THE POWEROID, AN EXTENSION OF THE MATHEMATICAL NOTION OF POWER.

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1. The essential property of the factorials occurring in finite differences is that, when the proper difference operator is applied to the factorial, we obtain the factorial of the preceding degree, while every factorial of degree > 0 vanishes for $x = 0$. It is this property which makes factorials suited to expansion purposes. Therefore the question naturally presents itself, to find the most general class of polynomials possessing this property. A still more general class has been considered by Aitken[1] who introduces a new operator at each step; but it appears that there is considerable advantage in investigating separately the case where all the operators are identical, because in that case we may avail ourselves of certain theorems belonging to the calculus of operations and thus obtain convenient explicit expressions for the polynomials.

It will be assumed in this paper that the reader is familiar with certain elementary notions belonging to the calculus of operations, such as the definition and general properties of omega-symbols and their sub-class the theta-symbols. For details, and for the notation which will be employed in this paper, the reader is referred to the work quoted below.[2]

Where nothing else is said, the object of the operations will be a polynomial,

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[1] A. C. AITKEN: On a generalization of formulae for polynomial interpolation. Journal of the Institute of Actuaries, Vol. LXI (1930), p. 107. See also Proceedings of the Edinburgh Mathematical Society, Series 2, Vol. I (1929), p. 199.

[2] J. F. STEFFENSEN: Interpolation (Baltimore 1927), § 2 and § 18, or the same articles in the Danish edition of 1925.

so that expansions in powers of theta-symbols are permitted.[1] All theta-symbols occurring in this paper will be of the form

$$\theta = \varphi(D) = \sum_{\nu=1}^{\infty} k_\nu D^\nu \qquad (k_1 \neq 0), \tag{1}$$

the expansion being convergent if the symbol of differentiation D is replaced by a sufficiently small *number*. In other terms: the function $\varphi(t)$ is assumed to be analytical at the origin.

If, therefore, $\Omega(z_1, z_2, \ldots z_n)$ can be developed in powers of $z_1, z_2, \ldots z_n$, and if $\theta_1, \theta_2, \ldots \theta_n$ are various theta-symbols, then $\Omega(\theta_1, \theta_2, \ldots \theta_n)$ is a well-defined operation, because Ω can be developed in powers of $\theta_1, \theta_2, \ldots \theta_n$ and, therefore, in powers of D. Ω itself is not always a theta-symbol, because its expansion may contain a constant term, but

$$\Omega(\theta_1, \theta_2, \ldots \theta_n) - \Omega(0, 0, \ldots 0)$$

is a theta-symbol which, however, will only be of the form (1), if the first power of D is not missing in its expansion.

After these preliminaries we proceed to prove that, θ being given by (1), a polynomial which we shall denote by $x^{\overline{r}|}$ of degree r exists, this polynomial being completely determined by having to satisfy the conditions

$$x^{\overline{0}|} = 1; \quad 0^{\overline{r}|} = 0 \qquad (r > 0); \tag{2}$$

$$\theta\, x^{\overline{r}|} = r\, x^{\overline{r-1}|} \qquad (r > 0). \tag{3}$$

Since $x^{\overline{0}|} = 1$ is already known, we may assume $r > 0$, in which case there is, according to (2), no constant term. We may, therefore, write $x^{\overline{1}|} = \alpha x$, and it follows from (3) and (1) that $k_1 \alpha = 1$, so that α is determined, since $k_1 \neq 0$.

We now proceed by induction. If, for a certain degree $r - 1$, the coefficients a_μ in

$$x^{\overline{r-1}|} = \sum_{\mu=1}^{r-1} a_\mu x^\mu$$

have been determined, then the coefficients b_ν in the polynomial of the following degree

[1] This important remark is due to J. L. W. V. Jensen: Sur une identité d'Abel et sur d'autres formules analogues. Acta mathematica, vol. 26 (1902), p. 314.

$$x^{\overline{r}|} = \sum_{\nu=1}^{r} b_\nu x^\nu$$

are completely determined by comparing the coefficients of x^μ in $r\,x^{\overline{r-1}|}$ and $\theta\,x^{\overline{r}|}$. Now

$$\theta\,x^{\overline{r}|} = \sum_{s=1}^{r} k_s D^s \sum_{\nu=1}^{r} b_\nu x^\nu$$

$$= \sum_{s=1}^{r} k_s \sum_{\nu=s}^{r} b_\nu \nu^{(s)} x^{\nu-s}$$

or, putting $\nu = \mu + s$,

$$\theta\,x^{\overline{r}|} = \sum_{s=1}^{r} k_s \sum_{\mu=0}^{r-s} b_{\mu+s} (\mu + s)^{(s)} x^\mu$$

$$= \sum_{\mu=0}^{r-1} x^\mu \sum_{s=1}^{r-\mu} k_s b_{\mu+s} (\mu + s)^{(s)}.$$

We therefore have

$$r\,a_\mu = \sum_{s=1}^{r-\mu} (\mu + s)^{(s)} k_s b_{\mu+s} \tag{4}$$

which is valid for $\mu = 0, 1, \ldots r - 1$, if we put $a_0 = 0$. Thus we may calculate in succession $b_r, b_{r-1}, \ldots b_2, b_1$ by the equations

$$\begin{aligned}
r\,a_{r-1} &= r\,k_1 b_r \\
r\,a_{r-2} &= (r-1) k_1 b_{r-1} + r^{(2)} k_2 b_r \\
&\cdots\cdots\cdots\cdots\cdots \\
r\,a_0 \quad &= k_1 b_1 + 2!\,k_2 b_2 + \cdots + r!\,k_r b_r,
\end{aligned}$$

the solution being possible and unique, since $k_1 \neq 0$.

To a given θ corresponds, then, a completely determined polynomial of degree r, satisfying the conditions (2) and (3). The simplest of these polynomials is obtained by putting $\theta = D$ and is evidently x^r. On account of the close analogy between the polynomials $x^{\overline{r}|}$ and the powers x^r we suggest for the former the name »poweroids».

It may be noted at once that

$$\theta^r x^{\overline{r}|} = r!; \tag{5}$$

and, since $\theta^r x^r = r!\,k_1^r$, it follows that the coefficient of x^r in the expansion of $x^{\overline{r}|}$ is k_1^{-r}.

Further, we have

$$\theta (x + a)^{\overline{r|}} = r (x + a)^{\overline{r-1|}}; \tag{6}$$

for, putting $x + a = z$, it is seen that $D_x = D_z$, so that $\theta_x = \theta_z$.

Any polynomial $f(x)$ of degree n can be expressed in the form

$$f(x) = \sum_{\nu=0}^{n} c_\nu \, x^{\overline{\nu|}},$$

since this is a polynomial of degree n with $n + 1$ arbitrary constants. In order to determine these, we observe that

$$\theta^r f(x) = \sum_{\nu=r}^{n} c_\nu \, \nu^{(r)} \, x^{\overline{\nu-r|}}$$

and hence, putting $x = 0$, $r! \, c_r = \theta^r f(0)$, so that

$$f(x) = \sum_{\nu=0}^{n} \frac{\theta^\nu f(0)}{\nu!} x^{\overline{\nu|}}. \tag{7}$$

This is the expansion of a polynomial in poweroids of x. The analogy with the corresponding Maclaurin expansion is obvious.

2. The solution of equations of the form (4) is, of course, not a suitable process for obtaining the poweroid corresponding to a given θ. But several direct expressions can be given for the poweroid. One of them is

$$x^{\overline{\nu|}} = \theta' \left(\frac{D}{\theta}\right)^{\nu+1} x^\nu \tag{8}$$

where we have put

$$\theta' = \frac{d\theta}{dD} = \varphi'(D). \tag{9}$$

In order to show that (8) satisfies the conditions (2) and (3), we observe that, applying the operation θ to (8), we get, for $\nu > 0$,

$$\begin{aligned} \theta \, x^{\overline{\nu|}} &= \theta' \left(\frac{D}{\theta}\right)^{\nu} D x^\nu \\ &= \nu \, \theta' \left(\frac{D}{\theta}\right)^{\nu} x^{\nu-1} = \nu \, x^{\overline{\nu-1|}}, \end{aligned}$$

or (3). Further, it follows from (8) that

$$x^{\overline{0}|} = \theta' \frac{D}{\theta} \cdot 1 = \frac{k_1 + 2k_2 D + 3k_3 D^2 + \cdots}{k_1 + k_2 D + k_3 D^2 + \cdots} \cdot 1$$

or, developing in powers of D,

$$x^{\overline{0}|} = \left(1 + \frac{k_2}{k_1} D + \cdots\right) \cdot 1 = 1.$$

Finally, we must prove that $0^{\overline{\nu}|} = 0$ for $\nu > 0$. This is equivalent to requiring that the development of $\theta' \left(\frac{D}{\theta}\right)^{\nu+1}$ in powers of D does not contain D^ν, or that the development of

$$\varphi'(t) \left(\frac{t}{\varphi(t)}\right)^{\nu+1} = -\frac{1}{\nu} t^{\nu+1} D \left(\frac{1}{\varphi(t)}\right)^\nu$$

does not contain t^ν. But this is obvious, because differentiation term by term of the expansion of $\left(\frac{1}{\varphi(t)}\right)^\nu$ does not lead to $\frac{1}{t}$.

Another expression for the poweroid which is sometimes preferable is

$$x^{\overline{\nu}|} = x \left(\frac{D}{\theta}\right)^\nu x^{\nu-1}. \tag{10}$$

It is seen at once that this expression satisfies the condition $0^{\overline{\nu}|} = 0$ for $\nu > 0$. Further, for $\nu = 0$, (10) becomes

$$x^{\overline{0}|} = x \left(\frac{D}{\theta}\right)^0 x^{-1};$$

but $\left(\frac{D}{\theta}\right)^0$ is the identical operation which may evidently be applied to x^{-1} although x^{-1} is not a polynomial. We therefore have $x^{\overline{0}|} = x \cdot x^{-1} = 1$. Writing, finally, for a moment, instead of (10),

$$Q_\nu(x) = x \left(\frac{D}{\theta}\right)^\nu x^{\nu-1},$$

it remains to prove that $\theta Q_\nu(x) = \nu Q_{\nu-1}(x)$. Now we have, by (1), $f(x)$ being a polynomial so that the series is, in reality, finite

$$\theta x f(x) = \sum_{\nu=1}^{\infty} k_\nu [x f^{(\nu)}(x) + \nu f^{(\nu-1)}(x)]$$

or

$$\theta x f(x) = x \theta f(x) + \theta' f(x). \tag{11}$$

Hence

$$\theta\, Q_\nu(x) = x\,\theta \left(\frac{D}{\theta}\right)^{\nu} x^{\nu-1} + \theta' \left(\frac{D}{\theta}\right)^{\nu} x^{\nu-1}$$

$$= x \left(\frac{D}{\theta}\right)^{\nu-1} (\nu - 1)\, x^{\nu-2} + \theta' \left(\frac{D}{\theta}\right)^{\nu} x^{\nu-1}$$

or, by (8),

$$\theta\, Q_\nu(x) = (\nu - 1)\, Q_{\nu-1}(x) + x^{\overline{\nu-1|}}$$

which may be written

$$\theta\, Q_\nu(x) = \nu\, Q_{\nu-1}(x) + [x^{\overline{\nu-1|}} - Q_{\nu-1}(x)].$$

Hence, if for a particular value of ν we have proved $Q_{\nu-1}(x) = x^{\overline{\nu-1|}}$, then we have $\theta\, Q_\nu(x) = \nu\, Q_{\nu-1}(x)$, and therefore also $Q_\nu(x) = x^{\overline{\nu|}}$. But we have evidently as initial value $Q_0(x) = 1 = x^{\overline{0|}}$, so that the proof by induction is complete.

If, instead of (1), we write

$$\theta = \frac{D}{\psi(D)} \qquad (\psi(0) \neq 0), \qquad (12)$$

(10) assumes the form

$$x^{\overline{\nu|}} = x\, \psi^{\nu}(D)\, x^{\nu-1} \qquad (\psi(0) \neq 0). \qquad (13)$$

In this expression $\psi(t)$ may be any function which is analytical at the origin and does not vanish there.

3. It is also possible to calculate the poweroids by recurrence. From (8) we obtain

$$x^{\nu} = \frac{dD}{d\theta} \left(\frac{\theta}{D}\right)^{\nu+1} x^{\overline{\nu|}}. \qquad (14)$$

Replacing now, in (10), ν by $\nu + 1$ and inserting the expression (14), we find the desired formula

$$x^{\overline{\nu+1|}} = x \frac{dD}{d\theta}\, x^{\overline{\nu|}}. \qquad (15)$$

It is on several occasions useful to write

$$x^{\overline{\nu|}-1} \equiv \frac{x^{\overline{\nu|}}}{x} \qquad (16)$$

and to consider the polynomial $x^{\overline{\nu+1|}-1}$ along with $x^{\overline{\nu|}}$, both of these having the degree ν. In this notation (15) may be written

$$\theta'\, x^{\overline{\nu+1|}-1} = x^{\overline{\nu|}}. \qquad (17)$$

It should be noted that θ' is not a theta-symbol, since its expansion in powers of D, or

$$\theta' = k_1 + 2\,k_2\,D + 3\,k_3\,D^2 + \cdots, \tag{18}$$

contains the non-vanishing constant term k_1. The degree of a polynomial is, therefore, not diminished by applying the operation θ' to it.

Also the operation θ, applied to $x^{\overline{\nu+1}|-1}$, leads to a simple result, viz.

$$\theta\, x^{\overline{\nu+1}|-1} = \nu\, x^{\overline{\nu}|-1}. \tag{19}$$

In order to prove this, we observe that, according to (10)

$$x^{\overline{\nu+1}|-1} = \left(\frac{D}{\theta}\right)^{\nu+1} x^{\nu} \tag{20}$$

whence

$$\theta\, x^{\overline{\nu+1}|-1} = \left(\frac{D}{\theta}\right)^{\nu} D\, x^{\nu}$$

$$= \nu \left(\frac{D}{\theta}\right)^{\nu} x^{\nu-1} = \nu\, x^{\overline{\nu}|-1}.$$

Any polynomial $f(x)$ of degree n may be expressed in the form

$$f(x) = \sum_{\nu=0}^{n} a_\nu\, x^{\overline{\nu+1}|-1}.$$

In order to determine the coefficients, we have first, by (19),

$$\theta^r f(x) = \sum_{\nu=r}^{n} a_\nu\, \nu^{(r)}\, x^{\overline{\nu+1-r}|-1}$$

and thereafter, by (17),

$$\theta'\,\theta^r f(x) = \sum_{\nu=r}^{n} a_\nu\, \nu^{(r)}\, x^{\overline{\nu-r}|}$$

so that, putting $x = 0$, we find $a_r = \frac{1}{r!}\,\theta'\,\theta^r f(0)$. The required expansion is, therefore,

$$f(x) = \sum_{\nu=0}^{n} \frac{\theta'\,\theta^\nu f(0)}{\nu!}\, x^{\overline{\nu+1}|-1} \tag{21}$$

which should be compared with (7).

4. In the expressions (8), (10) and (13), $x^{\nu\rceil}$ is obtained by operating on a power of x, that is, on the poweroid corresponding to the operator D. It is, however, also possible to obtain $x^{\nu\rceil}$ by operating on any other poweroid, if the operator corresponding to the latter is introduced. Let $x_I^{\nu\rceil}$ be such a poweroid and θ_I the corresponding operator. Then it can be proved that

$$x^{\nu\rceil} = \frac{d\,\theta}{d\,\theta_I}\left(\frac{\theta_I}{\theta}\right)^{\nu+1} x_I^{\nu\rceil}, \tag{22}$$

corresponding to (8), and

$$x^{\nu\rceil} = x\left(\frac{\theta_I}{\theta}\right)^{\nu} x_I^{\nu\rceil-1}, \tag{23}$$

corresponding to (10), or, putting $\theta = \dfrac{\theta_I}{\psi(\theta_I)}$,

$$x^{\nu\rceil} = x\,\psi^{\nu}(\theta_I)\,x_I^{\nu\rceil-1} \qquad (\psi(0) \neq 0), \tag{24}$$

corresponding to (13).

In order to prove (22) we observe that, applying (14) to θ_I and $x_I^{\nu\rceil}$, we have

$$x^{\nu} = \frac{d\,D}{d\,\theta_I}\left(\frac{\theta_I}{D}\right)^{\nu+1} x_I^{\nu\rceil}$$

and, inserting this in (8),

$$\begin{aligned} x^{\nu\rceil} &= \frac{d\,\theta}{d\,D}\left(\frac{D}{\theta}\right)^{\nu+1} \frac{d\,D}{d\,\theta_I}\left(\frac{\theta_I}{D}\right)^{\nu+1} x_I^{\nu\rceil} \\ &= \frac{d\,\theta}{d\,\theta_I}\left(\frac{\theta_I}{\theta}\right)^{\nu+1} x_I^{\nu\rceil}, \end{aligned}$$

or (22).

Further, applying (10) to θ_I and $x_I^{\nu\rceil}$, we obtain

$$x^{\nu-1} = \left(\frac{\theta_I}{D}\right)^{\nu} x_I^{\nu\rceil-1}$$

and, inserting this in (10)

$$\begin{aligned} x^{\nu\rceil} &= x\left(\frac{D}{\theta}\right)^{\nu}\left(\frac{\theta_I}{D}\right)^{\nu} x_I^{\nu\rceil-1} \\ &= x\left(\frac{\theta_I}{\theta}\right)^{\nu} x_I^{\nu\rceil-1}, \end{aligned}$$

or (23).

A more symmetrical form may be given to (23) if we divide by x on both sides; we then have

$$x^{\overline{\nu}\rceil - 1} = \left(\frac{\theta_I}{\theta}\right)^{\nu} x_I^{\overline{\nu}\rceil - 1}, \tag{25}$$

and it is seen that solution with respect to $x_I^{\overline{\nu}\rceil - 1}$ produces an expression of the same form, as is already the case if (22) is solved with respect to $x_I^{\overline{\nu}\rceil}$.

5. While the operation θ is perfectly determined, the inverse operation θ^{-1} is not, but introduces an arbitrary additive constant, owing to the presence of D^{-1} in the expansion of $\theta^{-1} = D^{-1}\psi(D)$. But among the possible interpretations of the »theta integration» θ^{-1} we may select one which is perfectly determined. We denote it by I (the first letter in the word Integration) and define it by

$$I = \bar{D}\psi(D) \tag{26}$$

where $\bar{D}$ stands for that particular determination of D^{-1} which means integration from 0 to x (or whatever the variable may be). In (26) the operation $\psi(D)$ must always be performed before $\bar{D}$.

Caution is thus required in handling the operation I. For instance, if the operator obtained by expanding θ^{-1} and replacing thereafter D^{-1} by $\bar{D}$ is called $\bar{\theta}$, then I and $\bar{\theta}$ are not identical. As a simple example, let $\psi(D) = 1 + D$, then $\bar{\theta} = \bar{D} + 1$, but $I = \bar{D} + \bar{D}D$, and $\bar{D}D$ cannot be identified with 1, since

$$\bar{D}Df(x) = \int_0^x f'(x)\,dx = f(x) - f(0).$$

On the other hand $D\bar{D} = 1$, because

$$D\bar{D}f(x) = D\int_0^x f(x)\,dx = f(x).$$

The trouble is evidently that D and $\bar{D}$ are not commutative; this is why we may not exchange $\bar{D}$ and $\psi(D)$ in (26); and if the operation I is repeated ν times, it must be written $[\bar{D}\psi(D)]^{\nu}$ and not $\bar{D}^{\nu}\psi^{\nu}(D)$.

It can now be proved that

$$x^{\overline{\nu}|} = \nu!\, I^\nu\, 1, \tag{27}$$

because this expression satisfies the conditions (2) and (3).

We have evidently $x^{\overline{0}|} = 1$; and, since the operator $\bar{D}$ which always comes after $\psi(D)$ introduces the factor x, we have $0^{\overline{\nu}|} = 0$ for $\nu > 0$. Further, we have $\theta I = 1$, because

$$\theta I = \frac{D}{\psi(D)} \bar{D}\,\psi(D) = \frac{1}{\psi(D)} D\,\bar{D}\,\psi(D)$$

$$= \frac{1}{\psi(D)} \cdot \psi(D) = 1.$$

Therefore

$$\theta\, x^{\overline{\nu}|} = \nu!\, \theta I^\nu\, 1 = \nu!\, \theta I \cdot I^{\nu-1}\, 1$$

$$= \nu!\, I^{\nu-1}\, 1 = \nu\, x^{\overline{\nu-1}|}.$$

This completes the proof of (27).

By means of (27) we may write (7) in the form

$$f(x) = \sum_{\nu=0}^{n} \theta^\nu f(0) \cdot I^\nu\, 1, \tag{28}$$

which is included in the form given by Aitken l. c., when all his theta operators are made identical.

By (27) we find immediately

$$x^{\overline{\nu}|} = \nu\, I x^{\overline{\nu-1}|}, \tag{29}$$

which, like (15), may be used for calculating the poweroids by recurrence. As direct expressions for the poweroid, (8) and (10) are preferable to (27), owing to the peculiarities discussed above of the operation I.

6. Instead of (7) we may write symbolically

$$E^x = \sum_{\nu=0}^{\infty} \frac{x^{\overline{\nu}|}}{\nu!} \theta^\nu \tag{30}$$

or

$$e^{xD} = \sum_{\nu=0}^{\infty} \frac{x^{\overline{\nu}|}}{\nu!} \theta^\nu. \tag{31}$$

On both sides of these equations $f(0)$ is, as usual, left out, and the summation extends only apparently to infinity, since the operations are only applied to polynomials.

Differentiating (31) with respect to D, and dividing by x, we find

$$e^{xD} = \sum_{\nu=0}^{\infty} \frac{x^{\overline{\nu+1}|-1}}{\nu!} \theta' \theta^{\nu}, \tag{32}$$

or the symbolical form of (21).

When D and θ are replaced by *numbers*, we shall always write t instead of D, and ζ instead of θ. Thus we have, by definition,

$$\left.\begin{aligned} \zeta = \varphi(t) &= \frac{t}{\psi(t)} \\ &= \sum_{\nu=1}^{\infty} k_\nu t^\nu \qquad (k_1 \neq 0). \end{aligned}\right\} \tag{33}$$

Since t can be expanded in powers of ζ, the same applies to e^{xt}, and (31) gives the *form* of this expansion which is

$$e^{xt} = \sum_{\nu=0}^{\infty} \frac{x^{\overline{\nu}|}}{\nu!} \zeta^{\nu}. \tag{34}$$

The circle of convergence of this power series in ζ extends at least to the nearest singular point for the inverse function to $\zeta = \varphi(t)$, but can go further, as the example $x = 1$, $t = \mathrm{Log}(1 + \zeta)$ shows. For our purpose it suffices to consider values of ζ inside the aforesaid minimum region of convergence, in which case (34) is convergent for all values of x, because x does not enter into the determination of that region. We may, then, consider e^{xt} as the *generating function* of the poweroids, when e^{xt} is expanded in powers of ζ.

We obtain, therefore, by Bürmann's theorem[1], the following two expressions for the poweroid

$$x^{\overline{\nu}|} = D^{\nu}_{t=0}\, e^{xt} \frac{d\zeta}{dt} \left(\frac{t}{\zeta}\right)^{\nu+1} \tag{35}$$

and

$$x^{\overline{\nu}|} = x\, D^{\nu-1}_{t=0}\, e^{xt} \left(\frac{t}{\zeta}\right)^{\nu} \qquad (\nu > 0), \tag{36}$$

corresponding to (8) and (10).

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[1] See, for instance, Hurwitz-Courant: Funktionentheorie, 3. Auflage, p. 135.

We may, however, also obtain (35) and (36) without resorting to Bürmann's theorem. Let $\Phi(t)$ be any function, analytical at the origin, say,

$$\Phi(t) = \sum_{\nu=0}^{\infty} c_\nu t^\nu.$$

The coefficient of t^ν in the expansion of $\Phi(t)e^{xt}$ is, then,

$$\sum_{s=0}^{\nu} \frac{x^s}{s!} c_{\nu-s} = \frac{1}{\nu!} \Phi(D) x^\nu,$$

so that we have[1]

$$\Phi(t) e^{xt} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \Phi(D) x^\nu. \tag{37}$$

We note *en passant* that this theorem contains as a particular case, obtained by putting $x = 0$, the so-called *secondary form* of Maclaurin's theorem, or

$$\Phi(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \Phi(D) 0^\nu \tag{38}$$

which implies that

$$\Phi(D) 0^\nu = \Phi^{(\nu)}(0). \tag{39}$$

Considering now the coefficient of $t^{\nu-1}$ in the expansion (37), it is seen that

$$\Phi(D) x^{\nu-1} = D_{t=0}^{\nu-1} \Phi(t) e^{xt}$$

whence, putting $\Phi(t) = \psi^\nu(t)$,

$$\psi^\nu(D) x^{\nu-1} = D_{t=0}^{\nu-1} \psi^\nu(t) e^{xt}.$$

Multiplying by x and comparing with (13), we have (36), since $\psi(t) = \frac{t}{\zeta}$ according to (33).

As regards (35), it is seen from (37) that

$$\Phi(D) x^\nu = D_{t=0}^\nu \Phi(t) e^{xt}.$$

Putting now

$$\Phi(t) = \psi^{\nu+1}(t) \cdot D \frac{t}{\psi(t)},$$

[1] Compare L. M. Milne-Thomson: The Calculus of Finite Differences, Chapter VI, where a similar generating function is considered.

so that

$$\Phi(D) = \theta' \left(\frac{D}{\theta}\right)^{\nu+1},$$

we obtain

$$\theta' \left(\frac{D}{\theta}\right)^{\nu+1} x^{\nu} = D^{\nu}_{t=0}\, \psi^{\nu+1}(t) \left(D \frac{t}{\psi(t)}\right) e^{xt}$$

which, on comparing with (8), is seen to be identical with (35).

Since (31), when applied to a polynomial, may be considered to contain only a finite number of terms, also when the left-hand side is developed in powers of D, it is allowed to differentiate with respect to x on both sides. If we differentiate r times and put, thereafter, $x = 0$, we find

$$D^r = \sum_{\nu=r}^{\infty} D^r_{x=0}\, x^{\overline{\nu}|} \cdot \frac{\theta^{\nu}}{\nu!}. \tag{40}$$

In particular, we have

$$D\, x^{\overline{\nu}|} = D\,(x \cdot x^{\overline{\nu}|-1}) = x\, D\, x^{\overline{\nu}|-1} + x^{\overline{\nu}|-1},$$

so that, for $\nu \geq 1$, $D_{x=0}\, x^{\overline{\nu}|} = 0^{\overline{\nu}|-1}$. Hence we obtain by (40) for $r = 1$

$$D = \sum_{\nu=1}^{\infty} 0^{\overline{\nu}|-1} \cdot \frac{\theta^{\nu}}{\nu!}. \tag{41}$$

This formula and (40) are the formulas for numerical differentiation of a polynomial. But since (41) is the inversion of (1), we see at the same time that the expansion

$$t = \sum_{\nu=1}^{\infty} 0^{\overline{\nu}|-1} \cdot \frac{\zeta^{\nu}}{\nu!} \tag{42}$$

is the inversion of $\zeta = \varphi(t)$ in this sense that (42) represents that branch of the inverse function which vanishes for $\zeta = 0$. For (33) is convergent in a region including $t = 0$, and the inverse series in a region including $\zeta = 0$; and the form of the coefficients does not depend on the fact that the object of the operations is a polynomial. It therefore only remains to investigate the exact region of convergence of (42), which may be done in each case by the usual methods.

It may be noted that already (34) itself yields a solution of the equation $\zeta = \varphi(t)$ with respect to t, and this solution has the advantage of containing an arbitrary parameter x of which we may dispose in various ways.

7. We shall now consider some applications of the preceding theory, beginning by showing how the well-known factorials $x^{(\nu)}$, $x^{(-\nu)}$ and $x^{[\nu]}$ may be obtained when the corresponding θ-operators $\triangle$, ∇ and δ are given.

Putting first $\theta = \frac{E^\beta - 1}{\beta}$, we have

$$D = \text{Log}\, E = \frac{1}{\beta} \text{Log}\,(1 + \beta\theta)$$

and hence

$$\frac{dD}{d\theta} = \frac{1}{1 + \beta\theta} = E^{-\beta}$$

so that, by (15),

$$x^{\overline{\nu+1}|} = x\,(x - \beta)^{\overline{\nu}|}.$$

Repeated application of this formula leads to the poweroid

$$x^{\overline{\nu}|} = x\,(x-\beta)\,(x - 2\beta)\ldots(x - \overline{\nu - 1}\,\beta),$$

$x^{\overline{0}|}$ having here and everywhere else, by definition, the value 1.

If, in particular, $\beta = 1$, we have $\theta = \triangle$ corresponding to $x^{\overline{\nu}|} = x^{(\nu)}$, while $\beta = -1$ produces $\theta = \nabla$ corresponding to $x^{\overline{\nu}|} = x^{(-\nu)}$.

If we introduce the slightly more general operator

$$\theta = E^\alpha \cdot \frac{E^\beta - 1}{\beta}, \tag{43}$$

we may employ (23), putting

$$\theta_I = \frac{E^\beta - 1}{\beta}, \quad x_I^{\overline{\nu}|} = x\,(x - \beta)\ldots(x - \overline{\nu - 1}\,\beta).$$

In that case $\frac{\theta_I}{\theta} = E^{-\alpha}$, so that we get from (23)

$$x^{\overline{\nu}|} = x\,E^{-\nu\alpha}(x-\beta)\,(x - 2\beta)\ldots(x - \overline{\nu - 1}\,\beta)$$

or

$$x^{\overline{\nu}|} = x\,(x - \nu\alpha - \beta)\,(x - \nu\alpha - 2\beta)\ldots(x - \nu\alpha - \overline{\nu - 1}\,\beta). \tag{44}$$

If $\alpha = 0$, $\beta = \pm 1$ we have the results already found for descending and ascending differences. If $\alpha = -\frac{1}{2}$, $\beta = 1$ we have $\theta = \delta$ corresponding to $x^{\overline{\nu}|} = x^{[\nu]}$, that is, the results for central differences.

It is easy to prove that

$$\frac{d\,\delta}{d\,D} = \Box \tag{45}$$

so that, according to (17),

$$\Box\, x^{[\nu+1]-1} = x^{[\nu]} \tag{46}$$

to which may be joined, according to (19),

$$\delta\, x^{[\nu+1]-1} = \nu\, x^{[\nu]-1}. \tag{47}$$

If, finally, we let $\beta \to 0$ in (43) and (44), we get

$$\theta = E^{\alpha} D, \quad x^{\overline{\nu}|} = x\,(x - \nu\,\alpha)^{\nu-1}. \tag{48}$$

This poweroid which has been considered first by Abel[1] and later on by Halphen[2] and Jensen[3] may be called »Abel's poweroid».

Writing t and ζ for D and θ, (43) becomes

$$\zeta = e^{\alpha t} \cdot \frac{e^{\beta t} - 1}{\beta}, \tag{49}$$

and a solution of this equation may, by (34) and (44), be obtained in the form

$$e^{xt} = \sum_{\nu=0}^{\infty} \frac{\zeta^{\nu}}{\nu!}\, x\,(x - \nu\,\alpha - \beta)(x - \nu\,\alpha - 2\,\beta) \ldots (x - \nu\,\alpha - \overline{\nu - 1}\,\beta). \tag{50}$$

If $\beta \to 0$, it is seen that the equation

$$\zeta = t\, e^{\alpha t} \tag{51}$$

has a solution of the form[4]

$$e^{xt} = \sum_{\nu=0}^{\infty} \frac{\zeta^{\nu}}{\nu!}\, x\,(x - \nu\,\alpha)^{\nu-1} \tag{52}$$

which is an expansion in Abel's poweroids of x and in powers of ζ.

Another form of the solution is obtained from (42).

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[1] Démonstration d'une expression de laquelle la formule binôme est un cas particulier. Oeuvres I, p. 102.

[2] Sur une série d'Abel. Bulletin de la société mathématique, X, p. 67.

[3] l. c., p. 307.

[4] HURWITZ-COURANT l. c., p. 141, where also the case $\beta = 1$ is treated by BÜRMANN's theorem.

In compound interest the following equation occurs[1]

$$\tau = \xi (1 - \xi)^n \tag{53}$$

where n is positive and integral, and τ and ξ are both comprised between 0 and 1. It arises from (49) by putting $\zeta = \tau$, $e^{-t} = 1 - \xi$, $\alpha = -n$, $\beta = -1$ and the equation (50) yields

$$(1 - \xi)^{-x} = \sum_{\nu=0}^{\infty} \frac{\tau^\nu}{\nu!} x (x + n\nu + \nu - 1)^{(\nu-1)} \tag{54}$$

from which the required root ξ is immediately found. In this expansion, x may be chosen arbitrarily, e. g. $x = 1$, and the circle of convergence in τ does not depend on which value of x is chosen; it is, in fact, easy to prove that (54) is convergent for $|\tau| < \frac{n^n}{(n+1)^{n+1}}$ which does not depend on x.

Instead of (50) we may use (42), leading to

$$\operatorname{Log} \frac{1}{1 - \xi} = \sum_{\nu=1}^{\infty} \binom{n\nu + \nu - 1}{\nu - 1} \frac{\tau^\nu}{\nu}, \tag{55}$$

having the same radius of convergence.

8. Several poweroids are related to polynomials already employed in analysis, such as, for instance, Hermite's, Laguerre's, Bernoulli's and Euler's polynomials. We give a few examples, considering first the operator

$$\theta = E^\alpha e^{\beta D^2} D \tag{56}$$

which may also be written

$$\theta = e^{\alpha D + \beta D^2} D. \tag{57}$$

The corresponding poweroid is found by (10), thus

$$x^{\overline{\nu}|} = x\, E^{-\nu\alpha} e^{-\nu\beta D^2} x^{\nu-1}$$
$$= x\, E^{-\nu\alpha} \sum_{s=0}^{\infty} \frac{(-\nu\beta)^s}{s!} D^{2s} x^{\nu-1},$$

whence

$$x^{\overline{\nu}|} = \sum_{s=0}^{\leq \frac{\nu-1}{2}} \frac{(\nu-1)^{(2s)}}{s!} (-\nu\beta)^s x (x - \nu\alpha)^{\nu-1-2s}. \tag{58}$$

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[1] J. F. STEFFENSEN: Rentesregning, p. 155.

This is an expansion in Abel's poweroids; but we may also express $x^{\nu\rceil}$ by an Hermite polynomial. Let Hermite's polynomials be defined by the expansion

$$e^{-zt-\frac{t^2}{2}} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}; \tag{59}$$

then

$$H_n(z) = \sum_{s=0}^{\leq \frac{n}{2}} (-1)^{n-s}\frac{n^{(2s)}}{s!\,2^s} z^{n-2s}, \tag{60}$$

and comparison with (58) shows that

$$x^{\nu\rceil} = x(2\nu\beta)^{\frac{\nu-1}{2}} H_{\nu-1}\left(\frac{\nu\alpha - x}{\sqrt{2\nu\beta}}\right) \tag{61}$$

From the known properties of Hermite's polynomials may, therefore, for instance, be concluded that the roots of $x^{\nu\rceil}$ are all real if α is real and $\beta > 0$.

The expansion of $x^{\nu\rceil}$ in powers of x is less simple than (58) or (61). We have, by (57) and (10),

$$x^{\nu\rceil} = x\,e^{-\nu\alpha D - \nu\beta D^2} x^{\nu-1}.$$

If now, in (59), we put

$$t = \sqrt{2\nu\beta}\cdot D, \quad z = \alpha\sqrt{\frac{\nu}{2\beta}},$$

we have

$$e^{-\nu\alpha D - \nu\beta D^2} = \sum_{n=0}^{\infty} \frac{(2\nu\beta)^{\frac{n}{2}}}{n!} H_n\left(\alpha\sqrt{\frac{\nu}{2\beta}}\right) D^n,$$

so that we find

$$x^{\nu\rceil} = \sum_{n=0}^{\nu-1} \binom{\nu-1}{n} (2\nu\beta)^{\frac{n}{2}} H_n\left(\alpha\sqrt{\frac{\nu}{2\beta}}\right) x^{\nu-n}. \tag{62}$$

In order to expand a polynomial in these poweroids by (7), $\theta^\nu f(0)$ is wanted. We have, by (56),

$$\theta^\nu = E^{\nu\alpha} \sum_{s=0}^{\infty} \frac{(\nu\beta)^s}{s!} D^{\nu+2s} \tag{63}$$

and by (57)

$$\theta^\nu = \sum_{n=0}^{\infty} \frac{(-2\nu\beta)^{\frac{n}{2}}}{n!} H_n\left(-\alpha\sqrt{\frac{-\nu}{2\beta}}\right) D^{\nu+n}. \tag{64}$$

The case $\alpha = 0$, $\beta = \frac{1}{2}$ is particularly simple. We have, then, $\theta = e^{\frac{D^2}{2}} D$ and, denoting the corresponding poweroid by $h_\nu(x)$, according to (61),

$$h_\nu(x) = x\,\nu^{\frac{\nu-1}{2}} H_{\nu-1}\left(\frac{-x}{\sqrt{\nu}}\right) \tag{65}$$

or, by (58),

$$h_\nu(x) = \sum_{s=0}^{\leq \frac{\nu-1}{2}} (-1)^s \frac{(\nu-1)^{(2s)}}{s!} \left(\frac{\nu}{2}\right)^s x^{\nu-2s} \tag{66}$$

where the coefficients are easily seen to be integers. The first few of these poweroids are

$$h_0(x) = 1,\ h_1(x) = x,\ h_2(x) = x^2,\ h_3(x) = x^3 - 3x,$$

$$h_4(x) = x^4 - 12x^2,\ h_5(x) = x^5 - 30x^3 + 75x,\ h_6(x) = x^6 - 60x^4 + 540x^2.$$

As an application of (7) we will expand x^n in the poweroids $h_\nu(x)$. We have by (63) for $\alpha = 0$, $\beta = \frac{1}{2}$

$$\theta^\nu x^n = \sum_{s=0}^{\infty} \frac{n^{(\nu+2s)}}{s!} \left(\frac{\nu}{2}\right)^s x^{n-\nu-2s}.$$

For $x = 0$ all terms vanish except that for which $n - \nu = 2s$. Hence, $\theta^\nu 0^n$ vanishes except when $n - \nu$ is an even number, and we find

$$\theta^{2\nu} 0^{2n} = \frac{(2n)!}{(n-\nu)!} \nu^{n-\nu},$$

$$\theta^{2\nu+1} 0^{2n+1} = \frac{(2n+1)!}{(n-\nu)!} \left(\frac{2\nu+1}{2}\right)^{n-\nu},$$

and finally

$$x^{2n} = \sum_{\nu=1}^{n} \frac{(2n)^{(2n-2\nu)}}{(n-\nu)!} \nu^{n-\nu} h_{2\nu}(x), \tag{67}$$

$$x^{2n+1} = \sum_{\nu=0}^{n} \frac{(2n+1)^{(2n-2\nu)}}{(n-\nu)!} \left(\frac{2\nu+1}{2}\right)^{n-\nu} h_{2\nu+1}(x). \tag{68}$$

The generating function for $h_\nu(x)$ according to (34) is not particularly simple, because the solution of the equation $\zeta = e^{\frac{t^2}{2}} t$ is not an elementary function of ζ.

9. Certain poweroids are related to the polynomials $G_n(x,y)$[1] which may be defined by their generating function

$$e^{yt-x(e^t-1)} = \sum_{n=0}^{\infty} G_n(x,y)\frac{t^n}{n!} \tag{69}$$

or by the relation

$$G_n(e^z, y) = e^{e^z - yz} D_z^n e^{yz - e^z}. \tag{70}$$

Their explicit expression is

$$G_n(x,y) = \sum_{s=0}^{n} (-1)^s \frac{x^s}{s!} \triangle^s y^n, \tag{71}$$

which shows that the degree is n, separately in x and y.

They satisfy the relations

$$D_x G_n(x,y) = G_n(x,y) - G_n(x, y+1), \tag{72}$$

$$D_y G_n(x,y) = n\, G_{n-1}(x,y) \tag{73}$$

and the recurrence formula

$$G_{n+1}(x,y) = y\, G_n(x,y) - x\, G_n(x, y+1). \tag{74}$$

Consider now the operator

$$\theta = \frac{1}{\alpha} \operatorname{Log}(1 + \alpha D); \tag{75}$$

we find

$$\zeta = \frac{1}{\alpha}\operatorname{Log}(1 + \alpha t), \quad t = \frac{e^{\alpha\zeta} - 1}{\alpha}.$$

Inserting this in (34), we have

$$e^{\frac{x}{\alpha}(e^{\alpha\zeta}-1)} = \sum_{\nu=0}^{\infty} \frac{x^{\overline{\nu}|}}{\nu!}\zeta^{\nu},$$

and comparison with (69) shows that

$$x^{\overline{\nu}|} = \alpha^{\nu} G_{\nu}\left(-\frac{x}{\alpha}, 0\right). \tag{76}$$

The two most important cases are obtained for $\alpha = 1$ and $\alpha = -1$; they are

$$\theta = \operatorname{Log}(1 + D), \quad x^{\overline{\nu}|} = G_{\nu}(-x, 0) \tag{77}$$

[1] J. F. Steffensen: Some recent researches in the theory of statistics and actuarial science, Cambridge 1930, p. 24, or the same author's Forsikringsmatematik, p. 442.

and

$$\theta = \operatorname{Log} \frac{1}{1 - D}, \quad x^{\overline{\nu}|} = (-1)^{\nu} G_{\nu}(x, 0). \tag{78}$$

We have, by (71),

$$G_{\nu}(-x, 0) = \sum_{s=0}^{\nu} \frac{\Delta^{s} 0^{\nu}}{s!} x^{s}; \tag{79}$$

the coefficients $\frac{\Delta^s 0^\nu}{s!}$ which are all positive are tabulated in »Interpolation», p. 55.

It can be proved that the roots of the poweroids (76) are all real if α is real. It suffices evidently to prove that the roots of $G_\nu(x, 0)$ are real. If now, in (70), we put $y = 0$, $e^z = x$ and observe that $D_z = x D_x$, we get

$$G_n(x, 0) = e^x (x D)^n e^{-x}. \tag{80}$$

In handling the operator $(x D)$ it must of course be remembered that the differentiation should be performed before the multiplication by x. Since e^{-x} vanishes for $x \to + \infty$, it is clear that the expression $(x D)^n e^{-x}$ always vanishes for $x = 0$ and $x \to + \infty$, if only $n > 0$. But since $(x D) e^{-x}$ vanishes at 0 and $+ \infty$, $(x D)^2 e^{-x}$ must vanish at a point between 0 and $+ \infty$, that is, the operation $(x D)$ has introduced one more root; and so we may continue, reaching the conclusion that $G_n(x, 0)$ has exactly n real roots, one being zero, the others positive, and all of them different from one another.

In order to expand $\theta^\nu = \alpha^{-\nu} \operatorname{Log}^\nu (1 + \alpha D)$ in powers of D, we observe that

$$e^{x \alpha^{-1} \operatorname{Log}(1 + \alpha D)} = (1 + \alpha D)^{\frac{x}{\alpha}}$$

or

$$\sum_{s=0}^{\infty} \frac{x^s}{s!} \alpha^{-s} \operatorname{Log}^s (1 + \alpha D) = \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{x}{\alpha}\right)^{(s)} \alpha^s D^s$$

which, when both sides are applied to $f(0)$, is an algebraical identity with only a finite number of terms, and may, therefore, be differentiated ν times with respect to x. If we do this and put afterwards $x = 0$, we find

$$\theta^\nu = \sum_{s=\nu}^{\infty} \frac{1}{s!} D^\nu 0^{(s)} \cdot \alpha^{s-\nu} D^s \tag{81}$$

or

$$\theta^\nu = \sum_{s=\nu}^{\infty} \frac{D^\nu 0^{(-s)}}{\nu!} \cdot \frac{(-\alpha)^{s-\nu}}{s^{(s-\nu)}} D^s, \tag{82}$$

the latter form being sometimes preferable, because the numbers $\frac{D^\nu 0^{(-s)}}{\nu!}$ which are all integral and positive are tabulated in »Interpolation», p. 57.

In the case $\nu = 1$ we have directly by expanding (75)

$$\theta = D - \frac{\alpha}{2} D^2 + \frac{\alpha^2}{3} D^3 - \cdots$$

The expansion of x^n in the poweroids (76) has simple coefficients. We have, according to (82),

$$\theta^\nu x^n = \sum_{s=\nu}^{n} \binom{n}{s} (-\alpha)^{s-\nu} D^\nu 0^{(-s)} \cdot x^{n-s}$$

whence

$$\theta^\nu 0^n = (-\alpha)^{n-\nu} D^\nu 0^{(-n)}$$

so that, by (7),

$$x^n = \alpha^n \sum_{\nu=0}^{n} (-1)^{n-\nu} \frac{D^\nu 0^{(-n)}}{\nu!} G_\nu\left(-\frac{x}{\alpha}, 0\right). \tag{83}$$

The same result may be obtained a little more quickly by the secondary form of Maclaurin's theorem, putting $\Phi(D) = \theta^\nu$ after having written n for ν in (39). As θ^ν has already been expanded in powers of D, we obtain immediately, by (81) or (82), the above expression for $\theta^\nu 0^n$.

Another poweroid connected with the polynomials $G_n(x, y)$ is obtained by putting $\psi(D) = e^{-\alpha\triangle}$ in (13). The result is

$$x^{\overline{\nu|}} = x e^{-\nu\alpha\triangle} x^{\nu-1}$$

$$= x \sum_{s=0}^{\nu-1} \frac{(-\nu\alpha)^s}{s!} \triangle^s x^{\nu-1}$$

so that we have, by (12) and (71),

$$\theta = e^{\alpha\triangle} D, \quad x^{\overline{\nu|}} = x\, G_{\nu-1}(\nu\alpha, x). \tag{84}$$

In order to expand

$$\theta^\nu = e^{\nu\alpha\triangle} D^\nu = e^{\nu\alpha(e^D-1)} D^\nu$$

in powers of D, we need only compare with (69), which shows at once that

$$\theta^\nu = \sum_{n=0}^{\infty} \frac{1}{n!} G_n(-\nu\alpha, 0) D^{\nu+n}. \tag{85}$$

The expansion of x^n in these poweroids is found in the same way as (83); the result is

$$x^n = \sum_{\nu=0}^{n} \binom{n}{\nu} G_{n-\nu}(-\nu\alpha, 0) \cdot x\, G_{\nu-1}(\nu\alpha, x). \tag{86}$$

We finally note that a solution of the equation

$$\zeta = e^{\alpha(e^t-1)}\, t \tag{87}$$

may be found by (34) in the form

$$e^{xt} = \sum_{\nu=0}^{\infty} \frac{\zeta^\nu}{\nu!} \cdot x\, G_{\nu-1}(\nu\alpha, x) \tag{88}$$

or by (42) in the form

$$t = \sum_{\nu=1}^{\infty} \frac{\zeta^\nu}{\nu!} G_{\nu-1}(\nu\alpha, 0). \tag{89}$$

10. We now consider an operator of the form

$$\theta = E^\alpha D(1 + \beta D)^\gamma. \tag{90}$$

The corresponding poweroid may be found by (23), employing (48) for θ_I and $x_I^{\nu\rceil}$. We find

$$\begin{aligned} x^{\nu\rceil} &= x(1+\beta D)^{-\nu\gamma}(x-\nu\alpha)^{\nu-1} \\ &= x\sum_{s=0}^{\nu-1}\binom{-\nu\gamma}{s}\beta^s D^s (x-\nu\alpha)^{\nu-1} \end{aligned}$$

or

$$x^{\nu\rceil} = \sum_{s=0}^{\nu-1}(-1)^s \binom{\nu\gamma+s-1}{s}(\nu-1)^{(s)}\beta^s \cdot x(x-\nu\alpha)^{\nu-1-s} \tag{91}$$

being an expansion in Abel's poweroids which seem most suited to this case.

The expansion of

$$\theta^\nu = E^{\nu\alpha} D^\nu (1+\beta D)^{\nu\gamma}$$

is best left as

$$\theta^\nu = E^{\nu\alpha}\sum_{s=0}^{\infty}\binom{\nu\gamma}{s}\beta^s D^{\nu+s}. \tag{92}$$

From (90) may be concluded that the equation

$$\zeta = t(1+\beta t)^\gamma e^{\alpha t} \tag{93}$$

is solved by (34) or (42) with the expression (91) for the poweroid.

Two particular cases call for special notice. One of them is obtained by putting $\alpha = 0$, $\beta = -1$, $\gamma = 1$. Denoting this poweroid by $p_\nu(x)$ we have

$$\theta = D - D^2, \quad p_\nu(x) = \sum_{s=0}^{\nu-1} \binom{\nu + s - 1}{s} (\nu - 1)^{(s)} x^{\nu - s}. \tag{94}$$

The first few $p_\nu(x)$ are

$p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 + 2x$, $p_3(x) = x^3 + 6x^2 + 12x$,

$p_4(x) = x^4 + 12x^3 + 60x^2 + 120x$,

$p_5(x) = x^5 + 20x^4 + 180x^3 + 840x^2 + 1680x$,

$p_6(x) = x^6 + 30x^5 + 420x^4 + 3360x^3 + 15120x^2 + 30240x$.

The generating function is, by (34),

$$e^{\frac{x}{2}(1 - \sqrt{1 - 4\zeta})} = \sum_{\nu=0}^{\infty} \frac{p_\nu(x)}{\nu!} \zeta^\nu; \tag{95}$$

it is found by putting $\zeta = t - t^2$, $t = \frac{1}{2}(1 - \sqrt{1 - 4\zeta})$, the sign of the root being negative, since $\zeta = 0$ corresponds to $t = 0$.

The expansion of θ^ν is

$$\theta^\nu = \sum_{s=0}^{\nu} (-1)^s \binom{\nu}{s} D^{\nu+s}, \tag{96}$$

and for x^n we find the expansion in $p_\nu(x)$

$$x^n = \sum_{\nu \geq \frac{n}{2}}^{n} (-1)^{n-\nu} \binom{n}{\nu} \nu^{(n-\nu)} p_\nu(x). \tag{97}$$

The second particular case of special interest corresponds to $\alpha = 0$, $\beta = -1$, $\gamma = -1$. Here we have, denoting this poweroid by $q_\nu(x)$,

$$\theta = \frac{D}{1 - D}, \qquad q_\nu(x) = \sum_{s=0}^{\nu-1} (-1)^s \binom{\nu}{s} (\nu - 1)^{(s)} x^{\nu - s}. \tag{98}$$

The first few $q_\nu(x)$ are

$q_0(x) = 1$, $q_1(x) = x$, $q_2(x) = x^2 - 2x$, $q_3(x) = x^3 - 6x^2 + 6x$,

$q_4(x) = x^4 - 12x^3 + 36x^2 - 24x$, $q_5(x) = x^5 - 20x^4 + 120x^3 - 240x^2 + 120x$,

$q_6(x) = x^6 - 30x^5 + 300x^4 - 1200x^3 + 1800x^2 - 720x$.

We may also write

$$q_\nu(x) = (-1)^{\nu-1} e^x D^{\nu-1} e^{-x} x^\nu, \tag{99}$$

showing that $q_\nu(x)$ can be expressed by a Laguerre polynomial[1]

$$q_\nu(x) = (-1)^{\nu-1} (\nu-1)!\, x\, L_{\nu-1}^{[1]}(x). \tag{100}$$

The roots of $q_\nu(x)$ are, therefore, real, none of them negative, and all differing from one another.

We have, in this case, $\zeta = \frac{t}{1-t}$, $t = \frac{\zeta}{1+\zeta}$, so that the generating function for $q_\nu(x)$ is

$$e^{\frac{x\zeta}{1+\zeta}} = \sum_{\nu=0}^{\infty} \frac{q_\nu(x)}{\nu!} \zeta^\nu. \tag{101}$$

The expansion of θ^ν is

$$\theta^\nu = \sum_{s=0}^{\infty} \binom{\nu+s-1}{s} D^{\nu+s} \tag{102}$$

and the expansion of x^n in the poweroids $q_\nu(x)$

$$x^n = \sum_{\nu=1}^{n} \binom{n}{\nu} (n-1)^{(n-\nu)} q_\nu(x). \tag{103}$$

11. Bernoulli's polynomials may be defined by

$$B_\nu(x) = \frac{D}{\triangle} x^\nu, \tag{104}$$

this definition being in agreement with the Nörlund definition now usually[2] adopted. For, from (104) we immediately obtain

$$\triangle B_\nu(x) = D x^\nu, \qquad D B_\nu(x) = \nu \frac{D}{\triangle} x^{\nu-1} = \nu B_{\nu-1}(x),$$

and these two relations determine the polynomial completely in Nörlund's sense[3]. No supplementary condition is necessary in using (104) as the definition, since the expansion of $\frac{D}{\triangle}$ in powers of D does not contain negative powers.

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[1] PÓLYA und SZEGÖ: Aufgaben und Lehrsätze aus der Analysis, II, p. 293.

[2] CHARLES JORDAN: Calculus of Finite Differences, p. 231, employs a slightly different definition.

[3] »Interpolation», p. 119.

The definition (104) suggests the generalization

$$B_\nu^{[\lambda]}(x) = \left(\frac{D}{\triangle}\right)^\lambda x^\nu \tag{105}$$

where the parameter λ may be any, real or complex, number. These polynomials may be shown to be identical with Nörlund's generalized Bernoulli polynomials, in the case where the intervals of the successive differencing operations become identical[1]. For if, in (37), we put

$$\Phi(t) = \left(\frac{t}{e^t - 1}\right)^\lambda,$$

we have

$$\left(\frac{t}{e^t - 1}\right)^\lambda e^{xt} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \left(\frac{D}{e^D - 1}\right)^\lambda x^\nu$$

or

$$\left(\frac{t}{e^t - 1}\right)^\lambda e^{xt} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{[\lambda]}(x), \tag{106}$$

which is the generating function for Nörlund's polynomials.

We may, therefore, either take the properties of the polynomials $B_\nu^{[\lambda]}(x)$ as known, or derive them directly from (105) as the definition, which is very easy. For instance, it follows immediately from (105) that

$$D B_\nu^{[\lambda]}(x) = \nu B_{\nu-1}^{[\lambda]}(x), \qquad \triangle B_\nu^{[\lambda]}(x) = \nu B_{\nu-1}^{[\lambda-1]}(x); \tag{107}$$

and if, in (10), we put $\theta = \triangle$, so that $x^{\overline{\nu}|} = x^{(\nu)}$, it is seen by comparison with (105) that

$$B_{\nu-1}^{[\nu]}(x) = (x-1)^{(\nu-1)}. \tag{108}$$

If now, in (13), we put $\psi(D) = \left(\frac{D}{\triangle}\right)^\lambda$, we find the poweroid

$$x^{\overline{\nu}|} = x B_{\nu-1}^{[\nu\lambda]}(x), \tag{109}$$

corresponding, according to (12), to the operator

$$\theta = \left(\frac{\triangle}{D}\right)^\lambda D. \tag{110}$$

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[1] N. E. Nörlund: Differenzenrechnung, p. 145. We prefer distinguishing the case of equal intervals by writing $B_\nu^{[\lambda]}(x)$ instead of $B_\nu^{(\lambda)}(x)$.

The expansion of

$$\theta^{\nu} = \left(\frac{e^{D}-1}{D}\right)^{\nu\lambda} D^{\nu}$$

is found by (106), putting $x = 0$, $B_{\nu}^{[\lambda]}(0) = B_{\nu}^{[\lambda]}$ and replacing λ by $-\nu\lambda$, t by D. The result is

$$\theta^{\nu} = \sum_{s=0}^{\infty} \frac{B_{s}^{[-\nu\lambda]}}{s!} D^{\nu+s}. \tag{111}$$

A table of $B_{\nu}^{[\lambda]}$ is given by Nörlund l. c., p. 459.

The expansion of x^n in these poweroids is

$$x^n = \sum_{\nu=0}^{n} \binom{n}{\nu} B_{n-\nu}^{[-\nu\lambda]} \cdot x\, B_{\nu-1}^{[\nu\lambda]}(x). \tag{112}$$

A solution of the equation $\zeta = \varphi(t)$, or

$$\zeta = \left(\frac{e^{t}-1}{t}\right)^{\lambda} t \tag{113}$$

is obtained, by (34), from

$$e^{xt} = \sum_{\nu=0}^{\infty} \frac{\zeta^{\nu}}{\nu!} x\, B_{\nu-1}^{[\nu\lambda]}(x) \tag{114}$$

or, by (42), in the form

$$t = \sum_{\nu=1}^{\infty} \frac{\zeta^{\nu}}{\nu!} B_{\nu-1}^{[\nu\lambda]}. \tag{115}$$

12. Corresponding remarks apply to Euler's polynomials

$$\mathcal{E}_{\nu}(x) = \frac{1}{1+\frac{\triangle}{2}} x^{\nu} \tag{116}$$

or, developing in powers of $\triangle$,

$$\mathcal{E}_{\nu}(x) = \sum_{s=0}^{\nu} \frac{(-1)^s}{2^s} \triangle^s x^{\nu}. \tag{117}$$

They may be generalized by

$$\mathcal{E}_{\nu}^{[\lambda]}(x) = \left(1+\frac{\triangle}{2}\right)^{-\lambda} x^{\nu} \tag{118}$$

or, developing in powers of $\triangle$,

$$\mathcal{E}_{\nu}^{[\lambda]}(x) = \sum_{s=0}^{\nu} \frac{(-1)^s}{2^s} \binom{\lambda+s-1}{s} \triangle^s x^{\nu}. \tag{119}$$

These polynomials, too, may be shown to be identical with Nörlund's generalization. Putting, in fact, in (37),

$$\Phi(t) = \left(\frac{2}{e^t + 1}\right)^\lambda,$$

we find

$$\left(\frac{2}{e^t + 1}\right)^\lambda e^{xt} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \left(\frac{2}{e^D + 1}\right)^\lambda x^\nu$$

or, since

$$\frac{2}{e^D + 1} = \frac{1}{1 + \frac{\Delta}{2}},$$

$$\left(\frac{2}{e^t + 1}\right)^\lambda e^{xt} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \mathscr{E}_\nu^{[\lambda]}(x), \tag{120}$$

being the generating function for Nörlund's generalization[1].

From (118) the known properties of $\mathscr{E}_\nu^{[\lambda]}(x)$ may easily be derived, such as, for instance,

$$D\,\mathscr{E}_\nu^{[\lambda]}(x) = \nu\,\mathscr{E}_{\nu-1}^{[\lambda]}(x), \tag{121}$$

$$\left(1 + \frac{\Delta}{2}\right) \mathscr{E}_\nu^{[\lambda]}(x) = \mathscr{E}_\nu^{[\lambda-1]}(x). \tag{122}$$

Putting now, in (13),

$$\psi(D) = \left(\frac{2}{e^D + 1}\right)^\lambda,$$

we obtain the poweroid

$$x^{\overline{\nu}|} = x\,\mathscr{E}_{\nu-1}^{[\nu\lambda]}(x) \tag{123}$$

with the operator

$$\theta = \left(\frac{e^D + 1}{2}\right)^\lambda D \tag{124}$$

which may sometimes with advantage be written

$$\theta = \left(1 + \frac{\Delta}{2}\right)^\lambda D. \tag{125}$$

The expansion of θ^ν in powers of D is found by (124) and (120) for $x = 0$, in a similar way as (111); the result is

$$\theta^\nu = \sum_{s=0}^{\infty} \frac{\mathscr{E}_s^{[-\nu\lambda]}(0)}{s!} D^{\nu+s}. \tag{126}$$

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[1] Differenzenrechnung, p. 143.

The expansion of x^n is

$$x^n = \sum_{\nu=0}^{n} \binom{n}{\nu} \mathcal{E}_{n-\nu}^{[-\nu\lambda]}(0) \cdot x\, \mathcal{E}_{\nu-1}^{[\nu\lambda]}(x). \tag{127}$$

A solution of the equation

$$\zeta = \left(\frac{e^t + 1}{2}\right)^{\lambda} t \tag{128}$$

follows from (34) in the form

$$e^{xt} = \sum_{\nu=0}^{\infty} \frac{\zeta^\nu}{\nu!}\, x\, \mathcal{E}_{\nu-1}^{[\nu\lambda]}(x) \tag{129}$$

or from (42) as

$$t = \sum_{\nu=1}^{\infty} \frac{\zeta^\nu}{\nu!} \mathcal{E}_{\nu-1}^{[\nu\lambda]}(0). \tag{130}$$

Another poweroid related to the generalized Euler polynomials is obtained by putting, in (13), $\psi(D) = \Box^{-\lambda}$, that is

$$x^{\overline{\nu}|} = x\, \Box^{-\nu\lambda} x^{\nu-1} = x \left(\frac{E^{\frac{1}{2}}}{1 + \frac{\Delta}{2}}\right)^{\nu\lambda} x^{\nu-1} \tag{131}$$

or

$$x^{\overline{\nu}|} = x\, \mathcal{E}_{\nu-1}^{[\nu\lambda]}\left(x + \frac{\nu\lambda}{2}\right), \qquad \theta = \Box^{\lambda} D. \tag{132}$$

Considering that the expansion of $\Box^{-\nu\lambda}$ contains only even powers of D, it is seen that $x^{\overline{2n}|}$ is an even, $x^{\overline{2n+1}|}$ an odd function of x.

In order to expand

$$\theta^\nu = \left(\frac{e^{\frac{D}{2}} + e^{-\frac{D}{2}}}{2}\right)^{\nu\lambda} D^\nu \tag{133}$$

we put $x = \frac{\lambda}{2}$ in (120), the result being

$$\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\lambda} = \sum_{\nu=0}^{\infty} \frac{t^{2\nu}}{2^{2\nu}(2\nu)!} \mathcal{E}_{2\nu}^{[\lambda]} \tag{134}$$

where, following Nörlund[1], we have put

$$\mathcal{E}_{2\nu+1}^{[\lambda]} = 0, \qquad \mathcal{E}_{2\nu}^{[\lambda]} = 2^{2\nu} \mathcal{E}_{2\nu}^{[\lambda]}\left(\frac{\lambda}{2}\right), \tag{135}$$

so that $\mathcal{E}_{2\nu}^{[\lambda]}$ is not the same thing as $\mathcal{E}_{2\nu}^{[\lambda]}(0)$.

[1] A table of $\mathcal{E}_{2\nu}^{[\lambda]}$ is given in NÖRLUND: Mémoire sur les polynomes de Bernoulli. Acta mathematica, 1920, p. 195.

We then find, comparing (133) with (134),

$$\theta^{\nu} = \sum_{s=0}^{\infty} \frac{\mathcal{E}_{2s}^{[-\nu\lambda]}}{2^{2s}\,(2s)!} D^{\nu+2s}. \tag{136}$$

The expansion of x^n is

$$x^n = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{1}{2^{n-\nu}} \mathcal{E}_{n-\nu}^{[-\nu\lambda]} \cdot x\, \mathcal{E}_{\nu-1}^{[\nu\lambda]}\left(x + \frac{\nu\lambda}{2}\right). \tag{137}$$

A solution of the equation

$$\zeta = \left(\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2}\right)^{\lambda} t \tag{138}$$

is obtained by

$$e^{xt} = \sum_{\nu=0}^{\infty} \frac{\zeta^{\nu}}{\nu!}\, x\, \mathcal{E}_{\nu-1}^{[\nu\lambda]}\left(x + \frac{\nu\lambda}{2}\right) \tag{139}$$

or by

$$t = \sum_{\nu=0}^{\infty} \frac{\zeta^{2\nu+1}}{(2\nu+1)!} \mathcal{E}_{2\nu}^{[(2\nu+1)\lambda]}\left(\frac{2\nu+1}{2}\lambda\right). \tag{140}$$

13. In addition to the expansions of x^n in various poweroids already dealt with, we shall now give some examples of the application of (7). All these expansions are algebraical identities, some of them generalizations of well-known theorems.

We note first that, for $f(x) = (x+t)^{\overline{n}|}$, we get

$$(x+t)^{\overline{n}|} = \sum_{\nu=0}^{n} \binom{n}{\nu} t^{\overline{n-\nu}|} x^{\overline{\nu}|}, \tag{141}$$

being a generalization of the binomial theorem which is obtained for $\theta = D$, $x^{\overline{\nu}|} = x^{\nu}$. But (141) contains also the corresponding formula for descending factorials[1]

$$(x+t)^{(n)} = \sum_{\nu=0}^{n} \binom{n}{\nu} t^{(n-\nu)} x^{(\nu)}, \tag{142}$$

for ascending factorials

$$(x+t)^{(-n)} = \sum_{\nu=0}^{n} \binom{n}{\nu} t^{(-n+\nu)} x^{(-\nu)}, \tag{143}$$

[1] Differenzenrechnung, p. 151.

for central factorials

$$(x + t)^{[n]} = \sum_{\nu=0}^{n} \binom{n}{\nu} t^{[n-\nu]} x^{[\nu]}, \tag{144}$$

for Abel's poweroids

$$(x + t)(x + t - n\alpha)^{n-1} = \sum_{\nu=0}^{n} \binom{n}{\nu} t\,(t - \overline{n-\nu}\,\alpha)^{n-\nu-1}\, x\,(x - \nu\alpha)^{\nu-1}, \tag{145}$$

and so on.

Together with (7) we may consider (21), putting $f(x) = (x + t)^{\overline{n+1|}-1}$. In that case we have, according to (17), $\theta' f(x) = (x + t)^{\overline{n|}}$, and therefore $\theta' \theta^{\nu} f(0) = = n^{(\nu)} t^{\overline{n-\nu|}}$, so that the identity

$$(x + t)^{\overline{n+1|}-1} = \sum_{\nu=0}^{n} \binom{n}{\nu} t^{\overline{n-\nu|}} x^{\overline{\nu+1|}-1} \tag{146}$$

results, corresponding to (141). The formulas corresponding to (142)—(145) may therefore be written down at sight, and we only note that, if $\theta = \delta$, then $\theta' = \Box$, so that (21) becomes

$$f(x) = \sum_{\nu=0}^{n} \frac{x^{[\nu+1]-1}}{\nu!} \Box\, \delta^{\nu} f(0). \tag{147}$$

If a polynomial $g_{\nu}(x)$ of degree ν satisfies the condition

$$g'_{\nu}(x) = \nu\, g_{\nu-1}(x) \tag{148}$$

for all ν, the expansion of $g_n(x + t)$ in Abel's poweroids is simple. We have, in fact, $\theta = E^{\alpha} D$, so that

$$\theta^{\nu} f(0) = E^{\nu\alpha} D^{\nu} g_n(t) = n^{(\nu)} g_{n-\nu}(t + \nu\alpha),$$

and therefore

$$g_n(x + t) = \sum_{\nu=0}^{n} \binom{n}{\nu} g_{n-\nu}(t + \nu\alpha) \cdot x\,(x - \nu\alpha)^{\nu-1}. \tag{149}$$

Thus, for instance, for $g_n(x + t) = (x + t)^n$, we obtain Abel's identity[1]

$$(x + t)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} (t + \nu\alpha)^{n-\nu} \cdot x\,(x - \nu\alpha)^{\nu-1}. \tag{150}$$

Putting, next, $g_n(x + t) = B_n^{[\lambda]}(x + t)$, we find

$$B_n^{[\lambda]}(x + t) = \sum_{\nu=0}^{n} \binom{n}{\nu} B_{n-\nu}^{[\lambda]}(t + \nu\alpha) \cdot x\,(x - \nu\alpha)^{\nu-1} \tag{151}$$

[1] Oeuvres, vol. I, p. 102.

and similarly

$$\mathcal{E}_n^{[\lambda]}(x+t) = \sum_{\nu=0}^{n} \binom{n}{\nu} \mathcal{E}_{n-\nu}^{[\lambda]}(t+\nu\alpha) \cdot x(x-\nu\alpha)^{\nu-1}, \tag{152}$$

generalizing the formulas, due to Nörlund[1],

$$B_n^{[\lambda]}(x+t) = \sum_{\nu=0}^{n} \binom{n}{\nu} B_{n-\nu}^{[\lambda]}(t)\, x^\nu \tag{153}$$

and

$$\mathcal{E}_n^{[\lambda]}(x+t) = \sum_{\nu=0}^{n} \binom{n}{\nu} \mathcal{E}_{n-\nu}^{[\lambda]}(t)\, x^\nu \tag{154}$$

which are obtained from (151) and (152) by putting $\alpha = 0$.

In the same way we may employ (73) for expanding $G_n(x, y+t)$ in Abel's poweroids of y, the result being

$$G_n(x, y+t) = \sum_{\nu=0}^{n} \binom{n}{\nu} G_{n-\nu}(x, t+\nu\alpha) \cdot y(y-\nu\alpha)^{\nu-1}. \tag{155}$$

Also the expansions of $G_n(x, y)$ in the poweroids $p_\nu(x)$ and $q_\nu(x)$ have simple coefficients. Writing (72) in the form

$$G_n(x, y+1) = (1-D_x)\, G_n(x, y), \tag{156}$$

we have

$$(1-D_x)^\nu\, G_n(x, y) = G_n(x, y+\nu)$$

so that, when $\theta = D(1-D)$, corresponding to the poweroid $p_\nu(x)$,

$$\theta^\nu\, G_n(x, y) = D_x^\nu\, G_n(x, y+\nu)$$

or, according to (71),

$$\theta^\nu\, G_n(x, y) = \sum_{s=0}^{n-\nu} (-1)^{s+\nu} \frac{x^s}{s!} \triangle^{s+\nu} (y+\nu)^n,$$

so that

$$\theta_{x=0}^\nu\, G_n(x, y) = (-1)^\nu \triangle^\nu (y+\nu)^n.$$

Hence we have, by (7),

$$G_n(x, y) = \sum_{\nu=0}^{n} (-1)^\nu \frac{\triangle^\nu (y+\nu)^n}{\nu!} p_\nu(x). \tag{157}$$

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[1] Differenzenrechnung, p. 133; Acta mathematica, 1920, p. 146.

The expansion in $q_\nu(x)$ is obtained by multiplying on both sides of (156) by $(1 - D_x)^{-1}$ and writing $y - 1$ for y, that is,

$$(1 - D_x)^{-1} G_n(x, y) = G_n(x, y - 1),$$

whence

$$(1 - D_x)^{-\nu} G_n(x, y) = G_n(x, y - \nu).$$

If, now, $\theta = \frac{D}{1 - D}$, $x^{\overline{\nu}|} = q_\nu(x)$, we find by (71)

$$\theta^\nu_{x=0} G_n(x, y) = (-1)^\nu \triangle^\nu (y - \nu)^n = (-1)^\nu \nabla^\nu y^n,$$

so that, by (7),

$$G_n(x, y) = \sum_{\nu=0}^{n} (-1)^\nu \frac{\nabla^\nu y^n}{\nu!} q_\nu(x). \tag{158}$$

14. So far we have assumed that $f(x)$ is a polynomial. If that is not the case, the first question to consider is: What is to be understood by $\theta f(x)$? This is as a rule clear from the initial definition of θ in each particular case. For instance, if we choose $\theta = E^\alpha D$, then $\theta f(x)$ means $f'(x + \alpha)$ on the sole condition that $f(x)$ possesses a derivate at the point $x + \alpha$. But it does not follow that all such transformations of θ are permitted as are legitimate when $f(x)$ is a polynomial, such as expansion of θ in powers of D, etc. It must first be proved that the series thus obtained converges when applied to $f(x)$, and represents $f'(x + \alpha)$, which evidently implies supplementary conditions. Thus, for instance, we are only allowed to put

$$E^\alpha D = e^{\alpha D} D = \sum_{\nu=0}^{\infty} \frac{\alpha^\nu}{\nu!} D^{\nu+1}$$

if

$$f'(x + \alpha) = \sum_{\nu=0}^{\infty} \frac{\alpha^\nu}{\nu!} f^{(\nu+1)}(x),$$

that is, if $f'(x + \alpha)$ can be expanded in powers of α.

In what follows we therefore assume that the definition of θ with which we start is adhered to in that sense that we only allow such transformations of θ as lead to the same results when applied to the function on which we operate, In other terms, the nature of the function $f(x)$ puts certain restrictions on the permissible transformations of θ.

When we introduce repetitions of the θ operation, it will be on the assumption that the existence of $\theta^\nu f(x)$ has been ascertained. Further, if θ has been defined by the expansion of $\varphi(D)$, it cannot without proof be assumed that θ^ν is the expansion of $\varphi^\nu(D)$ as in the case of a polynomial.

Keeping in mind these reservations, we proceed to show how the method by which Aitken has obtained his formula leads to a remainder term for (7) or (28), when $f(x)$ is not a polynomial. We have, by (26) and (12),

$$I\theta f(x) = \bar{D}\,\psi(D)\cdot\frac{D}{\psi(D)}f(x)$$

$$= \bar{D}\,Df(x) = f(x) - f(0)$$

whence

$$f(x) = f(0) + I\theta f(x). \tag{159}$$

Replacing $f(x)$ by $\theta^\nu f(x)$, we get

$$\theta^\nu f(x) = \theta^\nu f(0) + I\theta^{\nu+1}f(x),$$

and applying on both sides the operation I^ν which is evidently commutative with a constant, we find, since $\theta^\nu f(0)$ is a constant,

$$I^\nu\theta^\nu f(x) = \theta^\nu f(0)\cdot I^\nu 1 + I^{\nu+1}\theta^{\nu+1}f(x).$$

From this we obtain finally, by summation from $\nu = 0$ to $\nu = n$, the required result

$$f(x) = \sum_{\nu=0}^{n}\theta^\nu f(0)\cdot I^\nu 1 + I^{n+1}\theta^{n+1}f(x). \tag{160}$$

We shall write this in the form

$$f(x) = \sum_{\nu=0}^{n}\frac{\theta^\nu f(0)}{\nu!}x^{\nu\rceil} + R, \tag{161}$$

$$R = I^{n+1}\theta^{n+1}f(x). \tag{162}$$

As an example, let us consider the expansion in Abel's poweroids. In that case we have $\theta = E^\alpha D$, and $I = \bar{D}E^{-\alpha}$, so that

$$If(x) = \int_0^x f(t-\alpha)\,dt = \int_{-\alpha}^{x-\alpha} f(t)\,dt.$$

»I» therefore means integration from $-\alpha$ to $x-\alpha$, and we may at once write down

$$R = \int_{-\alpha}^{x-\alpha} \cdots \int_{-\alpha}^{x-\alpha} f^{(n+1)}(x + \overline{n+1}\,\alpha)\, d\,x^{n+1}. \tag{163}$$

For $\alpha = 0$ this reduces to the remainder term of Maclaurin's formula

$$R = \int_0^x \cdots \int_0^x f^{(n+1)}(x)\, d\,x^{n+1}$$

$$= \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t)\, d\,t$$

$$= \frac{x^{n+1}}{n!} \int_0^1 (1-t)^n f^{(n+1)}(t\,x)\, d\,t.$$

If $f^{(n+1)}(x)$ is continuous in a closed interval to which 0, x, $n\,\alpha$ and $x + n\,\alpha$ belong, we may apply the theorem of mean value to (163) and find

$$R = f^{(n+1)}(\xi)\, I^{n+1}\, 1$$

or

$$R = \frac{x\,(x - \overline{n+1}\,\alpha)^n}{(n+1)!} f^{(n+1)}(\xi), \tag{164}$$

ξ being a point of the aforesaid interval.

The result is, then, the expansion in Abel's poweroids

$$f(x) = \sum_{\nu=0}^{n} \frac{f^{(\nu)}(\nu\,\alpha)}{\nu!}\, x\,(x - \nu\,\alpha)^{\nu-1} + R \tag{165}$$

where the remainder term is given by (163) or (164), the latter expression being available for real variables only.

In the paper quoted above, Halphen has dealt with the conditions on which an analytical function may be expanded in an infinite series of Abel's poweroids. The corresponding question for factorials has been treated by Nörlund.[1] These investigations show that the question of expanding a function in an infinite series of poweroids is a delicate one which it seems necessary to attack separately for each poweroid. There is therefore a considerable field for further investigations on these lines.

[1] N. E. Nörlund: Leçons sur les séries d'interpolation, Paris 1926.