DEVELOPMENTS IN THE ANALYTIC THEORY OF ALGEBRAIC DIFFERENTIAL EQUATIONS.

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1. Introduction.

Our present purpose is to obtain results of an analytic character for differential equations algebraic in

$$y,\ y^{(1)},\ \ldots\ y^{(n)}, \tag{1.1}$$

y being the unknown to be determined in terms of a complex variable x; we thus consider the equation

$$F(x,\ y,\ y^{(1)},\ \ldots\ y^{(n)}) = 0, \tag{1.2}$$

arranged as a polynomial in the symbols (1.1). The coefficients of the various monomials

$$(y)^{i_0}(y^{(1)})^{i_1}\ \ldots\ (y^{(n)})^{i_n}, \tag{1.2 a}$$

involved in the first member of (1. 2), will be assumed to be series of the form

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0 + a_{-1} x^{-1} + a_{-2} x^{-2} + \cdots, \tag{1. 3}$$

convergent for $|x| \geqq \varrho (> 0)$ or, more generally, they will be assumed to be functions, analytic in suitable regions[1], extending to infinity, and asymptotic (at infinity) within these regions to series (possibly divergent for all $x \neq \infty$) of the form (1. 3). The subject, as formulated, is very vast.

Accordingly, *we shall examine the situation in the case when the equation* (1.2) *has formal solutions of the same type as occur in the case of the irregular singular point (for ordinary linear differential equations).* In the formal theory of the equation (1. 2) we replace the coefficients of the monomials (1. 2 a) by the series (of the form (1. 3)) to which these coefficients are asymptotic. It will be desirable first to carry out suitable formal developments and afterwards to proceed with considerations of analytic character.

At this stage one may appropriately say a few words about the classical problem of the *irregular singular point.* Let

$$F_\nu (x, y, y^{(1)}, \ldots y^{(n)})$$

be the homogeneous part of F of degree ν in $y, y^{(1)}, \ldots y^{(n)}$; thus

$$F_\nu = \sum f_\nu^{i_0, \cdots i_n} (x) (y)^{i_0} (y^{(1)})^{i_1} \ldots (y^{(n)})^{i_n}, \tag{1. 4}$$

where the summation is over non-negative integers $i_0, \ldots i_n$, with $i_0 + \cdots + i_n = \nu$. In particular,

$$F_0 = F_0(x) = f_0^{0, \cdots 0} (x), \tag{1. 4 a}$$

We have

$$F \equiv F_0 + F_1 + \cdots + F_\sigma. \tag{1. 5}$$

In the particular case of $\sigma = 1$ the equation (1. 2) will be of the form

$$F_1 (x, y, y^{(1)}, \ldots y^{(n)}) = - F_0 (x). \tag{1. 6}$$

This is a non-homogeneous linear ordinary differential equation[2] whose solution is based on that of

$$F_1 = 0. \tag{1. 6 a}$$

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[1] The precise details regarding the regions will be given in the sequel.

[2] In order that (1. 6) should be a differential equation it is necessary that not all the coefficients in F_1 should be identically zero.

It is the latter equation which presents the classical problem of the irregular singular point. *The complete solution of the irregular singular point problem, both from the point of view of asymptotic representation and exponential summability (Laplace integrals, convergent factorial series), has been given by* W. J. TRJITZINSKY[1]. For a concise statement of the pertinent results the reader is referred to an address given by TRJITZINSKY before the American Mathematical Society[2]. Of the earlier work involving asymptotic methods in the problem of the irregular singular point of fundamental importance is the work of G. D. BIRKHOFF (cf. reference in (T)), which relates to the particular case when the roots of the characteristic equation are distinct. With regard to the methods involving Laplace integrals and factorial series, highly significant work had been previously done by N. E. NÖRLUND and J. HORN[3].

The equation (1. 2) (with $F_0(x) \equiv 0$) is a special case of non-linear ordinary differential equations (single equation of n-th order or systems) of the type investigated by a considerable number of authors, including W. J. TRJITZINSKY[4], with respect to whose work (T_1)[4] the following statements can be appropriately made at this time.

The main purpose of the developments given in (T_1) was the analytic theory of the single n-th order ($n > 1$) non-linear ordinary differential equation[5]. This necessitated use of asymptotic methods. As a preliminary was given the detailed treatment of the first order problem, the methods used being of the asymptotic type; this asymptotic method was then extended to the general case of $n > 1$. It must be said, however, that on one hand when the equations are given asymptotically with respect to the unknown and the derivatives of the unknown, *the use of asymptotic methods in the development of the analytic theory is imperative.* On the other hand, *in the particular case of a first order equation, given in the non-*

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[1] TRJITZINSKY, Analytic theory of linear differential equations [Acta mathematica *62* (1934), 167—226].

TRJITZINSKY, Laplace integrals and factorial series in the theory of linear differential and linear difference equations [Transactions Amer. Math. Soc. *37* (1935), 80—146].

[2] TRJITZINSKY, Singular point problems in the theory of linear differential equations [Bulletin Amer. Math. Soc. (1938), 209—233], in the sequel referred to as (T).

[3] For references and some details cf. (T).

[4] TRJITZINSKY, Analytic theory of non-linear singular differential equations [Mémorial des Sciences Mathématiques, No 90 (1938), 1—81], in the sequel referred to as (T_1). Many references are given in this work.

TRJITZINSKY, Theory of non-linear singular differential systems [Transactions Amer. Math. Soc. *42* (1937), 225—321], in the sequel referred to as (T_2).

[5] Cf. for formulation given in (T_1).

asymptotic form[1], *use of asymptotic methods is not necessary*, the methods of the highly important paper of J. MALMQUIST[2] being entirely adequate for the complete analytic treatment of this case; the latter fact was overlooked in (T_1).

In (T_1) and (T_2) 'actual' solutions were obtained which (in suitable complex neighborhoods of the singular point in question) were of the form, whose essential components were of the same asymptotic character as that of the 'actual' solutions in the problem of the irregular singular point for linear differential equations. The non-linear problem, referred to in (T_1) and (T_2), has obviously a connection with our present problem.

We shall also give some developments of analytic character, along the lines indicated above, *for non linear algebraic differential equations containing a parameter*. The formulation of the latter problem is given in section 9.

The main results of the present work are embodied in Theorems 6. 1, 7. 1, 8. 1 *and* 10. 1.

2. Formal Developments.

In so far as the formal developments are concerned, the situation is somewhat analogous to that involved in a paper by O. E. LANCASTER[3], who gives partial formal results for difference equations. The analogy in the formal theory is to be expected. In view of our present main purpose with regard to developments of analytic character, it will be necessary to give in detail some formal results for differential equations.

In accordance with E. FABRY[4] the formal solutions for the irregular singular point are of the type

$$s(x) = e^{Q(x)} x^r \sigma(x), \tag{2. 1}$$

where

$$Q(x) = q_p x^{\frac{p}{k}} + q_{p-1} x^{\frac{p-1}{k}} + \cdots + q_1 x^{\frac{1}{k}} \tag{2. 1 a}$$

(integer $p \geqq 0$; $Q(x) \equiv 0$ for $p = 0$)

and

[1] The equation (with $n = 1$) being defined with the aid of convergent series.

[2] J. MALMQUIST, Sur les points singuliers des équations différentielles [Arkiv för mat., astronomi och fysik, K. Svens. Vet. *15* (1920), No 3].

[3] O. E. LANCASTER, Non-linear algebraic difference equations with formal solutions . . . Amer. Journ. of Math. LXI (1939), 187—209].

[4] Cf. (T; 210).

(2. 1 b) $$\sigma(x) = \sigma_0(x) + \sigma_1(x)\log x + \cdots + \sigma_\mu(x)\log^\mu x \qquad (\text{integer } \mu \geqq 0),$$

(2. 1 c) $$\sigma_\gamma(x) = \sigma_{\gamma,0} + \sigma_{\gamma,1} x^{-\frac{1}{k}} + \sigma_{\gamma,2} x^{-\frac{2}{k}} + \cdots;$$

here k ($\geqq 1$) is an integer. The series (2. 1 c) may diverge for all $x \neq \infty$.

Throughout this section, unless stated otherwise, the coefficients in F ((1. 2)) will be supposed to be series, convergent for $|x| > \varrho$, or divergent of the form (1. 3).

We recall the following definition of (T; 213).

Definition 2. 1. *Generically $\{x\}_q$ (q an integer $\geqq 0$), will denote an expression*

(2. 2) $$\varrho_0(x) + \varrho_1(x)\log x + \cdots + \varrho_q(x)\log^q x,$$

the $\varrho_j(x)$ being series, possibly divergent (for all $x \neq \infty$), of the form

(2. 2 a) $$\varrho_{j,0} + \varrho_{j,1} x^{-\frac{1}{k}} + \varrho_{j,2} x^{-\frac{2}{k}} + \cdots \qquad (k \text{ a positive integer}).$$

Let $s(x)$ be defined by an expression (2. 1). It is observed that

(2. 3) $$Q^{(1)}(x) = x^{\frac{p}{k}-1}\left[r_0 + r_1 x^{-\frac{1}{k}} + \cdots + r_{p-1} x^{-\left(\frac{p-1}{k}\right)}\right]$$

where, if $Q(x) \not\equiv 0$, one may take $r_0 \neq 0$, $p > 0$,

$$\frac{d}{dx}\{x\}_0 = x^{-1}\{x\}_0,\ \frac{d}{dx}[\{x\}_0 \log^j x] = x^{-1}[\{x\}_0 \log^j x + \{x\}_0 \log^{j-1} x]$$

(for $j > 0$) and

(2. 3 a) $$\sigma^{(1)}(x) = \frac{d}{dx}\{x\}_\mu = x^{-1}\{x\}_\mu.$$

In view of (2. 3) and (2. 3 a)

(2. 4) $$s^{(1)}(x) = e^{Q(x)} x^{r(1)}\{x\}_\mu \qquad \left[r(1) = r + \frac{p}{k} - 1\right].$$

Similarly, from (2. 4) we obtain

$$s^{(2)}(x) = e^{Q(x)} x^{r(2)}\{x\}_\mu \qquad \left[r(2) = r(1) + \frac{p}{k} - 1\right]$$

and, in general,

(2. 5) $$s^{(j)}(x) = e^{Q(x)} x^{r(j)}\{x\}_\mu,$$

where

(2. 5 a) $$r(j) = r + j\left(\frac{p}{k} - 1\right) \qquad (j = 0, 1, 2, \ldots).$$

Thus

$$(s(x))^{i_0}(s^{(1)}(x))^{i_1}\dots(s^{(n)}(x))^{i_n}=e^{\nu Q(x)}x^{r'}\{x\}_{\nu\mu}, \tag{2. 6}$$

$$r'=\nu r+\bar r,\quad \bar r=\left(\frac{p}{k}-1\right)\sum_{j=0}^{n} j\, i_j, \tag{2. 6 a}$$

provided $i_0+\cdots+i_n=\nu$.

Now, by hypothesis,

$$f_\nu^{i_0,\dots i_n}(x)=a_m^{\nu:i_0,\dots i_n}x^m+a_{m-1}^{\nu:i_0,\dots i_n}x^{m-1}+\cdots+a_0^{\nu:i_0,\dots i_n}+ \\ +a_{-1}^{\nu:i_0,\dots i_n}x^{-1}+a_{-2}^{\nu:i_0,\dots i_n}x^{-2}+\cdots=x^m\{x\}_0, \tag{2. 7}$$

where $m=m(\nu:i_0,\dots i_n)$. Whence, in consequence of (1. 4), (2. 6) and (2. 7),

$$F_\nu(x,s,s^{(1)},\dots s^{(n)})=e^{\nu Q(x)}x^{\nu r}\sum x^{m+\bar r}\{x\}_0\{x\}_{\nu\mu};$$

here the summation is with respect to $i_0,\dots i_n$ $(i_0+\cdots+i_n=\nu)$, while integers m and rational numbers $\bar r$ depend on $i_0,\dots i_n$. Clearly

$$F_\nu(x,s,s^{(1)},\dots s^{(n)})=e^{\nu Q(x)}x^{\nu r}f(\nu;x);\ f(\nu;x)=x^{m(\nu)}\{x\}_{\nu\mu}, \tag{2. 8}$$

where $m(\nu)=l(\nu)/k$ (integer $l(\nu)$; $\nu=1,\dots\sigma$).

If a series $s(x)$ satisfies the equation $F=0$, in consequence of (2. 8) one should have formally

$$F_0+\sum_{\nu=1}^{\sigma}e^{\nu Q(x)}x^{\nu r}f(\nu;x)=0, \tag{2. 9}$$

where by (1. 4 a) and (2. 7)

$$F_0=x^m\{x\}_0\qquad (m=m(0:0,\dots 0)=m(0)). \tag{2. 9 a}$$

It is accordingly inferred without difficulty that *if* $s(x)$ *satisfies* (1. 2) (formally), *while* $Q(x)\not\equiv 0$, *then necessarily* $F_0\equiv 0$[1] *and* $s(x)$ *satisfies each of the equations*

$$F_1(x,s,\dots)=0,\dots F_\sigma(x,s,\dots)=0. \tag{2. 10}$$

In fact, the coefficients in $Q(x), r$ and the coefficients in $\sigma(x)$ will have to satisfy each of the following σ formal relations

$$f(1;x)=0,\ f(2;x)=0,\dots f(\sigma;x)=0, \tag{2. 10 a}$$

in the sense that, when $f(\nu;x)$ is arranged in the form $x^{m(\nu)}\{x\}_{\nu\mu}$ (cf. (2. 8)), the coefficients in the various power series involved are all zero. On taking note of

[1] Throughout, a formal series will be said to be $\equiv 0$ provided all the coefficients are zero.

(2. 9) and of the form of F_0 and of the $f(\nu; x)$ it is observed that, *if* $s(x)$ *satisfies* (1. 2), *while* $Q(x) \equiv 0$ *and* r $(\neq 0)$ *is irrational, we shall have* $F_0 \equiv 0$ *and* $s(x)$ *will satisfy each of the equations* (2. 10).

Inasmuch as in the sequel it will be assumed that in the series $s(x)$, formally satisfying $F = 0$, $Q(x)$ is not identically zero or $Q(x) \equiv 0$, but r is irrational, *we may confine ourselves to homogeneous equations of degree* ν; *namely*, $F_\nu = 0$.

The following will be proved.

If the formal homogeneous equation of degree ν,

$$(2.\ 11) \qquad F_\nu(x, y, y^{(1)}, \ldots y^{(n)}) = 0 \qquad \textit{(actually of order } n\textit{)},$$

is satisfied by the general formal solution of a linear differential equation

$$(2.\ 12) \qquad L(x, y(x)) \equiv \sum_{i=0}^{\eta} f_i(x)\, y^{(i)}(x) = 0,$$

actually of order η $(< n)$ *and with*

$$(2.\ 12\ \text{a}) \qquad f_i(x) = x^{\eta(i)}\{x\}_0 \qquad (\eta(i)\ \textit{rational}),$$

then

$$(2.\ 13) \qquad F_\nu(x, y, \ldots y^{(n)}) \equiv \sum_{j=0}^{n-\eta} \left[\frac{d^j}{dx^j} L(x, y(x))\right] \Phi_j(x, y, y^{(1)}, \ldots y^{(\eta+j)})$$
$$[\equiv \Omega(x, y, \ldots y^{(n)})],$$

where the Φ_j *are homogeneous (of degree* $\nu - 1$*) in* $y, \ldots y^{(\eta+j)}$, *the coefficients being of the form* $x^{\lambda_1}\{x\}_0$ $(\lambda_1$ *rational).*

To establish this result form the expression

$$(2.\ 14) \qquad \psi = F_\nu - \Omega,$$

where Ω is of the form of the second member in (2. 13), the Φ_j for the present being undefined. We may write

$$(2.\ 15) \qquad \frac{d^j}{dx^j} L(x, y(x)) = \sum f_j(m_0, m_1, \ldots m_{\eta+j})\,(y)^{m_0}(y^{(1)})^{m_1} \ldots (y^{(\eta+j)})^{m_{\eta+j}}$$

(summation with respect to $m_0, \ldots m_{\eta+j}$, with $m_0 + \cdots + m_{\eta+j} = 1$); clearly the coefficients in (2. 15) are of the same form in x as the $f_i(x)$. Also

$$(2.\ 15\ \text{a}) \qquad \Phi_j = \sum \varphi_j(x; k_0, \ldots k_{\eta+j})\,(y)^{k_0} \ldots (y^{(\eta+j)})^{k_{\eta+j}}$$

(summation with respect to $k_0, \ldots k_{\eta+j}$, with $k_0 + \cdots + k_{\eta+j} = \nu - 1$). The φ_j are at our disposal; *we wish to select these expressions so that* ψ *of* (2. 14) *is of the form*

(2. 16) $$\psi = \psi(x, y, \ldots y^{(\eta-1)}).$$

with no derivatives of y *of order higher than* $\eta - 1$ *present.*

Substitution of (2. 15) and (2. 15 a) into the expression Ω will yield

(2. 17) $$\Omega = \sum_{j=0}^{n-\eta} \sum_{m_0, \ldots} \sum_{k_0, \ldots} f_j(m_0, \ldots m_{\eta+j}) \varphi_j(x; k_0, \ldots k_{\eta+j})(y)^{i_0}(y^{(1)})^{i_1} \ldots (y^{(\eta+j)})^{i_{\eta+j}},$$

where

(2. 17 a) $$i_\lambda = m_\lambda + k_\lambda;\ m_0 + \cdots + m_{\eta+j} = 1;\ k_0 + \cdots + k_{\eta+j} = \nu - 1.$$

We thus may write

$$\Omega = \sum_{j=0}^{n-\eta} \sum q_j(i_0, \ldots i_{\eta+j})(y)^{i_0}(y^{(1)})^{i_1} \ldots (y^{(\eta+j)})^{i_{\eta+j}},$$

where the second sum displayed is with respect to $i_0, \ldots i_{\eta+j}$, with

$$i_0 + \cdots + i_{\eta+j} = \nu,$$

and

(2. 17 b) $$q_j(i_0, \ldots i_{\eta+j}) = \sum_{m_0, \ldots} \sum_{k_0, \ldots} f_j(m_0, \ldots m_{\eta+j}) \varphi_j(x; k_0, \ldots k_{\eta+j}),$$

the summation in (2. 17 b) (with $i_0, \ldots i_{\eta+j}$ fixed) being subject to (2. 17 a). Thus, by (2. 14) and (1. 4)

(2. 18) $$\psi = \sum_{i_0, \ldots i_n} f_\nu^{i_0, \ldots i_n}(x)(y)^{i_0} \ldots (y^{(n)})^{i_n} - \Omega = \Gamma_n + \Gamma_{n-1} + \cdots + \Gamma_{\eta-1},$$

the expressions $\Gamma_n, \ldots \Gamma_{\eta-1}$ being characterised as follows. Γ_n consists of all the therms in $F_\nu - \Omega$ which contain $y^{(n)}$; Γ_{n-1} contains no $y^{(n)}$ but contains $y^{(n-1)}$; Γ_{n-2} contains no $y^{(n)}$ and no $y^{(n-1)}$ but contains $y^{(n-2)}$; and so on — finally, $\Gamma_{\eta-1}$ contains no $y^{(n)}, \ldots y^{(\eta)}$ but contains $y^{(\eta-1)}$[1]. Picking from Ω the terms for which $j = n - \eta$ and $i_n > 0$ we obtain

(2. 19) $$\Gamma_n = \sum [f_\nu^{i_0, \ldots i_n}(x) - q_{n-\eta}(i_0, \ldots i_n)](y)^{i_0} \ldots (y^{(n)})^{i_n}$$

(summation with respect to $i_0, \ldots i_n$; $i_0 + \cdots + i_n = \nu$; $i_n > 0$).

[1] When Γ_μ is said to contain $y^{(\mu)}$ it is implied that this is the case when certain particular choices of the φ_j are avoided.

To form Γ_{n-1} we select from F_ν the terms for which $i_n = 0$, $i_{n-1} > 0$; from Ω we choose terms for which

$$(j = n - \eta,\ i_n = 0,\ i_{n-1} > 0),\ (j = n - \eta - 1,\ i_{n-1} > 0);$$

thus

$$\Gamma_{n-1} = \sum [f_\nu^{i_0, \dots i_n}(x) - q_{n-\eta}(i_0, \dots i_n) - q_{n-\eta-1}(i_0, \dots i_{n-1})]\,(y)^{i_0} \dots (y^{(n)})^{i_n} \tag{2. 19 a}$$
$$(i_0 + \cdots + i_n = \nu;\ i_n = 0;\ i_{n-1} > 0).$$

Proceeding further, one similarly obtains

$$\Gamma_{n-2} = \sum [f_\nu^{i_0, \dots i_n}(x) - q_{n-\eta}(i_0, \dots i_n) - q_{n-\eta-1}(i_0, \dots i_{n-1}) - q_{n-\eta-2}(i_0, \dots i_{n-2})]\,(y)^{i_0} \dots (y^{(n)})^{i_n} \tag{2. 19 b}$$
$$(i_0 + \cdots + i_n = \nu;\ i_n = 0;\ i_{n-1} = 0;\ i_{n-2} > 0).$$

In general

$$\Gamma_{n-\sigma} = \sum [f_\nu^{i_0, \dots i_n}(x) - q_{n-\eta}(i_0, \dots i_n) - q_{n-\eta-1}(i_0, \dots i_{n-1}) - \cdots - q_{n-\eta-\sigma}(i_0, \dots i_{n-\sigma})]\,(y)^{i_0} \dots (y^{(n)})^{i_n} \tag{2. 19 c}$$
$$(i_0 + \cdots + i_n = \nu;\ i_n = 0;\ i_{n-1} = 0, \dots;\ i_{n-\sigma+1} = 0;\ i_{n-\sigma} > 0);$$

such expressions are formed for $\sigma = 0, 1, \dots, n - \eta$. The remaining expression $\Gamma_{\eta-1}$ will consist of all terms of $F_\nu - \Omega$, not contained in any of the $\Gamma_{n-\sigma}(0 \leqq \sigma \leqq n - \eta)$. The φ_j can be so chosen that

$$q_{n-\eta}(i_0, \dots i_n) + q_{n-\eta-1}(i_0, \dots i_{n-1}) + \cdots + q_{n-\eta-\sigma}(i_0, \dots i_{n-\sigma}) = f_\nu^{i_0, \dots i_n}(x) \tag{2. 20}$$
$$[\text{cf. (2. 17 b)};\ i_0 + \cdots + i_n = \nu;\ i_n = i_{n-1} = \cdots = i_{n-\sigma+1} = 0;\ i_{n-\sigma} > 0]$$

for $\sigma = 0, 1, \dots, n - \eta$.

Let $c(m, \nu)$ be the number of distinct sets of integers $i_0, i_1, \dots i_m$ such that

$$i_0 + \cdots + i_m = \nu;\ i_0 \geqq 0, \dots, i_m \geqq 0;$$

then

$$c(m, \nu) = c(m - 1, 0) + c(m - 1, 1) + \cdots + c(m - 1, \nu) \tag{2. 21}$$
$$(m = 1, 2, \dots;\ c(0, \nu) = 1).$$

The number of equations (2. 20) (with σ fixed) is the number of sets $(i_0, i_1, \dots i_{n-\sigma})$ for which $i_0 + \cdots + i_{n-\sigma} = \nu$ and $i_{n-\sigma} > 0$. The number of equations (2. 20) (with σ and $i_{n-\sigma}$ fixed) will be $c(n - \sigma - 1, \nu - i_{n-\sigma})$ and the total number (for a given σ) will be

$$c(n - \sigma - 1, 0) + c(n - \sigma - 1, 1) + \cdots + c(n - \sigma - 1, \nu - 1);$$

in view of (2. 21) the expression for this number may be written as

$$c(n-\sigma, \nu - 1).$$

Thus, *the total number of equations* (2, 20), formed for $\sigma = 0, \ldots n - \eta$, *will be*

$$c_\nu = \sum_{\sigma=0}^{n-\eta} c(n-\sigma, \nu - 1) \qquad \text{(cf. (2. 21))}. \tag{2. 21 a}$$

In consequence of (2. 17 b) the equations (2. 20) are linear non-homogeneous in the $\varphi_j(x; k_0, \ldots k_{\eta+j})$. Inasmuch as in (2. 17 b)

$$k_0 + \cdots + k_{\eta+j} = \nu - 1 \qquad (k_0 \geqq 0, \ldots, k_{\eta+j} \geqq 0)$$

it follows that, for j fixed, there are

$$c(\eta + j, \nu - 1)$$

expressions $\varphi_j(x; k_0, \ldots k_{\eta+j})$. To infer this it is necessary merely to note the statement preceding (2. 21). Accordingly, the total number of φ_j (for $j = 0, \ldots n - \eta$), involved in the equations (2. 20), is

$$c(\eta, \nu - 1) + c(\eta + 1, \nu - 1) + \cdots + c(n, \nu - 1).$$

The latter sum, however, is precisely the number c_ν of (2. 21 a). It is not dif ficult to see that *the equations* (2. 20) *are actually satisfied (formally) for a suitable choice of the* φ_j; clearly, the φ_j so chosen will be in the form of a product of a rational power of x by an expression $\{x\}_0$.

With the equations (2. 20) satisfied, (2. 18) will be reduced to

$$\psi = \Gamma_{\eta-1} = \psi(x, y, y^{(1)}, \ldots y^{(\eta-1)}), \tag{2. 22}$$

none of the $y^{(\lambda)}(\lambda \geqq \eta)$ being involved. From (2. 14) we then obtain

$$F_\nu = \Omega(x, y, \ldots y^{(n)}) + \psi(x, y, y^{(1)}, \ldots y^{(\eta-1)}), \tag{2. 23}$$

where Ω is of the form of the second member in (2. 13). According to the hypothesis of the assertion (to be proved) in connection with (2. 11), ... (2. 13), the equation $F_\nu = 0$ is satisfied by the general formal solution (containing η arbitrary constants) of (2. 12). In view of the definition of Ω by the second member of (2. 13) we shall have $\Omega = 0$ for the above mentioned general formal solution. Whence this solution must also satisfy the equation $\psi = 0$. Inasmuch as the latter equation is of order $\leqq \eta - 1$, the coefficients of the various monomials

(2. 23 a) $$(y)^{i_0} \dots (y^{(\eta-1)})^{i_{n-1}}$$

in ψ must be all formally zero. We thus have $F_\nu = \Omega$, which completes the proof of the assertion in question. Clearly, if the $f_i(x)$ in (2. 12) and the coefficients in F_ν are rational functions of x the same will be true of the coefficients in the Φ_j.

An examination of the steps involved from (2. 14) to (2. 23 a) leads to the following conclusion.

If the 'actual' homogeneous equation of order n and degree ν

(2. 24) $$F_\nu(x, y, y^{(1)}, \dots y^{(n)}) = 0$$

has coefficients asymptotic, in a region R extending to infinity, to series of the form (2. 7) *and if* (2. 24) *is satisfied by every 'actual' solution of an 'actual' linear differential equation*

(2. 25) $$L(x, y(x)) \equiv \sum_{i=0}^{\eta} f_i(x)\, y^{(i)}(x) = 0 \qquad (\eta < n),$$

where

(2. 25 a) $$f_i(x) \sim \zeta_i(x) = x^{\eta(i)}\{x\}_0 \qquad (\text{in } R;\ \eta(i) \text{ rational}),$$

then (2. 13) *will hold, the coefficients in the Φ_j being functions asymptotic in R to expressions of the form $x^{\lambda_1}\{x\}_0$ (λ_1 rational). The above assertion is made under the supposition that*

(2. 25 b) $$f_i^{(j)}(x) \sim \zeta_i^{(j)}(x) \qquad (\text{in } R;\ j = 1, 2, \dots n - \eta).$$

The truth of this statement follows, if we recall that the coefficients $\varphi_j(x; k_0, \dots k_{\eta+j})$, involved in the Φ_j, enter linearly in the system of equations (2. 20), while in (2 20) the coefficients of the φ_j are functions asymptotic in R to expressions of the form $x^\lambda\{x\}_0$ (λ rational).

3. Conditions for Existence of Formal Solutions.

In view of (1. 4) and (2. 7) the formal equation (1. 4) may be written as

(3. 1) $$F_\nu \equiv \sum_{i_1, \dots, i_\nu} x^{\eta(i_1, \dots i_\nu)} [b_0^{i_1, \dots i_\nu} + b_1^{i_1, \dots i_\nu} x^{-1} + \dots + b_\sigma^{i_1, \dots i_\nu} x^{-\sigma} + \dots] \cdot$$
$$\cdot\, y^{(i_1)} y^{(i_2)} \dots y^{(i_\nu)} = 0 \qquad (0 \leqq i_1, i_2, \dots i_\nu \leqq n),$$

where the $\eta(i_1, \dots i_\nu)$ are integers. We shall now examine conditions under which (3. 1) has a formal solution $s(x)$, as given by (2. 1), ... (2. 1 c) with $\mu = 0$ and $p > 0$; that is, a solution

$$s(x) = e^{Q(x)} x^r \sigma(x) \tag{3. 2}$$

with

$$\sigma(x) = \sigma_0(x) = \sigma_0 + \sigma_1 x^{-\frac{1}{k}} + \cdots + \sigma_m x^{-\frac{m}{k}} + \cdots \qquad (\sigma_0 \neq 0), \tag{3. 2 a}$$

$$Q(x) = h_0 x^{\frac{p}{k}} + \cdots + h_{p-1} x^{\frac{1}{k}} \qquad (h_0 \neq 0). \tag{3. 2 b}$$

Formally one then will have

$$\frac{d}{dx} s(x) = e^{Q(x)} x^r \left[\lambda(x) + \frac{d}{dx}\right] \sigma(x), \tag{3. 3}$$

where

$$\begin{aligned} &\lambda(x) = Q^{(1)}(x) + r x^{-1} = x^{\frac{p}{k}-1} w(x), \\ &w(x) = w_0 + w_1 x^{-\frac{1}{k}} + \cdots + w_p x^{-\frac{p}{k}}, \quad w_j = \lambda(j) h_j \qquad (j = 0, 1, \ldots p), \end{aligned} \tag{3. 3 a}$$

$$\lambda(j) = \frac{p-j}{k} (0 \leqq j \leqq p-1), \quad \lambda(p) = 1, \quad h_p = r. \tag{3. 3 b}$$

Consecutive applications of the operations involved in (3. 3) will yield

$$s^{(i)}(x) = e^{Q(x)} x^r \left[\lambda(x) + \frac{d}{dx}\right]^i \sigma(x),$$

which, in view of (3. 3 a) and (3. 3 b), can be put in the form

$$s^{(i)}(x) = e^{Q(x)} x^{r + i\left(\frac{p}{k}-1\right)} \sigma_i(x), \tag{3. 4}$$

$$\begin{aligned} \sigma_i(x) &= \left[w(x) + x^{1-\frac{p}{k}} \frac{d}{dx}\right]^i \sigma(x) = \left[w(x) + x^{1-\frac{p}{k}} \frac{d}{dx}\right] \sigma_{i-1}(x) \\ &= \sigma_0^{(i)} + \sigma_1^{(i)} x^{-\frac{1}{k}} + \sigma_2^{(i)} x^{-\frac{2}{k}} + \cdots. \end{aligned} \tag{3. 4 a}$$

Accordingly

$$\sigma_0(x) = \sigma(x), \quad \sigma_\gamma^{(0)} = \sigma_\gamma. \tag{3. 4 b}$$

It is observed that in (3. 4 a) the symbol

$$\left[w(x) + x^{1-\frac{p}{k}} \frac{d}{dx}\right]^i$$

cannot be symbolically expanded according to the binomial theorem. By (3. 4 a)

$$\sigma_\tau^{(i)} = \sum_{j=0}^{\tau} \lambda(j)\, h_j\, \sigma_{\tau-j}^{(i-1)} + q(\tau)\, \sigma_{\tau-p}^{(i-1)} \tag{3. 5}$$

$$[\tau = 0, 1, 2, \ldots;\ j \leqq p;\ \text{cf. } (3.\ 3\ \text{b}),\ (3.\ 4\ \text{b})],$$

where

$$q(\tau) = 0 \ (\text{for } \tau \leqq p),\quad q(\tau) = -\frac{\tau - p}{k} \qquad (\text{for } \tau > p). \tag{3. 5 a}$$

One may write (3. 5) in the form

$$y_\tau(i+1) = a\, y_\tau(i) + f_\tau(i) \qquad \left(a = \frac{p h_0}{k}\right), \tag{3. 6}$$

where

$$\begin{aligned} y_\tau(i) = \sigma_\tau^{(i)},\ f_\tau(i) &= \sum_{s=1}^{\tau} \lambda(s)\, h_s\, \sigma_{\tau-s}^{(i)} + q(\tau)\, \sigma_{\tau-p}^{(i)} \\ &= \sum_{j=1}^{\tau} \lambda(j)\, h_j\, y_{\tau-j}(i) + q(\tau)\, y_{\tau-p}(i) \qquad (\text{cf. } (3.\ 3\ \text{b});\ j \leqq p) \end{aligned} \tag{3. 6 a}$$

and, by (3. 4 b),

$$y_\tau(0) = \sigma_\tau \qquad (\tau = 0, 1, \ldots). \tag{3. 6 b}$$

If in (3. 6) $f_\tau(i)$ is thought of as known, the resulting difference equation gives the following solution for positive integral values i:

$$y_\tau(i) = a^i\, y_\tau(0) + \sum_{j=0}^{i-1} f_\tau(j)\, a^{i-1-j}. \tag{3. 7}$$

Accordingly, from (3. 5) we infer that

$$\sigma_\tau^{(i)} = a^i \sigma_\tau + \sum_{j=0}^{i-1} a^{i-1-j} \left[\sum_{s=1}^{\tau} \lambda(s)\, h_s\, \sigma_{\tau-s}^{(j)} + q(\tau)\, \sigma_{\tau-p}^{(j)}\right] \tag{3. 8}$$

(cf. (3. 6), (3. 3 b), (3. 5 a); $s \leqq p$). Consideration of (3. 8) leads to the conclusion that the $\sigma_r^{(i)}$ are of the form

$$\sigma_r^{(i)} = \sum_{\varrho=0}^{r} \lambda_{r,\varrho}^{(i)}\, \sigma_\varrho \tag{3. 9}$$

$$[i = 0, 1, 2, \ldots;\ \lambda_{r,\varrho}^{(0)} = 0\ (0 \leqq \varrho < r),\ \lambda_{r,r}^{(0)} = 1].$$

Substitution of this in (3. 8) will yield

$$\sum_{\varrho=0}^{\tau} \lambda_{\tau,\varrho}^{(i)} \sigma_\varrho = a^i \sigma_\tau + \sum_{j=0}^{i-1} a^{i-1-j} \sum_{s=1}^{\tau} \lambda(s) h_s \sum_{\varrho=0}^{\tau-s} \lambda_{\tau-s,\varrho}^{(j)} \sigma_\varrho + \sum_{j=0}^{i-1} a^{i-1-j} q(\tau) \sum_{\varrho=0}^{\tau-p} \lambda_{\tau-p,\varrho}^{(j)} \sigma_\varrho$$

$$(3.10) \quad = a^i \sigma_\tau + \sum_{j=0}^{i-1} a^{i-1-j} \left\{ \sigma_0 \sum_{s=1}^{\tau} \lambda(s) h_s \lambda_{\tau-s,0}^{(j)} + \sigma_1 \sum_{s=1}^{\tau-1} \lambda(s) h_s \lambda_{\tau-s,1}^{(j)} + \cdots \right.$$

$$\left. + \sigma_m \sum_{s=1}^{\tau-m} \lambda(s) h_s \lambda_{\tau-s,m}^{(j)} + \cdots + \sigma_{\tau-1} \lambda(1) h_1 \lambda_{\tau-1,\tau-1}^{(j)} \right\} + \sum_{j=0}^{i-1} a^{i-1-j} q(\tau) \sum_{\varrho=0}^{\tau-p} \lambda_{\tau-p,\varrho}^{(j)} \sigma_\varrho .$$

Here and in the sequel

$$(3.10\text{a}) \qquad \lambda(j) h_j = 0 \qquad (\text{for } j > p).$$

Comparing the coefficients of the σ_ϱ we obtain

$$(3.10\text{b}) \qquad \lambda_{\tau,\tau}^{(i)} = a^i . \qquad \left(a = \frac{p h_0}{k}\right),$$

$$(3.10\text{c}) \qquad \lambda_{\tau,\varrho}^{(i)} = \sum_{j=0}^{i-1} a^{i-1-j} \sum_{s=1}^{\tau-\varrho} \lambda(s) h_s \lambda_{\tau-s,\varrho}^{(j)} \qquad (\tau - p < \varrho < \tau),$$

$$(3.10\text{d}) \qquad \lambda_{\tau,\varrho}^{(i)} = \sum_{j=0}^{i-1} a^{i-1-j} \sum_{s=1}^{\tau-\varrho} \lambda(s) h_s \lambda_{\tau-s,\varrho}^{(j)} + \sum_{j=0}^{i-1} a^{i-1-j} q(\tau) \lambda_{\tau-p,\varrho}^{(j)}$$

$$(s \leqq p;\ 0 \leqq \varrho \leqq \tau - p;\ \text{cf. } (3.10\text{a}), (3.5\text{a}), (3.3\text{b}).$$

In view of (3. 9) it is noted that the $\lambda_{\tau,\varrho}^{(0)}$ are known. For $i = 1$ the relations (3. 10 b)—(3. 10 d) will serve to determine the $\lambda_{\tau,\varrho}^{(1)}$. In general, having obtained the

$$\lambda_{\tau,\varrho}^{(j)} \qquad (j = 1, 2, \ldots, i-1),$$

the $\lambda_{\tau,\varrho}^{(i)}$ $(0 \leqq \varrho \leqq \tau)$ will be given by (3. 10 b)—(3. 10 d), as formulated. Thus we observe that *the coefficients $\sigma_r^{(i)}$, involved in $\sigma_i(x)$ of (3. 4), are of the form (3. 9), where the $\lambda_{r,\varrho}^{(i)}$ can be determined with the aid of (3. 10 b)—(3. 10 d).*

By (3. 4)

$$s^{(i_1)}(x) s^{(i_2)}(x) \ldots s^{(i_\nu)}(x) = e^{\nu Q(x)} x^{\nu r} x^{\left(\frac{p}{k}-1\right)(i_1 + \cdots + i_\nu)} \prod_{\alpha=1}^{\nu} \sigma_{i_\alpha}(x)$$

$$(3.11) \qquad = e^{\nu Q(x)} x^{\nu r} x^{\left(\frac{p}{k}-1\right)(i_1 + \cdots + i_\nu)} \sum_{j=0}^{\infty} c_j^{i_1, \ldots i_\nu} x^{-\frac{j}{k}},$$

where

$$(3.11\text{a}) \qquad c_j^{i_1, \ldots i_\nu} = \sum_{j_1, \ldots j_\nu} \sigma_{j_1}^{(i_1)} \sigma_{j_2}^{(i_2)} \cdots \sigma_{j_\nu}^{(i_\nu)}$$

$$(j_1, \ldots j_\nu \geqq 0;\ j_1 + \cdots + j_\nu = j).$$

In consequence of (3. 11) and (3. 1) it is observed that $s(x)$ ((3. 2)) will be a formal solution of $F_\nu = 0$, if

$$(3.\ 12)\qquad \sum_{i_1,\ldots i_\nu=0}^{n} x^{\eta_{i_1},\ldots i_\nu} \sum_{\beta=0}^{\infty} b_\beta^{i_1,\ldots i_\nu} x^{-\beta} \sum_{j=0}^{\infty} c_j^{i_1,\ldots i_\nu} x^{-\frac{j}{k}} = 0,$$

where

$$(3.\ 12\ \text{a})\qquad \eta_{i_1,\ldots i_\nu} = \eta(i_1, \ldots i_\nu) + \left(\frac{p}{k} - 1\right)(i_1 + \cdots + i_\nu) = \frac{1}{k} l_{i_1,\ldots, i_\nu}$$

(integers $l_{i_1,\ldots i_\nu}$). For convenience we shall write

$$\sum_{\beta=0}^{\infty} b_\beta^{i_1,\ldots i_\nu} x^{-\beta} = \sum_{j=0}^{\infty} b_j(i_1, \ldots i_\nu) x^{-\frac{j}{k}},$$

with

$$(3.\ 13)\qquad \begin{aligned} &b_j(i_1, \ldots i_\nu) = 0 \quad \left(\text{when } \frac{j}{k} \neq \text{an integer}\right) \\ &b_{\beta k}(i_1, \ldots i_\nu) = b_\beta^{i_1,\ldots i_\nu} \qquad (\beta = 0, 1, 2, \ldots). \end{aligned}$$

From (3. 12) it is then deduced that

$$(3.\ 14)\qquad \sum_{i_1,\ldots i_\nu=0}^{n} x^{\frac{1}{k} l_{i_1,\ldots i_\nu}} \sum_{j=0}^{\infty} d_j^{i_1,\ldots i_\nu} x^{-\frac{j}{k}} = 0 \qquad (\text{cf. } (3.\ 12\ \text{a})),$$

where

$$(3.\ 14\ \text{a})\qquad d_j^{i_1,\ldots i_\nu} = \sum_{j_1+j_2=j} b_{j_1}(i_1, \ldots i_\nu) c_{j_2}^{i_1,\ldots i_\nu} \qquad (\text{cf. } (3.\ 13),\ (3.\ 11\ \text{a})).$$

In order that (3. 14) should be formally satisfied it is necessary that there should be at leaat two terms of the same degree ϱ in x, the other terms being all of degree $\leqq \varrho$. Thus, we should have

$$\frac{1}{k} l_{\alpha_1,\ldots\alpha_\nu} = \frac{1}{k} l_{\beta_1,\ldots\beta_\nu} = \varrho = \varrho\left(\frac{p}{k}\right)$$

for some particular distinct sets of values $(\alpha_1, \ldots \alpha_\nu)$, $(\beta_1, \ldots \beta_\nu)$, while

$$\frac{1}{k} l_{i_1,\ldots i_\nu} \leqq \varrho\left(\frac{p}{k}\right) \qquad (\text{for all sets } (i_1, \ldots i_\nu)).$$

In view of (3. 12 a) it is accordingly observed that one should have

$$(3.\ 15)\qquad \frac{p}{k} - 1 = -\frac{\eta(\beta_1, \ldots \beta_\nu) - \eta(\alpha_1, \ldots \alpha_\nu)}{(\beta_1 + \cdots + \beta_\nu) - (\alpha_1 + \cdots + \alpha_\nu)} \qquad \left(\frac{p}{k} > 0\right),$$

provided $\beta_1 + \cdots + \beta_\nu \neq \alpha_1 + \cdots + \alpha_\nu$, and

$$\text{(3. 15 a)} \qquad \eta(i_1, \ldots i_\nu) - \eta(\beta_1, \ldots \beta_\nu) \leqq -\left(\frac{p}{k} - 1\right)[(i_1 + \cdots + i_\nu) - (\beta_1 + \cdots + \beta_\nu)]$$

(for all sets $(i_1, \ldots i_\nu)$). This gives rise to a diagram of the Puiseux-type, in a way analogous to that of the case of non-linear algebraic difference equations. Thus, the number-pairs

$$\text{(3. 16)} \qquad (i_1 + \cdots + i_\nu,\ \eta(i_1, \ldots i_\nu))$$

we represent in the Cartesian (x, y) plane, where $x = i_1 + \cdots + i_\nu$ and $y = \eta(i_1, \ldots i_\nu)$. It is then observed that admissible values $\frac{p}{k} - 1$ (which will be taken rational, p and k being integers), such that (3. 15), (3. 15 a) hold, are defined as the negatives of the slopes of the rectilinear segments joining pairs of points (3. 16), with the understanding that only those segments are considered whose totality constitutes a polygonal line L concave downward, with no points (5. 16) above L. Inasmuch as we should have $\frac{p}{k} > 0$, *only those sides of the polygon L will give rise to admissible values $\frac{p}{k}$ whose slopes are less than unity.*

In the case when a vertex P of L is multiple, that is, when we have for at least two distinct sets $(\beta_1, \ldots \beta_\nu)$, $(\alpha_1, \ldots \alpha_\nu)$ the equalities

$$\text{(3. 17)} \qquad \beta_1 + \cdots + \beta_\nu = \alpha_1 + \cdots + \alpha_\nu, \quad \eta(\beta_1, \ldots \beta_\nu) = \eta(\alpha_1, \ldots \alpha_\nu),$$

one may choose for $\frac{p}{k} - 1$ any rational number $\alpha (> -1)$, provided that L lies to one side of the line through P with the slope $-\alpha$. We then shall have $\frac{p}{k} > 0$ and (3. 15 a) will be satisfied.

Suppose $\frac{p}{k}(> 0)$ is given by (3. 15) (or as described in the case of a multiple vertex). We proceed finding conditions under which the differential equation has a corresponding formal solution of the stated type. It is observed that (3. 14) can be arranged as follows:

$$\text{(3. 18)} \qquad x^{\frac{\lambda}{k}}[\delta_0 + \delta_1 x^{-\frac{1}{k}} + \delta_2 x^{-\frac{2}{k}} + \cdots] = 0,$$

where

$$\frac{\lambda}{k} = \varrho = \frac{1}{k} l_{\alpha_1, \ldots \alpha_\nu},$$

the integers λ, k being suitably chosen. Clearly one should have

$$\delta_j = 0 \qquad (j = 0, 1, \ldots). \tag{3. 18 a}$$

Subsequent developments will be considerably simplified if, corresponding to the value $\frac{p}{k}$ under consideration, we take note of the relations

$$\begin{gathered} \eta(\alpha_1, \ldots \alpha_\nu) + \left(\frac{p}{k} - 1\right)(\alpha_1 + \cdots + \alpha_\nu) = \eta(\beta_1, \ldots \beta_\nu) + \left(\frac{p}{k} - 1\right)(\beta_1 + \cdots + \beta_\nu) = \frac{\lambda}{k}, \\ \eta(i_1, \ldots i_\nu) + \left(\frac{p}{k} - 1\right)(i_1 + \cdots + i_\nu) \leqq \frac{\lambda}{k} \end{gathered} \tag{3. 19}$$

and write the differential equation (3. 1) in the form

$$F_\nu \equiv \sum_{i_1, \ldots i_\nu} x^{\frac{\lambda}{k} - \left(\frac{p}{k} - 1\right)(i_1 + \cdots + i_\nu)} \left[\sum_{\gamma=0}^{\infty} b'_\gamma(i_1, \ldots i_\nu)\, x^{-\frac{\gamma}{k}}\right] \cdot y^{(i_1)} \ldots y^{(i_\nu)} = 0. \tag{3. 20}$$

This is possible, inasmuch as in view of the second inequality (3. 19) one has

$$\frac{\lambda}{k} - \left(\frac{p}{k} - 1\right)(i_1 + \cdots + i_\nu) - \eta(i_1, \ldots i_\nu) = \frac{1}{k} w(i_1, \ldots i_\nu) \geqq 0, \tag{3. 20 a}$$

where $w(i_1, \ldots i_\nu)$ is an integer. By (3. 20 a) the $b'_\gamma(i_1, \ldots i_\nu)$ of (3. 20) are related with the $b_\gamma(i_1, \ldots i_\nu)$ of (3. 13) as follows:

$$b'_\gamma(i_1, \ldots i_\nu) = \begin{cases} 0 & (\gamma < w), \\ b_{\gamma - w}(i_1, \ldots i_\nu) & (w = w(i_1, \ldots i_\nu);\ \gamma \geqq w). \end{cases} \tag{3. 20 b}$$

According to this *the* $b'_0(i_1, \ldots i_\nu)$ *are those* $b_0(i_1, \ldots i_\nu) [= b_0^{i_1, \ldots i_\nu}]$ *for which* $w(i_1, \ldots i_\nu)$ *is zero*; thus, amongst the $b'_0(i_1, \ldots i_\nu)$ will be found in particular

$$b_0(\alpha_1, \ldots \alpha_\nu),\ b_0(\beta_1, \ldots, \beta_\nu). \tag{3. 21}$$

Substitution of (3. 11) in (3. 20) will yield, after division by $x^{\nu r} \exp. [\nu Q(x)]$,

$$x^{\frac{\lambda}{k}} \sum_{i_1, \ldots i_\nu} \sum_{\gamma=0}^{\infty} b'_\gamma(i_1, \ldots i_\nu)\, x^{-\frac{\gamma}{k}} \sum_{j=0}^{\infty} c_j^{i_1, \ldots i_\nu} x^{-\frac{j}{k}} = 0. \tag{3. 22}$$

Thus, the δ_i of (3. 18 a) (cf. (3. 18)) may be expressed as

$$\delta_i = \sum_{i_1, \ldots i_\nu} \sum_{t=0}^{i} b'_{i-t}(i_1, \ldots i_\nu)\, c_t^{i_1, \ldots i_\nu}. \tag{3. 22 a}$$

Hence, in view of (3. 11 a), the equations (3. 18 a) may be written in the form

$$\delta_i \equiv \sum_{i_1, \ldots i_\nu} \sum_{t=0}^{i} b'_{i-t}(i_1, \ldots i_\nu) \sum_{\tau_1+\cdots+\tau_\nu=t} \prod_{s=1}^{\nu} \sigma_{\tau_s}^{(i_s)} = 0 \qquad (i = 0, 1, \ldots). \tag{3. 23}$$

Furthermore, by virtue of (3. 9)

$$\delta_i \equiv \sum_{i_1, \ldots i_\nu} \sum_{t=0}^{i} b'_{i-t}(i_1, \ldots i_\nu) \sum_{\tau_1+\cdots+\tau_\nu=t} \prod_{s=1}^{\nu} \sum_{\varrho=0}^{\tau_s} \lambda_{\tau,\varrho}^{(i_s)} \sigma_\varrho \tag{3. 24}$$

(cf. (3. 10 b)—(3. 10 d)). By (3. 24), for $i = 0$, and by (3. 10 b)

$$\begin{aligned} \sigma_0^{-\nu} \delta_0 \equiv \sigma_0^{-\nu} \sum_{i_1, \ldots i_\nu} b'_0(i_1, \ldots i_\nu) \prod_{s=1}^{\nu} \lambda_{0,0}^{(i_s)} \sigma_0 \\ \equiv \sum_{i_1, \ldots, i_\nu} b'_0(i_1, \ldots i_\nu) \left(\frac{p h_0}{k}\right)^{i_1+\cdots+i_\nu} \equiv B_0\left(\frac{p h_0}{k}\right) \end{aligned} \tag{3. 25}$$

Thus the first equation (3. 23) will be satisfied if and only if h_0 is a root of the *characteristic equation*

$$B_0\left(\frac{p h_0}{k}\right) = 0, \tag{3. 26}$$

where $B_0(u)$ *is defined in* (3. 25).

From (3. 10 a)—(3. 10 d) we obtain

$$\lambda_{\tau, \tau-1}^{(i)} = i a^{i-1} \lambda(1) h_1 \qquad \left(a = \frac{p h_0}{k}\right), \tag{3. 27}$$

$$\lambda_{\tau, \tau-2}^{(i)} = c_{i,0} \lambda(2) h_2 a^{i-1} + c_{i,1} (\lambda(1) h_1)^2 a^{i-2}. \tag{3. 27 a}$$

By induction it is established that

$$\begin{gathered} \lambda_{\tau, \tau-m}^{(i)} = \sum_{q=1}^{m} c_{i, q-1} a^{i-q} \sum_{k_1+\cdots+k_q=m} (\lambda(k_1) h_{k_1}) \ldots (\lambda(k_q) h_{k_q}) = \Gamma_m^{(i)} \\ (m = 1, \ldots p-1), \end{gathered} \tag{3. 28}$$

where

$$p \geqq k_j \geqq 1; \; c_{i,0} = i; \; c_{i,\sigma} = \sum_{j=0}^{i-1} c_{j, \sigma-1}. \tag{3. 29}$$

Furthermore,

$$\lambda_{\tau, \tau-1}^{(i)} = q(\tau) c_{i,0} a^{i-1} + \Gamma_p^{(i)} \qquad \left(q(\tau) = \frac{\tau - p}{-k}\right) \tag{3. 30}$$

By induction it is inferred that

$$\lambda^{(i)}_{\tau,\,\tau-p-m} = \Gamma^{(i)}_{p+m} + q\left(\tau - \frac{m}{2}\right)\sum_{s=1}^{m}(s+1)\,c_{i,\,s}\,a^{i-s-1}.$$

$$(3.\ 31) \qquad \sum_{k_1+\cdots+k_s=m} (\lambda(k_1)\,h_{k_1})\ldots(\lambda(k_s)\,h_{k_s})$$

$$[\lambda(s)\,h_s = 0 \text{ (for } s > p);\ p \geqq k_j \geqq 1;\ m = 1, 2, \ldots \tau - p;\ \tau \geqq p + 1].$$

In view of (3. 24) it is then found that δ_1 contains $\lambda(1)\,h_1\,B_0^{(1)}(a) + B_1(a)\left(a = \frac{p\,h_0}{k}\right)$ as a factor. Accordingly, h_1 will be determined from the equation $\delta_1 = 0$, if a is a simple root of the characteristic equation (3. 26).

The subsequent expressions for the δ_j $(j = 2, 3, \ldots)$ are rather complicated. Suffice it to say that, while it is necessary that $\frac{p\,h_0}{k}$ should satisfy (3. 26), it is not necessary for the existence of a solution of the stated kind that $\frac{p\,h_0}{k}$ should be a simple root of (3. 26). On the other hand, *a condition requiring* $\frac{p\,h_0}{k}$ *to be a simple root of* (3. 26), *while sufficient in an extended variety of cases for the existence of a formal solution of the stated type, is sufficient not in all cases.*

Inasmuch as our main concern is with the analytic theory we shall not need any further details in this direction. It will be essential, however, to note the following.

With (3. 26) satisfied, $\delta_i\,(i > 0)$ is a function of $h_0, \ldots h_p,\ \sigma_0, \ldots \sigma_{i-1}$; thus,

$$\delta_i \equiv \delta_i(h_0, \ldots h_p;\ \sigma_0, \ldots \sigma_{i-1}),$$

δ_i being independent of $\sigma_i,\ \sigma_{i+1}, \ldots$.

Lemma 3. 1. *Consider the formal non-linear differential equation* $F_\nu = 0$ (3. 1). *Let* $\frac{p}{k}\,(> 0)$ *be an admissible value* (p, k *integers*) *formed in accordance with the text subsequent to* (3. 14 a) *up to* (3. 17). *If the equation* $F_\nu = 0$ *has a formal solution* (3. 2)—(3. 2 b) *with this value of* $\frac{p}{k}$, *then* h_0 *necessarily satisfies the characteristic equation* (3. 26) *and we have* δ_0 *given by* (3. 25), *while*

$$(3.\ 33) \qquad \delta_i \equiv \delta_i(h_0, \ldots h_p;\ \sigma_0, \ldots \sigma_{i-1}) = 0 \qquad (i > 0),$$

where the δ_i are defined by (3. 24); *the δ_i are the coefficients in the expansion* (3. 18) *of the first member of* (3. 14) (*cf.* (3. 11)—(3. 14 a)).

Examples of equations $F_\nu = 0$ (3. 1) which possess formal solutions (3. 2)—(3. 2 b) can be easily given. For instance, let $L(x, y) = 0$ be any equation of the form (2. 12), (2. 12 a) and satisfied by the given formal solutions; we may then take F_ν of the form (2. 13), assigning the coefficients in the φ_j arbitrarily of the form $x^{\lambda_1}\{x\}_0$ (λ_1 rational).

4. **A Transformation.**

Suppose that we have on hand a differential equation

$$F_\nu^* \equiv \sum_{i_1, \ldots i_\nu} x^{\eta(i_1, \ldots i_\nu)} b^{i_1, \ldots i_\nu}(x)\, y^{(i_1)} y^{(i_2)} \ldots y^{(i_\nu)} = 0 \tag{4. 1}$$

$$(0 \leqq i_1, i_2, \ldots i_\nu \leqq n;\ \eta(i_1, \ldots i_\nu) \text{ integers})$$

with coefficients $b^{i_1, \ldots i_\nu}(x)$ analytic (for $x \neq \infty$) in a region R, extending to infinity and bounded by two curves each with a limiting direction at infinity; moreover, suppose that

$$b^{i_1, \ldots i_\nu}(x) \sim \sum_{\gamma=0}^{\infty} b_\gamma^{i_1, \ldots i_\nu} x^{-\gamma} = \beta^{i_1, \ldots i_\nu}(x) \qquad (\text{in } R). \tag{4. 1 a}$$

With the 'actual' differential equation (4. 1) there is associated a formal equation

$$F_\nu \equiv \sum_{i_1, \ldots i_\nu} x^{\eta(i_1, \ldots i_\nu)} \beta^{i_1, \ldots i_\nu}(x)\, y^{(i_1)} y^{(i_2)} \ldots y^{(i_\nu)} = 0. \tag{4. 2}$$

In accordance with the previously established usage, we shall say that $s(x)$ is a formal solution of (4. 1) if it is a formal solution of (4. 2).

Suppose now that $s(x)$ of the form (3. 2)—(3. 2 b) is a formal solution of (4. 2) in accordance with Lemma (3. 1). *The main purpose of this paper is to examine the possibility that there should exist an 'actual' solution $y(x)$, analytic in a suitable subregion* (*extending to infinity*) *R' of R and satisfying in R' the equation* (4. 1) *as well as the asymptotic relation*

$$y(x) \sim s(x). \tag{4. 3}$$

As a preliminary to the investigation of this sort, we recall that corresponding to the side of the Puiseux diagram, to which the value $\frac{p}{k}$ (involved in (3. 2 b)) belongs, the formal equation (4. 2) has been put in the form (3. 20). The corresponding form for the actual equation will be

$$F_\nu^* \equiv \sum_{i_1, \ldots i_\nu} x^{\frac{\lambda}{k} - \left(\frac{p}{k} - 1\right)(i_1 + \cdots + i_\nu)} b'^{i_1, \ldots i_\nu}(x)\, y^{(i_1)} y^{(i_2)} \ldots y^{(i_\nu)} = 0, \tag{4. 4}$$

where the functions $b'^{i_1, \ldots i_\nu}(x)$ satisfy the relations

$$b'^{i_1, \ldots i_\nu}(x) \sim \beta'^{i_1, \ldots i_\nu}(x) = \sum_{\gamma=0}^{\infty} b'_\gamma(i_1, \ldots i_\nu)\, x^{-\frac{\gamma}{k}} \qquad (\text{in } R). \tag{4. 4 a}$$

On the basis of the form of $s(x)$, as given by (3. 2), we envisage the transformation

$$y(x) = e^{Q(x)} x^r [\sigma(t, x) + \varrho(x)], \tag{4. 5}$$

where

$$\sigma(t, x) = \sigma_0 + \sigma_1 x^{-\frac{1}{k}} + \cdots + \sigma_t x^{-\frac{t}{k}} \tag{4. 5 a}$$

and $\varrho(x)$ is the new variable. We have

$$\frac{d^i}{d x^i}[e^{Q(x)} x^r \varrho(x)] = e^{Q(x)} x^{r + i\left(\frac{p}{k} - 1\right)} \varrho_i(x) \tag{4. 6}$$

with

$$\varrho_i(x) = \left[w(x) + x^{1-\frac{p}{k}} \frac{d}{dx}\right]^i \varrho(x) \qquad (\text{cf. } (3. 3 \text{ a})). \tag{4. 6 a}$$

In particular

$$\varrho_i(x) = \left[w(x) + x^{1-\frac{p}{k}} \frac{d}{dx}\right] \varrho_{i-1}(x) \qquad (i = 1, 2, \ldots;\ \varrho_0(x) = \varrho(x)). \tag{4. 6 b}$$

On the other hand,

$$\frac{d^i}{d x^i}[e^{Q(x)} x^r \sigma(t, x)] = e^{Q(x)} x^{r + i\left(\frac{p}{k} - 1\right)} \sigma_i(t, x), \tag{4. 7}$$

$$\sigma_i(t, x) = \left[w(x) + x^{1-\frac{p}{k}} \frac{d}{dx}\right]^i \sigma(t, x) = \left[w(x) + x^{1-\frac{p}{k}} \frac{d}{dx}\right] \sigma_{i-1}(t, x) \tag{4. 7 a}$$

$$= \sigma_0^{(i)}(t) + \sigma_1^{(i)}(t)\, x^{-\frac{1}{k}} + \cdots + \sigma_\gamma^{(i)}(t)\, x^{-\frac{\gamma}{k}} + \cdots \qquad (\sigma_0(t, x) = \sigma(t, x)).$$

In section 3 the $\sigma_\gamma^{(i)}$ of (3. 4 a) have been computed explicitly in terms of the coefficients σ_ϱ of (3. 2 a). In view of (4. 5 a) it is inferred that the $\sigma_\gamma^{(i)}(t)$ of (4. 7 a) are the $\sigma_\gamma^{(i)}$ with the $\sigma_j\,(j > t)$ replaced by zeros. Thus

$$\sigma_\gamma^{(i)}(t) = \sigma_\gamma^{(i)} \quad [\text{with } \sigma_j\,(j > t) \text{ replaced by zeros}]. \tag{4. 8}$$

Whence in consequence of (3. 9)

$$\sigma_\gamma^{(i)}(t) = \sum_{\varrho=0}^{t} \lambda_{\gamma,\varrho}^{(i)}\,\sigma_\varrho \qquad (i = 0,\ 1,\ 2,\ \ldots;\ \varrho \leqq \gamma); \tag{4. 8 a}$$

here the $\lambda_{\gamma,\varrho}^{(i)}$ are precisely the constants so designated in (3. 9) and defined in (3. 9), (3. 10 b), (3. 10 c), (3. 10 d).

By (4. 5), (4. 6) and (4.7)

$$y^{(i)}(x) = e^{Q(x)}\,x^{r+i\left(\frac{p}{k}-1\right)}(\sigma_i(t,x) + \varrho_i(x)) \qquad (\text{cf. } (4.\ 6\ \text{b}),\ (4.\ 7\ \text{a}),\ (4.\ 8)). \tag{4. 9}$$

Furthermore

$$y^{(i_1)}(x)\,y^{(i_2)}(x)\,\ldots\,y^{(i_\nu)}(x) = e^{\nu\,Q(x)}\,x^{\nu r}\,x^{\left(\frac{p}{k}-1\right)(i_1+\cdots+i_\nu)}\prod_{\alpha=1}^{\nu}(\sigma_{i_\alpha}(t,x) + \varrho_{i_\alpha}(x)). \tag{4. 9 a}$$

Substituting in (4. 4) we get

$$F_\nu^* \equiv e^{\nu\,Q(x)}\,x^{\nu r + \frac{\lambda}{k}} \sum_{i_1,\ldots i_\nu} b'^{\,i_1\ldots i_\nu}(x) \prod_{\alpha=1}^{\nu}(\sigma_{i_\alpha}(t,x) + \varrho_{i_\alpha}(x)) = 0. \tag{4. 10}$$

Now, inasmuch as

$$\prod_{\alpha=1}^{\nu}(1 + c_\alpha) = 1 + \sum_{m=1}^{\nu}\sum_{j_1<j_2<\ldots<j_m} c_{j_1}c_{j_2}\ldots c_{j_m},$$

the above may be written as

$$\sum_{i_1,\ldots i_\nu} b'^{\,i_1\ldots i_\nu}(x)\prod_{\alpha=1}^{\nu}\sigma_{i_\alpha}(t,x)\left\{1 + \sum_{m=1}^{\nu}\sum_{j_1<\ldots<j_m}\frac{\varrho_{i_{j_1}}(x)\,\varrho_{i_{j_2}}(x)\ldots\varrho_{i_{j_m}}(x)}{\sigma_{i_{j_1}}(x)\,\sigma_{i_{j_2}}(x)\ldots\sigma_{i_{j_m}}(x)}\right\} = 0.$$

Accordingly ϱ satisfies

$$L(\varrho) + K(\varrho) = F(x), \tag{4. 11}$$

where

$$L(\varrho) \equiv \sum_{i_1,\ldots i_\nu} b'^{\,i_1\ldots i_\nu}(x)\prod_{\alpha=1}^{\nu}\sigma_{i_\alpha}(t,x)\sum_{j=1}^{\nu}\frac{\varrho_{i_j}(x)}{\sigma_{i_j}(x)}, \tag{4. 11 a}$$

$$K(\varrho) \equiv \sum_{i_1, \ldots i_\nu} b'^{i_1, \ldots i_\nu}(x) \prod_{\alpha=1}^{\nu} \sigma_{i_\alpha}(t, x) \sum_{m=2}^{\nu} \sum_{j_1 < \ldots < j_m} \frac{\varrho_{i_{j_1}}(x)}{\sigma_{i_{j_1}}(x)} \cdots \frac{\varrho_{i_{j_m}}(x)}{\sigma_{i_{j_m}}(x)}, \tag{4. 11 b}$$

$$F(x) = -\sum_{i_1, \ldots i_\nu} b'^{i_1, \ldots i_\nu}(x) \prod_{\alpha=1}^{\nu} \sigma_{i_\alpha}(t, x). \tag{4. 11 c}$$

In view of (4. 4 a) one may write for any $\tau > 0$

$$b'^{i_1, \ldots i_\nu}(x) = \sum_{\gamma=0}^{\tau} b'_\gamma(i_1, \ldots i_\nu) x^{-\frac{\gamma}{k}} + x^{-\frac{\tau+1}{k}} \beta_{i_1, \ldots i_\nu}(\tau, x), \tag{4. 12}$$

with

$$|\beta_{i_1, \ldots i_\nu}(\tau, x)| \leqq \beta_\tau \qquad (x \text{ in } R). \tag{4. 12 a}$$

Thus $F(x)$ of (4. 11 c) may be expressed as

$$F(x) = F_1(x) + F_2(x), \tag{4. 13}$$

$$F_1(x) = -\sum_{i_1, \ldots i_\nu} \sum_{\gamma=0}^{\tau} b'_\gamma(i_1, \ldots i_\nu) x^{-\frac{\gamma}{k}} \prod_{\alpha=1}^{\nu} \sigma_{i_\alpha}(t, x), \tag{4. 13 a}$$

$$F_2(x) = -x^{-\frac{\tau+1}{k}} \beta(t, \tau; x), \; \beta(t, \tau; x) = \sum_{i_1, \ldots i_\nu} \beta_{i_1, \ldots i_\nu}(\tau, x) \prod_{\alpha=1}^{\nu} \sigma_{i_\alpha}(t, x). \tag{4. 13 b}$$

We shall examine $F(x)$ closer. On taking account of (3. 11) it is inferred that

$$\prod_{\alpha=1}^{\nu} \sigma_{i_\alpha}(t, x) = \sum_{j=0}^{\infty} c_j^{i_1, \ldots i_\nu}(t) x^{-\frac{j}{k}}, \tag{4. 14}$$

where (compare with (3. 11 a))

$$c_j^{i_1, \ldots i_\nu}(t) = \sum_{j_1, \ldots j_\nu} \sigma_{j_1}^{(i_1)}(t) \sigma_{j_2}^{(i_2)}(t) \ldots \sigma_{j_\nu}^{(i_\nu)}(t) \qquad (j_1 + \cdots + j_\nu = j). \tag{4. 14 a}$$

By (4. 8 a) and (3. 9)

$$\sigma_\gamma^{(i)}(t) = \sigma_\gamma^{(i)} \qquad (0 \leqq \gamma \leqq t). \tag{4. 15}$$

Hence from (4. 14 a) it is deduced that

$$c_j^{i_1, \ldots i_\nu}(t) = c_j^{i_1, \ldots i_\nu} \qquad (0 \leqq j \leqq t). \tag{4. 16}$$

Substituting (4. 14) in $F_1(x)$ of (4. 13 a) one obtains

$$(4.\ 17)\qquad -F_1(x) = \delta_0(\tau, t) + \delta_1(\tau, t)\, x^{-\frac{1}{k}} + \cdots + \delta_i(\tau, t)\, x^{-\frac{i}{k}} + \cdots.$$

First of all we note that in view of the origin of $F_1(x)$ the series (4. 17) certainly converges for $|x| \geqq x_0$ (x_0 sufficiently great). If it is recalled how δ_i of (3. 25) was derived, it is concluded that

$$(4.\ 17\ \text{a})\qquad \delta_i(\tau, t) = \delta_i,$$

with the $\sigma_j^{(i_s)}$ replaced by $\sigma_j^{(i_s)}(t)$ and the $b'_\gamma(i_1, \ldots i_\nu)$ (for $\gamma > \tau$) replaced by zeros. Accordingly, by (4. 15) and (3. 23)

$$(4.\ 17\ \text{b})\qquad \delta_i(\tau, t) = \delta_i \qquad (0 \leqq i \leqq t),$$

provided we take $\tau \geqq t$.

The relations (4. 17 b) are of great importance for us, inasmuch as in consequence of the way the formal solution $s(x)$ has been defined

$$\delta_0 = 0,\ \delta_1 = 0,\ \delta_2 = 0, \ldots.$$

Thus, *with* $\tau \geqq t$, from (4. 17) it is deduced that

$$-F_1(x) = x^{-\frac{t+1}{k}} [\delta_{t+1}(\tau, t) + \delta_{t+2}(\tau, t)\, x^{-\frac{1}{k}} + \cdots].$$

On taking account of the convergence of the series (4. 17) we conclude that

$$(4.\ 18)\qquad |F_1(x)| \leqq |x|^{-\frac{t+1}{k}} F_1(t, \tau) \qquad (\text{in } R).$$

Furthermore, by (4. 13 b), (4. 12 a) and (4. 7 a) one has

$$(4.\ 18\ \text{a})\qquad |F_2(x)| \leqq |x|^{-\frac{t+1}{k}} F_2(t, \tau) \qquad (\text{in } R).$$

Thus, by (4. 13), (4. 18), (4. 18 a)

$$(4.\ 19)\qquad F(x) = x^{-\frac{t+1}{k}} F(t, x),$$

$$(4.\ 19\ \text{a})\qquad |F(t, x)| \leqq F_t \qquad (\text{in } R;\ \text{finite } F_t;\ \text{independent of } x).$$

The form of $L(\varrho)$ ((4. 11 a)) will be now determined. It is observed that $\varrho_0(x) = \varrho(x)$ and that in view of (4.6 b)

$$\varrho_i(x) = w_{i,0}(x)\,\varrho(x) + w_{i,1}(x)\,\varrho^{(1)}(x) + \cdots + w_{i,i}(x)\,\varrho^{(i)}(x), \tag{4. 20}$$

where $w_{0,0}(x) = 1$ and

$$w_{i,0}(x) = w(x)\,w_{i-1,0}(x) + x^{1-\frac{p}{k}}\,w^{(1)}_{i-1,0}(x),\quad w(x) = \sum_{j=0}^{p} \lambda(j)\,h_j\,x^{-\frac{j}{k}}, \tag{4. 20 a}$$

$$w_{i,m}(x) = w(x)\,w_{i-1,m}(x) + x^{1-\frac{p}{k}}\left(w^{(1)}_{i-1,m}(x) + w_{i-1,m-1}(x)\right) \quad (m = 1, 2, \ldots i-1), \tag{4. 20 b}$$

$$w_{i,i}(x) = x^{1-\frac{p}{k}}\,w_{i-1,i-1}(x). \tag{4. 20 c}$$

By (4. 11 a) and (4. 20)

$$L(\varrho) = \sum_{i_1,\ldots i_\nu} b'^{i_1,\ldots i_\nu}(x) \sum_{j=1}^{\nu} \varrho_{i_j}(x) \prod_{\alpha\neq j} \sigma_{i_\alpha}(t,x)$$

$$= \sum_{i_1,\ldots i_\nu} b'^{i_1,\ldots i_\nu}(x) \sum_{j=1}^{\nu} \sum_{\gamma=0}^{i_j} w_{i_j,\gamma}(x)\,\varrho^{(\gamma)}(x) \prod_{\alpha\neq j} \sigma_{i_\alpha}(t,x).$$

Thus

$$L(\varrho) = l_n(x)\,\varrho^{(n)}(x) + l_{n-1}(x)\,\varrho^{(n-1)}(x) + \cdots + l_0(x)\,\varrho(x), \tag{4. 21}$$

where

$$l_\gamma(x) = \sum_{i_1,\ldots i_\nu} b'^{i_1,\ldots i_\nu}(x) \sum_{j=1}^{\nu} w_{i_j,\gamma}(x)\,k^{i_j,\gamma} \prod_{\alpha\neq j} \sigma_{i_\alpha}(t,x) \quad \text{(cf. (4. 20 a)—(4. 20 c))} \tag{4. 21 a}$$

with

$$k^{i,\gamma} = 0 \quad (\text{for } i < \gamma), \qquad k^{i,\gamma} = 1 \quad (\text{for } i \geqq \gamma). \tag{4. 21 b}$$

It is observed that

$$w_{i,m}(x) = x^{m\left(1-\frac{p}{k}\right)}\,v_{i,m}(x) \qquad (m = 0, 1, \ldots i), \tag{4. 22}$$

where

$$v_{i,m}(x) = \text{polynomial in } x^{-\frac{1}{k}}, \qquad v_{i,i}(x) = 1. \tag{4. 22 a}$$

Whence (4. 21 a) may be put in the form

$$l_\gamma(x) = x^{\gamma\left(1-\frac{p}{k}\right)} \sum_{i_1,\ldots i_\nu} b'^{i_1,\ldots i_\nu}(x) \sum_{j=1}^{\nu} v_{i_j,\gamma}(x)\,k^{i_j,\gamma} \prod_{\alpha\neq j} \sigma_{i_\alpha}(t,x) \tag{4. 23}$$

[cf. (4. 20 a)—(4. 20 c), (4. 21 b), (4. 22)].

By (4. 4 a), (4. 7 a) and (4. 23)

$$(4.\ 24)\qquad l_\gamma(x)\, x^{-\gamma\left(1-\frac{p}{k}\right)} = \lambda_\gamma(x) \sim l_{\gamma,0}(t) + l_{\gamma,1}(t)\, x^{-\frac{1}{k}} + \cdots + l_{\gamma,j}(t)\, x^{-\frac{j}{k}} + \cdots \quad (\text{in } R).$$

The series in (4. 24) is the formal expansion of the expression

$$(4.\ 24\ \text{a})\qquad \sum_{i_1,\ldots i_\nu} \beta'^{i_1,\ldots i_\nu}(x) \sum_{j=1}^{\nu} v_{i_j,\gamma}(x)\, k^{i_j,\gamma} \prod_{\alpha\neq j} \sum_{s=0}^{\infty} \sigma_s^{(i_\alpha)}(t)\, x^{-\frac{s}{k}} \quad (\text{cf. } (4.\ 4\ \text{a}),\ (4.\ 21\ \text{b})).$$

It is observed that

$$(4.\ 24\ \text{b})\qquad l_{\gamma,j}(t) = l_{\gamma,j} \qquad (j = 0, i, \ldots t'),$$

where the $l_{\gamma,j}$ are independent of t, being the coefficients in the formal expansion of (4. 24 a) with the $\sigma_s^{(i)}(t)$ replaced by the $\sigma_s^{(i)}$, respectively; moreover, t' can be made arbitrarily great by a suitable choice of t. On taking account of (4. 24) one may write (4. 21) in the form

$$(4.\ 25)\qquad \begin{aligned} L(\varrho) \equiv x^{n\left(1-\frac{p}{k}\right)} [\lambda_n(x)\, \varrho^{(n)}(x) + \lambda_{n-1}(x)\, x^{\frac{p}{k}-1}\, \varrho^{(n-1)}(x) + \cdots \\ \cdots + \lambda_\gamma(x)\, x^{(n-\gamma)\left(\frac{p}{k}-1\right)}\, \varrho^{(\gamma)}(x) + \cdots + \lambda_0(x)\, x^{n\left(\frac{p}{k}-1\right)}\, \varrho(x)] \quad (\text{cf. } (4.\ 24). \end{aligned}$$

Let $v_{i,\gamma,0}$ denote the constant term in the polynomial $v_{i,\gamma}(x)$. Then by (4.22 a) we have

$$(4.\ 26)\qquad v_{n,n,0} = 1.$$

The constant $l_{n,0}(t)$ $(= l_{n,0})$, involved in $\lambda_n(x)$, is obtained from (4. 24 a) on noting that

$$(4.\ 26\ \text{a})\qquad \sigma_0^{(i_\alpha)}(t) = \sigma_0\, a^{i_\alpha} \qquad \left(a = \frac{p\, h_0}{k}\right)$$

and on taking account of (4. 4 a). Thus

$$l_{n,0} = \sum_{i_1,\ldots i_\nu} b_0'(i_1, \ldots i_\nu) \sum_{j=1}^{\nu} v_{i_j,n,0}\, k^{i_j,n} \prod_{\alpha\neq j} (\sigma_0\, a^{i_\alpha}) \qquad (\text{cf. } (4.\ 21\ \text{b})).$$

Whence, inasmuch as $k^{i_j,n} = 0$ for $i_j < n$ and $k^{n,n} = 1$, one has

$$l_{n,0} = \sum_{i_1,\ldots i_\nu} b_0'(i_1, \ldots i_\nu) \sum_{j=1}^{\nu} v_{n,n,0}\, k^{i_j,n}\, \sigma_0^{\nu-1}\, a^{i_1+\cdots+i_\nu-i_j}$$

and, finally,

$$l_{n,0} = \sigma_0^{\nu-1} \sum_{j=1}^{\nu} j \sum_{i_1, \ldots i_\nu}^{(j)} b'_0(i_1, \ldots i_\nu) \left(\frac{p h_0}{k}\right)^{i_1 + \cdots + i_\nu - n}; \tag{4. 27}$$

here the summation symbol with the superscript j is over the totality of all those sets $(i_1, \ldots i_\nu)$ which contain precisely j elements each equal to n.

At times the supposition will be made that $l_{n,0}$ ((4. 27)) is distinct from zero. This hypothesis depends only on those of the initial coefficients of the differential equation $F_\nu = 0$ which correspond to the Puiseux-diagram-segment associated with $\frac{p}{k}$. In this connection it is to be recalled that h_0 depends on the aforesaid coefficients only.

By (4. 25), if $l_{n,0} \not\equiv 0$, one will have

$$L(\varrho) \equiv x^{n\left(1-\frac{p}{k}\right)} \lambda_n(x) \left[\varrho^{(n)}(x) + b_1(x) x^{\frac{p}{k}-1} \varrho^{(n-1)}(x) + \cdots + b_n(x) x^{n\left(\frac{p}{k}-1\right)} \varrho(x)\right] \tag{4. 28}$$

(cf. (4. 24)),

where

$$b_\gamma(x) \sim b_{\gamma,0}(t) + b_{\gamma,1}(t) x^{-\frac{1}{k}} + \cdots \qquad \text{(in } R\text{)}. \tag{4. 28 a}$$

Here the $b_{\gamma,j}\,(0 \leqq j \leqq j')$ are independent of t; on the other hand, j' can be made arbitrarily great by a suitable choice of t.

In view of (4. 11 b), of (4. 20) and (4. 22)

$$\begin{aligned} K(\varrho) &= \sum_{i_1, \ldots i_\nu} b'^{i_1, \ldots i_\nu}(x) \sum_{m=2}^{\nu} \sum_{j_1 < \cdots < j_m} \varrho_{i_{j_1}}(x) \cdots \varrho_{i_{j_m}}(x) \prod_{\alpha=1}^{\nu}{}' \sigma_{i_\alpha}(t, x) \\ &= \sum_{i_1, \ldots i_\nu} b'^{i_1, \ldots i_\nu}(x) \sum_{m=2}^{\nu} \sum_{j_1 < \cdots < j_m} \left[\sum_{\gamma=0}^{i_{j_1}} v_{i_{j_1},\gamma}(x) \varrho^{(\gamma)}(x) x^{\gamma\left(1-\frac{p}{k}\right)}\right] \\ &\quad \cdot \left[\sum_{\gamma=0}^{i_{j_2}} v_{i_{j_2},\gamma}(x) \varrho^{(\gamma)}(x) x^{\gamma\left(1-\frac{p}{k}\right)}\right] \cdots \left[\sum_{\gamma=0}^{i_{j_m}} v_{i_{j_m},\gamma}(x) \varrho^{(\gamma)}(x) x^{\gamma\left(1-\frac{p}{k}\right)}\right] \prod_{\alpha=1}^{\nu}{}' \sigma_{i_\alpha}(t, x), \end{aligned} \tag{4. 29}$$

where the product symbol is with respect to $i_1, i_2, \ldots i_\nu$, omitting $i_{j_1}, i_{j_2}, \ldots i_{j_m}$. In consequence of (4. 22 a) from (4. 29) it is inferred that

$$K(\varrho) = K_2(\varrho) + K_3(\varrho) + \cdots + K_\nu(\varrho), \tag{4. 30}$$

where

$$(4.\ 30\text{ a})\qquad K_m(\varrho)=\sum_{m_0,\ldots m_n} k^{m_0,\cdots m_n}(t,x)\prod_{\alpha=0}^{n}(\varrho^{(\alpha)}(x))^{m_\alpha}\,x^{\alpha\left(1-\frac{p}{k}\right)m_\alpha}\qquad (m_0+\cdots+m_n=m).$$

In (4. 30 a) the $k_m^{m_0,\cdots}(t,x)$ are analytic in x for x in R $(x\neq\infty)$ and

$$(4.\ 30\text{ b})\qquad k_m^{m_0,\cdots m_n}(t,x)\sim\sum_{\gamma=0}^{\infty}k_{m,\gamma}^{m_0,\cdots m_n}(t)\,x^{-\frac{\gamma}{k}}\qquad(\text{in } R),$$

while the $k_{m,\gamma}^{m_0,\cdots m_n}(t)$ are independent of t for $\gamma\leqq\gamma'$ ($\gamma'\to\infty$ with t).

We formulate the preceding results as follows.

Lemma 4. 1. *Consider the actual differential equation $F_\nu^*=0$ ((4. 1)). Let $s(x)$ ((3. 2)—(3. 2 b)) be a formal solution of (4. 2) according to Lemma 3. 1. Let (4. 4) be the corresponding form for the equation $F_\nu^*=0$. The transformation (4. 5) (with (4. 5 a)) leads to the equation*

$$(4.\ 31)\qquad L(\varrho)+K(\varrho)=F(x)$$

for the new variable $\varrho(x)$. In (4. 31) the linear differential expression $L(\varrho)$ is given by (4. 25) (with (4. 24)); when $l_{n,0}$ of (4. 27) is not zero, one may put $L(\varrho)$ in the form (4. 28) (with (4. 28 a)). Moreover

$$K(\varrho)=K_2(\varrho)+\cdots+K_\nu(\varrho),$$

where $K_m(\varrho)\,(2\leqq m\leqq\nu)$ is a homogeneous differential expression of order not exceeding n and of degree m; $K_m(\varrho)$ may be expressed as in (4. 30 a) (with (4. 30 b)). The function $F(x)$ is analytic in $R\,(x\neq\infty)$ and is of the form (4. 19) (with (4. 19 a)).

5. Lemmas Preliminary to Existence Theorems.

To construct a solution, with appropriate properties, of (4. 31) we determine in succession functions

$$(5.\ 1)\qquad w_0(x),\ w_1(x),\ \ldots$$

by means of the relations

$$(5.\ 2)\qquad L(w_0)=F(x),\quad w_{-1}(x)=0,$$

$$(5.\ 2\text{ a})\qquad L(w_i)=-K(w_{i-1})+F(x)\qquad(i=1,2,\ldots).$$

Under suitable conditions $\lim w_i(x)$ will be a solution of (4. 31). Whe shall write

$$z_i(x) = w_i(x) - w_{i-1}(x) \qquad (i = 0, 1, \ldots); \tag{5. 3}$$

then

$$z_0(x) + \cdots + z_j(x) = w_j(x) \qquad (j = 0, 1, \ldots). \tag{5. 3 a}$$

The successive differential relations to be satisfied by the $z_i(x)$ are

$$L(z_0) = F(x), \ L(z_j(x)) = -K(w_{j-1}(x)) + K(w_{j-2}(x)) \qquad (j = 1, 2, \ldots). \tag{5. 4}$$

Under suitable convergence conditions the series

$$\varrho(x) = z_0(x) + z_1(x) + \cdots + z_j(x) + \cdots \tag{5. 5}$$

will represent a solution of (4. 31).

Unless stated otherwise it will be assumed that R covers the complete neighborhood of infinity; that is, that R consists of the region

$$0 \leqq \bar{x} \leqq 2\pi k; \ |x| \geqq x_0 (> 0) \qquad (\bar{x} = \text{angle of } x).$$

For the present it will be assumed that $l_{n,0} \neq 0$ (cf. (4. 27)). In this case $L(\varrho)$ is given by (4. 28). The equation

$$\frac{1}{\lambda_n(x)} x^{n\left(\frac{p}{k}-1\right)} L(\varrho) \equiv T(\varrho) \equiv \varrho^{(n)}(x) + b_1(x) x^{\frac{p}{k}-1} \varrho^{(n-1)}(x) + \cdots + b_n(x) x^{n\left(\frac{p}{k}-1\right)} \varrho(x) = 0 \tag{5. 6}$$

presents the general problem of the *irregular singular point* (for linear differential equations). It will be necessary to use some of the results of the complete analytic theory of this problem, developed by TRJITZINSKY[1].

The equation (5. 6) possesses n formally linearly independent formal solutions

$$s_i(x) = e^{Q_i(x)} x^{r_i} \sigma(i, x) \qquad (i = 1, 2, \ldots n) \tag{5. 7}$$

where

$$\sigma(i, x) = \{x\}_{\mu_i} \qquad (\text{cf. Definition } 2. 1) \tag{5. 7 a}$$

and

$$Q_i(x) = \text{polynomial in } x^{\frac{1}{k\nu_i}} \qquad (\text{integers } \nu_i \geqq 1). \tag{5. 7 b}$$

The power series involved in $\{x\}_{\mu_i}$ are series in $x^{1/(k\nu_i)}$. *We note also that the highest power in Q_i is $x^{\frac{p}{k}}$.* Now the $Q_i(x)$ depend only on a certain initial

[1] See the concise statement of the pertinent results, established by TRJITZINSKY, in (T) [cf. foot-note on p. 3].

number of the coefficients in the formal series to which the $b_\gamma(x)$ are asymptotic. Hence by taking t sufficiently great (as forthwith is done) *we have the* $Q_i(x)$ *in* (5. 7) *independent of* t. We recall the following definitions introduced in (T) (cf. pp. 213, 214).

A curve B will be said to be *regular* if it is simple and extends to infinity where it has a unique limiting direction.

A region R is *regular* if it is closed, extends to infinity, and is such that if x is in R then $|x| \geqq a > 0$; also the boundary of R is simple and consists of an arc γ of the circle $|x| = r_1$ and of two regular curves extending from different extremities of γ. In a generic sense

$$R(\theta_1, \theta_2) \tag{5. 8}$$

is to denote a regular region for which the two regular curves (parts of the boundary) have limiting directions θ_1 and θ_2, respectively.

We designate by $B_{i,j}$ a regular curve along which

$$\Re(Q_i(x) - Q_j(x)) = 0. \tag{5. 9}$$

Such curves are defined only provided $Q_i(x) - Q_j(x) \not\equiv 0$. We denote by

$$R_1, R_2, \ldots R_N \tag{5. 10}$$

the regular regions, separated by the $B_{i,j}$ curves (formed, whenever possible, for $i, j = 1, 2, \ldots n$), constructed so that interior no such region are there any $B_{i,j}$ curves. Any particular region R_k has the form $R(\theta_{k,1}, \theta_{k,2})$ $(\theta_{k,1} \leqq \theta_{k,2})$. We shall designate the regular curves, forming part of the boundary of R_k and possessing at infinity the limiting directions $\theta_{k,1}$ and $\theta_{k,2}$, by ${}_lB_k$ and ${}_rB_k$, respectively.

According to the Fundamental Existence Theorem, due to Trjitzinsky, the following may be stated for the equation (5. 6), with reference to any particular region R_k of the set (5. 10).

If $\theta_{k,1} = \theta_{k,2}$, equation (5. 6) will possess a full set of solutions

$$y_i(x) \qquad (i = 1, \ldots n), \tag{5. 11}$$

with elements $y_i(x)$ analytic in $R_k (x \neq \infty)$ and satisfying relations

$$y_i(x) \sim s_i(x) \qquad (\text{in } R_k;\ i = 1, \ldots n;\ \text{cf. } (5.\ 7)); \tag{5. 11 a}$$

that is,

$$y_i(x) = e^{Q_i(x)} x^{r_i} y(i, x), \tag{5. 12}$$

with

$$y(i, x) \sim \sigma(i, x) = \{x\}_{\mu_i} \qquad \text{(in } R_k). \tag{5. 12 a}$$

If $\theta_{k,1} < \theta_{k,2}$, there exist regular overlapping subregions of R_k,

$$_rR_k = R(\theta_{k,1}, \theta_{k,2}), \; {}_lR_k = R(\theta_{k,1}, \theta_{k,2}), \tag{5. 13}$$

with boundaries containing $_lB_k$ and $_rB_k$, respectively, so that there exist two full sets of solutions

$$_ry_i(x) \; (i = 1, \ldots n); \; {}_ly_i(x) \; (i = 1, \ldots n), \tag{5. 13 a}$$

for which[1]

$$_ry_i(x) \sim s_i(x) \; (i = 1, \ldots n; \text{ in } {}_rR_k), \tag{5. 13 b}$$

$$_ly_i(x) \sim s_i(x) \; (i = 1, \ldots n; \text{ in } {}_lR_k). \tag{5. 13 c}$$

In the sequel the symbol $(a_{i,j})$ will denote a matrix with $a_{i,j}$ in i-th row and j-th column $(i, j = 1, \ldots n)$. The determinant of $(a_{i,j})$ will be designated by $|(a_{i,j})|$.

In view of the definition of $T(\varrho)$, given in (5. 6), the equations (5. 4) may be written in the form

$$T(z_j(x)) = \beta_j(x) \qquad (j = 0, 1, \ldots), \tag{5. 14}$$

where

$$\beta_0(x) = \frac{1}{\lambda_n(x)} x^{n\left(\frac{p}{k}-1\right)} F(x), \tag{5. 14 a}$$

$$\beta_j(x) = \frac{1}{\lambda_n(x)} x^{n\left(\frac{p}{k}-1\right)} [-K(w_{j-1}(x)) + K(w_{j-2}(x))] \qquad (j = 1, 2, \ldots) \tag{5. 14 b}$$

(cf. (4. 19), (4. 30), (4. 30 a), (4. 30 b)).

Let us consider now a non homogeneous differential equation

$$T(\zeta(x)) = \beta(x), \tag{5. 15}$$

typical of any equation (5. 14). In view of our purposes it will be desirable to transform (5. 15) into a system.

First of all we note that the system, written in matrix form,

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[1] For details see TRJITZINSKY [Acta mathematica, *loc. cit.*]

$$Z^{(1)}(x) = Z(x)\,D(x),\ \ Z(x) = (\zeta_{i,j}(x)), \tag{5. 16}$$

where

$$D(x) = (d_{i,j}(x)) = \begin{Bmatrix} 0,\ 0,\ \ldots,\ -b_n(x)\,x^{n\left(\frac{p}{k}-1\right)} \\ 1,\ 0,\ \ldots,\ -b_{n-1}(x)\,x^{(n-1)\left(\frac{p}{k}-1\right)} \\ \cdots\cdots\cdots\cdots\cdots \\ 0,\ 0,\ \ldots\ 1,\ -b_1(x)\,x^{\frac{p}{k}-1} \end{Bmatrix}, \tag{5. 16 a}$$

is associated with the equation $T(\zeta(x)) = 0$ as follows. If $(\zeta_{i,j}(x))$ is a matrix solution of (5. 16) then $(\zeta_{i,j}(x)) = (\zeta_i^{(j-1)}(x))$ and the $\zeta_i(x)$ $(i = 1, \ldots n)$ will constitute a full set of solutions of $T(\zeta(x)) = 0$. On the other hand, *if* $\zeta_i(x)$ $(i=1, \ldots n)$ *constitute a full set of solutions of* $T(\zeta(x)) = 0$, *the matrix*

$$Z(x) = (\zeta_{i,j}(x)) = (\zeta_i^{(j-1)}(x)) \tag{5. 16 b}$$

will satisfy (5, 16). It is also observed that if a matrix

$$(\zeta_{i,j}(x))$$

satisfies the non homogeneous system

$$Z^{(1)}(x) = Z(x)\,D(x) + B(x),\ \ Z(x) = (\zeta_{i,j}(x)) \tag{5. 17}$$

(cf. (5. 16 a)), where $B(x) = (\beta_{i,j}(x))$ with

$$\beta_{i,j}(x) = 0\ \ (j < n),\ \ \beta_{i,n}(x) = \beta(x), \tag{5. 17 a}$$

then

$$\zeta_{i,j}^{(1)}(x) = \zeta_{i,j+1}(x)\ \ (j < n),\ \ \zeta_{i,j+1}(x) = \zeta_{i,1}^{(j)}(x) \tag{5. 17 b}$$

and

$$T(\zeta_{i,1}(x)) = \beta(x). \tag{5. 17 c}$$

That is, every function in the first column of the matrix solution $(\zeta_{i,j}(x))$ *of* (5. 17) *will satisfy the equation* $T(\zeta(x)) = \beta(x)$.

A solution of (5. 17) may be given in the form

$$Z(x) = W(x)\,Z_0(x)\ \ [Z(x) = (\zeta_{i,j}(x)),\ \ Z_0(x) = (\zeta_{i,j:0}(x)),\ \ W(x) = (w_{i,j}(x))], \tag{5. 18}$$

where $Z_0(x)$ satisfies (5. 16) and

$$W^{(1)}(x) = B(x)\,Z_0^{-1}(x). \tag{5. 18 a}$$

Let R denote any particular region referred to in the text from (5. 11) to (5. 13).

On taking account of the italicised statement in connection with (5. 16 b), the matrix $Z_0(x)$ in (5. 18) is formed according to (5. 16 b),

(5. 19) $$Z_0(x) = (\zeta_{i,j:0}(x)) = (y_i^{(j-1)}(x)),$$

where the $y_i(x)$ are from (5. 11 a) or from (5. 13 b), (5. 13 c), according to the character of R. Thus

(5. 19 a) $$y_i(x) = e^{Q_i(x)} x^{r_i} y(i, x), \; y(i, x) \sim \{x\}_{\mu_i} \qquad \text{(in } R\text{)}.$$

We also have

(5. 19 b) $$y_i^{(j-1)}(x) = e^{Q_i(x)} x^{r_i + (j-1)\left(\frac{p}{k} - 1\right)} y_{j-1}(i, x),$$

where $y_{j-1}(i, x) \sim \{x\}_{\eta_i}$ (in R). We proceed to determine the form of the ele. ments in the n-th row of the matrix

(5. 20) $$Z_0^{-1}(x) = (\bar{y}_{i,j}(x)).$$

In the determinant $|(y_i^{(j-1)}(x))|$ the logarithms, occurring in (5. 19 b), will of course disappear and we obtain

(5. 21) $$\Delta(x) = |(y_i^{(j-1)}(x))| = e^{Q_1(x) + \cdots + Q_n(x)} x^{r_1 + \cdots + r_n} x^{\frac{k'}{2}(n^2 - n) - \frac{\omega}{k}} d(x),$$

with integer $\omega \geqq 0$, $k' = \frac{p}{k} - 1$ and

$$d(x) \sim d_0 + d_1 x^{-\frac{1}{k}} + \cdots \qquad \text{(in } R;\ d_0 \neq 0\text{)}.$$

By (5. 20)

$$\Delta(x)\, \bar{y}_{n,j}(x) = (-1)^{n+j} \begin{vmatrix} y_1(x), \ldots y_{j-1}(x), & y_{j+1}(x), \ldots y_n(x) \\ \cdots & \cdots \\ y_1^{(n-2)}(x), \ldots y_{j-1}^{(n-2)}(x), & y_{j+1}^{(n-2)}(x), \ldots y_n^{(n-2)}(x) \end{vmatrix}$$

Whence, in view of (5. 19 b)

(5. 22) $$\Delta(x)\, \bar{y}_{n,j}(x) = e^{Q_1(x) + \cdots + Q_n(x) - Q_j(x)} x^{r_1 + \cdots + r_n - r_j} x^{\frac{k'}{2}(n^2 - 3n + 2)} d_{n,j}(x),$$

where

$$d_{n,j}(x) \sim \{x\}_{n(j)} \qquad \text{(in } R\text{)}.$$

Thus, in consequence of (5. 21) and (5. 22)

$$\bar{y}_{n,j}(x) = e^{-Q_j(x)} x^{-r_j} x^{-\omega_1} \bar{y}(n, j, x) \qquad \left(\omega_1 = k'(n-1) - \frac{\omega}{k}\right), \tag{5. 23}$$

with

$$\bar{y}(n, j, x) \sim \{x\}_{n(j)} \qquad \text{(in } R\text{)}. \tag{5. 23 a}$$

By (5. 17 a), (5. 20) and (5. 18 a)

$$w_{i,j}(x) = \int^{x} \beta(x)\, \bar{y}_{n,j}(x)\, dx \qquad \text{(cf. (5. 23))}. \tag{5. 24}$$

In view of (5. 18) and (5. 19) a solution of (5. 17) will accordingly be given by

$$\begin{aligned} Z(x) = (\zeta_{i,j}(x)) &= \left(\sum_{\lambda=1}^{n} w_{i,r}(x)\, \zeta_{\lambda, j:0}(x)\right) \\ &= \left(\sum_{\lambda=1}^{n} y_{\lambda}^{(j-1)}(x) \int^{x} \beta(x)\, \bar{y}_{n,\lambda}(x)\, dx\right) \qquad \text{(cf. (5. 19 b))}. \end{aligned} \tag{5. 25}$$

In consequence of the remark subsequent to (5. 17 c) it may be asserted that *the elements*

$$\zeta_{i,1}(x) = \sum_{\lambda=1}^{n} y_{\lambda}(x) \int^{x} \beta(x)\, \bar{y}_{n,\lambda}(x)\, dx = z(x)$$

will be independent of i and *will constitute a solution of* $T(z(x)) = \beta(x)$ (provided the integrations can be evaluated). The statement with respect to (5. 17), (5. 17 b) will be applicable, yielding from (5. 25) the following important further result

$$\zeta_{i,1}^{(j-1)}(x) = \sum_{\lambda=1}^{n} y_{\lambda}^{(j-1)}(x) \int^{x} \beta(x)\, \bar{y}_{n,\lambda}(x)\, dx = z^{(j-1)}\, x \qquad (j = 1, \ldots n). \tag{5. 26}$$

On taking account of (5. 19 b) and (5. 23) the following Lemma is inferred.

Lemma 5. 1. *Let* $T(z(x))$ *be the linear differential operator of* (5. 6). *Let* R *be a region of the text from* (5. 11) *to* (5. 13). *Provided the integrations involved can be evaluated, the equation*

$$T(z(x)) = \beta(x) \qquad (\beta(x) \textit{ defined in } R) \tag{5. 27}$$

will possess a solution $z(x)$ such that (5. 26) *will hold; that is,*

$$\text{(5. 27 a)} \qquad \begin{aligned} z^{(j-1)}(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x)} x^{r_\lambda+(j-1)\left(\frac{p}{k}-1\right)} y_{j-1}(\lambda, x) \\ \cdot \int^{x} \beta(x) e^{-Q_\lambda(x)} x^{-r_\lambda-\omega_1} \bar{y}(n, \lambda, x)\, dx \qquad (j = 1, \ldots n). \end{aligned}$$

In consequense of the asymptotic relation given subsequent to (5. 19 b), as well as of (5. 23 a), from (5. 27 a) we derive

$$\text{(5. 28)} \qquad |z^{(j-1)}(x)| < a^2 f \sum_{\lambda=1}^{n} |e^{Q_\lambda(x)} x^{r_\lambda+(j-1)\left(\frac{p}{k}-1\right)+\varepsilon}| \cdot \int^{x} |e^{-Q_\lambda(x)} x^{-r_\lambda-\omega_1-\beta+\varepsilon}||dx|$$

$(j = 1, \ldots n;\ \varepsilon > 0$, arbitrarily small; x in $R)$[1],

provided

$$\text{(5. 28 a)} \qquad \beta(x) = x^{-\beta} f(x),\ |f(x)| \leq f \qquad (\text{in } R).$$

In this connection it is understood that the integrals in (5. 28) exist along suitable paths; moreover, a may depend on ε.

We shall need the following Lemma.

Lemma 5. 2. *Let $C(x)$ be a polynomial in $x^{\frac{1}{k}}$. Let R be a region extending to infinity. Suppose*

$$\text{(5. 29)} \qquad \frac{\partial}{\partial |x|} \Re(C(x)) \leqq 0 \ (\text{in } R),\ \Re(\alpha) \leqq -1-\sigma\ (\sigma > 0);$$

then

$$\text{(5. 29 a)} \qquad \int_{\infty}^{x} |e^{C(u)} u^{\alpha}||du| \leqq \frac{1}{\sigma} |e^{C(x)} x^{\alpha+1}|$$

for all x in R which are such that the ray

$$\theta = \text{angle of } x,\ r \geqq |x| \qquad (\text{polar coordinates } \theta, r)$$

lies in R, the path of integration in (5. 29 a) *being along this ray. If in place of* (5. 29) *we have*

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[1] Use is made of inequalities $|y_{j-1}(\lambda, x)|$, $|\bar{y}(n, \lambda, x)| < a|x|^{\varepsilon}$, valid in R.

$$(5.\ 30)\qquad \frac{\partial}{\partial |x|}\Re(C(x)) > 0 \text{ (in } R), \ |e^{-C(x)}| \sim 0 \ (\textit{in } R)^{1},$$

then

$$(5.\ 30\text{ a})\qquad \int_c^x |e^{C(u)} u^\alpha|\,|du| < |e^{C(x)} x^{\alpha+1}| (|c| \geqq c(\alpha');\ \alpha' = -\Re(\alpha)),$$

provided $x(|x| \geqq |c|)$ *is on a ray* $\theta =$ *angle of* c, *extending into* R.

It is noted that, if the leading term in $C(x)$ is $g_q x^{\frac{q}{k}}$, then the asymptotic relation of (5. 30) will hold when

$$(5.\ 31)\qquad \cos\left(\bar{g}_q + \frac{q}{k}\theta\right) \geqq \xi > 0 \qquad (\text{in } R;\ \bar{g}_q = \text{angle of } g_q).$$

To establish the first part of the Lemma we write

$$(5.\ 32)\qquad |e^{C(u)} u^\alpha| = |e^{C(u)} u^{\alpha+1+\sigma}|\,|u^{-1-\sigma}| = |u^{-1-\sigma}|\, e^{H(u)},$$

where by (5. 29)

$$(5.\ 32\text{ a})\qquad \frac{\partial}{\partial |u|} H(u) = \frac{\partial}{\partial |u|}\Re(C(u)) + \frac{1}{|u|}\Re(\alpha + 1 + \sigma) \leqq 0 \qquad (\text{in } R).$$

Along the ray in question $H(u)$ is monotone non-increasing, on this ray $\exp. H(u)$ attains its upper bound at x. We have

$$\int_\infty^x |e^{C(u)} u^\alpha|\,|du| \leqq e^{H(x)} \int_\infty^x |u|^{-1-\sigma} |du| \qquad (x \text{ in } R).$$

The second member here is clearly identical with the last member in (5. 29 a).

To demonstrate (5. 30 a) we note that

$$(5.\ 33)\qquad |e^{C(u)} u^\alpha| = \exp. H_1(u),\ H_1(u) = \Re(C(u)) + \Re(\alpha \log u),$$

so that

$$\frac{\partial}{\partial |u|} H_1(u) = \frac{\partial}{\partial |u|}\Re(C(u)) + \frac{1}{|u|}\Re(\alpha).$$

With

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[1] The asymptotic relatiou here is in the sense that $\lim |x|^\gamma |\exp.(-C(x))| = 0$ (as $x \to \infty$ in R; all $\gamma > 0$).

$$C(u) = g_q x^{\frac{q}{k}} + \cdots + g_1 x^{\frac{1}{k}} \qquad (\bar{g}_j = \text{angle of } g_j)$$

and $\Re(\alpha) = -\alpha'$, it is inferred that

$$|u| \frac{\partial}{\partial |u|} H_1(u) = \sum_{i=1}^{q} |g_j| \frac{j}{k} |u|^{\frac{j}{k}} \cos\left(\bar{g}_j + \frac{j}{k}\theta\right) - \alpha'.$$

Hence for all u on the ray $\theta =$ angle of c ($|c| = c(\alpha')$) sufficiently great, with $|u| \geqq |c|$,

$$\frac{\partial}{\partial |u|} H_1(u) \geqq 0.$$

In view of (5. 33) this would imply that the upper bound of $|u^\alpha \exp. C(u)|$, for u on the path of integration in (5. 30 a), is attained at $u = x$. Thus, under the stated conditions

$$\int_c^x |e^{C(u)} u^\alpha| \, |du| \leqq |e^{C(x)} x^\alpha| \int_c^x |du| < |e^{C(u)} x^{\alpha+1}|.$$

The Lemma is accordingly established.

Definition 5. 1. *Let R denote any particular region referred to in the text from (5. 11) to (5. 13 c). We shall designate by R^* any regular subregion of R such that with respect to R^* the following will hold for every particular function*

(5. 34) $$q_\lambda(x) = -\frac{\partial}{\partial |x|} \Re(Q_\lambda(x)).$$

Either

(5. 34 a) $$q_\lambda(x) \leqq 0 \qquad (\textit{in } R^*)$$

or

(5. 34 b) $$q_\lambda(x) > 0 \ (\text{in } R^*), \ |e^{Q_\lambda(x)}| \sim 0 \ (\text{in } (R^*).$$

Given a region R, as specified in the above Definition, subregions R^* could be found as follows. We consider all regular curves extending into R along which at least one of the functions $q_\lambda(x)$ vanishes.[1] Interior each of the several

[1] Such curves are formed only corresponding to the functions $q_\lambda(x)$ which are not identically zero. A *regular* curve satisfying an equation $q_\lambda(x) = 0$ will have at infinity the limiting direction of a corresponding curve satisfying the equation $\Re(Q_\lambda^*(x)) = 0$.

regular subregions of R, into which R is subdivided by these curves, each of the functions

$$(5.35) \qquad q_1(x), \ldots q_n(x)$$

will maintain its sign. Consider any such particular subregion R'. If in R' all the $q_j(x) \leqq 0$, R' is a region R^*. If there are some $q_j(x)$, say

$$(5.36) \qquad q_{j_1}(x),\ q_{j_2}(x), \ldots q_{j_m}(x),$$

which are positive in R', one may take as R^* any subregion of R' within which

$$(5.36\,\mathrm{a}) \qquad e^{Q_{j_1}(x)},\ e^{Q_{j_2}(x)}, \ldots e^{Q_{j_m}(x)} \sim 0;$$

in the case when $R' = R(\theta_1, \theta_2)$ $(\theta_1 < \theta_2)$ conditions (5.36 a) will be satisfied in $R^* = R(\theta_1 + \varepsilon_1, \theta_2 - \varepsilon_1)$ $(\varepsilon_1 > 0$, suitably small).

In any case, at least when $R = R(\alpha_1, \alpha_2)$ $(\alpha_1 \neq \alpha_2)$ existence of subregions R^* of R is certainly assured; moreover, the parts of R which are not of the type R^* (for $|x| \geqq r_0$; r_0 suitably great) can be enclosed in a number of sectors the sum of whose angles can be taken arbitrarily small. Furthermore, these statements can still be made with the subregions R^* so chosen that, if x is a point in R^*, necessarily the ray

$$(5.37) \qquad \theta = \text{angle of } x,\ r \geqq |x| \qquad (\text{polar coordinatas } \theta, r)$$

will lie in R^*.

In the sequel *it will be always implied that a region R^* is so chosen that the statement in connection with* (5.37) *holds.*

Case I. *R^* is a region, as specified in Definition* 5.1, *such that*

$$(5.38) \qquad q_j(x) \leqq 0 \qquad (j = 1, \ldots n;\ in\ R^*;\ cf.\ (5.34)).$$

Case II. *R^* is a region, as described in Definition* 5.1, *such that $q_{j_1}(x)$, $q_{j_2}(x), \ldots q_{j_m}(x)$ are positive in R^*, while* (5.36 a) *holds in R^*.* In this case, as a matter of notation, entailing no loss of generality, one may write

$$(5.39) \qquad q_j(x) > 0\ (j = 1, \ldots m),\ q_j(x) \leqq 0\ (j = m + 1, \ldots n)$$

for x in R^* and

$$(5.39\,\mathrm{a}) \qquad e^{Q_j(x)} \sim 0 \qquad (j = 1, \ldots m;\ \text{in } R^*).$$

It will be sufficient to have (5. 39 a) satisfied to a finite (sufficiently great) number of terms. We then may assert the results of Theorems 7. 1, 8. 1 for some value $t(>0)$, but not necessarily for arbitrarily great values of t.

6. The First Existence Theorem.

Let us consider Case I (§ 5). We shall solve in succession the equations (5. 14). In view of (5. 14 a) and (4. 19), (4. 19 a)

$$\beta_0(x) = x^{-\beta_0} f_0(x),\ \beta_0 = \frac{t+1}{k} - n\left(\frac{p}{k} - 1\right), \tag{6. 1}$$

where

$$|f_0(x)| \leqq f_0 \qquad \text{(in } R^*\text{)}. \tag{6. 1 a}$$

We choose t in the transformation (4. 5) sufficiently great so that

$$\Re(-r_\lambda - \omega_1 - \beta_0 + \varepsilon) \leqq -1 - \sigma \qquad (\sigma > 0;\ \lambda = 1, \ldots n). \tag{6. 2}$$

With the aid of (6. 2) and of Lemma 5. 2 we obtain

$$\int_\infty^x |e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon}|\,|dx| \leqq \frac{1}{\sigma} |e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon + 1}| \tag{6. 3}$$

(in R^*). Whence in consequence of Lemma 5. 1 and of (5. 28)

$$|z_0^{(j-1)}(x)| < \frac{1}{\sigma} a^2 f_0 \sum_{\lambda=1}^{n} \left| e^{Q_\lambda(x)} x^{r_\lambda + (j-1)\left(\frac{p}{k} - 1\right) + \varepsilon} \right| |e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon + 1}|.$$

Thus

$$|z_0^{(j-1)}(x)| < z_0 |x|^{(j-1)\left(\frac{p}{k} - 1\right)} |x^{\alpha_0}| \qquad (j = 1, \ldots n;\ x \text{ in } R^*), \tag{6. 4}$$

where

$$\alpha_0 = 2\varepsilon - \omega_1 - \beta_0 + 1 \quad \text{(cf. (6. 1))}, \quad z_0 = \frac{n}{\sigma} a^2 f_0. \tag{6. 4 a}$$

It is supposed that t is taken sufficiently great so that $\alpha_0 < 1$. This is secured in consequence of (6. 6 c), below. Using the definition of $T(\varrho)$, given in (5. 6), we obtain

$$|z_0^{(n)}(x)| \leqq \sum_{i=0}^{n-1} \left| b_{n-i}(x) x^{(n-i)\left(\frac{p}{k} - 1\right)} \right| |z_0^{(i)}(x)| + |\beta_0(x)|.$$

Inasmuch as (4. 28 a) implies that

$$(6.\ 5) \qquad |b_j(x)| \leqq b \qquad (\text{in } R^*),$$

in consequence of (6. 1), (6. 1 a) and (6. 4) we obtain

$$(6.\ 5\ \text{a}) \qquad |z_0^{(n)}(x)| < n b z_0 \left| x^{n\left(\frac{p}{k}-1\right)+\alpha_0}\right| + f_0 |x|^{-\beta_0} \qquad (\text{in } R^*).$$

Thus

$$(6.\ 6) \qquad |z_0^{(n)}(x)| < c_0 |x|^{n\left(\frac{p}{k}-1\right)} |x|^{\alpha_0} \qquad (x \text{ in } R^*;\ |x| \geqq \varrho_1),$$

with

$$(6.\ 6\ \text{a}) \qquad c_0 = \text{max. of } z_0,\ n b z_0 + f_0 \varrho_1^{-n'};$$

here in view of (5. 23)

$$(6.\ 6\ \text{b}) \qquad n' = \beta_0 + \alpha_0 + n\left(\frac{p}{k} - 1\right) = 2\varepsilon + \frac{1}{k}(p + \omega) > 0.$$

The relation (6. 6 b) is secured in wiew of (5. 23). We take t so that

$$(6.\ 6\ \text{c}) \qquad \frac{t+1}{k} \geqq 2 n'.$$

Combining (6. 4) and (6. 6) it is inferred that

$$(6.\ 7) \qquad |z_0^{(i)}(x)| < c_0 |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0} \qquad (i = 0, \ldots n;\ x \text{ in } R^*),$$

where α_0 and c_0 are defined by (6. 4 a), (6. 6 a).

By (5. 14 b; $j = 1$)

$$\beta_1(x) = \frac{-1}{\lambda_n(x)} x^{n\left(\frac{p}{k}-1\right)} K(z_0(x)).$$

Thus, using (4. 30), (4. 30 b) and (6. 7) we obtain

$$|K_m(z_0(x))| \leqq \sum_{m_0 + \cdots + m_n = m} |k^{m_0, \ldots m_n}(t, x)| \prod_{\alpha=0}^{n} c_0^{m_\alpha} |x|^{\alpha_0 m_\alpha} \qquad (\text{in } R^*)$$

and

$$(6.\ 8) \qquad |k_m^{m_0 \cdots m_n}(t, x)| \leqq \bar{k} \qquad (\text{in } R^*);$$

furthermore

$$|K(z_0(x))| \leqq \sum_{m=2}^{\nu} |K_m(z_0(x))| \leqq \bar{k} \sum_{m=2}^{\nu} c_0^m |x|^{\alpha_0 m} q_m$$

(in R^*), where

(6. 9)
$$q_m = \sum_{m_0 + \cdots + m_n = m} 1;$$

thus, inasmuch as $\alpha_0 < 0$,

(6. 10)
$$|K(z_0(x))| \leqq \bar{k}\, k' \,|x|^{2\alpha_0} \qquad (\text{in } R^*;\ |x| \geqq 1),$$

(6. 10 a)
$$k' = \sum_{m=2}^{\nu} c_0^m q_m \qquad (\text{cf. } (6.\ 9), (6.\ 6\text{ a})).$$

Whence, with

$$\left|\frac{1}{\lambda_n(x)}\right| \leqq \lambda' \qquad (\text{in } R^*)$$

we obtain

(6. 11)
$$\beta_1(x) = x^{-\beta_1} f_1(x) \qquad \left(-\beta_1 = n\left(\frac{p}{k} - 1\right) + 2\alpha_0\right),$$

where

(6. 11 a)
$$|f_1(x)| \leqq f_1 \ (\text{in } R^*),\ f_1 = \bar{k}\, k' \lambda' \qquad (\text{cf. } (6.\ 10\text{ a}), (6.\ 8)).$$

In view of (5. 28), (5. 28 a) a solution $z_1(x)$ of the equation $T(z_1(x)) = \beta_1(x)$ will satisfy the inequalities (5. 28) with $\beta = \beta_1$ and $f = f_1$ (x in R^*). Application of Lemma 5. 2 will yield

(6. 12)
$$|z_1^{(j-1)}(x)| < z_1 |x|^{(j-1)\left(\frac{p}{k}-1\right)} |x|^{\alpha_1} \quad (j = 1, \ldots n;\ x \text{ in } R^*),$$

where

(6. 12 a)
$$\alpha_1 = 2\varepsilon - \omega_1 - \beta_1 + 1 \qquad \left(\text{cf. } (6.\ 11),\ z_1 = \frac{n}{\sigma} a^2 f_1\right).$$

In this connection it is understood that t is so chosen that

(6. 13)
$$\Re(-r_\nu - \omega_1 - \beta_1 + \varepsilon) \leqq -1 - \sigma \qquad (\sigma > 0;\ \lambda = 1, \ldots n),$$

which holds in consequence of the preceding. With the aid of (5. 6) and of (6. 12) we obtain the inequality, analogous to (6. 5 a),

(6. 14)
$$|z_1^{(n)}(x)| < n\, b\, z_1 \left|x^{n\left(\frac{p}{k}-1\right)+\alpha_1}\right| + f_1 |x|^{-\beta_1} \qquad (\text{cf. } (6.\ 11), (6.\ 11\text{ a})),$$

valid in R^*. Whence

(6. 14 a)
$$|z_1^{(n)}(x)| < c_1 |x|^{n\left(\frac{p}{k}-1\right)} |x|^{\alpha_1} \qquad (\text{in } R^*;\ |x| \geqq \varrho_1),$$

with

(6. 14 b)
$$c_1 = \text{max. of } z_1,\ n\, b\, z_1 + f_1 \varrho_1^{-n'};$$

here n' is from (6. 6 b). Together with (6. 14) this yields

(6. 15) $$|z_1^{(i)}(x)| < c_1 |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_1} \qquad (i = 0,\ 1,\ \ldots n;\ \text{in } R^*).$$

In consequence of (6. 4 a), (6. 1) and (5. 23)

(6. 16) $$\alpha_0 = n' - \frac{t+1}{k}, \quad \alpha_1 = n' + 2\,\alpha_0 = 3\,n' - 2\left(\frac{t+1}{k}\right),$$

where $n' (> 0)$ is given by (6. 6 b). Under (6. 6 c)

(6. 16 a) $$\alpha_1 \leqq \alpha_0 \leqq 0.$$

Let A be a number such that

$$c_0 \leqq A, \quad c_1 \leqq A^2.$$

We may replace c_0, c_1 in (6, 7), (6, 15) by A and A^2, respectively.

Suppose that for some $j \geqq 2$ we have

(6. 17) $$z_s^{(i)}(x) = x^{i\left(\frac{\varrho}{k}-1\right)} \zeta_{s,i}(x) \qquad (s = 0,\ 1,\ \ldots j-1;\ i = 0,\ 1,\ \ldots n),$$

(6. 17 a) $$|\zeta_{s,i}(x)| \leqq A^{s+1} |x|^{\alpha_s} \qquad (i = 0,\ 1,\ \ldots n;\ \text{in } R^*;\ |x| \geqq \varrho_1)$$

for $s = 0, \ldots j-1$: while for $s = 0, 1, \ldots j-1$

(6. 17 b) $$\alpha_s = (2\,s + 1)\,n' - (s+1)\left(\frac{t+1}{k}\right) \qquad (n' \text{ from (6. 6 b)}).$$

The statement with respect to (6. 17)—(6. 17 b) has been already established for $j = 2$ (in (6. 7), (6. 15), (6. 16)).

By (4. 30), (4. 30 a)

$$|K(w_{j-1}(x)) - K(w_{j-2}(x))| = |K(z_{j-1}(x) + w_{j-2}(x)) - K(w_{j-2}(x))| \leqq \sum_{m=2}^{\nu} |T_m(x)|,$$

where

$$T_m = K_m(z_{j-1}(x) + w_{j-2}(x)) - K_m(w_{j-2}(x)).$$

We have

$$w_{j-2}^{(\alpha)}(x) = z_0^{(\alpha)}(x) + \cdots + z_{j-2}^{(\alpha)}(x) = x^{\alpha\left(\frac{p}{k}-1\right)} w_{j-2,\alpha}(x),$$

with

(6. 18) $$w_{j-2,\alpha}(x) = \zeta_{0,\alpha}(x) + \cdots + \zeta_{j-2,\alpha}(x).$$

Moreover,

$$(6.\ 19)\quad T_m = \sum_{m_0+\cdots+m_n=m} k_m^{m_0,\ldots m_n}(t,\ x)\left[\prod_{\alpha=0}^{n}(w_{j-2,\alpha}(x)+\zeta_{j-1,\alpha}(x))^{m_\alpha} - \prod_{\alpha=0}^{n}(w_{j-2,\alpha}(x))^{m_\alpha}\right]$$

$$= \sum_{m_0+\cdots+m_n=m} k_m^{m_0,\ldots m_n}(t,\ x)\left[\prod_{\alpha=1}^{m}(w_{j-2,i_\alpha}(x)+\zeta_{j-1,i_\alpha}(x)) - \prod_{\alpha=1}^{m} w_{j-2,i_\alpha}(x)\right]$$

$$(0 \leqq i_1,\ \ldots\ i_m \leqq n).$$

Here sets of subscripts $(i_1,\ i_2,\ \ldots\ i_n)$ depend on the sets $(m_0,\ \ldots\ m_n)$. Now

$$(6.\ 19\,\text{a})\quad \prod_{\alpha=1}^{m}(w_{j-2,i_\alpha}(x)+\zeta_{j-1,i_\alpha}(x)) - \prod_{\alpha=1}^{m} w_{j-2,i_\alpha}(x) = \sum_{\gamma_1=1}^{m}\left(\prod_{s\neq\gamma_1} w_{j-2,i_s}(x)\right)\zeta_{j-1,i_{\gamma_1}}(x)$$

$$+\sum_{\gamma_1<\gamma_2=1}^{m}\left(\prod_{s\neq\gamma_1,\gamma_2} w_{j-2,i_s}(x)\right)\zeta_{j-1,i_{\gamma_1}}(x)\,\zeta_{j-1,i_{\gamma_2}}(x)+\cdots+\zeta_{j-1,i_1}(x)\,\zeta_{j-1,i_2}(x)\ldots\zeta_{j-1,i_m}(x).$$

On the other hand, by (6. 18) and (6. 17 a), (6. 17 b)

$$|w_{j-2,\alpha}(x)| \leqq \sum_{s=0}^{j-2} A^{s+1}|x|^{\alpha_s} = |x|^{\alpha_0} A \sum_{s=0}^{j-2} A^s |x|^{2sn'-s\left(\frac{t+1}{k}\right)}$$

$$< A|x|^{\alpha_0}\sum_{s=0}^{\infty}\left[A|x|^{2n'-\left(\frac{t+1}{k}\right)}\right]^s \qquad (\text{in } R^*).$$

Choosing t sufficiently great so that in R^*

$$(6.\ 19\,\text{b})\qquad A|x|^{2n'-\left(\frac{t+1}{k}\right)} \leqq \frac{1}{2},$$

one accordingly obtains

$$(6.\ 20)\qquad |w_{j-2,\alpha}(x)| < 2A|x|^{\alpha_0} \qquad (\text{in } R^*;\ \alpha = 0,\ 1,\ \ldots\ n).$$

Using (6. 8), (6. 17 a; $s = j-1$), (6. 20) and (6. 19), (6. 19 a), it is inferred that

$$(6.\ 21)\quad |T_m| < \bar{k}\,q_m\,[m|x|^{(m-1)\alpha_0+\alpha_{j-1}}(2A)^{m-1}A^j + \frac{1}{2}m(m-1)|x|^{(m-2)\alpha_0+2\alpha_{j-1}}$$

$$\cdot(2A)^{m-2}A^{2j} + \cdots + A^{mj}|x|^{m\alpha_{j-1}}]$$

$$= \bar{k}\,q_m\{[2A|x|^{\alpha_0} + A^j|x|^{\alpha_{j-1}}]^m - (2A|x|^{\alpha_0})^m\}$$

$$= \bar{k}\,q_m(2A)^m|x|^{\alpha_0 m}[(1+\sigma(j,\ x))^m - 1] \qquad (\text{in } R^*),$$

where q_m is given by (6. 9) and

$$\sigma(j,x) = \frac{1}{2} A^{j-1} |x|^{\alpha_{j-1}-\alpha_0} = \frac{1}{2}\left[A\,|x|^{2n'-\left(\frac{t+1}{k}\right)}\right]^{j-1}. \tag{6. 22}$$

Now, by a mean value theorem

$$(1+u)^m - 1 \leqq m(1+u)^{m-1}u \qquad \text{(for } u > 0\text{)};$$

hence

$$(1+\sigma(j,x))^m - 1 \leqq m(1+\sigma(j,x))^{m-1}\sigma(j,x). \tag{6. 23}$$

In view of (6. 19 b) and (6. 22)

$$\sigma(j,x) \leqq 2^{-j}.$$

Thus, by (6. 23) and (6. 22)

$$(1+\sigma(j,x))^m - 1 \leqq m\frac{1}{2}\left[A\,|x|^{2n'-\left(\frac{t+1}{k}\right)}\right]^{j-1}(1+2^{-j})^{m-1}$$
$$< m\,2^{m-2}\left[A\,|x|^{2n'-\left(\frac{t+1}{k}\right)}\right]^{j-1} \qquad \text{(in } R^*\text{)};$$

whence from (6. 21) we deduce

$$|T_m| < \bar{k}\, q_m\, m\, 2^{m-2}(2A)^m A^{j-1}|x|^{\alpha_0(m-1)+\alpha_{j-1}}. \tag{6. 24}$$

Furthermore, in consequence of the inequality subsequent to (6. 17 b)

$$|K(w_{j-1}(x)) - K(w_{j-2}(x))| < \bar{k}\sum_{m=2}^{\nu} m\, q_m\, 2^{2m-2}|x|^{(m-1)\alpha_0+\alpha_{j-1}}A^{m+j-1}$$
$$= \bar{k}\,|x|^{\alpha_0+\alpha_{j-1}}c' A^{j+1} \qquad \text{(in } R^*\text{)}, \tag{6. 25}$$

with c' denoting a number, independent of x and j, such that

$$\sum_{m=2}^{\nu} m\, q_m\, 2^{2m-2}(A\,|x|^{\alpha_0})^{m-2} \leqq c' \qquad \text{(in } R^*\text{)}. \tag{6. 25 a}$$

By virtue of the inequality $|1/\lambda_n(x)| \leqq \lambda'$ from (6. 25) and (5. 14 b) it is inferred that

$$|\beta_j(x)| < \lambda' c' \bar{k}\,|x|^{\alpha_0+\alpha_{j-1}+n\left(\frac{p}{k}-1\right)} A^{j+1} \qquad \text{(in } R^*\text{)}.$$

One accordingly may write

$$\beta_j(x) = x^{-\beta_j} f_j(x), \quad |f_j(x)| < f_j \qquad \text{(in } R^*\text{)}, \tag{6. 26}$$

where

$$-\beta_j = \alpha_0 + \alpha_{j-1} + n\left(\frac{p}{k}-1\right), \quad f_j = \lambda' c' \bar{k} A^{j+1}. \tag{6. 26 a}$$

By (5. 28), stated in connection with (5. 28 a), in consequence of the relation $T(z_j(x)) = \beta_j(x)$, from (6. 26) it is deduced that

$$|z_j^{(i)}(x)| < a^2 f_j \sum_{\lambda=1}^{n} \left| e^{Q_\lambda(x)} x^{r_\lambda + i\left(\frac{p}{k}-1\right)+\varepsilon} \right| \left| \int_{\infty}^{x} | e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_j + \varepsilon} | \, d x \right|$$

$i = 0, \ldots n-1$; in R^*). Lemma 5. 2 is applicable if t is chosen sufficiently great so that

$$\Re(-r_\lambda - \omega_1 - \beta_j + \varepsilon) \leqq -1 - \sigma \qquad (\sigma > 0;\ \lambda = 1, \ldots n). \tag{6. 27}$$

We then obtain

$$|z_j^{(i)}(x)| < z_j |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_j} \qquad (i = 0, 1, \ldots n-1) \tag{6. 28}$$

where

$$z_j = \frac{n}{\sigma} a^2 f_j, \qquad \alpha_j = 2\varepsilon - \omega_1 + 1 - \beta_j. \tag{6. 28 a}$$

Using the equation $T(z_j(x)) = \beta_j(x)$ and the definition of T given in (5. 6), from (6. 28) and (6. 26) it is inferred that

$$\begin{aligned} |z_j^{(n)}(x)| &\leqq \sum_{i=0}^{n-1} \left| b_{n-i}(x) x^{(n-i)\left(\frac{p}{k}-1\right)} \right| |z_j^{(i)}(x)| + |\beta_j(x)| \\ &< n b z_j |x|^{n\left(\frac{p}{k}-1\right)} |x|^{\alpha_j} + |x|^{-\beta_j} f_j \qquad (\text{in } R^*). \end{aligned} \tag{6. 29}$$

Now

$$\beta_j + n\left(\frac{p}{k} - 1\right) + \alpha_j = n' > 0$$

by (6. 6 b). Hence (6. 29) and (6. 28) imply

$$|z_j^{(i)}(x)| < c_j |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_j} \qquad (i = 0, 1, \ldots n;\ \text{in } R^*), \tag{6. 30}$$

where

$$c_j = \text{max. of } z_j,\ n b z_j + f_j \varrho_1^{-n'}, \tag{6. 30 a}$$

inasmuch as x is in R^* with $|x| \geqq \varrho_1$. Let us examine α_j, as given in (6. 28 a). In consequence of (6. 26 a), (6. 17 b) and (5. 23), as well as in view of the definition of n'

(6. 30 b) $$\alpha_j = (2j + 1)n' - (j + 1)\left(\frac{t + 1}{k}\right).$$

This is what we would obtain from (6. 17 b) *for* $s = j$.

Turning attention to (6. 30 a), in view of (6. 28 a) and (6. 26 a) it is concluded that

(6. 31) $$c_j = a' f_j = a' \lambda' c' \bar{k} A^{j+1},$$

where

(6. 31 a) $$a' = \text{max. of } \frac{n}{\sigma} a^2, \frac{n^2 b}{\sigma} a^2 + \varrho_1^{-n'}.$$

By taking $\varepsilon > 0$ and ϱ_1 suitably great one may secure a (from the inequalities of foot-note p. 35) to be as small as desired. Accordingly, a' of (6. 31 a) can be made so small that $a' \lambda' c' \bar{k} \leqq 1$. We then obtain $c_j \leqq A^{j+1}$, and one may take

(6. 31 b) $$c_j = A^{j+1}.$$

This completes the induction formulated in connection with (6. 17)—(6. 17 b).

Recalling the statement with respect to (5. 5), we conclude that the series

(6. 32) $$\varrho^{(i)}(x) = \sum_{s=0}^{\infty} z_s^{(i)}(x) = \sum_{s=0}^{\infty} x^{i\left(\frac{p}{k}-1\right)} \zeta_{s,i}(x) \qquad (i = 0, 1, \ldots n)$$

are absolutely and uniformly convergent for x in R^* ($|x| \geqq \varrho_1$; ϱ_1 sufficiently great). In fact, the series displayed in the last member of (6. 32) is dominated by

$$S(x) = \sum_{s=0}^{\infty} |x|^{i\left(\frac{p}{k}-1\right)} A^{s+1} |x|^{\alpha_s} = |x|^{i\left(\frac{p}{k}-1\right)} |x|^{n'-\left(\frac{t+1}{k}\right)} \sum_{s=0}^{\infty} \left[A |x|^{2n'-\left(\frac{t+1}{k}\right)}\right]^s$$

(in R^*; $|x| \geqq \varrho_1$); the latter series converges in the indicated region, inasmuch as (6. 19 b) holds. We have

(6. 33) $$|\varrho^{(i)}(x)| \leqq 2A |x|^{i\left(\frac{p}{k}-1\right)} |x|^{n'-\left(\frac{t+1}{k}\right)} \quad (\text{in } R^*; |x| \geqq \varrho_1; \text{ cf. } (6.\ 6\ \text{b}))$$

for $i = 0, 1, \ldots n$. Clearly *the function* $\varrho(x)$, *defined in* R^* *by the above limiting process and satisfying* (6. 33), *constitutes an 'actual' solution of the transformed differential equation* (4. 31) (*cf. Lemma* 4. 1).

Existence Theoreme 6. 1. *Consider the actual differential equation* (4.1). *Let* $s(x)$ (3. 2)—(3. 2 b) *be a formal solution of* (4. 2). *Let* (4. 4) (*with* (4. 4 a)) *be* *the corresponding form for the equation* $F_\nu^* = 0$. *Corresponding to* $s(x)$ *there is a linear differential expression*

$$T(\varrho(x)) \equiv \varrho^{(n)}(x) + b_1(x)\, x^{\frac{p}{k}-1}\, \varrho^{(n-1)}(x) + \cdots + b_n(x)\, x^{n\left(\frac{p}{k}-1\right)}\, \varrho(x)$$

$$[cf.\ (4.\ 28\,a),\ (4.\ 28),\ (4.\ 25),\ (4.\ 21)];$$

it is assumed that the number $l_{n,0}$ *of* (4. 27) *is distinct from zero. We let* R *denote a region of the text from* (5. 11) *to* (5. 13). *Let* R^* *denote a regular subregion of* R *for which* (5. 39; $j = 1, 2, \ldots n$) *holds,* (*cf. formulation of Case I in connection with* (5. 39), *as well as* (5. 34)).

Given an integer t, *however large* ($t \geqq t'$; t' *suitable great*), *there exists a solution* $y(x)$ *of* $F_\nu^* = 0$, *analytic in* R^* *and such that*

$$y^{(i)}(x) \sim s^{(i)}(x) \quad (x \text{ in } R^*;\ \text{to } n(t) \text{ terms};\ i = 0, \ldots n); \tag{6. 34}$$

here $n(t) \to \infty$, *as* $t \to \infty$. *More precisely, we have*

$$y^{(i)}(x) = \frac{d^i}{d x^i}\left[e^{Q(x)} x^r (\sigma(t, x) + \varrho(x))\right] \qquad (i = 0, 1, \ldots n), \tag{6. 35}$$

where

$$\sigma(t, x) = \sigma_0 + \sigma_1 x^{-\frac{1}{k}} + \cdots + \sigma_t x^{-\frac{t}{k}} \tag{6. 35 a}$$

and $\varrho(x)$ *is analytic in* R^* *and satisfies in* R^* *the inequalities* (6. 33).

We observe that the function $y(x)$, involved in the above Theorem may conceivably depend on t. The question whether $y(x)$ does actually depend on t is for the present left open. If $y(x)$ is independent of t, then the asymptotic relations (6. 34) will be in the ordinary sense; that is, to infinitely many terms.

7. The Second Existence Theorem.

We consider now Case II (cf. the end of section 5). Accordingly, in R^*,

$$q_j(x) > 0 \ (j = 1, \ldots m), \quad q_j(x) \leqq 0 \ (j = m + 1, \ldots n), \tag{7. 1}$$

and (5. 40 a) will hold; $q_j(x)$ is defined in (5. 34). All the integrations in this section will be along a portion of a fixed ray Γ in R^*, say

$$\theta = \theta_0. \tag{7. 2}$$

As in section 6 one has

(7. 3) $\beta_0(x) = x^{-\beta_0} f_0(x), \quad |f_0(x)| \leqq f_0 \text{ (on } \Gamma), \quad \beta_0 = \frac{t+1}{k} - n\left(\frac{p}{k} - 1\right).$

We choose t so that

(7. 4) $$\Re(-r_\lambda - \omega_1 - \beta_0 + \varepsilon) \leqq -2 \qquad (\lambda = m+1, \dots n).$$

Lemma 5. 2 may be then applied with $\sigma = 1$, yielding

(7. 5) $$\int_\infty^x |e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon}| \, |dx| \leqq |e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon + 1}|$$

(on Γ; $m < \lambda \leqq n$). In consequence of the second part of Lemma 5. 2

(7. 5 a) $$\int_{c_0}^x |e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon}| \, |dx| < |e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon + 1}|$$

(x on Γ; $|x| \geqq |c_0|$; $|c_0| = c_0(t)$ sufficiently great; $\lambda \leqq m$).

On noting that $T(z_0(x)) = \beta_0(x)$, from (5. 28) we infer

(7. 6) $$|z_0^{(j-1)}(x)| < z_0 |x|^{(j-1)\left(\frac{p}{k} - 1\right)} |x|^{\alpha_0} \qquad (j = 1, \dots n;\ x \text{ on } \Gamma;\ |x| \geqq c_0(t)),$$

with

(7. 6 a) $$\alpha_0 = 2\varepsilon - \omega_1 - \beta_0 + 1 = n' - \left(\frac{t+1}{k}\right), \quad z_0 = n\, a^2 f_0 \qquad \text{(cf. (6. 6 b))}.$$

As before, it is arranged to have $n' > 0$. By methods like those employed from (6. 4 a) to (6. 6 a) we now obtain

(7. 7) $$|z_0^{(n)}(x)| \leqq c_0 |x|^{n\left(\frac{p}{k} - 1\right)} |x|^{\alpha_0} \qquad (x \text{ on } \Gamma;\ |x| \geqq c_0(t)),$$

where

(7. 7 a) $$c_0 = \text{max. of } z_0,\ n\, b\, z_0 + f_0 (c_0(t))^{-n'}.$$

Thus

(7. 8) $$|z_0^{(i)}(x)| \leqq c_0 |x|^{i\left(\frac{p}{k} - 1\right)} |x|^{\alpha_0} \qquad (i = 0, \dots n;\ \text{on } \Gamma;\ |x| \geqq c_0(t)).$$

Now, it is noted that $\beta_1(x)$ is given by a formula subsequent to (6. 7). In consequence of (7. 8) we obtain the analogue of (6. 11), (6. 11 a)

(7. 9) $$\beta_1(x) = x^{-\beta_1} f_1(x), \quad |f_1(x)| \leqq f_1 \qquad (\text{on } \Gamma;\ |x| \geqq c_0(t) \geqq 1).$$

(7. 9 a) $$-\beta_1 = n\left(\frac{p}{k} - 1\right) + 2\,\alpha_0, \quad f_1 = \bar{k}\,k'\,\lambda \qquad (k' \text{ from } (6.\ 10\,\text{a})).$$

It is noted that, in view of (7. 6 a) and (7. 7 a), c_0 *and hence* k' *can be made arbitrarily small*, if we take $\varepsilon > 0$ and $c_0(t)$ suitably great.[1] When solving the equation $T(z_1(x)) = \beta_1(x)$, in view of (7. 9) and (5. 28) it is concluded that

(7. 10) $$\left|z_1^{(j-1)}(x)\right| < a^2 f_1 \sum_{\lambda=1}^{n} \left| e^{Q_\lambda(x)}\, x^{r_\lambda + (j-1)\left(\frac{p}{k} - 1\right) + \varepsilon} \right|$$
$$\cdot \int_l^x \left| e^{-Q_\lambda(x)}\, x^{-r_\lambda - \omega_1 - \beta_1 + \varepsilon} \right| \left| d\,x \right| \qquad (j = 1, \ldots n;\ x \text{ on } \Gamma;\ |x| \geqq c_0(t));$$

here $l = c_0$ for $1 \leqq \lambda \leqq m$ and $l = \infty$ for $m < \lambda \leqq n$. By Lemma 5. 2 (with $\sigma = 1$)

$$\int_\infty^x \left| e^{-Q_\lambda(x)}\, x^{-r_\lambda - \omega_1 - \beta_1 + \varepsilon} \right| \left| d\,x \right| \leqq \left| e^{-Q_\lambda(x)}\, x^{-r_\lambda - \omega_1 - \beta_1 + \varepsilon + 1} \right|$$

(on Γ; $|x| \geqq c_0(t)$; $m < \lambda \leqq n$). Thus

(7. 11) $$\int_\infty^x \left| e^{-Q_\lambda(x)}\, x^{-r_\lambda - \omega_1 - \beta_1 + \varepsilon} \right| \left| d\,x \right| \leq \gamma_1 \left| e^{-Q_\lambda(x)}\, x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon + 1} \right|$$
$$(x \text{ on } \Gamma;\ |x| \geqq c_0(t);\ m < \lambda \leqq n), \qquad \gamma_1 = (c_0(t))^{2n' - \left(\frac{t+1}{k}\right)}$$

inasmuch as $\beta_0 - \beta_1 = 2\,n' - (t+1)/k$. As before, we choose t so that $\beta_0 - \beta_1 < 0$. On the other hand, for $1 \leqq \lambda \leqq m$

$$\left| e^{-Q_\lambda(x)}\, x^{-r_\lambda - \omega_1 - \beta_1 + \varepsilon} \right| \leqq \gamma_1 \left| e^{-Q_\lambda(x)}\, x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon} \right|$$

for x on Γ ($|x| \geqq c_0(t)$). Thus, by (7. 5 a)

(7. 11 a) $$\int_{c_0}^x \left| e^{-Q_\lambda(x)}\, x^{-r_\lambda - \omega_1 - \beta_1 + \varepsilon} \right| \left| d\,x \right| < \gamma_1 \left| e^{-Q_\lambda(x)}\, x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon + 1} \right|$$
$$(\text{on } \Gamma;\ |x| \geqq c_0(t);\ \lambda = 1, \ldots m).$$

By virtue of (7. 10), (7. 11), (7. 11 a) it is inferred that

[1] One may arrange to have a as small as desired.

$$(7.\ 12)\qquad |z_1^{(j-1)}(x)| < \gamma_1 z_1 |x|^{(j-1)\left(\frac{p}{k}-1\right)} |x|^{\alpha_0}$$

$$(j = 1, \ldots n;\ x \text{ on } \Gamma;\ |x| \geqq c_0(t);\ z_1 = n a^2 f_1).$$

In consequence of (7. 12) and of the inequality obtained from the relation $T(z_1(x)) = \beta_1(x)$ and of (7. 9), one observes that

$$|z_1^{(n)}(x)| < n b \gamma_1 z_1 |x|^{n\left(\frac{p}{k}-1\right)+\alpha_0} + f_1 |x|^{n\left(\frac{p}{k}-1\right)+2\alpha_0}.$$

Thus

$$(7.\ 13)\qquad |z_1^{(i)}(x)| \leqq c_1 |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0} \qquad (x \text{ on } \Gamma;\ |x| \geqq c_0(t))$$

for $i = 0, \ldots n$, where

$$(7.\ 13\text{ a})\qquad c_1 = \text{max. of } \gamma_1 z_1,\ n b \gamma_1 z_1 + f_1 (c_0(t))^{n' - \left(\frac{t+1}{k}\right)} \quad (\text{cf. } (7.\ 12), (7.\ 9\text{ a}), (7.\ 11)).$$

For a suitable choice of $c_0(t)$ we have both c_0 and c_1 sufficiently small so that

$$(7.\ 14)\qquad |c_0| \leqq A,\ |c_1| \leqq A^2, \qquad 0 < A \leqq \frac{1}{2}.$$

Suppose now that for some $j \geqq 2$ *we have*

$$(7.\ 15)\qquad z_s^{(i)}(x) = x^{i\left(\frac{p}{k}-1\right)} \zeta_{s,i}(x) \qquad (s = 0, 1, \ldots j-1;\ i = 0, \ldots n),$$

$$(7.\ 15\text{ a})\qquad |\zeta_{s,i}(x)| \leqq A^{s+1} |x|^{\alpha_1} \qquad (i = 0, \ldots n;\ \textit{on } \Gamma;\ |x| \geqq c_0(t)$$

where α_0 *is from* (7. 6 a).

The relations (7. 15), (7. 15 a) have been established for $j = 2$ in (7. 8), (7. 13), (7. 14).

In view of (4. 30) and (4. 30 a)

$$|K(w_{j-1}(x)) - K(w_{j-2}(x))| \leqq \sum_{m=2}^{\nu} |T_m(x)|,$$

where

$$T_m(x) = K_m(z_{j-1}(x) + w_{j-2}(x)) - K_m(w_{j-2}(x)).$$

As before, we write

$$(7.\ 16)\qquad w_{j-2}^{(\alpha)}(x) = z_0^{(\alpha)}(x) + \cdots + z_{j-2}^{(\alpha)}(x) = x^{\alpha\left(\frac{p}{k}-1\right)} w_{j-2,\alpha}(x),$$

$$w_{j-2,\alpha}(x) = \zeta_{0,\alpha}(x) + \cdots + \zeta_{j-2,\alpha}(x).$$

In consequence of (7. 15 a) we now have

$$(7.\ 16\ \mathrm{a})\qquad |w_{j-2,\alpha}(x)| < |x|^{\alpha_0}(A + A^2 + \cdots) \leqq 2A|x|^{\alpha_0} \qquad (\alpha = 0, \ldots n)$$

for x on Γ ($|x| \geqq c_0(t)$).

By (6. 19), (6. 19 a) and (7. 16 a)

$$|T_m(x)| < \bar{k} \sum_{m_0+\cdots+m_n=m} \sum_{\gamma_1=1}^{m} |\zeta_{j-1, i_{\gamma_1}}(x)| (2A|x|^{\alpha_0})^{m-1}$$

$$+ \sum_{\gamma_1<\gamma_2=1}^{m} |\zeta_{j-1, i_{\gamma_1}}(x) \zeta_{i-1, i_{\gamma_2}}(x)| (2A|x|^{\alpha_0})^{m-2} + \cdots + |\zeta_{j-1, i_1}(x) \ldots \zeta_{j-1, i_m}(x)|.$$

Hence by virtue of (7. 15 a) (for $s = j - 1$)

$$|T_m(x)| < \bar{k}\, q_m [(A^j|x|^{\alpha_0} + 2A|x|^{\alpha_0})^m - (2A|x|^{\alpha_0})^m]$$

(on Γ; $|x| \geqq c_0(t)$; cf. (6. 9)). Whence by (7. 14)

$$|T_m(x)| < \bar{k}\, q_m (2A|x|^{\alpha_0})^m \left[\left(1 + \frac{1}{2}A^{j-1}\right)^m - 1\right]$$

$$(7.17)\quad < \bar{k}\, m\, q_m (2A|x|^{\alpha_0})^m \left(1 + \frac{1}{2}A^{j-1}\right)^{m-1} \frac{1}{2}A^{j-1} \leqq \left(\frac{5}{4}\right)^{m-1} \frac{1}{2}\bar{k}\, m\, q_m (2A|x|^{\alpha_0})^m A^{j-1}$$

and

$$(7.\ 18)\qquad |K(w_{j-1}(x)) - K(w_{j-2}(x))| < \bar{k}(2A|x|^{\alpha_0})^2 \bar{c} = 2\bar{k}\bar{c}A^{j+1}|x|^{2\alpha_0},$$

where $\bar{c}$ is a number independent of j and x, such that

$$(7.\ 18\ \mathrm{a})\qquad \sum_{m=2}^{\nu} \left(\frac{5}{4}\right)^{m-1} m\, q_m (2A|x|^{\alpha_0})^{m-2} \leqq \bar{c}$$

for x on Γ ($|x| \geqq c_0(t)$). Consequently, in view of the inequality $|1/\lambda_n(x)| \leqq \lambda'$, from (7. 18) and (5. 14 b) it is inferred that

$$|\beta_j(x)| < 2\lambda'\bar{k}\bar{c}|x|^{2\cdot\alpha_0 + n\left(\frac{p}{k}-1\right)} A^{j+1}.$$

Thus

$$(7.\ 19)\qquad \beta_j(x) = x^{-\beta_j} f_j(x),\ |f_j(x)| < f_j \qquad (\text{on } \Gamma),$$

$$(7.\ 19\ \mathrm{a})\qquad -\beta_j = 2\alpha_0 + n\left(\frac{p}{k} - 1\right) = -\beta_1,\ f_j = 2\lambda'\bar{k}\bar{c}A^{j+1} \qquad (\text{cf. } 7.\ 9\ \mathrm{a}).$$

In consequence of (5. 28) and in view of the relation $T(z_j(x)) = \beta_j(x)$

$$(7.\ 20)\quad |z_j^{(i)}(x)| < a^2 f_j \sum_{\lambda=1}^{n} |e^{Q_\lambda(x)} x^{r_\lambda + i\left(\frac{p}{k}-1\right)+\varepsilon}| \int_l^x |e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_j + \varepsilon}|\,|dx|$$

($i = 0, \ldots n-1$; on Γ; l as in (7. 10)). Inasmuch as, by (7. 19 a), $\beta_j = \beta_1$ it is concluded that the integrals displayed in (7. 20) are identical with those in (7. 10). Recalling (7. 11), (7. 11 a) one obtains

$$(7.\ 21)\qquad |z_j^{(i)}(x)| < \gamma_1 z_j |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0}$$
$$(i = 0, \ldots n-1;\ x \text{ on } \Gamma;\ |x| \geqq c_0(t);\ z_j = n a^2 f_j),$$

where γ_1 is from (7. 11) and f_j is from (7. 19 a). With the aid of (7. 21) and of the inequality

$$|z_j^{(n)}(x)| \leqq \sum_{i=0}^{n-1} |b_{n-i}(x) x^{(n-i)\left(\frac{p}{k}-1\right)}|\,|z_j^{(i)}(x)| + |\beta_j(x)|,$$

in view of (7. 19), (7, 19 a) it is deduced that

$$|z_j^{(n)}(x)| < n\, b\, \gamma_1 z_j |x|^{n\left(\frac{p}{k}-1\right)} |x|^{\alpha_0} + f_j |x|^{-\beta_1}.$$

Thus by (7. 9 a) and (7. 21)

$$(7.\ 22)\qquad |z_j^{(i)}(x)| \leqq c_j |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0} \qquad (x \text{ on } \Gamma;\ |x| \geqq c_0(t))$$

for $i = 0, 1, \ldots n$, where

$$c_j = \text{max. of } \gamma_1 z_j,\ n\, b\, \gamma_1 z_j + f_j (c_0(t))^{n' - \left(\frac{t+1}{k}\right)}$$

(compare with (7. 13 a)). In consequence of (7. 19 a), (7. 21), (7. 19 a)

$$(7.\ 22\text{ a})\qquad c_j = m' \lambda'' A^{j+1} \qquad (\lambda'' = 2\lambda' \bar{k} \bar{c};\ \text{cf. } (7.\ 18\text{ a})),$$

$$(7.\ 22\text{ b})\qquad m' = \text{max. of } \gamma_1 n a^2,\ n^2 b a^2 \gamma_1 + (c_0(t))^{n' - \left(\frac{t+1}{k}\right)}.$$

Inasmuch as γ_1 is given by (7. 11) and $n' - \left(\frac{t+1}{k}\right) < 0$, it is observed that m' of (7. 22 b) can be made arbitrarily small by choosing $c_1(t)$ suitably great. On the other hand, λ'' in (7. 22 a) does not increase indefinitely with $c_0(t)$. Thus, if we take $c_0(t)$ sufficiently great (but independent of j) so that

$$m' \lambda'' \leqq 1,$$

from (7. 22 a) we obtain

$$c_j \leqq A^{j+1}. \tag{7. 23}$$

In conjunction with (7. 22) the inequality (7. 23) implies that (7. 15), (7. 15 a) holds for $s = j$. Therefore *by induction it has been established that*

$$z_s^{(i)}(x) = x^{i\left(\frac{p}{k}-1\right)} \zeta_{s,i}(x) \quad (s = 0, 1, \ldots; i = 0, \ldots n), \tag{7. 24}$$

$$|\zeta_{s,i}(x) \leqq A^{s+1} |x|^{c_0} \quad (i = 0, \ldots n; \text{ on } \Gamma; |x| \geqq c_0(t)), \tag{7. 24 a}$$

provided $c_0(t)$ is taken sufficiently great.

The series

$$\varrho^{(i)}(x) = \sum_{s=0}^{\infty} x^{i\left(\frac{p}{k}-1\right)} \zeta_{s,i}(x) \qquad (i = 0, 1, \ldots n) \tag{7. 25}$$

converge absolutely and uniformly for x on Γ ($|x| \geqq c_0(t)$); moreover, in view of (7. 24 a)

$$|\varrho^{(i)}(x) \leqq 2A|x|^{i\left(\frac{p}{k}-1\right)} |x|^{n'-\left(\frac{t+1}{k}\right)} \qquad (\text{on } \Gamma; |x| \geqq c_0(t)) \tag{7. 26}$$

for $i = 0, 1, \ldots n$. *The function $\varrho(x)$ will be an 'actual' solution of the transformed equation referred to in Lemma* 4. 1.

Existence Theorem 7. 1. *Let $F_\nu^* = 0$ be an 'actual' differential equation, as given in* (4. 1). *Let $s(x)$* ((3. 2)—(3. 2 b)) *be a formal solution of* (4. 2). *We recall the fact that corresponding to $s(x)$ there is a linear differential expression $T(\varrho(x))$* (*cf.* (4. 28 a), (4. 28), (4. 25), (4. 21)). *We assume that $l_{n,0}$ of* (4. 27) $\neq 0$. *With R designating a region of the text in connection with* (5. 11)—(5. 13), *let R^* denote a subregion of R, as specified in Definition* 5. 1. *Thus, with suitable notation one may assert* (5. 40), (5. 40 a), *where* $q_j(x) = -\dfrac{\partial}{\partial |x|} \Re(Q_j(x))$.

Given an integer t ($t \geqq t'$; t' suitably great), however large, and given a fixed ray Γ, $\theta = \theta_0$, in R^*[1] *there exists a solution $y(x)$ of $F_\nu^* = 0$ analytic on Γ* ($|x| \geqq c_0(t)$; *$c_0(t)$ sufficiently great*) *and such that*

[1] Extending to infinity in R^*.

$$(7.\ 27) \qquad y^{(i)}(x) \sim s^{(i)}(x) \quad (x \text{ on } \Gamma;\ \text{to } n(t) \text{ terms};\ i = 0, \ldots n);$$

here $n(t) \to \infty$, *as* $t \to \infty$. *In detail, one has*

$$(7.\ 28) \qquad y^{(i)}(x) = \frac{d^i}{d x^i}\left[e^{Q(x)} x^r (\sigma(t, x) + \varrho(x))\right] \qquad (i = 0, \ldots n),$$

where $\sigma(t, x)$ *is given by* (6. 35 a) *and* $\varrho(x)$ *is analytic on* Γ (*for* $|x| \geqq c_0(t)$) *and satisfies on* Γ *the inequalities* (7. 26).

8. The Third Existence Theorem.

With

$$(8.\ 1) \qquad q_j(x) = -\frac{\partial}{\partial |x|} \Re\,(Q_j(x)) \qquad (j = 1, \ldots n),$$

where the $Q_j(x)$ are the polynomials involved in the text from (5. 6) to (5. 7 b), Theorem 6. 1 was concerned with existence results for $F_\nu^* = 0$ (4. 1) for x in a regular region R^*, in which $q_j(x) \leqq 0$ $(j = 1, \ldots n)$.

In Theorem 7. 1 we succeeded in obtaining existence results for $F_\nu^* = 0$ when x is *merely* on a ray Γ in a regular region R, in which some of the $q_j(x)$ are non-positive and others are positive; thus, $q_j(x) > 0$ $(j = 1, \ldots m)$, $q_j(x) \leqq 0$ $(j = m + 1, \ldots n)$, exp. $Q_j(x) \sim 0$ $(j - 1, \ldots m)$ in R^*.

We are now concerned with the possibility of proving existence of solutions of $F_\nu^* = 0$, under the same circumstances as in Theorem 7. 1, but for x in regular region R', in place of a ray Γ. We proceed to construct suitable regions R'. First, *let* R^* *denote a regular subregion of* R (R *from the text in conjunction with* (5. 7)—(5. 13 c)) *such that the* $q_j(x)$ *of* (8. 1) *do not change signs in* R^*; *as a matter of notation one then may write*

$$(8.\ 2) \qquad \begin{aligned} q_j(x) &> 0 \qquad (j = 1, \ldots m;\ \text{in } R^*), \\ q_j(x) &\leqq 0 \qquad (j = m + 1, \ldots n;\ \text{in } R^*). \end{aligned}$$

Take R^* *so that* exp. $Q_j(x) \sim 0$ $(j = 1, \ldots. m;$ *in* $R^*)$. *We let* R' *denote any regular subregion of* R^*, *such that interior* R' *there extend no regular curves*[1] *defined by the equations*

---

[1] If $Q_j(x) = q_{j,0}\, x^{\frac{\sigma}{k}} + \cdots + q_{j,\,\sigma-1}\, x^{\frac{1}{k}}$ $(q_{j,0} = |q_{j,0}|$ exp. $(\sqrt{-1}\,\bar{q}_{j,0}) \neq 0)$, then the regular curves $q_j(x) = 0$ (j fixed) will possess at infinity the limiting directions satisfying the equation $\cos\left(\bar{q}_{j,0} + \frac{\sigma}{k}\theta\right) = 0$; on the other hand, the regular curves $q(j, x) = 0$ (j fixed) will have at infinity directions θ for which $\sin\left(\bar{q}_{j,0} + \frac{\sigma}{k}\theta\right) = 0$.

$$q(j, x) \equiv -\frac{\partial}{\partial \theta} \Re(Q_j(x)) = 0 \qquad (j = 1, \ldots m;\ \theta = \text{angle of } x); \tag{8. 3}$$

moreover, R' is to be such that, if x represents a point in R', the ray $\theta =$ angle of x, $r \geqq |x|$ (θ, r polar coordinates), is in R',

With respect to the behaviour of the $Q_j(x)$ in R' we note the following. If $C(x) = - Q_j(x)$ $(m < j \leqq n)$, then by Lemma 5. 2

$$\int_{\infty}^{x} |e^{C(u)} u^{\alpha}| \, |du| \leqq |e^{C(x)} x^{\alpha+1}| \qquad (x \text{ in } R'), \tag{8. 4}$$

provided $\Re(\alpha) \leqq -2$ and the path of integration is along the ray $\theta =$ angle of x. If $C(x) = - Q_j(x)$ $(1 \leqq j \leqq m)$ one has

$$\frac{\partial}{\partial |x|} \Re(C(x)) = q_j(x) > 0, \ e^{-C(x)} \sim 0 \qquad (\text{in } R'); \tag{8. 5}$$

hence by Lemma 5. 2 we again have

$$\int_{c}^{x} |e^{C(u)} u^{\alpha}| \, |du| < |e^{C(x)} x^{\alpha+1}| \qquad (|c| \geqq c(\alpha');\ \alpha' = -\Re(\alpha)) \tag{8. 5 a}$$

for x $(|x| \geqq |c|)$ on the ray $\theta =$ angle of c. The function $C(x) = - Q_j(x)$ $(1 \leqq j \leqq m)$ is such that

$$\frac{\partial}{\partial \theta} \Re(C(x)) \equiv q(j, x) \tag{8. 6}$$

does not change sign in R'. Let the two regular curves (without common points) which form part of the boundary of R' be designated as T_1 and T_2. In view of the statement with reference to (8. 6), there exists a curve $T_{\nu(j)}$ $(\nu(j) = 1$ or $2)$ such that, when $\gamma(j)$ is a point on $T_{\nu(j)}$, $|\exp. C(x)|$ is monotone non-decreasing as x varies in R' from $\gamma(j)$ along an arc of the circle $r = |\gamma|$. With integration along an arc $r = |\gamma(j)|$ and c (in R') on this arc, we shall have

$$\int_{\gamma(j)}^{c} |e^{C(u)} u^{\alpha}| \, |du| \leqq B(\alpha'') \int_{\gamma(j)}^{c} |e^{C(u)} u^{-\alpha'}| \, |du|, \tag{8. 7}$$

where $\alpha = -\alpha' + \sqrt{-1}\,\alpha''$ and

(8. 7 a) $$B(\alpha'') = \text{upper bound in } R' \text{ of } e^{-\alpha''\theta};$$

moreover,

(8. 8) $$\int_{\gamma(j)}^{c} |e^{C(u)} u^{\alpha}|\,|du| \leqq B(\alpha'')|e^{C(c)} c^{-\alpha'}| \left|\int_{\gamma(j)}^{c} |du|\right| \leqq B' B(\alpha'')|e^{C(c)} c^{-\alpha'+1}|,$$

where B' is the upper bound of $|\theta_1 - \theta_2|$ (θ_1 = angle of x_1, θ_2 = angle of x_2) for all pairs of points x_1, x_2 in R'. With $j \leqq m$ and $C(x) = -Q_j(x)$ it will be understood that

(8. 9) $$\int_{\gamma(j)}^{x} e^{C(u)} u^{\alpha} du = \int_{\gamma(j)}^{c} e^{C(u)} u^{\alpha} du + \int_{c}^{x} e^{C(u)} u^{\alpha} du$$

(angle of c = angle of x; $|c| = |\gamma(j)|$; $|x| \geqq |\gamma(j)|$), where the integration from $\gamma(j)$ to c is within R' along an arc of the circle $r = |\gamma(j)|$ and the integration from c to x is along a rectilinear segment. By (8. 9), (8. 8) and (8. 5 a)

(8. 10) $$\left|\int_{\gamma(j)}^{x} e^{C(u)} u^{\alpha} du\right| < B' B(\alpha'')|e^{C(c)} c^{-\alpha'+1}| + |e^{C(x)} x^{\alpha+1}|$$

x in R'; $\alpha' = -\Re\,\alpha$), provided $|\gamma(j)|$ is selected sufficiently great. Inasmuch as $\Re\,(C(u))$ is monotone increasing along (c, x), from c to x, and $|c(x)| \leqq 1$, from (8. 10) we obtain

$$\left|\int_{\gamma(j)}^{x} e^{C(u)} u^{\alpha} du\right| < B' B(\alpha'')|e^{C(c)} c^{-\alpha'+1}| + B(\alpha'')|e^{C(x)} x^{-\alpha'+1}|$$
$$\leqq B(\alpha'')|e^{C(x)} x^{-\alpha'+1}|(1 + B') \qquad (x \text{ in } R';\ \alpha' = -\Re\,\alpha).$$

Thus, with $C(x) = -Q_j(x)$ $(j \leqq m)$,

(8. 11) $$\left|\int_{\gamma(j)}^{x} e^{C(u)} u^{\alpha} du\right| < D_j(\alpha'')|e^{C(x)} x^{\alpha+1}| \qquad (x \text{ in } R')$$

where

(8. 11 a) $$D_j(\alpha'') = \text{upper bound in } R' \text{ of } e^{\alpha''\theta} B(\alpha'')(1 + B')$$
$$[\theta = \text{angle of } x;\ x \text{ in } R';\ |x| \geqq \gamma(j);\ B(\alpha'') \text{ from (8. 7 a)}].$$

In consequence of (8. 4) and (8. 11) we have the following result.

Lemma 8. 1. *Consider the italicised statement in connection with* (8. 2), (8. 3). *We shall have*

$$\int_{\infty}^{x} |e^{-Q_j(u)} u^{\alpha}|\,|du| \leqq |e^{-Q_j(x)} x^{\alpha+1}| \tag{8. 12}$$

($j = m+1, \ldots n$; *x in R'*), *provided* $\Re(\alpha) \leqq -2$ *and the path of integration is along the ray* $\theta =$ *angle of* x. *Also*

$$\int_{\gamma(j)}^{x} |e^{-Q_j(u)} u^{\alpha}|\,|du| < D_j(\alpha'')\,|e^{-Q_j(x)} x^{\alpha+1}| \tag{8. 13}$$

($j = 1, \ldots m$; *x in R'*; $|x| \geqq \gamma(j)$; *$\gamma(j)$ sufficiently great*; $\alpha'' =$ *imaginary part of $D(\alpha'')$ from* (8. 11 a)). *In* (8. 13) *$\gamma(j)$ is a point as specified subsequent to* (8. 6) *and the path of integration is as described with respect to* (8. 9).

Note. In (8. 13) one may replace $D(\alpha'')$ by

$$D = \text{max. of } D_j(\alpha'') \text{ and } 1 \qquad (j = 1, \ldots m). \tag{8. 13 a}$$

As before, we arrange to have n' (cf. (6. 6 b)) > 0. We have (7. 3) in R' and t is chosen so that (7. 4) holds. On using (5. 28), from the equation $T(z_0(x)) = \beta_0(x)$ it is inferred that

$$|z_0^{(i)}(x)| < a^2 f_0 \sum_{\lambda=1}^{n} |e^{Q_\lambda(x)} x^{r_\lambda + i\left(\frac{p}{k}-1\right)+\varepsilon}|\,\left|\int^{x} |e^{-Q_\lambda(x)} x^{-r_\lambda - \omega_1 - \beta_0 + \varepsilon}|\,|dx|\right.$$

($i = 0, \ldots n-1$; in R'). *Integrations here and in the remainder of this section are along paths indicated in Lemma* 8. 1. Thus

$$|z_0^{(i)}(x)| \leqq D z_0 |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0} \quad (i = 0, \ldots n-1; \text{ in } R'), \tag{8. 14}$$

where z_0 and α_0 are from (7. 6 a) and $|x| \geqq \gamma_0(t)$ ($\gamma_0(t)$ sufficiently great). In consequence of the inequality subsequent to (6. 4 a) and of (6. 5) with the aid of (8. 14) it is deduced that

$$|z_0^{(i)}(x)| \leqq \gamma_0 |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0} \quad (i = 0, \ldots n; \text{ in } R'; |x| \geqq \gamma_0(t)), \tag{8. 15}$$

$$\gamma_0 = \text{max. of } D z_0,\ n b D z_0 + f_0(\gamma_0(t))^{-n'}. \tag{8. 15 a}$$

Using (8. 15) and repeating the argument from (6. 7) to (6. 10), it is concluded that

$$|\beta_1(x)| \leqq \lambda' \bar{k} |x|^{n\left(\frac{p}{k}-1\right)} \sum_{m=2}^{\nu} \gamma_0^m |x|^{\alpha_0 m} q_m.$$

Thus

$$\beta_1(x) = x^{-\beta_1} f_1(x), \quad |f_1(x)| \leqq f_1, \tag{8. 16}$$

$$-\beta_1 = n\left(\frac{p}{k} - 1\right) + 2\alpha_0, \quad f_1 = \lambda' \bar{k} \sum_{m=2}^{\nu} \gamma_0^m (\gamma_0(t))^{(m-2)\alpha_0} q_m. \tag{8. 16 a}$$

Inasmuch as, by (7. 6 a), $z_0 = n a^2 f_0$ and one may arrange to have a arbitrarily small, with $\gamma_0(t)$ sufficiently great, it is inferred that γ_0 *of* (8. 15 a) *can be made as small as desired; the same will be true of* f_1 *of* (8. 16 a).

Since $\beta_0 - \beta_1 = 2n' - (t+1)/k$ and $|x| \geqq \gamma_0(t)$, from (8. 16) it is deduced that

$$|\beta_1(x)| \leqq |x|^{-\beta_0} f_1 \gamma'', \quad \gamma'' = (\gamma_0(t))^{2n' - \left(\frac{t+1}{k}\right)} \tag{8. 17}$$

In consequence of the relation $T(z(x)) = \beta_1(x)$, of (8. 17) and of (5. 28) we obtain inequalities like those preceding (8. 14), with $z_0(x)$, f_0 replaced by $z_1(x)$ and $f_1\gamma''$, respectively. Accordingly, by virtue of Lemma 8. 1 it is observed that

$$|z_1^{(i)}(x)| \leqq D\gamma'' z_1 |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0}, \quad z_1 = n a^2 f_1 \tag{8. 18}$$

($i = 0, \ldots n-1$; in R'). As before, t is taken so that $2n' - (t+1)/k < 0$; accordingly, γ'' *can be made as small as desired by suitable choice of* $\gamma_0(t)$. With the aid of (8. 18) and (8. 17) it is concluded that

$$|z_1^{(n)}(x)| \leqq \sum_{i=0}^{n-1} |b_{n-i}(x) x^{(n-i)\left(\frac{p}{k}-1\right)}| |z_1^{(i)}(x)| + |\beta_1(x)|$$

$$\leqq |x|^{n\left(\frac{p}{k}-1\right)+\alpha_0} \{n b D \gamma'' z_1 + f_1 \gamma'' |x|^{-n'}\}.$$

Thus

$$|z_1^{(i)}(x)| \leqq \gamma_1 |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0} \qquad (i = 0, \ldots n; \text{ in } R'; \ |x| \geqq \gamma_0(t)), \tag{8. 19}$$

$$\gamma_1 = \text{max. of } D\gamma'' z_1, \ n b D \gamma'' z_1 + f_1 \gamma'' (\gamma_0(t))^{-n'}. \tag{8. 19 a}$$

Since γ'' of (8. 17) may be made as small as desired by suitable choice of $\gamma_0(t)$ and since γ_0 can be made as small as needed, we shall arrange to have

$$\gamma_0 \leqq A, \quad \gamma_1 \leqq A^2, \quad 0 < A \leqq \frac{1}{2}. \tag{8. 20}$$

Suppose that for some $j \geqq 2$

$$z_s^{(i)}(x) = x^{i\left(\frac{p}{k}-1\right)} \zeta_{s,i}(x) \qquad (s = 0, \ldots j-1;\ i = 0, \ldots n), \tag{8. 21}$$

$$|\zeta_{s,i}(x)| \leqq A^{s+1} |x|^{\alpha_0} \qquad (i = 0, \ldots n;\ \textit{in } R';\ |x| \geqq \gamma_0(t)). \tag{8. 21 a}$$

In consequence of (8. 21), (8. 21 a) we obtain (7. 18), (7. 18 a), valid in R'. Hence from (5. 14 b) we infer

$$|\beta_j(x)| < 2\lambda' \bar{k} \bar{c} |x|^{2\alpha_0 + n\left(\frac{p}{k}-1\right)} A^{j+1} \qquad (\text{in } R'), \tag{8. 22}$$

where $\bar{c}$ is from (7. 18 a) (with x in R'; $|x| \geqq \gamma_0(t)$). Whence

$$\beta_j(x) = x^{-\beta_1} f_j(x), \quad |f_j(x)| < f_j = 2\lambda' \bar{k} \bar{c} A^{j+1} \tag{8. 22 a}$$

and

$$|\beta_j(x)| < |x|^{-\beta_0} f_j \gamma'' \qquad (\text{cf. (8. 22 a), (8. 17); in } R'). \tag{8. 23}$$

By (5. 28), as applied to $T(z_j(x)) = \beta_j(x)$, by (8. 23) and view of Lemma (8. 1) it is deduced that

$$|z_j^{(i)}(x)| < D\gamma'' z_j |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0}, \qquad z_j = n a^2 f_j \tag{8. 24}$$

(cf. (8. 22 a); $i = 0, \ldots n-1$; in R'). In place of (8. 19), (8. 19 a) one now has

$$|z_j^{(i)}(x)| < \gamma_j |x|^{i\left(\frac{p}{k}-1\right)} |x|^{\alpha_0} \qquad (i = 0, \ldots n;\ \text{in } R';\ |x| \geqq \gamma_0(t)), \tag{8. 25}$$

where

$$\gamma_j = \text{max. of } D\gamma'' z_j, \quad n b D \gamma'' z_j + f_j \gamma'' (\gamma_0(t))^{-n'} \tag{8. 25 a}$$

(cf. (8. 24), (8. 17), (8. 22 a)). On taking account of (8. 24) and of (8. 22 a), it is seen that

$$\gamma_j = \mu f_j \gamma'' < 2\lambda' \bar{k} \bar{c} \mu \gamma'' A^{j+1} \qquad (\bar{c} \text{ from (7. 18 a); } x \text{ in } R';\ |x| \geqq \gamma_0(t))$$

where

$$\mu = \text{max. of } D n a^2, \quad n^2 b a^2 D + (\gamma_0(t))^{-n'}$$

In consequence of the definition of γ'', given in (8. 17), *one may choose* $\gamma_0(t)$ (independent of j) *so great that* $2\lambda' \bar{k} \bar{c} \mu \gamma'' \leqq 1$. We then have

$$\gamma_j < A^{j+1}. \tag{8. 26}$$

The inequalities (8. 25), (8. 26) imply that (8, 21), (8, 21 a) will hold for $s=j$. This completes the induction, and one may assert that the equations

$$T(z_j(x)) = \beta_j(x) \qquad (j = 0, 1, \ldots)$$

can be solved in succession in such a wise that

$$(8.\ 27) \qquad \begin{gathered} z_s^{(i)}(x) = x^{i\left(\frac{p}{k}-1\right)} \zeta_{s,i}(x), \quad |\zeta_{s,i}(x)| \leqq A^{s+1} |x|^{n'-\left(\frac{t+1}{k}\right)} \\ (s = 0, 1, \ldots;\ i = 0, \ldots n;\ x \text{ in } R';\ |x| \geqq \gamma_0(t)); \end{gathered}$$

here $\gamma_0(t)$ is to be suitably great; the $\zeta_{s,i}(x)$ are analytic in R'. With

$$\varrho(x) = z_0(x) + z_1(x) + \cdots,$$

one has (7. 25) and

$$(8.\ 27\ \text{a}) \qquad |\varrho^{(i)}(x)| \leqq 2A|x|^{i\left(\frac{p}{k}-1\right)} |x|^{n'-\left(\frac{t+1}{k}\right)} \qquad (\text{in } R';\ |x| \geqq \gamma_0(t)).$$

As before, $\varrho(x)$ will constitute an analytic solution of the transformed equation of Lemma 4. 1.

The above developments enable formulation of the following result.

Existence Theoreme 8. 1. *Let* $s(x)$ ((3. 2)—(3. 2 b)) *be a formal solution of* (4. 2) *and let* $F_\nu^* = 0$ *be the 'actual' differential equation* (4. 1). *Assume that* $l_{n,0}$ *of* (4. 27) $\neq 0$. *Designate by* R' *a region* as described in the italicised statement in connection with (8. 2), (8. 3).

Given an integer t ($t \geqq t'$; t' *suitably great*), *however large, there exists a solution* $y(x)$ *of* $F_\nu^*=0$ *analytic in* R' (*for* $|x| \geqq \gamma_0(t)$; $\gamma_0(t)$ *sufficiently great*), *such that*

$$(8.\ 28) \qquad y^{(i)}(x) \sim s^{(i)}(x) \qquad (x \textit{ in } R';\ \textit{to } n(t) \cdot \textit{terms};\ i = 0, 1, \ldots n),$$

where $n(t) \to \infty$ *with* t. *In particular*, *one has*

$$(8.\ 28\ \text{a}) \qquad y^{(i)}(x) = \frac{d^i}{dx^i}[e^{Q(x)} x^r (\sigma(t, x) + \varrho(x))] \qquad (i = 0, \ldots n);$$

here $\sigma(t, x)$ *is given by* (6. 35 a) *and* $\varrho(x)$ *satisfies* (8. 27 a).

It is observed that *existence of regions* R', referred to in the obove theorem, *is always assured*.

When the given algebraic differential equation has a formal solution $s(x)$ of the general type (2. 1)—(2. 1 c), we still shall have existence results of essentially the same form as presented in theorems 6. 1, 7. 1, 8. 1. These results can be obtained by the methods already used. Some additional, but not unsurmountable, difficulties are encountered in this connection. No new ideas are necessary in the indicated extension; accordingly, we shall not present the details involved in such a generalisation.

9. Preliminaries for Equations with a Parameter.

In this section and in section 10 use will be made of the following notation.

Generically $\{x, \lambda\}$ is to signify a series

$$(9.\ 1) \qquad \{x, \lambda\} = \sigma_0(x) + \sigma_1(x)\lambda^{-\frac{1}{k}} + \cdots + \sigma_\nu(x)\lambda^{-\frac{\nu}{k}} + \cdots \qquad (\text{integer } k > 0),$$

whose coefficients $\sigma_\nu(x)$ are, together with the derivatives of all orders, continuous on a real interval $(a \leqq x \leqq b)$; the series may diverge for any or all x on (a, b) for all $\lambda \neq \infty$.

$\Gamma(a, b\colon R)$ will denote the aggregate of the values of x and λ for which

$$(9.\ 2) \qquad a \leqq x \leqq b \quad \text{and} \quad \lambda \text{ is in } R,$$

where R is a region regular in the sense indicated preceding (5. 8).

Generically $[x, \lambda]_\alpha (x, \lambda$ in $\Gamma(a, b; R))$ is a function asymptotic in $\Gamma(a, b; R)$, to α terms, to a series $\{x, \lambda\}$; this will be expressed by writing

$$(9.\ 3) \qquad [x, \lambda]_\alpha \underset{\alpha}{\backsim} \{x, \lambda\} \qquad (x, \lambda \text{ in } \Gamma(a, b;\ R)).$$

We shall denote by $[x, \lambda]$ a function $\sim \{x, \lambda\}$ to any number of terms, however great. A relation (9. 3) will signify that

$$(9.\ 3\text{ a}) \qquad [x, \lambda]_\alpha = \sigma_0(x) + \sigma_1(x)\lambda^{-\frac{1}{k}} + \cdots + \sigma_{\alpha-1}(x)\lambda^{-\frac{\alpha-1}{k}} + \sigma_\alpha(x, \lambda)\lambda^{-\frac{\alpha}{k}},$$

$$(9.\ 3\text{ b}) \qquad |\sigma_\alpha(x, \lambda)| < b_\alpha \qquad (x, \lambda \text{ in } \Gamma(a, b;\ R)).$$

With the above notation in view we shall consider the algebraic differential equation

$$(9.\ 4) \qquad F(x, \lambda, y) \equiv \sum_{i_0, \ldots} f^{i_0, \ldots i_n}(x, \lambda)(y)^{i_0}(y^{(1)})^{i_1} \ldots (y^{(n)})^{i_n} = 0$$

$(0 \leqq i_0, \ldots i_n;\ i_1 + \cdots + i_n \leqq \nu)$, *where the coefficients are of the form*

$$f^{i_0, \dots i_n}(x, \lambda) = \lambda^{m(i_0, \dots i_n)}[x, \lambda] \qquad (x, \lambda \text{ in } \Gamma(a, b; R)) \tag{9. 5}$$

(the $m(i_0, \dots i_n)$ integers), the symbol involved in the second member in (9. 5) *having the generic significance indicated above.* Without any loss of generality one may arrange to have only integral powers of λ involved in $[x, \lambda]$ of (9. 5). Amongst functions of the form (9. 5) are obviously included polynomials in λ, whose coefficients are functions of x indefinitely differentiable on (a, b).

The particular case of (9. 4), when $\nu = 1$, that is, when the equation is linear is of considerable importance, as it contains as special instances a number of classical equations and problems. Important earlier work for the linear case of problem (9. 4) has been previously done by G. D. Birkhoff, R. Langer, J. D. Tamarkin.[1] A theory, complete from a certain point of view, of the linear equation (9. 4) has been given by Trjitzinsky;[2] the results of his work (T_3) will be widely used in the sequel for the purpose of solution of the following analytic problem.

In the case when (9. 4) *has a formal solution*

$$s(x, \lambda) = e^{Q(x, \lambda)}\{x, \lambda\} \quad [cf. \ (9. 1); \ x \text{ on } (a, b)], \tag{9. 6}$$

where

$$Q(x, \lambda) = q_0(x)\lambda^{\frac{h}{k}} + q_1(x)\lambda^{\frac{h-1}{k}} + \dots + q_{h-1}(x)\lambda^{\frac{1}{k}} \tag{9. 6 a}$$

[the $q_j(x)$ indefinitely differentiable on (a, b); $h > 0$; $q_0^{(1)}(x) \neq 0$], to construct regions $\Gamma(a', b'; R)$ [(a', b') sub interval of (a, b); cf. definition in connection with (9. 2)] *and 'actual' solutions $y(x, \lambda)$ such that*

$$y(x, \lambda) \sim s(x, \lambda) \qquad (x, \lambda \text{ in } \Gamma(a', b'; R)) \tag{9. 6 b}$$

to a number of terms.

Formal solutions of type (9. 6) are of interest because it is known that every n-th order homogeneous linear equation (9. 4) has a full set of formal solutions of the type (9. 6). Of course, some or all of the $Q^{(1)}(x, \lambda)$ may be zero.[3]

By a reasoning of the type used before it follows that, inasmuch as we consider the case when (9. 4) has a formal solution (9. 6) with $Q^{(1)}(x, \lambda) \not\equiv 0$,

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[1] For references see (T, footnote 4).

[2] Trjitzinsky, Theory of linear differential equations containing a parameter [Acta mathematica, *67* (1936), 1—50], in the sequel referred to as (T_3). Also see (T; pp. 215—219).

[3] In sections 9, 10 all the derivations are with respect to x.

we should confine ourselves to the homogeneous equation of degree, say, ν. Thus, *the equation under consideration will be*

$$F_\nu(x, \lambda; y) \equiv \sum_{i_1, \ldots i_\nu} \lambda^{\eta(i_1, \ldots i_\nu)} b^{i_1, \ldots i_\nu}(x, \lambda) y^{(i_1)} y^{(i_2)} \ldots y^{(i_\nu)} = 0 \tag{9. 7}$$

$$[0 \leqq i_1, \ldots i_\nu \leqq n; \text{ the } \eta(i_1, \ldots i_\nu) \text{ integers}],$$

where

$$b^{i_1, \ldots i_\nu}(x, \lambda) = [x, \lambda] \qquad (x, \lambda \text{ in } \Gamma(a, b; R)). \tag{9. 7 a}$$

The corresponding formal equation will be

$$F_\nu^*(x, \lambda; y) \equiv \sum_{i_1, \ldots i_\nu} \lambda^{\eta(i_1, \ldots i_\nu)} \beta^{i_1, \ldots i_\nu}(x, \lambda) y^{(i_1)} \ldots y^{(i_\nu)} = 0 \tag{9. 8}$$

$(0 \leqq i_1, \ldots i_\nu \leqq n)$, where

$$\beta^{i_1, \ldots i_\nu}(x, \lambda) = \{x, \lambda\} \qquad (x \text{ on } (a, b)). \tag{9. 8 a}$$

In accordance with (9. 8 a)

$$\beta^{i_1, \ldots i_\nu}(x, \lambda) = \sum_{m=0}^{\infty} b_m^{i_1, \ldots i_\nu}(x) \lambda^{-m}, \tag{9. 8 b}$$

the $b_m^{i_1, \ldots i_\nu}(x)$ being indefinitely differentiable for $a \leqq x \leqq b$.

By reasoning of the type employed in section 2 the following is established.

If the equation (9. 7) (*actually of order* n) *is satisfied by the general 'actual' solution of the 'actual' linear differential equation*

$$L(x, \lambda; y) \equiv \sum_{i=0}^{\eta} f_i(x, \lambda) y^{(i)} = 0 \qquad (f_n(x, \lambda) \not\equiv 0), \tag{9. 9}$$

where

$$f_i(x, \lambda) \sim \zeta_i(x, \lambda) = \lambda^{\eta(i)} \{x, \lambda\} \qquad (\eta(i) \text{ integers; in } \Gamma(a, b; R)), \tag{9. 9 a}$$

$$f_i^{(j)}(x, \lambda) \sim \zeta_i^{(j)}(x) \qquad (j = 1, \ldots n - \eta; \text{ in } \Gamma(a, b; R)), \tag{9. 9 b}$$

then

$$F_\nu(x, \lambda; y) \equiv \sum_{j=0}^{n-\eta} \left[\frac{d^j}{d x^j} \Gamma(x, \lambda; y)\right] \Phi_j(x, \lambda; y, \ldots y^{(\eta+j)}), \tag{9. 10}$$

the Φ_j *being homogeneous* (*of degree* $\nu - 1$) *in* $y, \ldots y^{(\eta+j)}$ *with coefficients of the* *form* $\lambda^\gamma [x, \lambda]$ (*integer* γ; x, λ *in* $\Gamma(a', b'; R)$; (a', b') *a sub interval of* (a, b)). The same will hold with respect to (9. 8), with 'actual' replaced by 'formal' and $[x, \lambda]$ replaced by $\{x, \lambda\}$.

With the above in view, it is easy to give examples of equations (9. 7), *having one or more formal solutions of the type* (9. 6), (9. 6 a).

Consider now a series $s(x, \lambda)$ of the form (9. 6)

$$(9.\ 11) \qquad s(x, \lambda) = e^{Q(x, \lambda)} \sigma(x, \lambda), \quad \sigma(x, \lambda) = \sum_{j=0}^{\infty} \sigma_j(x) \lambda^{-\frac{j}{k}} \quad (Q(x, \lambda) \text{ from } (9.\ 6\text{ a})).$$

Differentiating formally one obtains

$$(9.\ 12) \qquad s^{(1)}(x, \lambda) = e^{Q(x, \lambda)} \lambda^{\frac{h}{k}} \sigma_1(x, \lambda);$$

$$(9.\ 12\text{ a}) \qquad \sigma_1(x, \lambda) = w(x, \lambda) \sigma(x, \lambda) + \lambda^{-\frac{h}{k}} \sigma^{(1)}(x, \lambda),$$
$$w(x, \lambda) = q_0^{(1)}(x) + \cdots + q_{h-1}^{(1)}(x) \lambda^{-\frac{h-1}{k}}$$

From this it is inferred that

$$(9.\ 13) \qquad s^{(i)}(x, \lambda) = e^{Q(x, \lambda)} \lambda^{\frac{ih}{k}} \sigma_i(x, \lambda),$$

$$(9.\ 13\text{ a}) \qquad \sigma_i(x, \lambda) = w(x, \lambda) \sigma_{i-1}(x, \lambda) + \lambda^{-\frac{h}{k}} \sigma_{i-1}^{(1)}(x, \lambda) \qquad (i = 1, 2, \ldots),$$

where

$$(9.\ 14) \quad \sigma_0(x, \lambda) = \sigma(x, \lambda), \quad \sigma_i(x, \lambda) = \sigma_{0, i}(x) + \sigma_{1, i}(x) \lambda^{-\frac{1}{k}} + \cdots \qquad (\sigma_{j, 0}(x) = \sigma_j(x))$$

and

$$(9.\ 14\text{ a}) \quad \sigma_{0, i}(x) = q_0^{(1)}(x) \sigma_{0, i-1}(x), \quad \sigma_{1, i}(x) = q_0^{(1)}(x) \sigma_{1, i-1}(x) + q_1^{(1)}(x) \sigma_{0, i-1}(x), \ \ldots,$$
$$\sigma_{h-1, i}(x) = q_0^{(1)}(x) \sigma_{h-1, i-1}(x) + \cdots + q_{h-1}^{(1)}(x) \sigma_{0, i-1}(x);$$

$$(9.\ 14\text{ b}) \quad \sigma_{m, i}(x) = \sigma_{m-h, i-1}^{(1)}(x) + [q_0^{(1)}(x) \sigma_{m, i-1}(x) + \cdots + q_{h-1}^{(1)}(x) \sigma_{m-h+1, i-1}(x)]$$

for $m \geqq h$. Thus

$$(9.\ 15) \qquad \sigma_{0, i}(x) = (q_0^{(1)}(x))^i \sigma_0(x) \qquad (i = 1, 2, \ldots),$$

and, for $\delta = 0, 1, \ldots h - 1$,

$$(9.\ 15\text{ a}) \qquad \sigma_{\delta, i}(x) = \sum_{\gamma=0}^{\delta} a(\delta, i; \gamma) \sigma_\gamma(x) \qquad (i = 1, 2, \ldots),$$

where the $a(\ldots)$ are polynomials in $q_0^{(1)}(x), \ldots q_\delta^{(1)}(x)$. Moreover, in view of (9. 14 b) (for $m = h$) and (9. 15), (9. 15 a), one has

$$\sigma_{h,\,i}(x) = \beta_0(h,\,i)\,\sigma_0^{(1)}(x) + \sum_{\gamma=0}^{h} a(h,\,i;\;\gamma)\,\sigma_\gamma(x) \tag{9. 16}$$

$[\beta(h,\,i),\; a(\ldots)$ polynomials in $q_0^{(1)}(x), \ldots q_{h-1}^{(1)}(x),\; q_0^{(2)}(x)]$.

Using (9. 16) and the preceding relations, we obtain

$$\sigma_{h+1,\,i}(x) = \beta_0(h+1,\,i)\,\sigma_0^{(1)}(x) + \beta_1(h+1,\,i)\,\sigma_1^{(1)}(x) + \sum_{\gamma=0}^{h+1} a(h+1,\,i;\;\gamma)\,\sigma_\gamma(x), \tag{9. 16 a}$$

where $\beta_0(\ldots),\ \beta_1(\ldots),\ a(\ldots)$ are polynomials in

$$q_0^{(1)}(x), \ldots q_{h-1}^{(1)}(x);\quad q_0^{(2)}(x),\; q_1^{(2)}(x).$$

In consequence of (9. 14 a)—(9. 16 a) by induction we infer that, for $0 \leqq \delta \leqq h - 1$,

$$\begin{aligned}\sigma_{h+\delta,\,i}(x) = {}& \beta_0(h+\delta,\,i)\,\sigma_0^{(1)}(x) + \cdots \\ & + \beta_\delta(h+\delta,\,i)\,\sigma_\delta^{(1)}(x) + \sum_{\gamma=0}^{h+\delta} a(h+\delta,\,i;\;\gamma)\,\sigma_\gamma(x),\end{aligned} \tag{9. 17}$$

where the coefficients $\beta_j(\ldots),\ a(\ldots)$ are polynomials in

$$q_0^{(1)}(x), \ldots q_{h-1}^{(1)}(x);\quad q_0^{(2)}(x),\; q_1^{(2)}(x), \ldots q_\delta^{(2)}(x).$$

We next obtain

$$\sigma_{2h,\,i}(x) = \beta_0(2h,\,i)\,\sigma_0^{(2)}(x) + \sum_{\gamma=0}^{2h} a(2h,\,i;\;\gamma)\,\sigma_\gamma(x) + \sum_{\gamma=0}^{h} a_1(2h,\,i;\;\gamma)\,\sigma_\gamma^{(1)}(x) \tag{9. 18}$$

$[\beta_0(\ldots),\ a(\ldots),\ a_1(\ldots)$ polynomials in $q_j^{(1)}(x),\ q_j^{(2)}(x),\ q_0^{(3)}(x)]$.

By induction in a larger sense it is finally deduced that

$$\begin{aligned}\sigma_{m,\,i}(x) = {}& \beta_0(m,\,i)\,\sigma_0^{(t)}(x) + \beta_1(m,\,i)\,\sigma_1^{(t)}(x) + \cdots + \beta_\delta(m,\,i)\,\sigma_\delta^{(t)}(x) \\ & + \sum_{\gamma=0}^{th+\delta} a(m,\,i;\,\gamma)\,\sigma_\gamma(x) + \sum_{\gamma=0}^{(t-1)h+\delta} a_1(m,\,i;\;\gamma)\,\sigma_\gamma^{(1)}(x) + \sum_{\gamma=0}^{(t-2)h+\delta} a_2(m,\,i;\;\gamma)\,\sigma_\gamma^{(2)}(x) + \cdots \\ & + \sum_{\gamma=0}^{h+\delta} a_{t-1}(m,\,i;\;\delta)\,\sigma_\gamma^{(t-1)}(x)\end{aligned} \tag{9. 19}$$

$[m = th + \delta;\ \beta_j(\ldots),\ a_j(\ldots)$ polynomials in $q_j^{(1)}(x), \ldots q_j^{(t)}(x)\,(j = 0, \ldots h-1)$, $q^{(t+1)}(x), \ldots q_\delta^{(t+1)}(x)]$

for $t = 1, 2, \ldots$ and $\delta = 0, 1, \ldots h - 1$.

By (9. 13) and (9. 14)

$$s^{(i_1)} s^{(i_2)} \dots s^{(i_\nu)} = e^{\nu Q(x,\lambda)} \lambda^{(i_1 + \cdots + i_\nu)\frac{h}{k}} \sum_{j=0}^{\infty} c_j^{i_1, \dots i_\nu}(x) \lambda^{-\frac{j}{k}}, \tag{9. 20}$$

$$c_j^{i_1, \dots i_\nu}(x) = \sum_{j_1, \dots j_\nu} \sigma_{j_1, i_1}(x) \sigma_{j_2, i_2}(x) \dots \sigma_{j_\nu, i_\nu}(x) \qquad (j_1, \dots j_\nu \geqq 0;\ j_1 + \cdots + j_\nu = j). \tag{9. 20 a}$$

If $s(x, \lambda)$ is a formal solution of (9. 8) one must have

$$F_\nu^*(x, \lambda;\ s(x, \lambda)) \equiv e^{\nu Q(x,\lambda)} \sum_{i_1, \dots i_\nu} \lambda^{\eta_{i_1, \dots i_\nu}} \sum_{j=0}^{\infty} d_j^{i_1, \dots i_\nu}(x) \lambda^{-\frac{j}{k}} = 0, \tag{9. 21}$$

where

$$\eta_{i_1, \dots i_\nu} = \eta(i_1, \dots i_\nu) + (i_1 + \cdots + i_\nu)\frac{h}{k} = \frac{1}{k} l_{i_1, \dots i_\nu} \qquad (l_{i_1, \dots i_\nu} \text{ integers}) \tag{9. 21 a}$$

and

$$d_j^{i_1, \dots i_\nu}(x) = \sum_{m+t=j} b_m(i_1, \dots i_\nu;\ x)\, c_t^{i_1, \dots i_\nu}(x), \tag{9. 21 b}$$

the $b_m(i_1, \dots i_\nu;\ x)$ being defined by the relations

$$\begin{aligned} & b_m(i_1, \dots i_\nu;\ x) = 0 \qquad \left(\text{for } \frac{m}{k} \neq \text{ an integer}\right), \\ & b_{\beta k}(i_1, \dots i_\nu;\ x) = b_\beta^{i_1, \dots i_\nu}(x) \qquad (\beta = 0, 1, \dots;\ \text{cf. } (9.\ 8\,\text{b})). \end{aligned} \tag{9. 21 c}$$

One should select h/k so that there are at least two terms of the same degree ϱ in λ, the other terms being all of degree $\leqq \varrho$. Thus, h/k must be so selected that for some particular two distinct sets $(\alpha_1, \dots \alpha_\nu)$, $(\beta_1, \dots \beta_\nu)$

$$\frac{1}{k} l_{\alpha_1, \dots \alpha_\nu} = \frac{1}{k} l_{\beta_1, \dots \beta_\nu} = \varrho,$$

while

$$\frac{1}{k} l_{i_1, \dots i_\nu} \leqq \varrho \qquad (\text{for all sets } (i_1, \dots i_\nu)).$$

Thus, provided $\beta_1 + \cdots + \beta_\nu \neq \alpha_1 + \cdots + \alpha_\nu$,

$$\frac{h}{k} = -\frac{\eta(\beta_1, \dots \beta_\nu) - \eta(\alpha_1, \dots \alpha_\nu)}{(\beta_1 + \cdots + \beta_\nu) - (\alpha_1 + \cdots + \alpha_\nu)}, \tag{9. 22}$$

while

$$(9.\ 22\ \mathrm{a})\qquad \eta(i_1,\dots i_\nu)-\eta(\beta_1,\dots\beta_\nu)\leqq -\frac{h}{k}[(i_1+\cdots+i_\nu)-(\beta_1+\cdots+\beta_\nu)]$$

(for all sets $(i_1,\dots i_\nu)$). It is important to note that *admissible values of* h/k *will arise only if the second member in* (9. 22) *is positive.* We represent the number pairs $(i_1+\cdots+i_\nu)$, $\eta(i_1,\dots i_\nu)$ in the Cartesian (x, y) plane, with $x=i_1+\cdots+i_\nu$ and $y=\eta(i_1,\dots i_\nu)$. *There arises a diagram* L *of Puiseux-type* precisely as described in the text from (3. 15 a) to (3. 17). The polygonal line L is concave downward. *The admissible values of* $\frac{h}{k}$ *are found amongst the negatives of the slopes of the rectilinear segments constituting* L. Inasmuch as one should have $\frac{h}{k}>0$, *only those sides of* L *will give rise to admissible values* $\frac{h}{k}$, *whose slopes are negative.*

In the case when for at least two distinct sets $(\beta_1,\dots\beta_\nu)$, $(\alpha_1,\dots\alpha_\nu)$ one has

$$\beta_1+\cdots+\beta_\nu=\alpha_1+\cdots+\alpha_\nu,\quad \eta(\beta_1,\dots\beta_\nu)=\eta(\alpha_1,\dots\alpha_\nu),$$

that is, when there is a vertex P of L which is *'multiple'*, we may take for $\frac{h}{k}$ any *positive* rational number α, provided that L lies to one side of the line through P with the slope $-\alpha$. One then will have $\frac{h}{k}>0$ (h, k integers) and (9. 22 a) will be satisfied.

Suppose $\frac{h}{k}$ is selected as an admissible value according to the above, either given by (9. 22) or as indicated above in connection with a 'multiple' vertex of L. One may then arrange (9. 21) formally as

$$(9.\ 23)\qquad F_\nu^*(x,\lambda;\ s(x,\lambda))\equiv e^{\nu Q(x,\lambda)}\lambda^{\frac{r}{k}}\left[\delta_0(x)+\delta_1(x)\lambda^{-\frac{1}{k}}+\delta_2(x)\lambda^{-\frac{2}{k}}+\cdots\right]=0,$$

where $\frac{r}{k}=\varrho$. Thus, if $s(x,\lambda)$ is a formal solution of $F_\nu^*=0$, necessarily

$$(9.\ 24)\qquad \delta_i(x)=0\qquad (i=0,1,\dots).$$

Corresponding to the value $\frac{h}{k}$ under consideration we write the equation $F_\nu^*=0$ [(9. 8), (9. 8 b)] as follows

$$(9.\ 25)\qquad \begin{aligned} F_\nu^* &\equiv \sum_{i_1,\ldots i_\nu} \lambda^{\eta(i_1,\ldots i_\nu)} \sum_{m=0}^{\infty} b_m(i_1,\ldots i_\nu;\ x)\,\lambda^{-\frac{m}{k}} y^{(i_1)}\ldots y^{(i_\nu)} \\ &\equiv \sum_{i_1,\ldots i_\nu} \lambda^{\frac{r'}{k}-\frac{h}{k}(i_1+\cdots+i_\nu)} \sum_{\gamma=0}^{\infty} b'_\gamma(i_1,\ldots i_\nu;\ x)\,\lambda^{-\frac{\gamma}{k}} y^{(i_1)}\ldots y^{(i_\nu)} = 0 \end{aligned}$$

(cf. (9. 21 c), (9. 8 b)), where

$$(9.\ 25\ \mathrm{a})\qquad b'_\gamma(i_1,\ldots i_\nu;\ x) = \begin{cases} 0 & (0 \leqq \gamma < w), \\ b_{\gamma-w}(i_1,\ldots i_\nu;\ x) & (\gamma \geqq w = w(i_1,\ldots i_\nu)) \end{cases}$$

with

$$(9.\ 25\ \mathrm{b})\qquad \frac{1}{k}w(i_1,\ldots i_\nu) = \frac{r}{k} - \frac{h}{k}(i_1+\cdots+i_\nu) - \eta(i_1,\ldots i_\nu) \geqq 0.$$

In view of (9. 25), (9. 25 b) the equations (9. 24) are expressible in the form

$$(9.\ 26)\qquad \delta_i(x) \equiv \sum_{i_1,\ldots i_\nu} \sum_{t=0}^{i} b'_{i-t}(i_1,\ldots i_\nu;\ x)\,c_t^{i_1,\ldots i_\nu}(x) = 0 \qquad (\text{cf. } (9.\ 20\ \mathrm{a})),$$

By (9. 26), (9. 20 a) and (9. 15)

$$(9.\ 26\ \mathrm{a})\qquad \delta_0(x) \equiv \sigma_0^\nu(x)\,E(x;\ q_0^{(1)}(x)) \equiv \sigma_0^\nu(x)\sum_{i_1,\ldots i_\nu} b'_0(i_1,\ldots i_\nu;\ x)\,(q_0^{(1)}(x))^{i_1+\cdots+i_\nu} = 0.$$

We thus see that of importance is the *characteristic equation* $E(x;\ q_0^{(1)}(x)) = 0$, which must be satisfied by $q_0^{(1)}(x)$. There is a characteristic equation like (9. 26 a) corresponding to every side, with a negative slope of the polygon L, as well as corresponding to some lines through the 'multiple' vertices of L. It is recalled that $q_0(x)$ is the leading coefficient in the polynomial $Q(x, \lambda)$.

We shall not go through any further formal details except to note that, in view of (9. 26), (9. 20 a), (9. 15), (9. 15 a) and (9. 19),

$$(9.\ 27)\qquad \delta_i(x) \equiv \delta_i(q_0^{(1)},\ldots q_{h-1}^{(1)};\ \sigma_0(x),\ldots \sigma_i(x)) \qquad (i = 0, 1, \ldots),$$

with a number of derivatives of $q_j^{(1)}(x)\,(j = 0,\ldots h-1)$ and of $\sigma_j(x)\,(j = 0,\ldots i)$ involved. The $\sigma_{i+\beta}(x)\,(\beta = 1, 2, \ldots)$ do not enter in the expression for δ_i.

Lemma 9. 1. *Consider the formal non linear differential equation* (9. 8), (9. 8 b). *Let* $\frac{h}{k}$ *(h, k positive integers) be an admissible value in accordance with* *the text from* (9. 21 c) *to* (9. 23). *If the equation* $F_\nu^* = 0$ *has a formal solution*

$$s(x, \lambda) = e^{Q(x, \lambda)}\sigma(x, \lambda) \left[Q(x, \lambda) = q_0(x)\lambda^{\frac{h}{k}} + \cdots + q_{h-1}(x)\lambda^{\frac{1}{k}}\right], \tag{9. 28}$$

where

$$\sigma(x, \lambda) = \sigma_0(x) + \sigma_1(x)\lambda^{-\frac{1}{k}} + \cdots, \tag{9. 28 a}$$

for x *in* (a, b), *then necessarily* $q_0^{(1)}$ *satisfies the characteristic equation* (9. 26 a), *associated with the side of the Puiseux-polygon to which* $\frac{h}{k}$ *belongs; moreover, the* $\delta_i(x)$ *in the formal expansion* (9. 23) *will be of the form described in connection with* (9. 27); *we have* $\delta_i(x) = 0$ $(i = 0, 1, \ldots)$.

The 'actual' differential equation $F_\nu(x, \lambda;\ y) = 0$ (9. 7) may be brought to the form corresponding to (9. 25). Thus,

$$F_\nu(x, \lambda;\ y) \equiv \sum_{i_1, \ldots i_\nu} \lambda^{\frac{r}{k} - \frac{h}{k}(i_1 + \cdots + i_\nu)} b'^{\,i_1, \ldots i_\nu}(x, \lambda)\, y^{(i_1)} \ldots y^{(i_\nu)} = 0, \tag{9. 29}$$

where

$$b'^{\,i_1, \ldots i_\nu}(x, \lambda) \sim \beta'^{\,i_1, \ldots i_\nu}(x, \lambda) = \sum_{\gamma=0}^{\infty} b'_\gamma(i_1, \ldots i_\nu;\ x)\lambda^{-\frac{\gamma}{k}} \quad (\text{in } \Gamma(a, b;\ R)). \tag{9. 29 a}$$

Basing on (9. 28), (9. 28 a), we make use of the transformation

$$y(x, \lambda) = e^{Q(x, \lambda)}[\sigma(t;\ x, \lambda) + \varrho(x, \lambda)], \tag{9. 30}$$

where

$$\sigma(t;\ x, \lambda) = \sigma_0(x) + \sigma_1(x)\lambda^{-\frac{1}{k}} + \cdots + \sigma_t(x)\lambda^{-\frac{t}{k}}; \tag{9. 30 a}$$

$\varrho(x, \lambda)$ being the new variable. One has

$$\frac{d^i}{dx^i}[e^{Q(x, \lambda)}\varrho(x, \lambda)] = e^{Q(x, \lambda)}\lambda^{i\frac{h}{k}}\varrho_i(x, \lambda), \tag{9. 31}$$

where

$$\varrho_i(x, \lambda) = \left[w(x, \lambda) + \lambda^{-\frac{h}{k}}\frac{d}{dx}\right]\varrho_{i-1}(x, \lambda) \quad (i = 1, 2, \ldots;\ \varrho_0(x, \lambda) = \varrho(x, \lambda)) \tag{9. 31 a}$$

with $w(x, \lambda)$ from (9. 12 a). Moreover,

$$(9.\ 32) \qquad \frac{d^i}{dx^i}[e^{Q(x,\lambda)}\sigma(t;\ x,\ \lambda)] = e^{Q(x,\lambda)}\lambda^{i\frac{h}{k}}\sigma_i(t;\ x,\ \lambda),$$

$$(9.\ 32\text{ a}) \qquad \begin{aligned}\sigma_i(t;\ x,\ \lambda) &= \left[w(x,\ \lambda) + \lambda^{-\frac{h}{k}}\frac{d}{dx}\right]\sigma_{i-1}(t;\ x,\ \lambda)\\ &= \sigma_{0,\,i}(t;\ x) + \sigma_{1,\,i}(t;\ x)\lambda^{-\frac{1}{k}} + \cdots + \sigma_{\gamma,\,i}(t;\ x)\lambda^{-\frac{\gamma}{k}} + \cdots\end{aligned}$$

$[\sigma_0(t;\ x, \lambda) = \sigma(t;\ x, \lambda)]$. In consequence of (9. 30), (9. 32 a) and (9. 13 a) it is inferred that $\sigma_{\gamma,i}(t; x)$ is $\sigma_{\gamma,i}(x)$ (cf. (9. 14)) with the $\sigma_j(x)\,(j > t)$ replaced by zeros. Hence, by virtue of (9. 15 a) and (9. 19),

$$(9.\ 32\text{ b}) \qquad \sigma_{\gamma,\,i}(t;\ x) = \sigma_{\gamma,\,i}(x) \qquad (i = 0,\ 1,\ \ldots;\ \chi = 0,\ 1,\ \ldots\ t).$$

In view of (9. 30), (9. 31) and (9. 32)

$$y^{(i)}(x,\ \lambda) = e^{Q(x,\lambda)}\lambda^{i\frac{h}{k}}[\sigma_i(t;\ x,\ \lambda) + \varrho_i(x,\ \lambda)]$$

and

$$y^{(i_1)}\ldots y^{(i_\nu)} = e^{\nu Q(x,\lambda)}\lambda^{\frac{h}{k}(i_1+\cdots+i_\nu)}\prod_{\alpha=1}^{\nu}[\sigma_{i_\alpha}(t;\ x,\ \lambda) + \varrho_{i_\alpha}(x,\ \lambda)].$$

Substituting this into the 'actual' equation (9. 29) we obtain

$$(9.\ 33) \quad F_\nu(x, \lambda; y) \equiv e^{\nu Q(x,\lambda)}\lambda^{\frac{r}{k}}\sum_{i_1,\ldots i_\nu} b'^{i_1,\ldots,i_\nu}(x,\ \lambda)\prod_{\alpha=1}^{\nu}[\sigma_{i_\alpha}(t;\ x,\ \lambda) + \varrho_{i_\alpha}(x,\ \lambda)] = 0$$

(cf. (9. 29 a)). Using developments of the type employed subsequent to (4. 10) it is now inferred that $\varrho(x, \lambda)$ satisfies

$$(9.\ 34) \qquad L(\varrho) + K(\varrho) = F(x,\ \lambda),$$

where

$$(9.\ 34\text{ a}) \qquad L(\varrho) \equiv \sum_{i_1,\ldots i_\nu} b'^{i_1,\ldots i_\nu}(x,\ \lambda)\prod_{\alpha=1}^{\nu}\sigma_{i_\alpha}(t;\ x,\ \lambda)\sum_{j=1}^{\nu}\frac{\varrho_{i_j}(x,\ \lambda)}{\sigma_{i_j}(t;\ x,\ \lambda)},$$

$$(9.\ 34\text{ b}) \qquad \begin{aligned}K(\varrho) \equiv \sum_{i_1,\ldots i_\nu} b'^{i_1,\ldots i_\nu}(x,\ \lambda)\prod_{\alpha=1}^{\nu}\sigma_{i_\alpha}(t;\ x,\ \lambda)\sum_{m=2}^{\nu}\ \sum_{j_1<\cdots<j_m}\\ \cdot\frac{\varrho_{i_{j_1}}(x,\ \lambda)}{\sigma_{i_{j_1}}(t;\ x,\ \lambda)}\cdots\frac{\varrho_{i_{j_m}}(x,\ \lambda)}{\sigma_{i_{j_m}}(t;\ x,\ \lambda)},\end{aligned}$$

$$(9.\ 34\text{ c}) \qquad F(x,\ \lambda) = -\sum_{i_1,\ldots i_\nu} b'^{i_1,\ldots i_\nu}(x,\ \lambda)\prod_{\alpha=1}^{\nu}\sigma_{i_\alpha}(t;\ x,\ \lambda).$$

Now, the asymptotic relations (9. 29 a) imply in particular that

$$(9.\ 35)\qquad b'^{i_1,\dots i_\nu}(x,\lambda)=\sum_{\gamma=0}^{t} b'_\gamma(i_1,\dots i_\nu;\,x)\,\lambda^{-\frac{\gamma}{k}}+\lambda^{-\frac{t+1}{k}}\beta_{i_1,\dots i_\nu}(t;\,x,\lambda),$$

$$(9.\ 35\ \mathrm{a})\qquad |\beta_{i_1,\dots i_\nu}(t;\,x,\lambda)|\leqq\beta_t\qquad (x,\lambda \text{ in } \Gamma(a,b;\,R)).$$

Hence, by (9. 34 c), $F(x,\lambda)=F_1+F_2$, where

$$(9.\ 36)\qquad F_1(x,\lambda)=-\sum_{i_1,\dots i_\nu}\sum_{\gamma=0}^{t} b'_\gamma(i_1,\dots i_\nu;\,x)\,\lambda^{-\frac{\gamma}{k}}\prod_{\alpha=1}^{\nu}\sigma_{i_\alpha}(t;\,x,\lambda),$$

$$F_2(x,\lambda)=-\lambda^{-\frac{t+1}{k}}\beta(t;\,x,\lambda),\quad \beta(t;\,x,\lambda)=\sum_{i_1,\dots i_\nu}\beta_{i_1,\dots i_\nu}(t;\,x,\lambda)\prod_{\alpha=1}^{\nu}\sigma_{i_\alpha}(t;\,x,\lambda).$$

By the same method as involved from (4. 14) to (4. 18) and using Lemma 9. 1 we now obtain

$$(9.\ 37)\qquad |F_1(x,\lambda)|\leqq|\lambda|^{-\frac{t+1}{k}}F_1(t)\qquad (x,\lambda \text{ in } \Gamma(a,b;\,R)).$$

Similarly by (9. 36) and (9. 35 a) it is deduced that

$$(9.\ 37\ \mathrm{a})\qquad |F_2(x,\lambda)|\leqq|\lambda|^{-\frac{t+1}{k}}F_2(t)\qquad (x,\lambda \text{ in } \Gamma(a,b;\,R)).$$

Whence

$$(9.\ 38)\qquad F(x,\lambda)=\lambda^{-\frac{t+1}{k}}F(t;\,x,\lambda),$$

$$(9.\ 38\ \mathrm{a})\qquad |F(t;\,x,\lambda)|\leqq F_t\qquad (\text{in } \Gamma(a,b;\,R)),$$

where F_t is independent of x and λ.

Using (9. 31 a) one finds

$$(9.\ 39)\quad \varrho_i(x,\lambda)=w_{i,0}(x,\lambda)\,\varrho(x,\lambda)+w_{i,1}(x)\,\varrho^{(1)}(x,\lambda)+\cdots+w_{i,i}(x,\lambda)\,\varrho^{(i)}(x,\lambda),$$

where $w_{0,0}(x,\lambda)=1$ and

$$(9.\ 39\ \mathrm{a})\quad w_{i,0}(x,\lambda)=w(x,\lambda)\,w_{i-1,0}(x,\lambda)+\lambda^{-\frac{h}{k}}w^{(1)}_{i-1,0}(x,\lambda),\quad w(x,\lambda)=\sum_{j=0}^{h-1}q_j^{(1)}(x)\,\lambda^{-\frac{j}{k}},$$

$$(9.\ 39\ \mathrm{b})\quad w_{i,m}(x,\lambda)=w(x,\lambda)\,w_{i-1,m}(x,\lambda)+\lambda^{-\frac{h}{k}}[w^{(1)}_{i-1,m}(x,\lambda)+w_{i-1,m-1}(x,\lambda)]$$
$$(m=1,2,\dots i-1),$$

(9. 39 c) $$w_{i,i}(x, \lambda) = \lambda^{-\frac{h}{k}} w_{i-1,i-1}(x, \lambda).$$

By virtue of (9, 39 a)—(9. 39 c)

(9. 40) $$w_{i,m}(x, \lambda) = \lambda^{-m\frac{h}{k}} v_{i,m}(x) \qquad (m = 0, \ldots i),$$

(9. 40 a) $v_{i,m}(x, \lambda) =$ polynomial in $\lambda^{-\frac{1}{k}} = [x, \lambda]$, $v_{i,i}(x, \lambda) = 1$ (in $\Gamma(a, b;\ R)$).

In consequence of (9. 39), (9. 40) and (9. 34 a)

(9. 41) $$L(\varrho) = l_n(x, \lambda)\,\varrho^{(n)} + l_{n-1}(x, \lambda)\,\varrho^{(n-1)} + \cdots + l_0(x, \lambda)\,\varrho,$$

where

(9. 41 a) $$l_\gamma(x, \lambda) = \lambda^{-\gamma\frac{h}{k}} \sum_{i_1, \ldots i_\nu} b'^{i_1, \ldots, i_\nu}(x, \lambda) \sum_{j=1}^{\nu} v_{i_j,\gamma}(x, \lambda)\, k^{i_j,\gamma} \prod_{\alpha \neq j} \sigma_{i_\alpha}(t;\ x, \lambda)$$

($k^{i,\gamma}$ from (4. 21 b)). One has

(9. 42) $$p_\gamma(x, \lambda) = \lambda^{\gamma\frac{h}{k}} l_\gamma(x, \lambda) \sim p_{\gamma,0}(t;\ x) + p_{\gamma,1}(t;\ x)\,\lambda^{-\frac{1}{k}} + \cdots + p_j(t;\ x)\,\lambda^{-\frac{j}{k}} + \cdots$$

for x, λ in $\Gamma(a, b;\ R)$. The series last displayed in (9. 42) is the formal expansion in powers of $\lambda^{-\frac{1}{k}}$ of

(9. 42 a) $$\sum_{i_2, \ldots i_\nu} \beta'^{i, \ldots i_\nu}(x, \lambda) \sum_{j=1}^{\nu} v_{i_j,\gamma}(x, \lambda)\, k^{i_j,\gamma} \prod_{\alpha \neq j} \sum_{s=0}^{\infty} \sigma_{s,i_\alpha}(t;\ x)\,\lambda^{-\frac{s}{k}}$$

(cf. (4. 21 b), (9. 29 a), (9. 40), (9. 32 a)). Hence

(9. 42 b) $$p_{\gamma,j}(t;\ x) = p_{\gamma,j}(x) \qquad (j = 0, \ldots t';\ t' \to \infty \text{ with } t),$$

where the second members are independent of t. Thus, $L(\varrho)$ may be expressed as

(9. 43) $$L(\varrho) \equiv \lambda^{-n\frac{h}{k}} \left[p_n(x, \lambda)\,\varrho^{(n)} + p_{n-1}(x, \lambda)\,\lambda^{\frac{h}{k}} \varrho^{(n-1)} + \cdots + p_0(x, \lambda)\,\lambda^{n\frac{h}{k}} \varrho \right]$$

(cf. (9. 42), (9. 42 b)). We shall now obtain explicitly $p_{n,0}(t;\ x) = p_{n,0}(x)$. Since $w_{0,0}(x, \lambda) = 1$, in consequence of (9. 39 c) we obtain $v_{n,n}(x, \lambda) = 1$. It is noted that $p_{n,0}(x)$ is the term free of λ in the formal expansion in powers of $\lambda^{-\frac{1}{k}}$ of (9 42 a) (for $\gamma = n$). Thus, in view of (9. 15) and (4. 21 b)

$$(9.\ 44)\qquad p_{n,0}(x) = \sigma_0^{\nu-1}(x) \sum_{j=1}^{\nu} j \sum_{i_1, \ldots i_\nu}^{(j)} b_0'(i_1, \ldots i_\nu; x)\,(q_0^{(1)}(x))^{i_1 + \cdots + i_\nu - n},$$

where the summation symbol with the superscript j is over all sets $(i_1, \ldots i_\nu)$ containing precisely j elements each equal to n.

Case 9. 45. *There is a closed sub interval (a', b') of (a, b) in which $p_{n,0}(x)$ of* (9. 44) *does not vanish.*

Case 9. 46. $p_{n,0}(x) = p_{n,1}(x) = \cdots = p_{n,w-1}(x) = 0$ *(x on (a, b); $w > 0$), while $p_{n,w}(x)$ (which is the coefficient of $\lambda^{-\frac{w}{k}}$ in the expansion of* (9. 42 a; for $\gamma = n$)) *is not identically zero. In this case let (a', b') be a closed sub interval of (a, b) in which $p_{n,w}(x)$ does not vanish.*

If Case 9. 46 is on hand we choose t sufficiently great so that the $p_{n,j}(x)$ $(j = 0, \ldots w)$ are independent of t.

In the Case 9. 45 one may write $L(\varrho)$ in the form

$$(9.\ 47)\qquad L(\varrho) \equiv \lambda^{-n\frac{h}{k}} p_n(x, \lambda)\, T(\varrho) \quad (\text{cf. } (9.\ 42);\ p_n(x, \lambda),\ p_n^{-1}(x, \lambda) = [x, \lambda]),$$

$$(9.\ 47\ \text{a})\qquad T(\varrho) \equiv \varrho^{(n)} + b_1(x, \lambda)\,\lambda^{\frac{h}{k}} \varrho^{(n-1)} + \cdots + b_n(x, \lambda)\,\lambda^{n\frac{h}{k}} \varrho,$$

where

$$(9.\ 47\ \text{b})\qquad b_\gamma(x, \lambda) = [x, \lambda] \sim b_{\gamma,0}(t; x) + b_{\gamma,1}(t; x)\,\lambda^{-\frac{1}{k}} + \cdots \qquad (\text{in } \Gamma(a', b'; R));$$

here the $b_{\gamma,j}(t; x)$ $(0 \leqq j \leqq j';\ j' \to \infty$ with $t)$ are independent of t.

In the Case 9. 46

$$(9.\ 48)\qquad L(\varrho) \equiv \lambda^{-\frac{1}{k}(nh+w)} \bar{p}_n(x, \lambda)\, T(\varrho),$$

where

$$(9.\ 48\ \text{a})\qquad \bar{p}_n(x, \lambda) = [x, \lambda] \sim p_{n,w}(x) + \cdots,\ \frac{1}{\bar{p}_n(x, \lambda)} = [x, \lambda] \qquad (\text{in } \Gamma(a'\ b';\ R)]$$

and

$$(9.\ 48\ \text{b})\qquad T(\varrho) \equiv \varrho^{(n)} + \bar{b}_1(x, \lambda)\,\lambda^{\frac{1}{k}(h+w)} \varrho^{(n-1)} + \cdots + \bar{b}_n(x, \lambda)\,\lambda^{\frac{1}{k}(nh+w)} \varrho$$

with

$$(9.\ 48\ \text{c})\qquad \bar{b}_\gamma(x, \lambda) = [x, \lambda] \sim \bar{b}_{\gamma,0}(t; x) + \bar{b}_{\gamma,1}(t; x)\,\lambda^{-\frac{1}{k}} + \cdots \qquad (\text{in } \Gamma(a', b'; R)),$$

the $\bar{b}_{\gamma,j}(t; x)$ $(0 \leqq j \leqq j_1;\ j_1 \to \infty$ with $t)$ being independent of t.

By (9. 34 b), (9. 39) and (9. 40) we get an analogue of (4. 29), (4. 30) and (4. 30 a). More precisely,

$$K(\varrho) = K_2(\varrho) + K_3(\varrho) + \cdots + K_\nu(\varrho), \tag{9. 49}$$

where

$$K_m(\varrho) = \sum_{m_0, \ldots m_n} k_m^{m_0, \ldots m_n}(t;\, x,\, \lambda) \prod_{\alpha=0}^{n} (\varrho^{(\alpha)})^{m_\alpha} \lambda^{-\alpha m_\alpha \frac{h}{k}} \qquad (m_0 + \cdots + m_n = m) \tag{9. 49 a}$$

with

$$k_m^{m_0, \ldots m_n}(t;\, x,\, \lambda) = [x,\, \lambda] \sim \sum_{\gamma=0}^{\infty} k_{m,\gamma}^{m_0, \ldots m_n}(t;\, x) \lambda^{-\frac{\gamma}{k}} \qquad (\text{in } \Gamma(a,\, b;\, R)), \tag{9. 49 b}$$

the coefficients in the series last displayed being independent of t for $\gamma \leqq \gamma'$ ($\gamma' \to \infty$ with t).

Lemma 9. 2. *Suppose that* $s(x, \lambda)$ (9. 28) *is a formal solution for* x *on* (a, b) *of the formal non linear homogeneous differential equation* (9. 8), (9. 8 b), *in accordance with Lemma* 9. 1. *Let* $F_\nu = 0$ (9. 29) *be the corresponding form of the 'actual' differential equation. The transformation*

$$y = e^{Q(x, \lambda)} [\sigma(t;\, x,\, \lambda) + \varrho(x,\, \lambda)] \qquad (cf.\ (9.\ 30),\ (9.\ 30\ \mathrm{a}))$$

will yield the equation

$$L(\varrho) + K(\varrho) = F(x,\, \lambda). \tag{9. 50}$$

In the Case 9. 45 $L(\varrho)$ *is given by* (9. 47)—(9. 47 b). *In the Case* (9. 46) $L(\varrho)$ *is given by* (9. 48)—(9. 48 c). $K(\varrho)$ *is of the form* (9. 49)—(9. 49 c) *and the function* $F(x, \lambda)$ *satisfies* (9. 38)—(9. 38 a).

10. The Fourth Existence Theorem.

With $T(\varrho)$ from (9. 47) or (9. 48), as the case may be, consider the equation

$$T(\varrho) = 0. \tag{10. 1}$$

In accordance with the existence theorems established by Trjitzinsky[1] for linear differential equations containing a parameter a sub-interval (a_1, b_1) of (a', b') can be found and a *regular* sub region R_1 of R so that the equation (10. 1) possesses a full set of solutions $y_i(x, \lambda)$ ($i = 1, \ldots n$) of the form

[1] (T_8).

$$y_i(x, \lambda) = e^{Q_i(x, \lambda)} \eta_i(x, \lambda), \tag{10. 2}$$

where

$$\eta_i(x, \lambda) = [x, \lambda]_\alpha \underset{\alpha}{\sim} \eta_{i,0}(x) + \eta_{i,1}(x)\lambda^{-\frac{1}{\nu_i k}} + \eta_{i,2}(x)\lambda^{-\frac{2}{\nu_i k}} + \cdots = {}_i\sigma(x, \lambda) \tag{10. 2 a}$$

for x, λ in $\Gamma(a_1, b_1; R_1)$. In (10. 2) the $Q_i(x, \lambda)$ are polynomials in $\lambda^{\frac{1}{\nu_i k}}$ (integers $\nu_i > 0$) with coefficients indefinitely differentiable for $a_1 \leqq x \leqq b_1$. The highest possible power of λ in $Q_i(x, \lambda)$ is $\lambda^{\frac{h}{k}}$ (in the Case 9. 45) and $\lambda^{\frac{1}{k}(h+w)}$ in the Case 9. 46. By choosing t sufficiently great *we arrange to have the* $Q_i(x, \lambda)$, as well as the $\eta_{i,j}(x)$ $(0 \leqq j \leqq j'; \ j' \to \infty$ with $t)$, *independent of* t. *The region* R_1 *is such that no function*

$$\Re\,(Q_i^{(1)}(x, \lambda) - Q_j^{(1)}(x, \lambda)) \qquad (i, j = 1, \ldots n) \tag{10. 3}$$

changes sign for λ *in* R_1 *and for* $a_1 \leqq x \leqq b_1$. *Such sub regions* R_1 *of* R *can always be constructed,* taking, if necessary, $b_1 - a_1$ sufficiently small.

Given α, however large, the solution referred to in (10. 2), (10. 2 a) can be so constructed that

$$y_i^{(j-1)}(x, \lambda) \underset{\alpha}{\sim} \frac{d^{j-1}}{d\,x^{j-1}}[e^{Q(x, \lambda)}\,{}_i\sigma(x, \lambda)] \qquad (\text{in } \Gamma(a_1, b_1; R)) \tag{10. 4}$$

for $j = 1, \ldots n$ and

$$y_i^{(j-1)}(x, \lambda) = e^{Q_i(x, \lambda)} \lambda^{(j-1)\frac{h'}{k}} \eta_{i, j-1}(x, \lambda) \qquad (h' = h \text{ or } h + w), \tag{10. 4 a}$$

$$\eta_{i,j-1}(x, \lambda) = [x, \lambda]_\alpha \qquad (\text{in } \Gamma(a_1, b_1; R_1);\ j = 1, \ldots n). \tag{10. 4 b}$$

The determinant of the matrix $(y_i^{(j-1)}(x, \lambda))$ $(i, j = 1, \ldots n)$ is

$$\Delta(x, \lambda) = |(y_i^{(j-1)}(x, \lambda))| = \exp.\left[-\lambda^{\frac{1}{k}h'} \int^x c_1(x, \lambda)\,d\,x\right],$$

where $c_1(x, \lambda)$ is $b_7(x, \lambda)$ (9. 47 b) or $\bar{b}_7(x, \lambda)$ (9. 48 c) and where the 'constant' of integration may depend on λ and is to be suitably chosen. Together with (10. 4 a) this implies that

$$\Delta(x, \lambda) = e^{Q_1(x, \lambda) + \cdots + Q_n(x, \lambda)} \lambda^{\frac{1}{2}(n^2-n)\frac{h'}{k} - \frac{\omega}{k}} d(x, \lambda) \qquad (\text{integer } \omega \geq 0), \tag{10. 5}$$

$$d(x, \lambda) = [x, \lambda] \sim d_0(x) + d_1(x)\lambda^{-\frac{1}{k}} + \cdots \qquad (\text{in } \Gamma(a_1, b_1; R_1);\ d_0(x) \not\equiv 0). \tag{10. 5 a}$$

It is noted that $d_0(x)$ *of* (10. 5 a) *does not vanish on* (a_1, b_1). *Thus*

$$\frac{1}{d(x,\lambda)} = [x, \lambda] \qquad (\text{in } \Gamma(a_1, b_1; R_1)). \tag{10. 5 b}$$

Define the $\bar{y}_{i,j}(x, \lambda)$ by the matrix relation

$$(\bar{y}_{i,j}(x,\lambda)) = (y_i^{(j-1)}(x,\lambda))^{-1}. \tag{10. 6}$$

One has

$$\Delta(x,\lambda)\,\bar{y}_{n,j}(x,\lambda)(-1)^{n+j} = \begin{vmatrix} y_1 & ,\ldots y_{j-1}, & y_{j+1} & ,\ldots y_n \\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ y_1^{(n-2)}, & \ldots y_{j-1}^{(n-2)}, & y_{j+1}^{(n-2)}, & \ldots y_n^{(n-2)} \end{vmatrix},$$

which in consequence of (10. 4 a) yields

$$\Delta(x,\lambda)\,\bar{y}_{n,j}(x,\lambda) = e^{Q_1(x,\lambda)+\cdots+Q_n(x,\lambda)-Q_j(x,\lambda)}\,\lambda^{\frac{1}{2}(n^2-3n+2)\frac{h'}{k}}[x,\lambda]_\alpha \quad (\text{in } \Gamma(a_1, b_1; R_1)). \tag{10. 6 a}$$

By (10. 6 a), (10. 5) and (10. 5 b) one finally obtains

$$\bar{y}_{n,j}(x,\lambda) = e^{-Q_j(x,\lambda)}\,\lambda^{-\omega_1}\,\bar{y}(n, j; x, \lambda) \qquad \left(\omega_1 = \frac{h'}{k}(n-1) - \frac{\omega}{k}\right) \tag{10. 7}$$

with

$$\bar{y}(n, j; x, \lambda) = [x, \lambda]_\alpha \qquad (\text{in } \Gamma(a_1, b_1; R_1)). \tag{10. 7 a}$$

It is to be recalled that for a solution z of the equation $T(z) = \beta$ ($T(z)$ from (5. 6)) we have previously obtained (5. 26). Adapting that result to the equation

$$T(z) = \beta(x, \lambda) \qquad (T \text{ from } (10. 1)), \tag{10. 8}$$

we conclude that, provided the integrations can be carried out, a solution $z(x, \lambda)$ of (10. 8) will satisfy

$$z^{(j-1)}(x,\lambda) = \sum_{\tau=1}^{n} y_\tau^{(j-1)}(x,\lambda) \int^x \beta(x,\lambda)\,\bar{y}_{n,\tau}(x,\lambda)\,dx \qquad (j = 1, \ldots n),$$

where the $\bar{y}_{n,\tau}(x, \lambda)$ are given by (10. 7), (10. 7 a) and the $y_\tau^{(j-1)}(x, \lambda)$ are of the form (10. 4 a), (10. 4 b). Accordingly, it is observed that *for a solution* $z(x, \lambda)$ *of* (10. 8) *one has*, for $j = 1, \ldots n$,

$$z^{(j-1)}(x,\lambda)=\sum_{\tau=1}^{n} e^{Q_\tau(x,\lambda)}\lambda^{(j-1)\frac{h'}{k}}\eta_{\tau,j-1}(x,\lambda)\int^{x} e^{-Q_\tau(x,\lambda)}\lambda^{-\omega_1}\cdot\bar{y}(n,\tau;u,\lambda)\beta(u,\lambda)\,du$$

$$\left[\omega_1=\frac{h'}{k}(n-1)-\frac{\omega}{k};\ \text{cf. } (10.\ 5), (10.\ 7\,\text{a}), (10.\ 4\,\text{b})\right]; \tag{10. 9}$$

here $h'=h$ *(in Case* 9. 45) *and* $h'=h+w$ *(in Case* 9. 46).

We shall now proceed to construct an appropriate solution of the transformed equation (9. 50). Unless stated otherwise we shall consider the Case 9. 46, when $L(\varrho)$ is expressible by (9. 48).

A solution of (9. 50) will be given in a form of a convergent series

$$\varrho(x,\lambda)=z_0(x,\lambda)+z_1(x,\lambda)+\cdots, \tag{10. 10}$$

whose terms are suitable determined functions satisfying

$$L(z_0)=F(x,\lambda), \tag{10. 11}$$

$$L(z_j)=-K(w_{j-1})+K(w_{j-2}) \qquad (j=1,2,\ldots;\ w_{-1}=0) \tag{10. 12}$$

with

$$w_j(x,\lambda)=z_0(x,\lambda)+z_1(x,\lambda)+\cdots+z_j(x,\lambda) \qquad (j=0,1,\ldots). \tag{10. 12 a}$$

By (9. 48) the equations (10. 11), (10. 12) may be put in the form

$$T(z_j)=\beta_j(x,\lambda) \qquad (j=0,1,\ldots), \tag{10. 13}$$

where

$$\beta_0(x,\lambda)=(\bar{p}_n(x,\lambda))^{-1}\lambda^{\frac{1}{k}(nh+w-t-1)}F(t;x,\lambda) \qquad (\text{cf. } (9.\ 38\ \text{a})), \tag{10. 13 a}$$

$$\beta_j(x,\lambda)=(\bar{p}_n(x,\lambda))^{-1}\lambda^{\frac{1}{k}(nh+w)}\left[-K(w_{j-1})+K(w_{j-2})\right] \tag{10. 13 b}$$

$(j=1,2,\ldots)$. In view of (9. 48 a)

$$\left|\frac{1}{\bar{p}_n(x,\lambda)}\right|\leqq p \qquad (x,\lambda \text{ in } \Gamma(a_1,b_1;R_1)). \tag{10. 14}$$

Hence by (10. 13 a)

$$\beta_0(x,\lambda)=\lambda^{-\beta_0}\gamma_0(x,\lambda), \qquad \beta_0=\frac{1}{k}(t+1)-\frac{1}{k}(nh+w), \tag{10. 15}$$

$$|\gamma_0(x,\lambda)|\leqq\gamma_0 \qquad (\text{in } \Gamma(a_1,b_1;R_1)); \tag{10. 15 a}$$

t is taken so that $\beta_0>0$.

By virtue of (10. 4 b) and (10. 7 a)

$$|\eta_{\tau, j-1}(x, \lambda)|, \ |\bar{y}(n, \tau; x, \lambda)| \leqq p_1 \qquad (\text{in } \Gamma(a_1, b_1; R_1)).$$

Hence in consequence of (10. 9) it is inferred that for a solution $z(x, \lambda)$ of the equation

$$(10.\ 16) \qquad T(z(x, \lambda)) = \lambda^{-\beta} \gamma(x, \lambda) \qquad [|\gamma(x, \lambda)| \leqq \gamma \text{ in } \Gamma(a_1, b_1; R_1)],$$

where $\beta > 0$, we have

$$(10.\ 16\,\text{a}) \qquad |z^{(j-1)}(x, \lambda)| \leqq p_1^2 \gamma |\lambda|^{(j-n)\frac{h'}{k} + \frac{\omega}{k} - \beta} \sum_{\tau=1}^{n} \int_{c_1}^{x} |e^{Q_\tau(x, \lambda) - Q_\tau(u, \lambda)}| \, |du|$$

for $j = 1, 2, \ldots n$ and x, λ in $\Gamma(a_1, b_1; R_1)$, provided $\gamma(x, \lambda)$ is integrable in x for x on (a_1, b_1). In (10. 16 a) c_1 is a_1 or b_1 (see (10. 18), below).

Let R_2 be a regular sub region of R_1 such that no function

$$(10.\ 17) \qquad \Re(Q_j^{(1)}(x, \lambda)) \quad (j = 1, \ldots n;\ \textit{cf. Def. of } R_1 \textit{ with respect to } (10.\ 3))$$

changes sign for x in R_2 and for $a_1 \leqq x \leqq b_1$. Regions R_2 will exist in all cases at least for $b_1 - a_1 (> 0)$ sufficiently small.

We shall take

$$(10.\ 18) \qquad c_1 = \begin{cases} a_1 & (\text{when } \Re\, Q_\tau^{(1)}(x, \lambda) \leqq 0 \text{ in } \Gamma(a_1, b_1; R_2)), \\ b_1 & (\text{when } \Re\, Q_\tau^{(1)}(x, \lambda) \geqq 0 \text{ in } \Gamma(a_1, b_1; R_2)). \end{cases}$$

Then the integral displayed in (10. 16 a) will satisfy

$$(10.\ 19) \qquad \int_{c_1}^{x} |e^{Q_\tau(x, \lambda) - Q_\tau(u, \lambda)}| \, |du| \leqq b_1 - a_1 \qquad (\text{in } \Gamma(a_1, b_1; R_2)).$$

Lemma 10. 1. *For a solution $z(x, \lambda)$ of the equation*

$$T(z(x, \lambda)) = \lambda^{-\beta} \gamma(x, \lambda) \qquad [|\gamma(x, \lambda)| \leqq \gamma \textit{ in } \Gamma(a_1, b_1; R_2)],$$

where β is real and R_2 is defined in connection with (10. 17), *one has, for x, λ in $\Gamma(a_1, b_1; R_2)$ and for $j = 1, \ldots n$,*

$$(10.\ 20) \qquad |z^{(j-1)}(x, \lambda)| \leqq n_1 \gamma |\lambda|^{-\beta + \frac{h'}{k}(j-n) + \frac{\omega}{k}} \qquad (n_1 = n(b_1 - a_1) p_1^2),$$

provided $\gamma(x, \lambda)$ is integrable in x for x on (a_1, b_1). This result holds with $h' = h + w$ in the Case 9. 46 and with $h' = h$ in the Case 9. 45.

Let b' be a number, independent of x and λ, such that

$$|b_i(x, \lambda)| \leqq b' \qquad (i = 1, \ldots n; \text{ in } \Gamma(a_1, b_1; R_2)), \tag{10. 21}$$

in the Case 9. 45, and such that

$$|\bar{b}_i(x, \lambda)| \leqq b' \qquad (i = 1, \ldots n; \text{ in } \Gamma(a_1, b_1; R_2)), \tag{10. 21 a}$$

in the Case 9. 46. In consequence of (9. 47 a) and (9. 48 b) for the solution referred to in Lemma 10. 1 one will have

$$|z^{(n)}(x, \lambda)| \leqq \sum_{i=1}^{n} b' |\lambda|^{\frac{1}{k}(ih+w)} |z^{(n-i)}(x, \lambda)| + \gamma |\lambda|^{-\beta} \tag{10. 22}$$

(in $\Gamma(a_1, b_1; R_2)$); here w is to be replaced by zero in the Case 9. 45. By (10. 20) and (10. 22)

$$|z^{(n)}(x, \lambda)| \leqq \gamma |\lambda|^{-\beta} + \sum_{i=1}^{n} b' n_1 \gamma |\lambda|^{-(i-1)\frac{w}{k} + h_2 - \beta} \qquad \left(h_2 = \frac{1}{k}(h' + \omega) > 0\right). \tag{10. 23}$$

Inasmuch as in R_2 $|\lambda| \geqq \lambda_0$, where for simplicity one may take $\lambda_0 \geqq 1$, it is concluded that

$$|z^{(n)}(x, \lambda)| \leqq n_2 \gamma |\lambda|^{-\beta + h_2} \qquad (n_2 = n n_1 b' + 1;\ h_2 \text{ from } (10. 23)) \tag{10. 24}$$

for x, λ in $\Gamma(a_1, b_1; R_2)$. Using (10. 24) and Lemma 10. 1 we obtain

Lemma 10. 2. *For the solution $z(x, \lambda)$ referred to in Lemma 10. 1, we have for $i = 0, 1, \ldots n$*

$$|z^{(i)}(x, \lambda)| \leqq n' \gamma |\lambda|^{-\beta + \frac{h'}{k}(i-n) + h_2} \tag{10. 25}$$

$$\left[h_2 = \frac{1}{k}(h + w + \omega);\ h' = h + w;\ x, \lambda \text{ in } \Gamma(a_1, b_1; R_2)\right]$$

in the Case 9. 46. In (10. 25) n' is the greater of the numbers n_1 and $n_2 = n n_1 b' + 1$. In the Case 9. 45 the same result may be asserted with w replaced by zero.

On taking account of (10. 15), (10. 15 a) with the aid of Lemma 10. 2 we obtain a solution $z_0(x, \lambda)$ of the first equation (10. 13) such that

$$|z_0^{(i)}(x, \lambda)| \leqq \zeta_0 |\lambda|^{-\frac{1}{k}(t+1) + \frac{h' i}{k} + h_0} \qquad (\text{in } \Gamma(a_1, b_1; R_2)), \tag{10. 26}$$

for $i = 0, \ldots n$, where

(10. 26 a) $$\zeta_0 = n' \gamma_0, \quad h_0 = h_2 - (n - 1)\frac{w}{k}.$$

In view of (9. 49 b) there exists a constant $\bar{k}$ independent of $m_0, \dots m_n$, x, λ so that

(10. 27) $$| k_m^{m_0, \dots m_n}(t; x, \lambda) | \leqq \bar{k} \qquad (\text{in } \Gamma(a_1, b_1; R_2)).$$

It is noted that $K_m(\varrho)$ is given by (9. 49 a) in both Cases 9. 45, 9. 46. By (10. 26) and (10. 27)

$$| K_m(z_0(x, \lambda)) | \leqq \bar{k}\, \zeta_0^m\, | \lambda |^{-\frac{m}{k}(t+1)+m h_0} \sum_{m_0+\cdots+m_n=m} | \lambda |^{\frac{w}{k}(m_1+2m_2+\cdots+n m_n)}$$

Now, for $m_0, \dots m_n \geqq 0$ and $m_0 + \cdots + m_n = m$ the greatest value of $m_1 + 2 m_2 + \cdots + n m_n$ is $n m$. Thus, with $|\lambda| \geqq 1$, one has

$$\sum_{m_0+\cdots+m_n=m} | \lambda |^{\frac{w}{k}(m_1+2m_2+\cdots+n m_n)} \leqq | \lambda |^{\frac{w}{k} n m} q_m,$$

where

(10. 28) $$q_m = \sum_{m_0+\cdots+m_n=m} 1$$

and

(10. 29) $$| K_m(z_0(x, \lambda)) | \leqq \bar{k}\, \zeta_0^m\, q_m\, | \lambda |^{-\frac{2}{k}(t+2)+2h_0+2\frac{w}{k}n} \quad (\text{in } \Gamma(a_1, b_1; R_2);\ m=2, \dots \nu),$$

provided we take t so that

(10. 29 a) $$-\frac{1}{k}(t+1) + h_0 + \frac{w}{k} n \quad \left[= -\frac{1}{k}(t+1) + h_2 + \frac{w}{k} \right] \leqq 0.$$

By virtue of (9. 49)

(10. 29 b) $$| K(z_0(x, \lambda)) | \leqq \bar{k}\, k_0\, | \lambda |^{-\frac{2}{k}(t+1)+2h_0+2\frac{w}{k}n} \qquad \left(k_0 = \sum_{m=2}^{\nu} \zeta_0^m q_m \right)$$

in $\Gamma(a_1, b_1; R_2)$ and, by (10. 13 b) and (10. 14)

(10. 30) $$| \beta_1(x, \lambda) | \leqq p\, | \lambda |^{\frac{1}{k}(nh+w)}\, | K(z_0(x, \lambda)) | \leqq \gamma_1 | \lambda |^{-\beta_1} \qquad (\text{in } \Gamma(a_1, b_1; R_2)),$$

where

(10. 30 a) $$\gamma_1 = p\, \bar{k}\, k_0, \quad \beta_1 = \frac{2}{k}(t+1) - 2h_0 - 2\frac{w}{k}n$$
$$- \frac{1}{k}(nh + w) = 2\beta_0 - 2h_2 + \frac{1}{k}(nh - w).$$

In consequence of (10. 30) and Lemma 10. 2 there exists a solution $z_1(x, \lambda)$ of the equation (10. 13; $j = 1$) satisfying

$$(10.\ 31) \qquad |z_1^{(i)}(x, \lambda)| \leqq \zeta_0 |\lambda|^{-\beta_1 + \frac{h'}{k}(i-n)+h_2} \qquad (\zeta_1 = n'\gamma_1;\ \text{in } \Gamma(a_1, b_1; R_2))$$

for $i = 0, \ldots n$.

We choose t *so that in addition to* (10. 29 a) *the inequality*

$$(10.\ 32) \qquad d' = -\frac{1}{k}(t+1) + 2h_2 + 2\frac{w}{k} < 0 \qquad \left(h_2 = \frac{1}{k}(h+w+\omega)\right)$$

is satisfied.

By (10. 26) and (10. 31) we have

$$(10.\ 33) \qquad |z_0^{(i)}(x, \lambda)| \leqq \zeta_0 |\lambda|^{d_i}, \quad |z_1^{(i)}(x, \lambda)| \leqq \zeta_1 |\lambda|^{d_i} |\lambda|^{d'}$$

$$\left[i = 0, \ldots n;\quad d_i = -\frac{1}{k}(t+1) + \frac{h'}{k}\, i + h_0;\quad \text{in } \Gamma(a_1, b_1; R_2)\right].$$

We take λ, *in* R_2, *so that* $|\lambda| \geqq \lambda_0$ $(\geqq 1)$, *where* λ_0 *is sufficiently great so that*

$$(10.\ 34) \qquad \zeta_1 |\lambda|^{d'} \leqq \zeta_0\, \varrho \qquad (\text{for } |\lambda| \geqq \lambda_0;\ d' \text{ from } (10.\ 32))$$

where ϱ *is some fixed number such that* $0 < \varrho < 1$. With $\lambda_0 > 1$ one may secure (10. 34) taking t sufficiently great. Whence (10. 33) will yield

$$(10.\ 35) \qquad z_0^{(i)}(x, \lambda) = \lambda^{d_i} z_{0,i}(x, \lambda), \quad z_1^{(i)}(x, \lambda) = \lambda^{d_i} z_{1,i}(x, \lambda),$$

where

$$(10.\ 35\text{ a}) \qquad |z_{0,i}(x, \lambda)| \leqq \zeta_0, \qquad |z_{1,i}(x, \lambda)| \leqq \zeta_0\, \varrho$$

$$\left[i = 0, \ldots n;\quad d_i = -\frac{1}{k}(t+1) + \frac{1}{k}h'\, i + h_0;\ \text{in } \Gamma(a_1, b_1, R_2)\right].$$

With a view to proof by induction *a supposition is now made that for some* $j \geqq 2$ *we have*

$$(10.\ 36) \qquad z_s^{(i)}(x, \lambda) = \lambda^{d_i} z_{s,i}(x, \lambda) \quad (s = 0, 1, \ldots j-1;\ i = 0, \ldots n),$$

$$(10.\ 36\text{ a}) \qquad |z_{s,i}(x, \lambda)| \leqq \zeta_0\, \varrho^s \qquad (i = 0, \ldots n;\ \text{in } \Gamma(a_1, b_1; R_2))$$

for $s = 0, 1, \ldots j-1$.

On writing

$$(10.\ 37) \qquad w_s^{(i)}(x, \lambda) = z_0^{(i)}(x, \lambda) + \cdots + z_s^{(i)}(x, \lambda) = \lambda^{d_i} w_{s,i}(x, \lambda),$$

one has

(10. 37 a) $\quad |w_{s,i}(x,\lambda)| \leqq \varrho_0 = \dfrac{\zeta_0}{1-\varrho} \quad (s=0, \ldots j-1;\ i=0, \ldots n;$ in $\Gamma(a_1, b_1; R_2))$.

By (10. 13 b) and (10. 14)

$$|\beta_j(x,\lambda)| \leqq p\,|\lambda|^{\frac{1}{k}(nh+w)}\,|K(w_{j-2}+z_{j-1}) - K(w_{j-2})| \leqq p\,|\lambda|^{\frac{1}{k}(nh+w)} \sum_{m=2}^{\nu} |T_m|,$$

where (compare with (6. 19))

$$T_m = \sum_{m_0+\cdots+m_n=m} k_m^{m_0, \ldots m_n}(t;x,\lambda)\left[\prod_{\alpha=0}^{n}(w_{j-2}^{(\alpha)} + z_{j-1}^{(\alpha)})^{m_\alpha}\lambda^{-\alpha m_\alpha \frac{h}{k}} - \prod_{\alpha=0}^{n}(w_{j-2}^{(\alpha)})^{m_\alpha}\lambda^{-\alpha m_\alpha \frac{h}{k}}\right]$$

$$= \sum_{m_0, \ldots} k_m^{m_0, \ldots m_n}(t;x,\lambda)\,\lambda^{f(m_0, \ldots m_n)}\left[\prod_{\alpha=0}^{n}(w_{j-2,\alpha} + z_{j-1,\alpha})^{m_\alpha} - \prod_{\alpha=0}^{n}(w_{j-2,\alpha})^{m_\alpha}\right],$$

where

$$f(m_0, \ldots m_n) = m\left[-\frac{1}{k}(t+1) + h_0\right] + \frac{w}{k}\sum_{\alpha=1}^{n}\alpha\, m_\alpha. \tag{10. 38}$$

Thus

$$T_m = \sum_{m_0+\cdots+m_n=m} k_m^{m_0, \ldots m_n}(t;x,\lambda)\,\lambda^{f(m_0, \ldots m_n)}\left\{\prod_{\alpha=1}^{m}(w_{j-2,i_\alpha} + z_{j-1,i_\alpha}) - \prod_{\alpha=1}^{m} w_{j-2,i_\alpha}\right\} \tag{10. 39}$$

(sets $(i_1, \ldots i_m)$ depending on $(m_0, \ldots m_n)$).

The difference of products involved above can be expressed as

$$\{\cdots\} = \sum_{\gamma_1=1}^{m} z_{j-1,i_{\gamma_1}} \prod_{s\neq\gamma_1} w_{j-2,i_s} + \sum_{\gamma_1<\gamma_2=1}^{m} z_{j-1,i_{\gamma_1}} z_{j-1,i_{\gamma_2}} \prod_{s\neq\gamma_1,\gamma_2} w_{j-2,i_s}$$
$$+ \cdots + z_{j-1,i_1} z_{j-1,i_2} \ldots z_{j-1,i_m}.$$

In view of (10. 36 a) and (10. 37 a) this differences satisfies

$$|\{\cdots\}| \leqq \sum_{\gamma_1=1}^{m} \varrho_0^{m-1}(\zeta_0\varrho^{j-1}) + \sum_{\gamma_1<\gamma_2=1}^{m}\varrho_0^{m-2}(\zeta_0\varrho^{j-1})^2 + \cdots + (\zeta_0\varrho^{j-1})^m$$

$$= (\varrho_0 + \zeta_0\varrho^{j-1})^m - \varrho_0^m = \varrho_0^m\left[\left(1 + \frac{\zeta_0}{\varrho_0}\varrho^{j-1}\right)^m - 1\right] \quad (\text{in } \Gamma(a_1, b_1; R_2)).$$

With the aid of the inequality subsequent to (6. 22) we finally obtain

$$|\{\cdots\}| \leqq \varrho_0^m m \left(1 + \frac{\zeta_0}{\varrho_0}\varrho^{j-1}\right)^{m-1} \frac{\zeta_0}{\varrho_0}\varrho^{j-1},$$

which by virtue of (10. 39), (10. 38) and (10. 27) implies that

$$|T_m| \leqq \bar{k} \sum_{m_0+\cdots+m_n=m} |\lambda|^{j(m_0,\ldots m_n)} |\{\cdots\}|$$

$$\leqq |\lambda|^{m\left[-\frac{1}{k}(t+1)+h_0\right]} \bar{k}\, \varrho_0^m m \left(1 + \frac{\zeta_0}{\varrho_0}\varrho^{j-1}\right)^{m-1} \frac{\zeta_0}{\varrho_0}\varrho^{j-1} \sum_{m_0,\ldots} |\lambda|^{\frac{w}{k}(m_1+2m_2+\cdots+nm_n)}$$

$$\leqq t_m \varrho^j |\lambda|^{m\left[-\frac{1}{k}(t+1)+h_0\right]} |\lambda|^{\frac{w}{k}nm},$$

where

(10. 40) $$t_m = \bar{k}\, q_m \varrho_0^m m (2-\varrho)^{m-1}\left(\frac{1-\varrho}{\varrho}\right) \qquad (q_m \text{ from } (10.\ 28)).$$

The above is asserted for x, λ ($|\lambda| \geqq 1$) in $\Gamma(a_1, b_1; R_2)$. By virtue of (10. 29 a)

$$|T_m| \leqq t_m \varrho^j |\lambda|^{2\left[-\frac{1}{k}(t+1)+h_0+\frac{w}{k}n\right]} \qquad (m = 2, \ldots \nu),$$

which implies in consequence of the inequality subsequent to (10. 37 a) that

(10. 41) $$|\beta_j(x, \lambda)| \leqq \gamma_j |\lambda|^{-\beta_j} \qquad (\text{in } \Gamma(a_1, b_1; R_2)),$$

where

(10. 41 a) $$\gamma_j = (t_2 + t_3 + \cdots + t_\nu)\, p\, \varrho^j, \quad -\beta_j = \frac{1}{k}(nh + w) - \frac{2}{k}(t+1) + 2h_0 + \frac{w}{k}n;$$

it is noted that

(10. 41 b) $$\beta_j = \beta_1 \qquad (\beta_1 \text{ from } (10.\ 30\ a)).$$

Applying Lemma 10. 2 to the equation $T(z_j) = \beta_j(x, \lambda)$ (cf. (10. 41)), a solution $z_j(x, \lambda)$ is obtained for which

(10. 42) $$|z_j^{(i)}(x, \lambda)| \leqq n' \gamma_j |\lambda|^{-\beta_1 + \frac{h'}{k}(i-n)+h_2} = n' \gamma_j |\lambda|^{d_i} |\lambda|^{d'}$$

$$\left[i = 0, \ldots n;\ h_2 = \frac{1}{k}(h + w + \omega);\ \text{in } \Gamma(a_1, b_1; R_2)\right].$$

In (10. 42) d_i, d' are from (10. 33) and (10. 32).

We take λ, in R_2, with $|\lambda| \geqq \lambda_0$, where λ_0 is so great that

(10. 43) $$(t_2 + \cdots + t_\nu) n' p |\lambda|^{d'} \leqq \zeta_0 \qquad (for\ |\lambda| \geqq \lambda_0).$$

One may choose λ_0 independent of j.

Substituting γ_j from (10. 41 a) in (10. 42) and applying (10. 43), we derive

(10. 44) $$z_j^{(i)}(x, \lambda) = \lambda^{d_i} z_{j,i}(x, \lambda) \qquad (i = 0, \ldots n),$$

with

(10. 44 a) $$|z_{j,i}(x, \lambda)| \leqq \zeta_0 \varrho^j \qquad (i = 0, \ldots n; \text{ in } \Gamma(a_1, b_1; R_2)).$$

It is clear that *equations* $T(z_j) = \beta_j(x, \lambda)$ *can be solved in succession so as to determine functions* $z_j(x, \lambda)$ $(j = 0, 1, \ldots)$ *for which* (10. 44), (10. 44 a) *may be asserted for*

$$j = 0, 1, \ldots; \quad i = 0, \ldots n.$$

Moreover, we shall have

$$|\beta_j(x, \lambda)| \leqq |\lambda|^{-\beta_1} \gamma_j \qquad (\gamma_j = (t_2 + \cdots + t_\nu) p \varrho^j)$$

for $j = 1, 2, \ldots$ and for x, λ in $\Gamma(a_1, b_1; R_2)$. The $\beta_j(x, \lambda)$ will be integrable in x for x on (a_1, b_1).

In terms of the above functions $z_j(x, \lambda)$ one may now form the series (10. 10). One will have

(10. 45) $$\varrho^{(i)}(x, \lambda) = z_0^{(i)}(x, \lambda) + z_1^{(i)}(x, \lambda) + \cdots = \lambda^{d_i} \sum_{s=0}^{\infty} z_{s,i}(x, \lambda)$$

$(i = 0, \ldots n)$. *The function* $\varrho(x, \lambda)$ *will be a solution of the transformed equation* (9. 50) *and will satisfy*

(10. 46) $$|\varrho^{(i)}(x, \lambda)| \leqq \frac{\zeta_0}{1 - \varrho} |\lambda|^{d_i} \qquad (i = 0, \ldots n; \text{ in } \Gamma(a_1, b_1; R_2)),$$

with $d_i = -\frac{1}{k}(t + 1) + \frac{h'}{k} i + h_0$. By (10. 26 a) and (10. 23)

(10. 46 a) $$d_i = -\frac{1}{k}(t + 1) + \frac{h'}{k}(i + 1) + \frac{1}{k}[\omega - (n - 1) w] \qquad (\omega \text{ from } (10.\ 5)).$$

By virtue of Lemma 9. 2 and of the result just formulated it is possible to assert the following theorem.

Existence Theorem 10. 1. *Suppose* $s(x, \lambda)$ (9. 28), (9. 28 a) *is a formal solution for* x *on* (a, b) *of the formal non linear homogeneous differential equation* (9. 8),

(9. 8 b) (*cf.* *Lemma* 9. 1). *Correspondingly the formal equation may be written as* (9. 25), (9. 25 a): *the »actual» equation* $F_\nu = 0$ *may be expressed as* (9. 29), (9. 29 a). *Associated with* $s(x, \lambda)$ *the non linear problem has the linear equation* $T(\varrho) = 0$ (10. 1), *whose solutions involve polynomials* $Q_i(x, \lambda)$ (*independent of* t, *if* t *is sufficiently great*) (*cf.* (10. 2)). *We note that existence of solutions of* $T(\varrho) = 0$ *of form* (10. 2), (10. 2 a) *is asserted for* x, λ *in* $\Gamma(a_1, b_1; R_1)$ (*notation of the early part of section* 9); (a_1, b_1) *is a closed sub interval of* (a, b); R_1 *is a regular sub region of* R *such that no function* (10. 3) *changes sign for* x, λ *in* $\Gamma(a_1, b_1; R_1)$. *We let* R_2 *be a regular sub region of* R_1 *so that no function* $\Re\, Q_j^{(1)}(x, \lambda)$ *changes sign for* x, λ *in* $\Gamma(a_1, b_1; R_2)$.

In the Case 9. 45 (a_1, b_1) *is to be chosen so that* $p_{n,0}(x)$ *of* (9. 44) *does not vanish for* $a_1 \leqq x \leqq b_1$.

In the Case 9. 46 *we choose* (a_1, b_1) *so that* $p_{n,w}(x)$ *does not vanish for* $a_1 \leqq x \leqq b_1$.

Given an integer t ($t \geqq t'$; t' *suitably great*), *however large, there exists a solu, tion* $y(x, \lambda)$ *of* $F_\nu = 0$, *defined for* x, λ ($|\lambda| \geqq \lambda_0$; λ_0 *suitably great*) *in* $\Gamma(a_1, b_1; R_2)$, *such that*

$$y^{(i)}(x, \lambda) \sim s^{(i)}(x, \lambda) \qquad [x, \lambda \text{ in } \Gamma(a_1, b_1; R_2);\ \text{to } n(t) \text{ terms};\ i = 0, \ldots n], \tag{10. 47}$$

where $n(t) \to \infty$ *with* t. *More specifically, one has*

$$y^{(i)}(x, \lambda) = \frac{d^i}{d x^i}\left[e^{Q(x, \lambda)}(\sigma(t; x, \lambda) + \varrho(x, \lambda))\right] \tag{10. 47 a}$$

($i = 0, 1, \ldots n$) *with*

$$\sigma(t; x, \lambda) = \sigma_0(x) + \sigma_1(x)\lambda^{-\frac{1}{k}} + \cdots + \sigma_t(x)\lambda^{-\frac{t}{k}} \tag{10. 47 b}$$

and $\varrho(x, \lambda)$ *satisfies in* $\Gamma(a_1, b_1; R_2)$ *the relations* (10. 46), (10. 46 a).

In the above $h' = h + w$, *where* $w = 0$ *in the Case* 9. 45.

Briefly, the essence of the developments of this work is as follows.

When the given 'actual' non linear homogeneous n-th order algebraic differential equation $F_\nu = 0$ has a formal solution s of the same type as occurs in the corresponding linear case, one can always construct regions R and 'actual solutions' y_t of $F_\nu = 0$ for which

$$y_t^{(i)} \sim s^{(i)} \qquad (i = 0, \ldots n;\ \text{in } R;\ \text{to } n(t) \text{ terms};\ n(t) \to \infty \text{ with } t).$$

Essentially, the regions are determined by the character of a certain linear problem associated with $F_\nu = 0$.