq \lim D_{B_n}(\mathfrak{x}) = d,$$

as easily seen by the Dirichlet principle.

By a well-known theorem of potential theory[2], the harmonic vectors $\mathfrak{y}_n$ having an equally bounded Dirichlet integral are equicontinuous in every closed subdomain of B and therefore permit a subsequence converging to a limiting harmonic vector $\mathfrak{x}$. Because of the equi-continuity of $\mathfrak{y}_n$ on C_2 we may suppose that $\mathfrak{y}_n$ tends to $\mathfrak{x}$ also on the boundary C_2, so that $\mathfrak{x}$ maps C_2 monotonically on Γ.

Our theorem of § 1, 2 and the concluding remark there implies that the vector $\mathfrak{x}$ has its boundary corresponding to C_1 on M. Hence $\mathfrak{x}$ is admissible, and $D(\mathfrak{x}) \geq d$. But as in part I we have

$$D(\mathfrak{x}) \leq \lim D(\mathfrak{x}_n) = d.$$

Therefore we have $D(\mathfrak{x}) = d$ and consequently $\mathfrak{x}$ is a solution of the variational problem.

That $\mathfrak{x}$ is a minimal surface is seen exactly as in part I.

A general case in which our sufficient condition is satisfied is that in which the simply-connected minimal surface through Γ meets the surface M.

2. The Transversality Condition.

We prove for the free boundary on M a relation which expresses in a weak sense the orthogonality between M and the minimal surface. For this, we suppose M to have a continuous tangent plane. We further suppose that we can transform the x-space in the neighborhood of M by transformations

$$x_i' = x_i + \varepsilon\, \xi_i(x_1, \ldots x_m;\, \varepsilon)$$

depending on a small parameter ε and having piecewise-continuous derivatives with respect to the coordinates x_μ and ε; so that M is transformed into itself, while everywhere else, in particular near Γ, the functions ξ_i are zero. We write $\xi_i(x_1, \ldots x_m;\, 0) = \xi_i$ and combine the x_i' as a vector $\mathfrak{y}$, the ξ_i as a vector ξ.

[1] There need not be continuous boundary values of $\mathfrak{x}_n$ defined on the unit circle C_1.

[2] See e.g. Hurwitz-Courant, Funktionentheorie, 3d ed., p. 461.

By the substitution of the components $x_i = x_i(u, v)$ of the minimizing vector these vectors $\mathfrak{y}$ become admissible vectors in B. Now, since certainly

$$D(\mathfrak{y}) \geq D(\mathfrak{x}),$$

we obtain in the usual way

$$D(\mathfrak{x}, \xi) = \frac{1}{2} \iint_B (\mathfrak{x}_u \xi_u + \mathfrak{x}_v \xi_v)\, d\,u\, d\,v = 0,$$

or, if L_ε is a piecewise-smooth curve in B tending to C_1 for $\varepsilon \to 0$ and including a domain B_ε with C_2,

$$\iint_{B_\varepsilon} (\mathfrak{x}_u \xi_u + \mathfrak{x}_v \xi_v)\, du\, dv \to 0 \qquad \text{for } \varepsilon \to 0.$$

By Green's formula and $\triangle \mathfrak{x} = 0$ we have

$$\int_{L_\varepsilon} \xi \frac{\partial \mathfrak{x}}{\partial \nu}\, d\,s \to 0,$$

where $\frac{\partial}{\partial \nu}$ means differentiation along the normal to L_ε and s the arc length on L_ε. If S is our minimal surface, L'_ε the image of L_ε on S, then this formula becomes, if now interpreted on L'_ε,

$$\int_{L'_\varepsilon} \xi \frac{\partial \mathfrak{x}}{\partial \nu}\, d\,s \to 0,$$

where again $\frac{\partial}{\partial \nu}$ means differentiation in S normal to L'_ε and s the arc length on L'_ε. For, our integral is invariant under conformal mapping and $\mathfrak{x}$ maps B conformally on S.

Since $\frac{\partial \mathfrak{x}}{\partial \nu}$ on S is a unit vector, tangent to S, and ξ an arbitrary tangential vector field near M, this equation expresses what may be called a weak condition of orthogonality between M and S. The curves L'_ε here may be chosen as any sequence of piecewise-smooth curves on S, so that the area on S between L'_ε and M tends to zero with ε.[1]

[1] The method of this section can be applied to the discussion of the transversality condition in other twodimensional variational problems.

§ 3. Remarks.

1. Critical Analysis of the Result.

It should be observed that, to the generality of our method, a lack of desirable detailed information corresponds. E. g., we learn nothing concerning the question, under what conditions the free boundary of S is a continuous curve, or an analytic curve, or under what conditions S may be analytically extended beyond the boundary. Answers to such questions, even in the case of analytic boundaries Γ or M, have not yet been given. Nor do we know how to replace our statement of weak orthogonality by one of actual orthogonality for sufficiently smooth surfaces M. In these directions our insight into the problems is far from being satisfactory, except for straight or plane boundaries.

The following remarks illustrate the fact that our assumption of mere continuity of M is not sufficient to ensure smooth behavior of $\mathfrak{x}$ on the free boundary: Suppose, first, M to be not bounded. It may then be that the solutions are of necessity not bounded. To exemplify this, we consider the problem of a simply-connected minimal surface whose boundary is partly a Jordan arc, and partly free on a surface M. For M we choose a plane, $z = 0$, slightly deformed along a groove as follows: We remove from the plane around the x-axis a strip bounded by the curves $y = \pm e^{-x^2}$ and replace this part by a surface whose cross-section for $x = a$ is given by two straight segments

$$z = \frac{-y + b}{c} \text{ for } y \geq 0; \qquad z = \frac{y + b}{c} \text{ for } y < 0,$$

where $b = e^{-a^2}$ and $c = \frac{\sqrt{1 + a^2}}{16} e^{-a^2}$. The area of this cross-section is

$$\frac{16 e^{-a^2}}{\sqrt{1 + a^2}}$$

while the area of the removed part of the plane from $x = a$ to $x = +\infty$ is for $a > 0$

$$2 \int_a^\infty e^{-x^2} dx < \frac{1}{a} e^{-a^2}.$$

For every positive a the latter area is less than the former. If, therefore, we take as the given Jordan arc simply the straight segment

$$z = 0, \quad x = a, \quad |y| \leq e^{-a^2},$$

then the corresponding minimal surface is the infinite plane spike consisting of the removed part of the plane with $x > a$. The example can be generalized to show that there are cases where infinitely many such infinite spikes occur in the solution. Such phenomena are not restricted to manifolds M which extend to infinity. It is easily seen that one can carve out similar grooves from any closed, smooth surface winding asymptotically around asymptotic curves. The surfaces M thus obtained lead to minimal surfaces with boundaries on M which are not continuous curves.

2. **Other Types of Problems.**

The most interesting among other problems with free boundaries are those in which the entire boundary is free on a given closed surface not of genus zero, e. g. on a torus. Then, apart from the topological character of the minimal surface, e. g. *simple-connectivity*, also topological data relative to M must be prescribed, such as linking numbers between curves interlocking with M or with curves M_ν on the minimal surface S which are near to the boundary. The result to be expected is that minimal surfaces of a prescribed type exist if, with the same boundary conditions, the lower limit d for this topological type is actually less than that for any lower or degenerate type, provided lower topological type is properly defined.

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