ON SOME TYPES OF FUNCTIONAL SPACES.

A Contribution to the Theory of Almost Periodic Functions.

BY

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WITH

AN APPENDIX.

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Preface.

In the present paper we shall study some types of functional spaces, the S^p-spaces, the W^p-spaces and the B^p-spaces ($p \geqq 1$) as well as the almost periodic subspaces of these spaces which were met with when generalising the theory of the almost periodic functions. The spaces of S-type, W-type and B-type will be treated separately.

In a later paper FØLNER will study the »ensemble« of all the types of spaces mentioned. In the investigation of this ensemble certain methods of constructing examples, developed in the present common paper, will be employed, though in a modified and generalised form as in different respects more properties have to be demanded of the constructed functions. In order to avoid repetitions and to make the latter paper more perspicuous, these generalisations of the examples will be treated by FØLNER in an appendix to the present paper.

Contents.

Introduction.

Throughout the paper we operate with (complex) LEBESGUE measurable functions of a real variable; therefore in the following by the word »function« we shall always mean a LEBESGUE measurable function. By the p-integral of a function $f(x)$ from a to b we shall mean $\int\limits_a^b |f(x)|^p\,dx$, whether this integral is finite or infinite. We call a function $f(x)$, defined on the whole x-axis, p-integrable if the p-integral of $f(x)$ extended over any finite interval is finite.

The different types of generalised almost periodic functions, the S^p-a. p., W^p-a. p. and B^p-a. p. functions, can on the one hand be interpreted as generalisations of the ordinary almost periodic functions and on the other as generalisations of the p-integrable periodic functions.

An ordinary almost periodic function (in the following often shortly denoted as an o. a. p. function) is a continuous complex function $f(x)$, defined for $-\infty < x < \infty$, which, corresponding to every $\varepsilon > 0$, has a relatively dense set of translation numbers $\tau = \tau(\varepsilon)$. A set is called relatively dense if there exists a length L such that any interval $\alpha < x < \alpha + L$ of this length contains at least one number of the set, and a number τ is called a translation number belonging to ε if it satisfies the inequality $|f(x+\tau) - f(x)| \leqq \varepsilon$ for all x.

The main theorem in the theory of the almost periodic functions states that *an almost periodic function can also be characterised as a function which may be approximated, uniformly for all x, by trigonometric polynomials, i. e. sums of the form*

$$\sum_{n=1}^{N} a_n\, e^{i\lambda_n x}$$

where the a_n are arbitrary complex numbers and the λ_n are real numbers.

It is this last property of the almost periodic functions which is used at their generalisation, the uniform convergence being only replaced by other limit notions. These limit notions are introduced by means of a distance notion; to two arbitrary functions a distance is ascribed, and a sequence of functions $f_n(x)$ is called convergent to the function $f(x)$, if the distance of $f_n(x)$ and $f(x)$ tends to zero for $n \to \infty$. Incidentally we remark that the uniform convergence for all x originates from the (ordinary) distance

$$D_0[f(x), g(x)] = \underset{-\infty < x < \infty}{\text{u. b.}} |f(x) - g(x)|.$$

Concerning the p-integrable periodic functions with a fixed period $b-a>0$ a similar main theorem is valid as for the almost periodic functions. $f(x)$ and $g(x)$ being two arbitrary periodic functions with the given period, we define the p-distance for $p \geqq 1$ by

$$D_p[f(x), g(x)] = \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)-g(x)|^p\, dx},$$

and we call a sequence $f_n(x)$ of periodic functions with the given period p-convergent to $f(x)$, if $D_p[f(x), f_n(x)] \to 0$ for $n \to \infty$. Then the main theorem states that *to any p-integrable periodic function $f(x)$ with the period $b-a$ a sequence of trigonometric polynomials with the period $b-a$ can be found which p-converges to $f(x)$.* (The converse theorem is here obvious.)

A comprehensive treatment of the generalised almost periodic functions was given in a paper by Besicovitch and Bohr: Almost Periodicity and General Trigonometric Series, Acta mathematica, vol. 57. We shall use some facts deduced in that paper, in each case quoting in detail the results we shall employ. On the one hand we shall use some simple relations for the different distances — they will be quoted in this introduction — and on the other hand certain properties of the generalised almost periodic functions which will be cited in Chapter I. In the following the paper in question will be quoted as I.

While the periodic functions may be considered as functions given in a finite interval, the period interval, in the theory of the almost periodic functions we principally have to operate with (i. e. in some way or other to take mean values over) the infinite interval $-\infty < x < \infty$. Desiring to transfer the p-distance mentioned above from a finite to an infinite interval we may choose among several different possibilities each of which presents its special peculiarity and its special interest. Within the set of all (measurable) functions we introduce for every $p \geqq 1$ three such distances which we denote, after Stepanoff, Weyl and Besicovitch, by

$$D_{S_L^p}[f(x), g(x)], \quad D_{W^p}[f(x), g(x)] \quad \text{and} \quad D_{B^p}[f(x), g(x)].$$

Stepanoff's distance is given by

$$D_{S_L^p}[f(x), g(x)] = \underset{-\infty<x<\infty}{\text{u. b.}} \sqrt[p]{\frac{1}{L}\int_x^{x+L} |f(\xi)-g(\xi)|^p\, d\xi}.$$

Here L is a fixed positive number; its value is unessential (L may for instance be chosen equal to 1) since given the two positive numbers L_1 and L_2 there exist two positive numbers k_1 and k_2, depending only on L_1 and L_2 and not on $f(x)$ and $g(x)$, such that (I, p. 221)

$$k_1 D_{S^p_{L_1}}[f(x), g(x)] \leqq D_{S^p_{L_2}}[f(x), g(x)] \leqq k_2 D_{S^p_{L_1}}[f(x), g(x)].$$

On account of these latter inequalities the distances $D_{S^p_L}$ corresponding to different L are said to be equivalent.

Concerning the BESICOVITCH distance the mean value is at once extended over the whole interval $-\infty < x < \infty$, viz.

$$D_{B^p}[f(x), g(x)] = \overline{\lim_{T \to \infty}} \sqrt[p]{\frac{1}{2T} \int_{-T}^{T} |f(x) - g(x)|^p \, dx}.$$

Finally the WEYL distance is an »intermediate thing« between the two distances cited above. Like STEPANOFF, WEYL considers a fixed length L which he however lets increase to ∞, viz.

$$D_{W^p}[f(x), g(x)] = \lim_{L \to \infty} \underset{-\infty < x < \infty}{\text{u. b.}} \sqrt[p]{\frac{1}{L} \int_{x}^{x+L} |f(\xi) - g(\xi)|^p \, d\xi} = \lim_{L \to \infty} D_{S^p_L}[f(x), g(x)].$$

It is easy to prove that the limit always exists for $L \to \infty$.

As immediately seen, all these distances are generalisations of the distance D_p; for, if $f(x)$ and $g(x)$ are periodic functions with the period h, we have

$$D_p[f(x), g(x)] = D_{S^p_h}[f(x), g(x)] = D_{W^p}[f(x), g(x)] = D_{B^p}[f(x), g(x)].$$

Instead of S^p_1 we simply write S^p, and similarly we omit p, if $p = 1$, in the symbols S^p_L, W^p and B^p. Frequently it is convenient to use a symbol which may represent an arbitrary one of the symbols S^p_L, W^p and B^p; in this case we use the symbol G or, if we want to emphasise the exponent p, the symbol G^p. We observe that the common symbol for S_L, W and B is G^1.

A sequence of functions $f_n(x)$ is called G-convergent to the function $f(x)$, if $D_G[f(x), f_n(x)] \to 0$ for $n \to \infty$, and we write

$$f_n(x) \xrightarrow{G} f(x).$$ [1]

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[1] $f_n(x) \xrightarrow{S^p_{L_1}} f(x)$ means the same as $f_n(x) \xrightarrow{S^p_{L_2}} f(x)$, the distances $D_{S^p_L}$ being equivalent for different values of L.

A function is called a G-a. p. function, if there exists a sequence of trigonometric polynomials which G-converges to the function. The set of G-a. p. functions is called the G-a. p. set.[1] For every $p \geqq 1$ we have thus introduced the three important sets:

the S^p-a. p. set, the W^p-a. p. set and the B^p-a. p. set.

Concerning particular G^p-a. p. functions, besides the o. a. p. functions and the p-integrable periodic functions, we mention the G^p-limit periodic functions; a function $f(x)$ is called G^p-limit periodic, if there exists a sequence of p-integrable periodic functions (generally without a common period) which G^p-converges to $f(x)$.

For every $p \geqq 1$ the inequalities

$$D_O[f(x), g(x)] \geqq D_{S_L^p}[f(x), g(x)] \geqq D_{W^p}[f(x), g(x)] \geqq D_{B^p}[f(x), g(x)]$$

are valid (I, p. 222); hence denoting the set of o. a. p. functions as the o. a. p. set, we have for each $p \geqq 1$:

the o. a. p. set $\subseteqq$ the S^p-a. p. set $\subseteqq$ the W^p-a. p. set $\subseteqq$ the B^p-a. p. set.

Further we have for $1 \leqq p_1 < p_2$ (I, p. 222)

$$D_{G^{p_1}}[f(x), g(x)] \leqq D_{G^{p_2}}[f(x), g(x)].$$

(This inequality is a consequence of Hölder's inequality quoted below.) Hence it holds for $1 \leqq p_1 < p_2$ that

the G^{p_1}-a. p. set $\supseteqq$ the G^{p_2}-a. p. set.

Just as the distance $D_O[f(x), g(x)]$ originates from an O-norm $D_O[f(x)] = D_O[f(x), 0]$, and the distance $D_p[f(x), g(x)]$ from a p-norm $D_p[f(x)] = D_p[f(x), 0]$, every one of our distances $D_G[f(x), g(x)]$ originates from a G-norm $D_G[f(x)] = D_G[f(x), 0]$. Obviously the G-norm satisfies, like the O-norm and the p-norm, the relation

(1) $$D_G[a f(x)] = |a| \cdot D_G[f(x)] \qquad (a \text{ a complex number});$$

further it satisfies the inequality (I, p. 222)

(2) $$D_G[f(x) + g(x)] \leqq D_G[f(x)] + D_G[g(x)]$$

[1] The $S_{L_1}^p$-a. p. set is identical with the $S_{L_2}^p$-a. p. set, as the distances $D_{S_L^p}$ are equivalent for different values of L, and it is called the S^p-a. p. set. The functions in the S^p-a. p. set are called S^p-a. p. functions.

which is equivalent to the Triangle Rule

$$D_G[f(x), g(x)] \leqq D_G[f(x), h(x)] + D_G[h(x), g(x)].$$

(This inequality is a consequence of MINKOWSKI's inequality quoted below.)

As a trigonometric polynomial is bounded, its G-norm is finite, and in consequence of the Triangle Rule the same is valid for any G-a. p. function. A function with finite G-norm is called a G-function. The set of all G-functions is called the G-set[1], and we have

the G-a. p. set $\subseteqq$ the G-set.

It is important to observe that *the S^p-set and the W^p-set are identical for each* $p \geqq 1$; that the S^p-set $\subseteqq$ the W^p-set is an immediate consequence of the inequality $D_{S^p} \geqq D_{W^p}$, and the converse is involved by the equation $D_{W^p} = \lim\limits_{L \to \infty} D_{S^p_L}$ which shows that if D_{W^p} is finite then $D_{S^p_L}$ will be finite for sufficiently large L (and therefore for all L). We emphasise that the analogue is not valid for the a. p. sets; in fact the S^p-a. p. set is a proper subset of the W^p-a. p. set.

The G-sets satisfy similar relations as the G-a. p. sets, and on account of the same distance relations:

For every $p \geqq 1$ is

the S^p-set (W^p-set) $\subseteqq B^p$-set,

and for $1 \leqq p_1 < p_2$ is

the G^{p_1}-set $\supseteqq$ the G^{p_2}-set.

In our G-sets we shall have to consider the so-called G-fundamental sequences. A sequence of functions $f_n(x)$ from the G-set is called a G-fundamental sequence, if $D_G[f_n(x), f_m(x)] \to 0$ when n and m, independently of each other, tend to ∞. Further we shall use the notion G-closed. We call a set of functions G-closed, if each function which is the G-limit of a sequence of functions from the set belongs itself to the set. On account of the Triangle Rule the G-set and the G-a. p. set are obviously G-closed.

Leaving out of account that the G-distance between two different functions may be zero, the G-set is organised as a linear metric space because of (1) and (2). It is easily shown that the same holds for the G-a. p. set. Firstly, the product of a G-a. p. function $f(x)$ by a constant is again a G-a. p. function;

[1] The $S^p_{L_1}$-set is identical with the $S^p_{L_2}$-set, as the distances $D_{S^p_L}$ are equivalent for different values of L, and it is called the S^p-set. The functions in the S^p-set are called S^p-functions.

for, if $s_n(x)$ is a sequence of trigonometric polynomials G-converging to $f(x)$, the sequence $a \cdot s_n(x)$ of trigonometric polynomials will G-converge to $a \cdot f(x)$, since

$$D_G[af(x),\, as_n(x)] = |a| \cdot D_G[f(x),\, s_n(x)].$$

Secondly, the sum of two G-a. p. functions $f^{(1)}(x)$ and $f^{(2)}(x)$ is again a G-a. p. function; for, if $s_n^{(1)}(x)$ is a sequence of trigonometric polynomials G-converging to $f^{(1)}(x)$, and $s_n^{(2)}(x)$ is a sequence of trigonometric polynomials G-converging to $f^{(2)}(x)$, the sequence $s_n^{(1)}(x) + s_n^{(2)}(x)$ consisting of trigonometric polynomials will G-converge to $f^{(1)}(x) + f^{(2)}(x)$, as

$$D_G[f^{(1)}(x) + f^{(2)}(x),\, s_n^{(1)}(x) + s_n^{(2)}(x)] = D_G[(f^{(1)}(x) - s_n^{(1)}(x)) + (f^{(2)}(x) - s_n^{(2)}(x))]$$
$$\leqq D_G[f^{(1)}(x),\, s_n^{(1)}(x)] + D_G[f^{(2)}(x),\, s_n^{(2)}(x)].$$

In the proofs of theorems on G-a. p. functions it is often convenient, instead of, as above, using the definition itself, to employ the following simple theorem: *A G-a. p. function can also be characterised as a function which is the G-limit of o. a. p. functions (and not just of trigonometric polynomials).* The proof is immediate. In fact, a function which can be approximated by o. a. p. functions must belong to the G-a. p. set, as the o. a. p. set $\subseteqq$ G-a. p. set, and the G-a. p. set is G-closed.

To pass from the G-set to a proper linear metric space where the distance between two different points is > 0 (and not only $\geqq 0$), an equivalence relation ($\sim$) between the G-functions is introduced in the following obvious way:

$$f(x) \sim g(x) \quad \text{if} \quad D_G[f(x),\, g(x)] = 0.$$

Then the G-set falls into classes of equivalent functions. Each of these classes is called a G-point. Evidently two functions of the G-set belong to the same G-point, if and only if they differ from each other by a function of the G-norm 0. Such functions of the G-norm 0 are called G-zero functions. Now a distance (again denoted by D_G) is introduced in the following manner: Let $\mathfrak{A}$ and $\mathfrak{B}$ be two arbitrary G-points; then we define $D_G[\mathfrak{A}, \mathfrak{B}]$ by the equation

$$D_G[\mathfrak{A}, \mathfrak{B}] = D_G[f(x),\, g(x)],$$

where $f(x)$ and $g(x)$ are arbitrary representatives of $\mathfrak{A}$ and $\mathfrak{B}$; this definition is evidently unique. The multiplication of a G-point by a constant, and the addition of two G-points being defined by means of representatives, it is plain that (1) and (2) are still satisfied, if we consider G-points instead of G-functions. And moreover the G-distance between two different G-points is always > 0.

Thus the set of G-points is organised as a linear metric space by the distance D_G. We denote it as **the G-space.**

If one function in a G-point is G-a. p., all functions of the point are G-a. p. functions, and the point is called a G-a. p. **point**. The set of the G-a. p. points, organised by the distance D_G, forms a linear subspace of the G-space. It is called **the G-a. p. space.**

Now we have introduced all the spaces which we shall investigate in the following, viz. for every $p \geqq 1$:

$$\begin{aligned}
&\text{the } S^p\text{-a. p. space} \subseteqq \text{the } S^p\text{-space},\\
&\text{the } W^p\text{-a. p. space} \subseteqq \text{the } W^p\text{-space},\\
&\text{the } B^p\text{-a. p. space} \subseteqq \text{the } B^p\text{-space}.
\end{aligned}$$

If one function in a G-point is G-limit periodic, so are all functions of the point, and the point is called a G-**limit periodic point.**

If a G-point contains a periodic function, it is called a **periodic G-point** (but of course it is not true, that all the functions of a periodic G-point are periodic functions).

We say that a sequence $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ of G-points G-**converges** to the G-point $\mathfrak{A}$ if $D_G[\mathfrak{A}, \mathfrak{A}_n] \to 0$ or, what is equivalent, if an (arbitrary) sequence of representatives $f_1(x), f_2(x), \ldots$ of $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ is G-convergent to a representative $f(x)$ of $\mathfrak{A}$.

A sequence $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ of G-points is called a G-**fundamental sequence** if $D_G[\mathfrak{A}_n, \mathfrak{A}_m] \to 0$ for n and m tending to ∞ or, what is equivalent, if a sequence of representatives $f_1(x), f_2(x), \ldots$ of $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ is a G-fundamental sequence.

As well known, a metric space is called **complete** if every fundamental sequence of the space is convergent; otherwise it is called **incomplete.**

A subset of a metric space is called **closed** (relatively to the latter) if every point of the space which is the limit of points of the subset belongs itself to the subset. Evidently, *the G-a. p. space is closed* (relatively to the G-space).

Concerning the Stepanoff distance, it is easy to see that, for any $p \geqq 1$, a function is a S^p-zero function only in the trivial case when it is 0 »almost everywhere« (i. e. except in a set of measure 0); consequently, for every p, an S^p-point consists of essentially only one function. In the two other cases (the W^p and B^p) the set of zero functions is considerably more comprehensive, and most comprehensive for $p = 1$; thus, while it is only a mathematical subtlety to speak of S^p-points instead of S^p-functions, it is of decisive importance to distinguish between G-points and G-functions in case of the W^p- and B^p-spaces.

To deduce the relations for the different distances (about which we referred to I) two very important inequalities are used, Hölder's inequality and Minkowski's inequality. As we later on shall apply these inequalities repeatedly, we quote them here in the introduction.

Hölder's inequality. *Let p and q be two positive numbers satisfying the condition*

$$\frac{1}{p}+\frac{1}{q}=1,$$

and $f(x)$ and $g(x)$ two complex functions, defined in the interval (a, b); then we have

$$\frac{1}{b-a}\int_a^b |f(x)\,g(x)|\,dx \leqq \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)|^p\,dx}\cdot\sqrt[q]{\frac{1}{b-a}\int_a^b |g(x)|^q\,dx}.$$

As the inequality can be reduced by $\frac{1}{b-a}$, *the corresponding inequality for integrals* (instead of mean values) *is also valid.*

We emphasise a special case of Hölder's inequality (which is obtained by replacing $f(x)$, $g(x)$ and p by $|f(x)|^{p_1}$, 1 and $\frac{p_2}{p_1}$ respectively), viz:

For $1 \leqq p_1 < p_2$ *is*

$$\sqrt[p_1]{\frac{1}{b-a}\int_a^b |f(x)|^{p_1}\,dx} \leqq \sqrt[p_2]{\frac{1}{b-a}\int_a^b |f(x)|^{p_2}\,dx}.$$

Minkowski's inequality. *Let $f(x)$ and $g(x)$ be two complex functions, defined in the interval (a, b), and $p \geqq 1$; then the inequality*

$$\sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)+g(x)|^p\,dx} \leqq \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)|^p\,dx}+\sqrt[p]{\frac{1}{b-a}\int_a^b |g(x)|^p\,dx}$$

holds. As before the corresponding integral inequality is also valid.

Obviously the inequality can also be written in the form

$$\sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)+g(x)|^p\,dx} \geqq \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)|^p\,dx}-\sqrt[p]{\frac{1}{b-a}\int_a^b |g(x)|^p\,dx}.$$

Besides trivial facts concerning LEBESGUE integrals we shall have to use

Fatou's theorem. *Let $f(t, n)$ be a non-negative function, given for all t in a finite interval (a, b) and for all positive integral values of n. Then we have*

$$\int_a^b \varliminf_{n\to\infty} f(t,n)\,dt \leqq \varliminf_{n\to\infty} \int_a^b f(t,n)\,dt.$$

Before passing to a summary of the paper we will here gather different remarks of a general character which, like the inequalities and the theorem above, will be of importance later on.

To begin with we introduce the notion of minimum of two complex functions $f(x)$ and $g(x)$, defined for all x, viz.

$$\min\,[f(x), g(x)] = \begin{cases} f(x) \text{ for the } x \text{ satisfying } |f(x)| \leqq |g(x)| \\ g(x) \text{ for the } x \text{ satisfying } |g(x)| < |f(x)|. \end{cases}$$

The little »lack of beauty« that $\min\,[f(x), g(x)]$ is not symmetric in $f(x)$ and $g(x)$ is of no importance whatsoever.

The definition of $\min\,[f(x), g(x)]$ involves immediately the following inequalities

$$|\min\,[f(x), g(x)]| \leqq |f(x)|, \qquad |\min\,[f(x), g(x)]| \leqq |g(x)|$$

and

$$|\min\,[f(x), g(x)] - f(x)| \leqq |g(x) - f(x)|, \qquad |\min\,[f(x), g(x)] - g(x)| \leqq |f(x) - g(x)|.$$

A G-point considered as a set of functions is G-closed, since, $f_1(x), f_2(x), \ldots$ being a sequence of functions of a G-point with the G-limit $f(x)$, we have

$$D_G\,[f(x), f_1(x)] = D_G\,[f(x), f_n(x)] \to 0,$$

so that $D_G\,[f(x), f_1(x)] = 0$, i. e. $f(x)$ belongs to the G-point.

A G-point considered as a set of functions is closed with respect to the minimum-operation, since, $f_1(x)$ and $f_2(x)$ being two functions of a G-point, we have $|\min\,[f_1(x), f_2(x)] - f_1(x)| \leqq |f_1(x) - f_2(x)|$ and consequently

$$D_G\,[\min\,[f_1(x), f_2(x)], f_1(x)] \leqq D_G\,[f_1(x), f_2(x)] = 0,$$

so that $\min\,[f_1(x), f_2(x)]$ lies also in the G-point.

Frequently it is convenient to use the distance

$$D^*_{B^p}[f(x), g(x)] =$$

$$\max\left[\overline{\lim_{T\to\infty}}\sqrt[p]{\frac{1}{T}\int_0^T |f(x)-g(x)|^p\,dx},\ \overline{\lim_{T\to\infty}}\sqrt[p]{\frac{1}{T}\int_{-T}^0 |f(x)-g(x)|^p\,dx}\right]$$

instead of the distance $D_{B^p}[f(x), g(x)]$. The two distances are equivalent, as

$$\sqrt[p]{\frac{1}{2}}\cdot D^*_{B^p}[f(x), g(x)] \leqq D_{B^p}[f(x), g(x)] \leqq D^*_{B^p}[f(x), g(x)].$$

The new distance originates from the norm $D^*_{B^p}[f(x)] = D^*_{B^p}[f(x), 0]$ which satisfies (1) and (2).

Finally we give a summary of the content of the paper.

In Chapter I the needed properties of the generalised almost periodic functions are quoted, inter alia certain translation properties and the approximation by Bochner-Fejér polynomials are treated.

In Chapter II the completeness or incompleteness of the different spaces is investigated. It is shown, what has been known in the main, that the S^p-space and the S^p-a. p. space as well as the B^p-space and the B^p-a. p. space are complete for every $p \geqq 1$, and, what has not been known before, that the W^p-space and the W^p-a. p. space are incomplete for every $p \geqq 1$. It is of special importance that the B^2-a. p. space is complete, as this involves the validity of Besicovitch's Theorem which is the analogue of the famous theorem of Riesz-Fischer on 2-integrable periodic functions.

In Chapter III, which has the character of an insertion, three theorems are proved which will be applied in the following Chapters.

Chapter IV deals with the mutual relations of the S^p-spaces and the S^p-a. p. spaces, Chapter V with the mutual relations of the W^p-spaces and the W^p-a. p. spaces, and Chapter VI with the mutual relations of the B^p-spaces and the B^p-a. p. spaces. The investigations of the spaces of W-type and those of B-type are essentially similar in some respects, but show also characteristic differences.

The paper consist partly of theorems and partly of counter examples. Many of the examples are simple and more or less trivial. The especially strong and substantial examples are called main examples. Most of the functions constructed in our examples are piecewise constant and change between the value 0 and values >0. Hence they have a graph like the function outlined in Fig. 1. The graph thus consists of rectangles which stand with one side on the x-axis. These rectangles are called towers, and the function is given by indicating the size of the towers and their position on the x-axis. The size of a tower may be given by its »height« k and »breadth« b, but it is often given by prescribing the p-integral of the tower for two different values p_1 and p_2 ($\geqq 1$); from these values k and b can immediately be calculated; for the p_1-integral of the tower is $I_1 = b\,k^{p_1}$ and the p_2-integral $I_2 = bk^{p_2}$, and hence

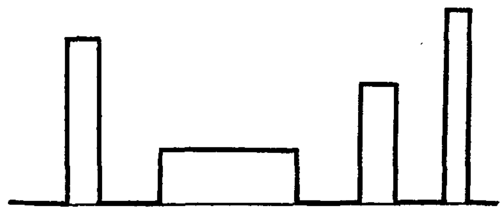

Fig. 1.

$$k = \left(\frac{I_2}{I_1}\right)^{\frac{1}{p_2-p_1}} \quad \text{and} \quad b = \left(\frac{I_1^{p_2}}{I_2^{p_1}}\right)^{\frac{1}{p_2-p_1}};$$

for an arbitrary p_3 ($\geqq 1$) the p_3-integral becomes

$$I_3 = b k^{p_3} = I_1^{\frac{p_2-p_3}{p_2-p_1}} \cdot I_2^{\frac{p_3-p_1}{p_2-p_1}}.$$

When indicating a tower by the values of its p-integral for two different values of p, the integral corresponding to the smaller of these values is always chosen less than the other integral (i. e. the height k of the tower is always chosen >1); then the p-integral is a steadily increasing function of p which tends to ∞ for $p \to \infty$. The position of a tower is generally indicated by the number with which the center of the lowest side coïncides. The tower is said to stand on this number. Sometimes we speak about a tower as placed or standing on an interval. This means that the tower stands on the centre of the interval and does not protrude beyond the interval.

All our examples of G-a. p. functions are chosen among the G-limit periodic functions.

CHAPTER I.

The Generalised Almost Periodic Functions.

In this chapter some well known properties of the generalised almost periodic functions are quoted which will be applied in our later investigations. We also remind of the proofs of some of the theorems.

Already in the introduction we have mentioned that the product of a G-a. p. function by a constant and the sum of two G-a. p. functions are again G-a. p. functions. Moreover it is valid that the product of a G-a. p. function by an ordinary almost periodic function is again a G-a. p. function.

If $f(x)$ is G-a. p., the modulus $|f(x)|$ and the function

$$(f(x))_N = \begin{cases} f(x) & \text{for } |f(x)| \leqq N \\ N\dfrac{f(x)}{|f(x)|} & \text{for } |f(x)| \geqq N \end{cases}$$

which originates from the function $f(x)$ by »cutting it off« at the positive number N are again G-a. p. This is a corollary of the following theorem: *Let $f(x)$ be a G-a. p. function and $\Phi(z)$ be a function defined in the whole complex plane (or, in case of a real function $f(x)$, on the real axis) with a bounded difference quotient; then $\Phi(f(x))$ is G-a. p.* The proof is immediate: Let $f_n(x)$ be a sequence of o. a. p. functions which G-converges to $f(x)$; then $\Phi(f_n(x))$ too is a sequence of o. a. p. functions on account of the uniform continuity of $\Phi(z)$; further

$$\Phi(f_n(x)) \xrightarrow{G} \Phi(f(x))$$

since the inequality

$$|\Phi(f(x)) - \Phi(f_n(x))| \leqq K|f(x) - f_n(x)|$$

involves

$$D_G[\Phi(f(x)),\ \Phi(f_n(x))] \leqq K D_G[f(x),\ f_n(x)] \to 0.$$

If $f(x)$ is G-a. p., the function $(f(x))_N$ will G-converge to $f(x)$ for $N \to \infty$.

Proof. Let $\varepsilon > 0$ be given. We choose a trigonometrical polynomial $s(x)$ such that $D_G[f(x), s(x)] < \frac{\varepsilon}{2}$, and use the estimation

$$D_G[f(x),\ (f(x))_N] \leqq D_G[f(x),\ s(x)] + D_G[s(x),\ (f(x))_N].$$

For $N \geqq$ u. b. $|s(x)| = K$ we have $s(x) = (s(x))_N$, and hence on account of the inequality

$$|(f(x))_N - (g(x))_N| \leqq |f(x) - g(x)|$$

(the validity of which is seen by help of a simple geometrical consideration) we get

$$D_G[s(x), (f(x))_N] = D_G[(s(x))_N, (f(x))_N] \leqq D_G[s(x), f(x)].$$

Thus we have for $N \geqq K$

$$D_G[f(x), (f(x))_N] \leqq 2 D_G[f(x), s(x)] < \varepsilon.$$

Just as for the ordinary almost periodic functions and the p-integrable periodic functions there exists a theory of Fourier series for the G-a. p. functions. Each G-a. p. function $f(x)$ has a mean value

$$M\{f(x)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\,dx = \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} f(x)\,dx = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{0} f(x)\,dx,$$

and the function

$$a(\lambda) = M\{f(x)e^{-i\lambda x}\}$$

of the real variable λ is different from 0 for at most an enumerable set of values of λ; these are called the Fourier exponents of $f(x)$ and denoted in one order or another by $\Lambda_1, \Lambda_2, \ldots$. The values of $a(\lambda)$, belonging to the Fourier exponents $\Lambda_1, \Lambda_2, \ldots$, are called the Fourier coefficients of $f(x)$ and denoted by $A_1, A_2, \ldots$ respectively. With the function $f(x)$ is associated the Fourier series $\Sigma A_n e^{i\Lambda_n x}$, and we write

$$f(x) \sim \Sigma A_n e^{i\Lambda_n x}.$$

Sometimes it is convenient to include certain »improper« terms $A_n e^{i\Lambda_n x}$ (at most an enumerable number) where $A_n = M\{f(x)e^{-i\Lambda_n x}\} = 0$. For such a term Λ_n is called an improper Fourier exponent and A_n $(=0)$ an improper Fourier coefficient belonging to the exponent Λ_n.

All the functions in a G-a. p. point have the same Fourier series which is called the Fourier series of the G-a. p. point. Corresponding to the uniqueness theorem of the o. a. p. functions the following uniqueness theorem is valid for the G-a. p. functions: *Two different G-a. p. points cannot have the same Fourier series.*

We emphasise that the Fourier series is formed in exactly the same way for all our types of generalised almost periodic functions. For a p-integrable periodic function it is easy to prove that it has the same Fourier series in the ordinary sense as in the G-a. p. sense.

Concerning the connection between a G-a. p. function and its Fourier series the usual rules on addition, and on multiplication by a constant are valid. If $f_m(x)$ is a sequence of G-a. p. functions G-converging to the G-a. p. function $f(x)$, the Fourier series of $f(x)$ can be obtained by a formal limit process from the Fourier series of $f_m(x)$.

The Bochner-Fejér method of summation, of great importance in the theory of the o. a. p. functions, can be transferred to the G-a. p. functions (I, § 12—13). Starting from the Fourier series of a G-a. p. function $f(x)$ *a sequence of trigonometric polynomials* $\sigma_q(x)$, *the Bochner-Fejér polynomials, can be found which* G-*converges to* $f(x)$. By means of kernels $K^q(t)$ which are non-negative trigonometric polynomials with the mean value 1 these Bochner-Féjer polynomials $\sigma_q(x)$ can be represented in the form

$$\sigma_q(x) = \underset{t}{M}\{f(x+t)\,K^q(t)\}.$$

To establish the approximation-properties of the Bochner-Fejér polynomials the following inequality which can be deduced from the representation above is of decisive importance:

$$D_G[\sigma_q(x)] \leqq D_G[f(x)].$$

It is essential for our later applications that this inequality holds even if G does not just denote the type of almost periodicity of the function $f(x)$. Certainly, in the proof of the inequality in question given in I, where G has an arbitrary fixed meaning, it was assumed that $f(x)$ was a. p. in the G-sense, but in fact it was only used in the proof that $f(x)$ was almost periodic in one sense or another and not just in the G-sense.

Concerning a W^p-a. p. function $f(x)$ the Bochner-Fejér sequence $\sigma_q(x)$ does not only W^p-converge to $f(x)$, but this W^p-convergence takes place with a certain »uniformity«; in fact, to every $\varepsilon > 0$ there can be determined an L_0 and a Q such that

$$D_{S_L^p}[f(x), \sigma_q(x)] \leqq \varepsilon \qquad \text{for } L \geqq L_0 \text{ and } q \geqq Q.$$

In the case of the W^2-a. p. functions, R. Schmidt (Math. Ann. Bd. 100) was the first to indicate approximating trigonometric polynomials with this property.

The Bochner-Fejér polynomials $\sigma_q(x)$ have the form

$$\sigma_q(x) = \sum_{n=1}^{N(q)} k_n^{(q)} A_n e^{i\Lambda_n x}$$

where the Λ_n are Fourier exponents and the A_n the corresponding Fourier coefficients of $f(x)$, and $N(q) \to \infty$ for $q \to \infty$. The factor $k_n^{(q)}$ satisfies the inequality $0 \leqq k_n^{(q)} \leqq 1$ and tends to 1 for fixed n and $q \to \infty$.

For a B^2-a. p. function $f(x)$ with the Fourier series $\Sigma A_n e^{i\Lambda_n x}$ the Parseval equation holds:

$$M\{|f(x)|^2\} = \Sigma |A_n|^2.$$

As cited in the introduction, for the B^2-a. p. functions the theorem of Besicovitch which is the analogue to Riesz-Fischer's Theorem on 2-integrable periodic functions is valid: *An arbitrary given trigonometric series* $\Sigma A_n e^{i\Lambda_n x}$ *is the Fourier series of a* B^2*-a. p. function if and only if* $\Sigma |A_n|^2$ *is convergent.* The proof of this theorem relies on the completeness of the B^2-a. p. space; we shall return to it in Chapter II.

Just as the o. a. p. functions, also the generalised almost periodic functions can (as shown in I) be characterised in two different ways, viz. on the one hand by their approximation by means of trigonometric polynomials, and on the other by translation properties. In the present paper the generalised almost periodic functions have been defined by their approximation properties. As to the S^p-a. p. and W^p-a. p. functions, however, also their translation properties will be needed for some of our investigations. In the two following theorems we shall state these translation properties which can easily be deduced from our definitions of the S^p-a. p. and W^p-a. p. functions.

Theorem 1. *An* S^p*-a. p. function* $f(x)$ *possesses, to every* $\varepsilon > 0$, *a relatively dense set of* S_L^p*-translation numbers (L arbitrary fixed), i. e. of numbers* τ *with the property*

$$D_{S_L^p}[f(x+\tau), f(x)] \leqq \varepsilon.$$

Proof. Let $\varphi(x)$ be an o. a. p. function such that

$$D_{S_L^p}[f(x), \varphi(x)] < \frac{\varepsilon}{3},$$

and let τ be an (ordinary) translation number of $\varphi(x)$ belonging to $\frac{\varepsilon}{3}$. Then we have

$$|\varphi(x+\tau) - \varphi(x)| \leqq \frac{\varepsilon}{3}$$

for all x and hence a fortiori

$$D_{S_L^p}[\varphi(x+\tau),\ \varphi(x)] \leqq \frac{\varepsilon}{3}\cdot$$

By means of the Triangle Rule we obtain

$$D_{S_L^p}[f(x+\tau),\ f(x)] \leqq$$
$$D_{S_L^p}[f(x+\tau),\ \varphi(x+\tau)] + D_{S_L^p}[\varphi(x+\tau),\ \varphi(x)] + D_{S_L^p}[\varphi(x),\ f(x)] \leqq \varepsilon.$$

Consequently τ is an S_L^p-translation number of $f(x)$ belonging to ε. As the τ's form a relatively dense set, the theorem is proved.

Theorem 2. *A W^p-a. p. function $f(x)$ has, to every $\varepsilon > 0$ and for L sufficiently large (i. e. $L \geqq L_0(\varepsilon)$), a relatively dense set of S_L^p-translation numbers.*

Proof. Let $\varphi(x)$ be an o. a. p. function so that

$$D_{W^p}[f(x),\ \varphi(x)] < \frac{\varepsilon}{3}\cdot$$

Since $D_{W^p} = \lim_{L\to\infty} D_{S_L^p}$, we have for a sufficiently large L, i. e. for $L \geqq L_0(\varepsilon)$, that

$$D_{S_L^p}[f(x),\ \varphi(x)] < \frac{\varepsilon}{3}\cdot$$

It follows as in the proof of Theorem 1 that every (ordinary) translation number τ of $\varphi(x)$ belonging to $\frac{\varepsilon}{3}$ is an S_L^p-translation number of $f(x)$ belonging to ε for any $L \geqq L_0$.

In this paper, among the G-a. p. functions, we shall particularly consider the G-limit periodic functions, as all our G-almost periodic examples will be chosen among the latter functions. Therefore we finish this Chapter I by some remarks on G-limit periodic functions.

We begin by showing that a G-limit periodic function can also be characterised as a G-a. p. function whose Fourier exponents are rational multiples of one and the same real number.

1°. Let $f(x)$ be a G-a. p. function with Fourier exponents which are rational multiples of a number d. Since the exponents (in finite number) of any Bochner-Fejér polynomial $\sigma_q(x)$ are Fourier exponents of $f(x)$ and therefore integral multiples of a number d_q, it is evident that each $\sigma_q(x)$ is a periodic function $\left(\text{with period } \frac{2\pi}{d_q}\right)\cdot$ Hence $f(x)$ being the G-limit of the sequence $\sigma_q(x)$ is a G-limit periodic function.

2°. Let then $f(x)$ be a G-limit periodic function and $f_1(x), f_2(x), \ldots$ a sequence of p-integrable periodic functions, with periods $h_1, h_2, \ldots$, which G-converges to $f(x)$. We shall prove that all the Fourier exponents of $f(x)$ are rational multiples of a single number d.

We may assume that the Fourier series of $f(x)$ does not only consist of the constant term, since in this particular case the theorem is obviously valid. Then there exists a Fourier exponent $\Lambda_n \neq 0$. If A_n ($\neq 0$) denotes the Fourier coefficient of $f(x)$ belonging to this exponent Λ_n, and $A_n^{(m)}$ the (proper or improper) Fourier coefficient of $f_m(x)$ belonging to the exponent Λ_n, the coefficient $A_n^{(m)}$ tends to A_n for $m \to \infty$. Hence $A_n^{(m)} \neq 0$ for m sufficiently large, i. e. for $m \geqq m_0 = m_0(n)$. The exponent Λ_n thus being a proper Fourier exponent of $f_m(x)$ for $m \geqq m_0$, we have

$$\Lambda_n = \frac{2\pi}{h_m}\nu_m \ (\nu_m \text{ integral}) \quad \text{and thus} \quad h_m = \frac{2\pi}{\Lambda_n}\nu_m \quad \text{for } m \geqq m_0.$$

Consequently for $m \geqq m_0$ the periods h_m are integral multiples of the number $g = \frac{2\pi}{\Lambda_n}$, i. e.

$$h_m = \nu_m g \qquad \text{for } m \geqq m_0.$$

The Fourier exponents of $f_m(x)$ for $m \geqq m_0$ are thus to be found among the numbers $\frac{2\pi}{h_m}\mu = \frac{2\pi}{g}\cdot\frac{\mu}{\nu_m}$ (μ integral) so that they are all rational multiples of the number $d = \frac{2\pi}{g}$. Finally the same must be valid for the Fourier exponents of the function $f(x)$ itself, since the Fourier series of $f(x)$ can be obtained as the formal limit of the Fourier series of $f_m(x)$ for $m \to \infty$.

Remark. We saw in 1° that the Bochner-Fejér polynomials of a G-limit periodic function are periodic functions. We shall add a remark concerning the periods of the Bochner-Fejér polynomials of a G-limit periodic function $f(x)$ which, as in 2°, is given as the G-limit of a sequence of p-integrable periodic functions $f_1(x), f_2(x), \ldots$ with periods $h_1, h_2, \ldots$. In fact we shall show that any Bochner-Fejér polynomial of $f(x)$

$$\sigma(x) = a_0 + a_1 e^{i\Lambda_1 x} + a_2 e^{i\Lambda_2 x} + \cdots + a_N e^{i\Lambda_N x}$$

has the number h_m as a period for m sufficiently large. For, as we saw in 2°, for each $\Lambda_n \neq 0$ we have for $m \geqq m_0 = m_0(n)$

$$\Lambda_n = \frac{2\pi}{h_m} \nu_{n,m} \qquad (\nu_{n,m} \text{ integral});$$

hence for $m \geqq \max\,[m_0(1), m_0(2), \ldots, m_0(N)]$ each of the exponents of our $\sigma(x)$ is an integral multiple of $\frac{2\pi}{h_m}$, and h_m therefore a period of $\sigma(x)$.

For the generalised limit periodic functions the theorems 1 and 2 can be sharpened; we choose a formulation which is just adapted to our applications.

Theorem 1 a. *Let $f(x)$ be an S^p-a. p. function, and $f_1(x), f_2(x), \ldots$ a sequence of 1-integrable periodic functions, with the periods $h_1, h_2, \ldots$, which G^1-converges to $f(x)$. Let further $\varepsilon > 0$ be arbitrarily given. Then for fixed L, and m sufficiently large, i. e. for $m \geqq m_0(\varepsilon, L)$, all integral multiples of h_m are S_L^p-translation numbers of $f(x)$ belonging to ε.*

We observe at once that $f(x)$, as a G^1-limit periodic function, has a Fourier series of »limit periodic form« and is therefore not only S^p-a. p., but also S^p-limit periodic.

Proof. Let $\sigma(x)$ be a Bochner-Fejér polynomial of $f(x)$ for which

$$D_{S_L^p}[f(x), \sigma(x)] < \frac{\varepsilon}{2}.$$

In consequence of the remark above, $\sigma(x)$ has the period h_m for m sufficiently large. Then, for each such m, every integral multiple νh_m of h_m will be an S_L^p-translation number of $f(x)$ belonging to ε, since on account of the Triangle Rule

$$D_{S_L^p}[f(x+\nu h_m), f(x)] \leqq$$

$$D_{S_L^p}[f(x+\nu h_m), \sigma(x+\nu h_m)] + D_{S_L^p}[\sigma(x+\nu h_m), \sigma(x)] + D_{S_L^p}[\sigma(x), f(x)] =$$

$$2\, D_{S_L^p}[f(x), \sigma(x)] < \varepsilon.$$

Theorem 2 a. *Let $f(x)$ be a W^p-a. p. function, and $f_1(x), f_2(x), \ldots$ a sequence of 1-integrable periodic functions, with the periods $h_1, h_2, \ldots$, which G^1-converges to $f(x)$. Let further $\varepsilon > 0$ be arbitrarily given. Then for m and L sufficiently large, i. e. for $m \geqq m_0(\varepsilon)$ and $L \geqq L_0(\varepsilon)$, all integral multiples of h_m are S_L^p-translation numbers of $f(x)$ belonging to ε.*

We observe at once that $f(x)$, as a G^1-limit periodic function, has a Fourier series of limit periodic form and is therefore not only W^p-a. p., but also W^p-limit periodic.

Proof. Let $\sigma(x)$ be a BOCHNER-FEJÉR polynomial of $f(x)$ for which

$$D_{W^p}[f(x), \sigma(x)] < \frac{\varepsilon}{2}\cdot$$

Since $D_{W^p} = \lim\limits_{L \to \infty} D_{S^p_L}$, we have for L sufficiently large $(L \geqq L_0 = L_0(\varepsilon))$

$$D_{S^p_L}[f(x), \sigma(x)] < \frac{\varepsilon}{2}\cdot$$

For m sufficiently large $(m \geqq m_0 = m_0(\varepsilon))$ the BOCHNER-FEJÉR polynomial $\sigma(x)$ has the period h_m, and as in the proof of Theorem 1 a we conclude that every integral multiple of h_m is an S^p_L-translation number of $f(x)$ belonging to ε.

CHAPTER II.

The Completeness or Incompleteness of the Different Spaces.

§ 1.

The Completeness of the S^p- and the S^p-a. p. Spaces.

In this paragraph we prove the following

Theorem. *The S^p-space and the S^p-a. p. space are complete for every $p \geqq 1$.*

Proof. It is sufficient to show that the S^p-space is complete, since this involves, the S^p-a. p. space being a closed subspace of the S^p-space, that every S^p-fundamental sequence of the S^p-a. p. space S^p-converges to a point of the S^p-space and therefore also to a point of the S^p-a. p. space. Thus we only have to show that every S^p-fundamental sequence of S^p-points is S^p-convergent or, what is equivalent, that every S^p-fundamental sequence of S^p-functions is S^p-convergent. Let then $f_1(x), f_2(x), \ldots$ be an S^p-fundamental sequence of S^p-functions, i. e. a sequence of S^p-functions for which $D_{S^p}[f_n(x), f_m(x)] \to 0$ when n and m tend to ∞. We shall prove that there exists a function $f(x)$ such that $D_{S^p}[f(x), f_m(x)] \to 0$ for $m \to \infty$. This function $f(x)$ will automatically be an S^p-function, as the S^p-set is S^p-closed.

We begin by determining an increasing sequence of positive integers $n_1 < n_2 < \cdots$ so that

$$D_{S^p}[f_n(x), f_m(x)] \leqq \frac{1}{2^\nu} \quad \text{for} \quad n \geqq n_\nu, \ m \geqq n_\nu \quad (\nu = 1, 2, \ldots).$$

Hence in particular

$$D_{S^p}[f_{n_\nu}(x), f_{n_{\nu+1}}(x)] \leqq \frac{1}{2^\nu} \qquad (\nu = 1, 2, \ldots).$$

Let

$$g_q(x) = \sum_{\nu=1}^{q} |f_{n_{\nu+1}}(x) - f_{n_\nu}(x)|,$$

then we have $0 \leqq g_1(x) \leqq g_2(x) \leqq \cdots$ so that $\lim\limits_{q \to \infty} g_q(x)$ exists (as finite or infinite) for every x. Further we have for each q, on account of the Triangle Rule,

$$D_{S^p}[g_q(x)] \leqq D_{S^p}[|f_{n_2}(x) - f_{n_1}(x)|] + \cdots + D_{S^p}[|f_{n_{q+1}}(x) - f_{n_q}(x)|] =$$

$$D_{S^p}[f_{n_1}(x), f_{n_2}(x)] + \cdots + D_{S^p}[f_{n_q}(x), f_{n_{q+1}}(x)] \leqq$$

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^q} < 1;$$

in particular

$$\int_\mu^{\mu+1} (g_q(x))^p \, dx < 1$$

is valid for every integer μ. Then we have for every integer μ

$$\int_\mu^{\mu+1} \lim_{q \to \infty} (g_q(x))^p \, dx = \lim_{q \to \infty} \int_\mu^{\mu+1} (g_q(x))^p \, dx \leqq 1.$$

Hence $\lim\limits_{q \to \infty} g_q(x)$ is finite for almost all x. Thus the series

$$\sum_{\nu=1}^{\infty} (f_{n_{\nu+1}}(x) - f_{n_\nu}(x))$$

is absolutely convergent, in particular convergent, for almost all x, which shows that the sequence $f_{n_1}(x), f_{n_2}(x), \ldots$ is convergent (to a finite limit) for almost all x. We shall see that the function

$$f(x) = \lim_{\nu \to \infty} f_{n_\nu}(x)$$

fulfils our demands. Let $\varepsilon > 0$ be arbitrarily given. Then m_0 can be determined such that $D_{S^p}[f_n(x), f_m(x)] \leqq \varepsilon$ for $n \geqq m_0$ and $m \geqq m_0$; if further ν_0 is chosen so large that $n_{\nu_0} \geqq m_0$ we have $D_{S^p}[f_{n_\nu}(x), f_m(x)] \leqq \varepsilon$ for $\nu \geqq \nu_0$ and $m \geqq m_0$, and consequently

$$\int_x^{x+1} |f_{n_\nu}(\xi) - f_m(\xi)|^p \, d\xi \leqq \varepsilon^p \quad \text{for all} \quad x,\ \nu \geqq \nu_0,\ m \geqq m_0.$$

Since $|f_{n_\nu}(\xi) - f_m(\xi)| \to |f(\xi) - f_m(\xi)|$ for almost all ξ when $\nu \to \infty$, we get for every x and $m \geqq m_0$ by Fatou's Theorem

$$\int_x^{x+1} |f(\xi) - f_m(\xi)|^p \, d\xi \leqq \varliminf_{\nu \to \infty} \int_x^{x+1} |f_{n_\nu}(\xi) - f_m(\xi)|^p \, d\xi \leqq \varepsilon^p;$$

hence $D_{S^p}[f(x), f_m(x)] \leqq \varepsilon$ for $m \geqq m_0$, i. e. $D_{S^p}[f(x), f_m(x)] \to 0$ for $m \to \infty$.

This proof of the completeness of the S^p-space is an immediate transferring of a well-known proof of the theorem that a fundamental sequence of p-integrable periodic functions $f_1(x), f_2(x), \ldots$ with the period h is p-convergent. Besides, this last theorem can on its side easily be derived from the theorem above concerning S^p-functions. Indeed, such a sequence of periodic functions $f_n(x)$ is at the same time an S^p-fundamental sequence and will therefore S^p-converge to an S^p-function $f(x)$, and from this function $f(x)$ we can immediately find a function $g(x)$, periodic with the period h, which is the p-limit of our sequence $f_n(x)$. We can simply use the periodic function $g(x)$ which in the period interval $0 \leqq x < h$ coïncides with $f(x)$. In fact this function $g(x)$ is a p-integrable function with the period h, and

$$D_p[g(x), f_n(x)] = \sqrt[p]{\frac{1}{h}\int_0^h |g(x) - f_n(x)|^p \, dx} =$$

$$\sqrt[p]{\frac{1}{h}\int_0^h |f(x) - f_n(x)|^p \, dx} \leqq D_{S_h^p}[f(x), f_n(x)] \to 0$$

for $n \to \infty$.

§ 2.

The Completeness of the B^p- and the B^p-a. p. Spaces.

In this paragraph we prove the following

Theorem. *The B^p-space and the B^p-a. p. space are complete for every $p \geqq 1$.*

Proof. As the B^p-a. p. space is a closed subspace of the B^p-space, it is sufficient (just as in the S-case in § 1) to prove the theorem for the B^p-space. Thus we have to show that every B^p-fundamental sequence of B^p-points is B^p-convergent or, which is equivalent, that every B^p-fundamental sequence of B^p-functions is B^p-convergent. Let then $f_1(x), f_2(x), \ldots$ be a B^p-fundamental sequence of B^p-functions, i. e. a sequence of B^p-functions so that there exists a sequence of positive numbers ε_n tending to 0 for which the inequality

$$(D^*_{B^p}[f_n(x), f_{n+q}(x)])^p < \varepsilon_n$$

holds for all n and $q > 0$. (We prefer here to use the distance $D^*_{B^p}$ instead of the distance D_{B^p}). We shall prove that a function $f(x)$ can be found such that

$$D^*_{B^p}[f(x), f_n(x)] \to 0 \qquad \text{for } n \to \infty.$$

This function $f(x)$ will automatically be a B^p-function, as the B^p-set is B^p-closed. Our construction of $f(x)$ is principally the same as that which Besicovitch used in the proof of his theorem concerning the Fourier series of B^2-a. p. functions; the following arrangement of the proof is due to B. Jessen. We will construct a function $f(x)$ such that

$$(D^*_{B^p}[f(x), f_n(x)])^p \leqq 2\,\varepsilon_n \qquad \text{for all } n.$$

As the construction is analogous for $x > 0$ and $x < 0$, we confine ourselves to state it for $x > 0$. Starting from the assumptions

$$\overline{\lim_{T \to \infty}} \frac{1}{T} \int_0^T |f_n(x) - f_{n+q}(x)|^p \, dx < \varepsilon_n \qquad \text{for all } n \text{ and } q > 0, \tag{1}$$

the task is to construct $f(x)$ so that

$$\overline{\lim_{T \to \infty}} \frac{1}{T} \int_0^T |f(x) - f_n(x)|^p \, dx \leqq 2\,\varepsilon_n \qquad \text{for all } n. \tag{2}$$

The construction of $f(x)$ is indicated in Fig. 2, and we shall show that the occurring positive numbers $T_1 < T_2 < \cdots$ can be chosen so that

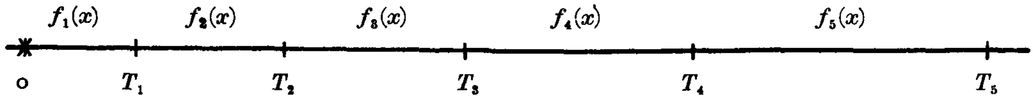

Fig. 2.

$$\frac{1}{T}\int_0^T |f(x) - f_n(x)|^p\,dx < 2\,\varepsilon_n \qquad \text{for } T > T_n \text{ and all } n \tag{3}$$

which obviously involves (2). To this purpose we first set up a number of conditions, arranged in certain groups, for the numbers $T_1, T_2, \ldots$ which involve (3) (and thereby (2)); afterwards, by help of (1), we shall show that these conditions can be satisfied simultaneously.

Group 1. The inequality (3) is satisfied for $n = 1$, if

$$\frac{1}{T}\int_0^T |f_2(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \text{for } T > T_1 \quad \boxed{T_1}$$

$$\frac{1}{T}\int_0^T |f_3(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \text{for } T > T_2 \quad \boxed{T_2}$$

$$\frac{1}{T}\int_0^T |f_4(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \text{for } T > T_3 \quad \boxed{T_3}$$

etc. etc.

and further

$$\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} |f_2(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \boxed{T_2(T_1)}$$

$$\frac{1}{T_3 - T_2}\int_{T_2}^{T_3} |f_3(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \boxed{T_3(T_2)}$$

etc. etc.

For, if $T > T_1$ lies between T_m and T_{m+1}, we have

$$\int_0^T |f(x) - f_1(x)|^p\,dx = \int_0^{T_1} + \int_{T_1}^{T_2} + \cdots + \int_{T_{m-1}}^{T_m} + \int_{T_m}^{T} |f(x) - f_1(x)|^p\,dx <$$

$$0(T_1 - 0) + \varepsilon_1(T_2 - T_1) + \cdots + \varepsilon_1(T_m - T_{m-1}) + \varepsilon_1 T < 2\,\varepsilon_1 T.$$

Group 2. The inequality (3) is satisfied for $n = 2$, if

$$\frac{1}{T}\int_0^T |f_3(x) - f_2(x)|^p\,dx < \varepsilon_2 \qquad \text{for } T > T_2 \quad \boxed{T_2}$$

$$\frac{1}{T}\int_0^T |f_4(x) - f_2(x)|^p\,dx < \varepsilon_2 \qquad \text{for } T > T_3 \quad \boxed{T_3}$$

etc. etc.

and further

$$\frac{1}{T_2}\int_0^{T_1} |f_1(x) - f_2(x)|^p\,dx < \varepsilon_2 \qquad \boxed{T_2(T_1)}$$

$$\frac{1}{T_3 - T_2}\int_{T_2}^{T_3} |f_3(x) - f_2(x)|^p\,dx < \varepsilon_2 \qquad \boxed{T_3(T_2)}$$

$$\frac{1}{T_4 - T_3}\int_{T_3}^{T_4} |f_4(x) - f_2(x)|^p\,dx < \varepsilon_2 \qquad \boxed{T_4(T_3)}$$

etc. etc.

For, if $T > T_2$ lies between T_m and T_{m+1}, we have

$$\int_0^T |f(x) - f_2(x)|^p\,dx = \int_0^{T_2} + \int_{T_2}^{T_3} + \cdots + \int_{T_{m-1}}^{T_m} + \int_{T_m}^{T} |f(x) - f_2(x)|^p\,dx <$$

$$\varepsilon_2(T_2 - 0) + \varepsilon_2(T_3 - T_2) + \cdots + \varepsilon_2(T_m - T_{m-1}) + \varepsilon_2 T < 2\,\varepsilon_2 T.$$

Group 3. Correspondingly it is seen that the inequality (3) is satisfied for $n = 3$, if

$$\frac{1}{T}\int_0^T |f_4(x) - f_3(x)|^p\,dx < \varepsilon_3 \qquad \text{for } T > T_3 \quad \boxed{T_3}$$

$$\frac{1}{T}\int_0^T |f_5(x) - f_3(x)|^p\,dx < \varepsilon_3 \qquad \text{for } T > T_4 \quad \boxed{T_4}$$

etc. etc.

and further

$$\frac{1}{T_3}\left[\int_0^{T_1} |f_1(x) - f_3(x)|^p\,dx + \int_{T_1}^{T_2} |f_2(x) - f_3(x)|^p\,dx\right] < \varepsilon_3 \qquad \boxed{T_3(T_1, T_2)}$$

$$\frac{1}{T_4 - T_3} \int_{T_3}^{T_4} |f_4(x) - f_3(x)|^p \, dx < \varepsilon_3 \qquad \boxed{T_4(T_3)}$$

$$\frac{1}{T_5 - T_4} \int_{T_4}^{T_5} |f_5(x) - f_3(x)|^p \, dx < \varepsilon_3 \qquad \boxed{T_5(T_4)}$$

etc. etc.

etc.

After each condition the T_n concerned are indicated in a rectangle. A composed indication like $T_2(T_1)$ means that the condition be understood as a claim to T_2 after T_1 having been chosen. We observe that, in consequence of (1), every condition is satisfied for all sufficiently large values of the number T_n in question. Since we have only a finite number of conditions for every T_n, and since the composed conditions have the form $T_n(\ldots)$ where the T's in the bracket have lower indices than n, it is obvious that the numbers $T_1, T_2, \ldots$ can be chosen successively so that all the conditions are satisfied.

We finish the paragraph by showing how the theorem of Besicovitch concerning B^2-a. p. functions can be deduced from the completeness of the B^2-a. p. space. From the Parseval equation for a B^2-a. p. function it results immediately that a necessary condition for a trigonometric series $\sum_1^\infty A_n e^{i\Lambda_n x}$ to be the Fourier series of a B^2-a. p. function is that $\sum_1^\infty |A_n|^2$ is convergent. Besicovitch's Theorem states that this condition is also sufficient.

Let then $\sum_1^\infty A_n e^{i\Lambda_n x}$ be a trigonometric series for which $\sum_1^\infty |A_n|^2$ is convergent. We shall prove that the series is the Fourier series of a B^2-a. p. function. We consider the sum of the first n terms of the series

$$s_n(x) = A_1 e^{i\Lambda_1 x} + A_2 e^{i\Lambda_2 x} + \cdots + A_n e^{i\Lambda_n x}.$$

From the Parseval equation for an o. a. p. function in the (trivial) case where it is a trigonometric polynomial we have

$$D_{B^2}[s_n(x), s_{n+q}(x)] = D_{B^2}[A_{n+1} e^{i\Lambda_{n+1} x} + \cdots + A_{n+q} e^{i\Lambda_{n+q} x}] = \sqrt{\sum_{\nu=n+1}^{n+q} |A_\nu|^2};$$

therefore, $\Sigma |A_n|^2$ being convergent, the sequence $s_n(x)$ is a B^2-fundamental sequence and thus (on account of the completeness of the B^2-a. p. space) B^2-converges to a B^2-a. p. function $f(x)$. The given series $\sum_1^\infty A_n e^{i\Lambda_n x}$ must be the Fourier series of this function $f(x)$, since the Fourier series of $f(x)$ can be obtained as the formal limit of the Fourier series of $s_n(x)$ (i. e. $s_n(x)$ itself) for $n \to \infty$.

Incidentally the proof shows that the Fourier series of a B^2-a. p. function B^2-converges to the function.

§ 3.

The Incompleteness of the W^p- and the W^p-a. p. Spaces. Main Example 1.

In this last paragraph we finally prove the following

Theorem. *The W^p-space and the W^p-a. p. space are incomplete for every $p \geqq 1$.*

As the W^p-a. p. space is a closed subspace of the W^p-space it is sufficient to show that the W^p-a. p. space is incomplete, since a W^p-fundamental sequence of W^p-a. p. points which is not W^p-convergent to any W^p-a. p. point is neither W^p-convergent to any W^p-point. Thus we have to prove that for every $p \geqq 1$ there exists a W^p-fundamental sequence of W^p-a. p. points which is not W^p-convergent, or, in other terms, that there exists a W^p-fundamental sequence of W^p-a. p. functions which is not W^p-convergent. We give a single example which can be used for all p by constructing *a sequence $F_1(x)$, $F_2(x)$, ... of W^p-a. p. functions which is a W^p-fundamental sequence for every $p \geqq 1$, but which is not W^p-convergent for any p.* In order to show that the sequence is not W^p-convergent for any p, it is sufficient to show that the sequence is not W^p-convergent for $p = 1$; for a sequence W^p-converging to $F(x)$ for some p or other would also W-converge to $F(x)$, since $D_W[F(x), F_n(x)] \leqq D_{W^p}[F(x), F_n(x)]$.

Main example 1. Let $m_1, m_2, \ldots$ be a sequence of integers $\geqq 2$, and let

$$h_1 = m_1, \quad h_2 = m_1 m_2, \quad h_3 = m_1 m_2 m_3, \ldots.$$

For $n = 1, 2, \ldots$ we put

$$f_n(x) = \begin{cases} 1 & \text{for } \nu h_n - \frac{1}{2} \leqq x \leqq \nu h_n + \frac{1}{2} \quad (\nu = 0, \pm 1, \pm 2, \ldots) \\ 0 & \text{for all other } x. \end{cases}$$

The function $f_1(x)$ thus consists of towers of breadth 1 and height 1 placed on all the numbers $\equiv 0 \pmod{h_1}$, the function $f_2(x)$ of towers of the same kind placed on all the numbers $\equiv 0 \pmod{h_2}$, etc. The function $f_n(x)$ is periodic with the period h_n.

Further we put

$$F_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

(see Fig. 3 where $m_1 = m_2 = m_3 = 2$ and $n = 3$).

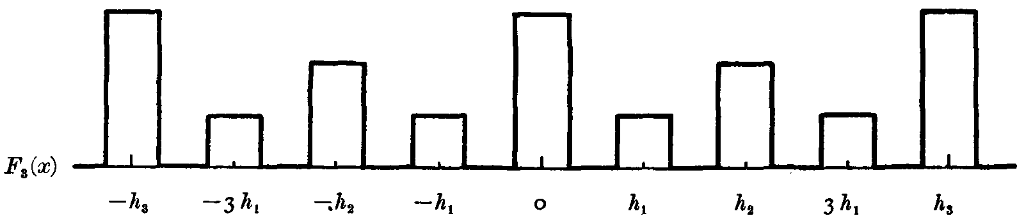

Fig. 3.

$F_1(x)$ thus consists of towers of breadth 1 and height 1 placed on all the numbers $\equiv 0 \pmod{h_1}$.

$F_2(x)$ consists partly of towers of breadth 1 and height 1 placed on all the numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$, and partly of towers of breadth 1 and height 2 placed on all numbers $\equiv 0 \pmod{h_2}$.

$F_3(x)$ consists partly of towers of breadth 1 and height 1 placed on all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$, partly of towers of breadth 1 and height 2 placed on all the numbers $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$, and finally of towers of breadth 1 and height 3 placed on all numbers $\equiv 0 \pmod{h_3}$.

$\vdots$

The function $F_n(x)$ is not only a W^p-a. p. function for every p, but moreover a bounded periodic function with the period h_n.

We begin by showing that $F_1(x)$, $F_2(x)$, ... is a W^p-fundamental sequence for every $p \geqq 1$, i. e. that to any $\varepsilon > 0$ there exists an $N = N(\varepsilon, p)$ such that $D_{W^p}[F_n(x), F_{n+q}(x)] < \varepsilon$ for $n \geqq N$ and $q > 0$. Since $(F_{n+q}(x) - F_n(x))^p$ is periodic (with the period h_{n+q}), we have

$$D_{W^p}[F_n(x), F_{n+q}(x)] = \sqrt[p]{M\{(F_{n+q}(x) - F_n(x))^p\}} =$$
$$\sqrt[p]{M\{(f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+q}(x))^p\}}.$$

Hence in consequence of MINKOWSKI's inequality

$$D_{W^p}[F_n(x), F_{n+q}(x)] \leqq \sqrt[p]{M\{(f_{n+1}(x))^p\}} + \sqrt[p]{M\{(f_{n+2}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}} =$$

$$\sqrt[p]{\frac{1}{h_{n+1}}} + \sqrt[p]{\frac{1}{h_{n+2}}} + \cdots + \sqrt[p]{\frac{1}{h_{n+q}}} =$$

$$\sqrt[p]{\frac{1}{m_1 m_2 \ldots m_{n+1}}} + \sqrt[p]{\frac{1}{m_1 m_2 \ldots m_{n+2}}} + \cdots + \sqrt[p]{\frac{1}{m_1 m_2 \ldots m_{n+q}}} \leqq$$

$$\sqrt[p]{\frac{1}{2^{n+1}}} + \sqrt[p]{\frac{1}{2^{n+2}}} + \cdots + \sqrt[p]{\frac{1}{2^{n+q}}},$$

where the right-hand side is less than the remainder R_n after the nth term of the convergent geometrical series

$$\sum_1^\infty \frac{1}{2^{\frac{n}{p}}}$$

and hence is $< \varepsilon$ for $n \geqq N = N(\varepsilon, p)$.

Next, we shall prove that the sequence $F_n(x)$ is not W-convergent. Roughly speaking, the reason is that the periodic function $F_n(x)$ (of the increasing sequence $F_n(x)$) has arbitrarily high towers for n sufficiently large which prevents its W-distance from a fixed W-function from tending to 0. Indirectly, we assume that there exists a function $F(x)$ such that

$$F_n(x) \xrightarrow{W} F(x).$$

$F(x)$ being a W-function or, what is equivalent, an S-function, the norm $D_S[F(x)]$ is finite, i. e. a constant K can be found so that

$$\int_x^{x+1} |F(t)|\, dt < K \qquad \text{for all } x.$$

We choose a fixed $N > K$, and consider $F_n(x)$ for $n \geqq N$. For the distance $D_W[F(x), F_n(x)]$ we have

$$D_W[F(x), F_n(x)] = D_W[F(x) - F_n(x)] \geqq D_B[F(x) - F_n(x)] =$$

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |F(x) - F_n(x)|\, dx \geqq \overline{\lim_{m\to\infty}} \frac{1}{(2m+1) h_N} \int_{-(m+\frac{1}{2})h_N}^{(m+\frac{1}{2})h_N} |F(x) - F_n(x)|\, dx.$$

Integrating only over a part of the interval $\left(-\left(m+\frac{1}{2}\right)h_N, \left(m+\frac{1}{2}\right)h_N\right)$, and dropping the non-negative contributions from the rest of the interval, we get

$$D_W[F(x), F_n(x)] \geqq \overline{\lim_{m\to\infty}} \frac{1}{(2m+1)h_N} \sum_{\nu=-m}^{m} \int\limits_{\nu h_N-\frac{1}{2}}^{\nu h_N+\frac{1}{2}} |F(x)-F_n(x)|\, dx.$$

Now, since $n \geqq N$, we have $F_n(x) \geqq N$ in every one of the $2m+1$ intervals

$$\left(\nu h_N - \frac{1}{2},\ \nu h_N + \frac{1}{2}\right),$$

and hence

$$\int\limits_{\nu h_N-\frac{1}{2}}^{\nu h_N+\frac{1}{2}} |F(x)-F_n(x)|\, dx \geqq \int\limits_{\nu h_N-\frac{1}{2}}^{\nu h_N+\frac{1}{2}} F_n(x)\, dx - \int\limits_{\nu h_N-\frac{1}{2}}^{\nu h_N+\frac{1}{2}} |F(x)|\, dx > N - K.$$

Thus we finally get for $n \geqq N$

$$D_W[F(x), F_n(x)] \geqq \frac{1}{h_N}(N-K),$$

where the right-hand side is a (perhaps »very small«) positive constant independent of n, and this contradicts the assumption that

$$D_W[F(x), F_n(x)] \to 0 \qquad \text{for } n \to \infty.$$

As in main example 1, in all the main examples of the paper (as well as of the appendix) a sequence of functions $F_1(x)$, $F_2(x)$, ... of bounded periodic functions with the periods $h_1 = m_1$, $h_2 = m_1 m_2$, ... is considered where $m_1, m_2, \ldots$ are integers $\geqq 2$. In most of the main examples further claims are put to these numbers concerning the rapidity with which they tend to ∞. In main examples 1 and 2 (and main example IV of the appendix), however, no such claim is made to the numbers $m_1, m_2, \ldots$, and we might as well have chosen them all equal to 2; in order to get the greatest possible analogy between our main examples, we have preferred not to make such a specialisation.

CHAPTER III.

Two Theorems on G^p-Functions and a Theorem on Periodic G-Points.

We begin by stating two theorems on the behaviour of G^p-functions for fixed G and variable p (of course $p \geqq 1$), the first theorem dealing with G^p-a. p. functions, the other with G^p-zero functions.

Theorem 1. *If a function is G^1-a. p. and belongs to the G^{p_0}-set for a $p_0 > 1$, it is G^p-a. p. for $p < p_0$.*

A bounded function being a G^p-function for all p, the theorem has the following

Corollary. *A bounded G^1-a. p. function is G^p-a. p. for all p.*

We next turn to the G^p-zero functions. As regards the S^p-zero functions we have already mentioned the (trivial) fact that these functions for each p are just those functions which are equal to 0 almost everywhere. In reality, the following theorem on G^p-zero functions therefore only deals with the cases $G = W$ and $G = B$, but of course it also holds for $G = S$.

Theorem 2. *If a function is a G^1-zero function and belongs to the G^{p_0}-set for a $p_0 > 1$, it is a G^p-zero function for $p < p_0$.*

Evidently we have (of the same reasons as above) the following

Corollary. *A bounded G^1-zero function is a G^p-zero function for all p.*

The proofs of the two theorems are based on Hölder's inequality. Let p_1 be an arbitrary number, $1 < p_1 < p_0$, and $f(x)$ an arbitrary function. In Hölder's inequality we replace $f(x)$ and $g(x)$ by $|f(x)|^{\frac{1}{p}}$ and $|f(x)|^{\frac{p_0}{q}}$, where the two positive numbers p and q are determined so that $\frac{1}{p} + \frac{p_0}{q} = p_1$ and $\frac{1}{p} + \frac{1}{q} = 1$, i. e.

$$\frac{1}{p} = \frac{p_0 - p_1}{p_0 - 1}, \quad \frac{1}{q} = \frac{p_1 - 1}{p_0 - 1}.$$

We then obtain

$$\frac{1}{b-a}\int_a^b |f(x)|^{p_1}\,dx \leqq \left[\frac{1}{b-a}\int_a^b |f(x)|\,dx\right]^{\frac{p_0-p_1}{p_0-1}} \cdot \left[\frac{1}{b-a}\int_a^b |f(x)|^{p_0}\,dx\right]^{\frac{p_1-1}{p_0-1}}$$

and, letting the interval (a, b) vary and »passing to the limit« in accordance with the definitions of the different distances, we get for $G = S$, W or B the inequality

$$(1) \qquad (D_{G^{p_1}}[f(x)])^{p_1} \leqq (D_{G^1}[f(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (D_{G^{p_0}}[f(x)])^{\frac{p_1-1}{p_0-1}p_0}.$$

Proof of Theorem 1. Let $f(x)$ be a function satisfying the assumptions of Theorem 1, i. e. a G^1-a. p. function and a G^{p_0}-function for a $p_0 > 1$. Let $\sigma_q(x)$ be a Bochner-Fejér sequence of $f(x)$. Then $D_{G^1}[f(x), \sigma_q(x)] \to 0$ for $q \to \infty$ and (as mentioned in Chapter I)

$$D_{G^{p_0}}[\sigma_q(x)] \leqq D_{G^{p_0}}[f(x)].$$

For an arbitrary p_1 between 1 and p_0 we have because of (1)

$$(D_{G^{p_1}}[f(x), \sigma_q(x)])^{p_1} \leqq (D_{G^1}[f(x), \sigma_q(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (D_{G^{p_0}}[f(x), \sigma_q(x)])^{\frac{p_1-1}{p_0-1}p_0} \leqq$$

$$(D_{G^1}[f(x), \sigma_q(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (D_{G^{p_0}}[f(x)] + D_{G^{p_0}}[\sigma_q(x)])^{\frac{p_1-1}{p_0-1}p_0} \leqq$$

$$(D_{G^1}[f(x), \sigma_q(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (2\, D_{G^{p_0}}[f(x)])^{\frac{p_1-1}{p_0-1}p_0}$$

where the right-hand side tends to 0 for $q \to \infty$, since $D_{G^1}[f(x), \sigma_q(x)] \to 0$. Consequently $D_{G^{p_1}}[f(x), \sigma_q(x)] \to 0$ so that $f(x)$ is a G^{p_1}-a. p. function.

Proof of theorem 2. Let $f(x)$ be a function satisfying the assumptions of Theorem 2, i. e. a G^1-zero function and a G^{p_0}-function for a $p_0 > 1$. For an arbitrary p_1 between 1 and p_0 we have because of (1)

$$(D_{G^{p_1}}[f(x)])^{p_1} \leqq (D_{G^1}[f(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (D_{G^{p_0}}[f(x)])^{\frac{p_1-1}{p_0-1}p_0} = 0,$$

i. e. $D_{G^{p_1}}[f(x)] = 0$.

Remark. Using the theory of Fourier series (in particular the uniqueness theorem) we may consider Theorem 2 as a special case of Theorem 1. In fact, a G^p-zero function being the same as a G^p-a. p. function with the Fourier series 0, the function $f(x)$ of Theorem 2 is on account of Theorem 1 a G^p-a. p. function for $p < p_0$, and having the Fourier series 0 it is therefore a G^p-zero function for $p < p_0$.

Now we pass to a theorem of a somewhat different character which will be useful for us later on.

Theorem on the periodic points. *If $f(x)$ is a* 1*-integrable periodic function, and if (for a $p>1$) the G^1-point »around« $f(x)$ contains any G^p-function at all, then the function $f(x)$ itself is p-integrable.*

In other words: A 1-integrable periodic function $f(x)$ is a G^p-function for all those p for which there exist G^p-functions in the (periodic) G^1-point around $f(x)$.

Proof. Since $D_{B^p} \leqq D_{G^p}$, a G^p-function is also a B^p-function, and the G^1-point around $f(x)$ is contained in the B-point around $f(x)$. Therefore it is sufficient to prove the theorem for $G = B$. Hence we assume that there exists a B-zero function $j(x)$ such that $f(x) + j(x)$ is a B^p-function, and we have to prove that $f(x)$ is p-integrable. Let the period of $f(x)$ be $b-a$, and let $T = \nu(b-a)$ where ν is a positive integer. Using first the inequality

$$|f(x) + g(x)| \geqq |(f(x))_N + (g(x))_N|$$

and afterwards Minkowski's inequality, we have for an arbitrary fixed $N > 0$

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T} |f(x) + j(x)|^p\,dx} \geqq \sqrt[p]{\frac{1}{2T}\int_{-T}^{T} |(f(x))_N + (j(x))_N|^p\,dx} \geqq$$

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T} |(f(x))_N|^p\,dx} - \sqrt[p]{\frac{1}{2T}\int_{-T}^{T} |(j(x))_N|^p\,dx}.$$

Now, as $|(j(x))_N|$ is $\leqq N$ as well as $\leqq |j(x)|$ we have $|(j(x))_N|^p \leqq N^{p-1}\,|j(x)|$. Hence

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T} |f(x) + j(x)|^p\,dx} \geqq$$

$$\sqrt[p]{\frac{1}{b-a}\int_{a}^{b} |(f(x))_N|^p\,dx} - N^{1-\frac{1}{p}}\sqrt[p]{\frac{1}{2T}\int_{-T}^{T} |j(x)|\,dx}.$$

Letting $\nu \to \infty$, we get, since $j(x)$ is a B-zero function,

$$D_{B^p}[f(x) + j(x)] \geqq \sqrt[p]{\frac{1}{b-a}\int_{a}^{b} |(f(x))_N|^p\,dx}.$$

Finally, letting $N \to \infty$, we get the inequality

$$D_{B^p}[f(x) + j(x)] \geqq \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)|^p\,dx} = D_{B^p}[f(x)]$$

which shows in particular that $f(x)$ is p-integrable.

We add two remarks on the periodic G-points.

1°. A periodic G-point contains essentially only one periodic function, or precisely speaking: *Two periodic functions in a G-point are identical almost everywhere.* For they have the same Fourier series in almost periodic and therefore in periodic sense; consequently they have a common period, and further they are identical almost everywhere because of the uniqueness theorem on p-integrable periodic functions with a fixed period. A period of some periodic function in a periodic G-point is called a period of the G-point.

2°. Every periodic G-point with the period h has a Fourier series of the form

$$\sum_{-\infty}^{\infty} A_n e^{i\frac{2\pi}{h}nx},$$

where all the Fourier exponents are integral multiples of the number $\frac{2\pi}{h}$. We shall prove that the converse is also true, i. e. *that every G-point which has a Fourier series of the form*

$$\sum_{-\infty}^{\infty} A_n e^{i\frac{2\pi}{h}nx}$$

is a periodic G-point with the period h. Let $\sigma_q(x)$ be a Bochner-Fejér sequence of the Fourier series. All the Fourier exponents being integral multiples of the number $\frac{2\pi}{h}$, the Bochner-Fejér polynomials are periodic with the period h. The sequence $\sigma_q(x)$ being G-convergent is in particular a G-fundamental sequence. As all the $\sigma_q(x)$ are periodic with the period h, we have

$$D_p[\sigma_{q_1}(x), \sigma_{q_2}(x)] = D_G[\sigma_{q_1}(x), \sigma_{q_2}(x)] \qquad (G = S_h^p,\ W^p,\ B^p).$$

Thus $\sigma_q(x)$ is also a p-fundamental sequence and therefore p-converges to a p-integrable periodic function $f(x)$ with the period h. Since, on account of

$$D_G[f(x), \sigma_q(x)] = D_p[f(x), \sigma_q(x)] \qquad (G = S_h^p, W^p, B^p),$$

the Bochner-Fejér sequence $\sigma_q(x)$ also G-converges to $f(x)$, the function $f(x)$ belongs to our G-a. p. point.

We remind in this connection of the fact (stated in Chapter I), that the G-limit periodic functions can be characterised as G-a. p. functions with Fourier series of the form

$$\Sigma A_n e^{i d r_n x}$$

where all the Fourier exponents are rational multiples of a number d. Evidently the same characterisation holds for the G-limit periodic points.

The theorem on the periodic points involves in particular that the upper bound P_1 for the p for which »the periodic representative« $f(x)$ of a periodic G^1-point is p-integrable is equal to the upper bound P_2 of the p for which the G^1-point contains G^p-functions. It may be of interest to show that this (more special) result can also be derived by help of Fourier series. Indirectly, we assume that the first upper bound P_1 is less than the other P_2. We choose p_1 so that $P_1 < p_1 < P_2$. Then there exists a G^{p_1}-function $g(x)$ in the G^1-point. Let now p_2 be chosen so that $P_1 < p_2 < p_1$. The function $g(x)$, lying in the periodic G^1-point, is G^1-a. p. and, being also a G^{p_1}-function, is simultaneously a G^{p_2}-a. p. function in consequence of Theorem 1. The Fourier series of the function $g(x)$ being that of the periodic G^1-point has the form

$$\Sigma A_n e^{i \frac{2\pi}{h} n x}.$$

The G^{p_2}-a. p. point around $g(x)$ having the same Fourier series is therefore, in consequence of Remark 2°, a periodic G^{p_2}-point and thus contains a p_2-integrable periodic function $h(x)$. The two periodic functions $f(x)$ and $h(x)$ both lying in our G^1-point must, in consequence of Remark 1°, be equal almost everywhere. Consequently $f(x)$ (as $h(x)$) is a p_2-integrable function, in contradiction to $p_2 > P_1$

CHAPTER IV.

The Mutual Relations of the S^p-Spaces and the S^p-a. p. Spaces.

§ 1.

Introduction.

Since $D_{G^p} \geqq D_{G^1}$ for $p \geqq 1$, every G^p-function is also a G^1-function, and every G^p-zero function is also a G^1-zero function. Consequently every G^p-point is entirely contained in a G^1-point. In the S-case however, as mentioned above, the S^p-zero functions have an especially simple character, being the same for every p, namely the functions which are 0 almost everywhere. Consequently *every S^p-point is itself an S-point* (and not only contained in an S-point). We start from an S-point and will investigate its behaviour as regards the S^p-spaces and the S^p-a. p. spaces. We call an S-point **alive at the time p_1 as to the S^p-spaces**, if the S-point is an S^{p_1}-point. Otherwise it is said to be **dead at the time p_1 as to the S^p-spaces**. If we know, whether an S-point is alive or dead at the time p_1 as to the S^p-spaces, we say that we know the **behaviour** of the S-point at the time p_1 as to the S^p-spaces. If an S-point is alive at one date, it is also alive at all the previous dates. The upper bound P of all p for which the S-point is alive is called the **lifetime** of the S-point as to the S^p-spaces. Beforehand, nothing can be said about the behaviour of the S-point at its moment of death (i. e. at the time P). If the S-point is S-a. p., we can, analogously, consider its lifetime as to the S^p-a. p. spaces and its behaviour as to the S^p-a. p. spaces in the moment of death. In consequence of Theorem 1, Chapter III *an S-a. p. point has the same lifetime as to the S^p-spaces and as to the S^p-a. p. spaces.* In the following two paragraphs we shall state all the possibilities which may occur.

§ 2.

S-Points which are not S-a. p. Points.

We consider an arbitrary S-point which is not S-a. p. and denote, as above, its lifetime as to the S^p-spaces by P. It will be proved by examples that the following possibilities (which are all those imaginable beforehand) may occur:

1. The lifetime $P = \infty$.
2. The lifetime P is arbitrary finite, $1 \leqq P < \infty$.
   2 a. The point is dead at the time P as to the S^p-spaces $(P > 1)$.
   2 b. The point is alive at the time P as to the S^p-spaces $(P \geqq 1)$

Example to 1.

We define $f(x)$ for $-\infty < x < \infty$ by

$$f(x) = \begin{cases} 1 & \text{for } 0 \leqq x \leqq 1 \\ 0 & \text{for all other } x. \end{cases}$$

Obviously, $f(x)$ being bounded is an S^p-function for every $p \geqq 1$. And that $f(x)$ is not S-a. p. is an immediate consequence of Theorem 1 of Chapter I, as $f(x)$ has no relatively dense set of S-translation-numbers belonging for instance to $\frac{1}{2}$, the equality $D_S[f(x+\tau), f(x)] = 1$ being valid for $|\tau| \geqq 1$.

Thus the S-point around $f(x)$ is not S-a. p. and has the lifetime $P = \infty$.

Example to 2 a.

P being an arbitrary number, $1 < P < \infty$, we define $f(x)$ for $-\infty < x < \infty$ by

$$f(x) = \begin{cases} \left(\dfrac{1}{x}\right)^{\frac{1}{P}} & \text{for } 0 < x \leqq 1 \\ 0 & \text{for all other } x. \end{cases}$$

The function $f(x)$ is an S^p-function for $p < P$ but not for $p = P$, since

$$(D_{S^p}[f(x)])^p = \int_0^1 \left(\frac{1}{x}\right)^{\frac{p}{P}} dx$$

and $\int_0^1 \left(\frac{1}{x}\right)^{\alpha} dx$ is convergent for $\alpha < 1$ and divergent for $\alpha = 1$. Further $f(x)$ is not S-a. p., as

$$D_S[f(x+\tau), f(x)] = \int_0^1 \left(\frac{1}{x}\right)^{\frac{1}{P}} dx \,(> 0) \quad \text{for} \quad |\tau| \geqq 1.$$

Thus the S-point around $f(x)$ is not S-a. p., has the lifetime P and is dead at the time P.

Example to 2 b.

P being an arbitrary number, $1 \leqq P < \infty$, we define $f(x)$ for $-\infty < x < \infty$ by

$$f(x) = \begin{cases} \left(\frac{1}{x(\log x)^2}\right)^{\frac{1}{P}} & \text{for } 0 < x \leqq a < 1 \\ 0 & \text{for all other } x. \end{cases}$$

The function $f(x)$ is an S^p-function for $p = P$, but not for $p > P$, since

$$(D_{S_a^p}[f(x)])^p = \frac{1}{a}\int_0^a \left(\frac{1}{x(\log x)^2}\right)^{\frac{p}{P}} dx$$

and $\int_0^a \left(\frac{1}{x(\log x)^2}\right)^{\alpha} dx$ is convergent for $\alpha = 1$ and divergent for $\alpha > 1$. Further $f(x)$ is not S-a. p. as

$$D_{S_a}[f(x+\tau), f(x)] = \frac{1}{a}\int_0^a \left(\frac{1}{x(\log x)^2}\right)^{\frac{1}{P}} dx \, (> 0) \quad \text{for} \quad |\tau| \geqq a.$$

Thus the S-point around $f(x)$ is not S-a. p., has the lifetime P and is alive at the time P.

§ 3.

S-a. p. Points. Main Example 2.

Next we consider the S-a. p. points. As mentioned in § 1, each such point has the same lifetime as to the S^p-spaces and the S^p-a. p. spaces. We will show that the following possibilities (which are all those imaginable beforehand) may occur:

1. The lifetime $P = \infty$.
2. The lifetime P is arbitrary finite, $1 \leqq P < \infty$.
   - 2 a. The point is dead at the time P as to the S^p-spaces ($P > 1$).
   - 2 b. The point is alive at the time P as to the S^p-spaces.
     - 2 b α. The point is alive at the time P as to the S^p-a. p. spaces ($P \geqq 1$).
     - 2 b β. The point is dead at the time P as to the S^p-a. p. spaces ($P > 1$).

The case 2 b β, i. e. that of an S-a. p. point which is an S^P-point but not an S^P-a. p. point, is the only not trivial one.

Example to 1.

Let $f(x)$ be a bounded periodic function. Then the S-point around $f(x)$ has the demanded properties.

Example to 2 a.

Let P be arbitrarily given, $1 < P < \infty$. We consider the periodic function $f(x)$ with the period 1, given in the period interval $0 < x \leqq 1$ by $f(x) = \left(\frac{1}{x}\right)^{\frac{1}{P}}$. Then the S-point around $f(x)$ has the demanded properties.

Example to 2 bα.

Let P be arbitrarily given, $1 \leqq P < \infty$. We consider the periodic function $f(x)$ with the period $a < 1$ which is given in the period interval $0 < x \leqq a$ by $f(x) = \left(\frac{1}{x(\log x)^2}\right)^{\frac{1}{P}}$. Then the S-point around $f(x)$ has the demanded properties.

Example to 2 bβ. **Main example 2.**

P being an arbitrarily given number, $1 < P < \infty$, we shall indicate *a function $F(x)$ which is S-a. p. (even S-limit periodic) and an S^P-function, but not an S^P-a. p. function.* The S-point around $F(x)$ is then of the type desired.

Let $m_1, m_2, \ldots$ be arbitrary integers $\geqq 2$, and $(1 >)\ \varepsilon_1 > \varepsilon_2 > \cdots$ a decreasing sequence tending to 0. In this main example, by a tower of type n we shall understand a tower with the 1-integral ε_n and the P-integral 1. The breadth b_n of a tower of type n is then $\varepsilon_n^{\frac{P}{P-1}}$ so that $b_n \to 0$ for $n \to \infty$ and $b_n < 1$ for all n.

We put

$$h_1 = m_1,\ h_2 = m_1 m_2,\ h_3 = m_1 m_2 m_3, \ldots$$

and, as in main example 1, we construct a sequence $F_1(x), F_2(x), \ldots$ of bounded periodic functions with the periods $h_1, h_2, \ldots$. The construction appears from the following array (compare with main example 1).

$F_1(x)$: On all numbers $\equiv 0 \pmod{h_1}$ is placed a tower of type 1.

$F_2(x)$: On all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$ is placed a tower of type 1.
. . . . $\equiv 0 \pmod{h_2}$ . . . . 2.

$F_3(x)$: On all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$ is placed a tower of type 1.
. . . . $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$ . . . . 2.
. . . . $\equiv 0 \pmod{h_3}$ . . . . 3.

$\vdots$

$F_n(x)$: On all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$ is placed a tower of type 1.
. . . . $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$ . . . . 2.
. . . . $\equiv 0 \pmod{h_3}$ but $\not\equiv 0 \pmod{h_4}$ . . . . 3.
$\vdots$
. . . . $\equiv 0 \pmod{h_{n-1}}$ but $\not\equiv 0 \pmod{h_n}$ . . . . $n-1$
. . . . $\equiv 0 \pmod{h_n}$ . . . . n.
$\vdots$

(See Fig. 4 where $m_1 = m_2 = m_3 = 2$ and $n = 3$). Since $b_n < 1 \leqq \frac{h_1}{2} < h_1$, two towers never overlap.

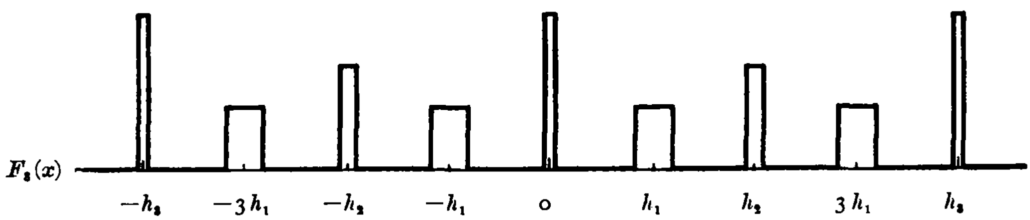

Fig. 4.

The function $F_n(x)$ is obviously a bounded periodic function with the period h_n. We shall show that $F_n(x)$ S-converges to a function $F(x)$ whose construction appears from the following array.

$F(x)$: On all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$ is placed a tower of type 1.
. . . . $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$ . . . . 2.
. . . . $\equiv 0 \pmod{h_3}$ but $\not\equiv 0 \pmod{h_4}$ . . . . 3.
$\vdots$
. . . . $\equiv 0 \pmod{h_n}$ but $\not\equiv 0 \pmod{h_{n+1}}$ . . . . n.
$\vdots$

The function $F(x)$ differs from $F_n(x)$ only as regards the towers on the numbers $\equiv 0 \pmod{h_{n+1}}$. As the breadth of the tower placed on 0 in $F_n(x)$ tends to 0 for $n \to \infty$, while in $F(x)$ there is no tower on 0, it is plain that

$$F(x) = \lim_{n \to \infty} F_n(x) \qquad \text{for all } x \neq 0.$$

We shall prove that

$$F_n(x) \xrightarrow{S} F(x).$$

On the numbers $\equiv 0 \pmod{h_{n+1}}$, in $F_n(x)$ there are standing towers of the type n, while in $F(x)$, with exception of the number 0, there are standing towers of the types $n+1, n+2, \ldots$. Hence, denoting by $n+q(m)$ $(q(m) \geqq 1)$ the type of the tower in $F(x)$ placed on a number $m \equiv 0 \pmod{h_{n+1}}$ but $\neq 0$, we have for $m \equiv 0 \pmod{h_{n+1}}$

$$\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|\,dx \leqq \int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} F(x)\,dx + \int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} F_n(x)\,dx =$$

$$\begin{cases} \varepsilon_n \text{ for } m = 0 \\ \varepsilon_{n+q(m)} + \varepsilon_n \text{ for } m \equiv 0 \pmod{h_{n+1}} \text{ but } \neq 0, \end{cases}$$

while for $m \equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_{n+1}}$

$$\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|\,dx = 0.$$

Thus we have

$$\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|\,dx < 2\,\varepsilon_n \quad \text{for all} \quad m \equiv 0 \pmod{h_1}.$$

Since an arbitrary interval of the length h_1 is contained in an interval

$$m - \frac{h_1}{2} \leqq x \leqq m + \frac{3\,h_1}{2} \quad (m \equiv 0 \pmod{h_1}),$$

we get

$$\int_{x-\frac{h_1}{2}}^{x+\frac{h_1}{2}} |F(t) - F_n(t)|\,dt < 4\,\varepsilon_n \quad \text{for all } x,$$

so that $D_{S_{h_1}}[F(x), F_n(x)] \leqq \frac{4\,\varepsilon_n}{h_1}$ which tends to 0 for $n \to \infty$. Hence $F_n(x) \xrightarrow{S} F(x)$, and $F(x)$ is therefore an S-a. p. function.

The function $F(x)$ is obviously an S^P-function, as all the towers of $F(x)$ have the P-integral 1 and therefore

$$\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} (F(x))^P\,dx \leqq 1 \qquad \text{for all } m \equiv 0 \pmod{h_1}$$

and hence

$$\int_{x-\frac{h_1}{2}}^{x+\frac{h_1}{2}} (F(t))^P \, dt \leqq 2 \qquad \text{for all } x,$$

so that

$$D_{S^P_{h_1}}[F(x)] \leqq \sqrt[P]{\frac{2}{h_1}}.$$

Finally $F(x)$ is **not an** S^P-**a. p. function.** Otherwise by Theorem 1 a, Chapter I, the function $F(x)$ being the S-limit function of the periodic functions $F_n(x)$ with the periods h_n, the number h_n should be a »fine« $S^P_{h_1}$-translation number of $F(x)$ for »large« n. This, however, is impossible. In fact, by the translation h_n the interval $-\frac{h_1}{2} \leqq x \leqq \frac{h_1}{2}$ containing 0 is translated into the interval $-\frac{h_1}{2} + h_n \leqq x \leqq \frac{h_1}{2} + h_n$ containing h_n, and in the first interval $F(x)$ has no tower while in the second it has a tower with the P-integral 1, so that

$$D_{S^P_{h_1}}[F(x + h_n),\ F(x)] \geqq \sqrt[P]{\frac{1}{h_1}}$$

for every h_n.

Besides, in order to prove that $F(x)$ is not S^P-a. p., we could have confined ourselves to apply the general Theorem 1 of Chapter I instead of the Theorem 1 a (dealing with limit periodic functions). In fact, for every τ with a modulus $\geqq \frac{h_1}{2}$ we have

$$D_{S^P_{h_1}}[F(x + \tau),\ F(x)] \geqq \sqrt[P]{\frac{1}{h_1}},$$

since the interval $-\frac{3}{4}h_1 \leqq x \leqq \frac{3}{4}h_1$ of length $\frac{3}{2}h_1$ in which $F(x)$ has no tower will be translated by τ into an interval containing at least one of the towers of $F(x)$.

We remark, that the function $F(x)$ constructed above is of similar character as a type of examples of o. a. p. functions treated by TOEPLITZ (Mathematische Annalen, Bd. 98).

CHAPTER V.

The Mutual Relations of the W^p-Spaces and the W^p-a. p. Spaces.

§ 1.

Introduction.

In this Chapter we shall study the mutual relations of the W^p-spaces and the W^p-a. p. spaces, p ranging over all values $\geqq 1$. Of all the W^p-points $(p \geqq 1)$ the W-points are the most comprehensive, and every W^p-point is contained in a W-point. We therefore consider an arbitrary W-point and shall investigate how this W-point behaves as to the W^p-spaces and the W^p-a. p. spaces. First we consider the W^p-points $(p > 1)$ contained in our W-point, but subsequently also the single functions of the W-point. In our characterisation of the W^p-points and of the functions in the W-point only the W^p-spaces and the W^p-a. p. spaces are applied (and not the other types of spaces). Before carrying out our investigations we must have some knowledge about the W-zero functions.

§ 2.

W-zero Functions.

Let $f(x)$ be a W-zero function. We denote the upper bound of the p for which $f(x)$ is a W^p-function by P. In consequence of Theorem 2, Chapter III the function $f(x)$ is a W^p-zero function for $p<P$ so that P can also be defined as the upper bound of those p for which $f(x)$ is a W^p-zero function. We will show that the following possibilities (which are all those imaginable beforehand) may be realised:

1. $P=\infty$.
2. P arbitrary finite, $1 \leqq P < \infty$.
   - 2 a. $f(x)$ is not a W^P-function $(P > 1)$.
   - 2 b. $f(x)$ is a W^P-function.
     - 2 b α. $f(x)$ is a W^P-zero function $(P \geqq 1)$.
     - 2 b β. $f(x)$ is not a W^P-zero function $(P > 1)$.

Example to 1.

A quite obvious example is $f(x) = 0$ for all x. Another example is a function which is bounded and tends to 0 for $x \to \pm\infty$.

Example to 2 a.

$$f(x) = \begin{cases} \left(\frac{1}{x}\right)^{\frac{1}{p}} & \text{for } 0 < x \leqq 1 \\ 0 & \text{for all other } x. \end{cases}$$

Example to 2 b α.

$$f(x) = \begin{cases} \left(\frac{1}{x(\log x)^2}\right)^{\frac{1}{p}} & \text{for } 0 < x \leqq a < 1 \\ 0 & \text{for all other } x. \end{cases}$$

Example to 2 b β.

We construct a function $f(x)$ in the following way: Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers $\leqq 1$ which tends to 0. On the number n $(n = 1, 2, \ldots)$ a tower with the 1-integral ε_n and the P-integral 1 is placed. As the breadths of the towers are $\leqq 1$, they do not overlap. $f(x)$ is a W-zero function as

$$\int_x^{x+1} f(t)\, dt \to 0 \quad \text{for} \quad x \to \infty.$$

Further $f(x)$ is a W^p-function, but not a W^p-zero function as

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} (f(x))^p\, dx = 1 \quad \text{for} \quad n = 1, 2, \ldots.$$

There exists an infinite number of such functions which do not differ by W^p-zero functions (i. e. do not belong to the same W^p-point), for instance the functions $a \cdot f(x)$, where a is an arbitrary complex number $\neq 0$ and $f(x)$ the function constructed above.

It may be observed that a function $f(x)$ which is a W-zero function and a W^p- but not W^p-zero function can never be a W^p-a. p. function. In fact, if it was W^p-a. p., it would, as it has the Fourier series 0, be a W^p-zero function.

§ 3.

W-Points in General.

In this paragraph we shall state the laws for the W^p-points ($p > 1$) and the functions in a W-point. A single proof belonging to this investigation will be postponed to § 6 because of its particular character. In § 4 and § 5 examples are given which serve as existence proofs for the different types of W-points.

We consider an arbitrary W-point. We call the point alive at the time p_1 as to the W^p-spaces, if it contains at least one W^{p_1}-point, or, what is equivalent, if it contains at least one W^{p_1}-function; otherwise the W-point is said to be dead at the time p_1. If the W-point is W-a. p., we define in an analogous way the meaning of the point being alive or dead at the time p_1 as to the W^p-a. p. spaces. If we know, whether the W-point is alive or dead at the time p_1 as to the W^p-spaces (W^p-a. p. spaces) we say that we know its behaviour at the time p_1 as to the W^p-spaces (W^p-a. p. spaces). If the W-point is alive at one date, it is alive at all the previous dates. By the lifetime P of the W-point as to the W^p-spaces we understand the upper bound of those p for which the W-point is alive as to the W^p-spaces. If the W-point is W-a. p. in an analogous way its lifetime as to the W^p-a. p. spaces is defined. In consequence of Theorem 1, Chapter III *a W-a. p. point has the same lifetime as to the W^p-spaces and as to the W^p-a. p. spaces.* Beforehand, we cannot say anything about the behaviour of the W-point at the moment of death as to the W^p-spaces and eventually the W^p-a. p. spaces.

At first we study the W^p-points in our W-point. Let $p > 1$ be arbitrarily given. The set of all the W^p-functions in the W-point, if such functions exist, divides into a set of W^p-points. These W^p-points are called the p-descendants of the W-point. In consequence of example 2 bβ of § 2, if there is one p-descendant, there will be an infinite number of them, since the sum of any function $a f(x)$, $a \neq 0$, of this example with p instead of P (or rather the W^p-point around this function) and a fixed p-descendant is again a p-descendant. Let p_1 and p_2 be two numbers, $1 < p_1 < p_2$. We consider a p_1-descendant of the W-point. The set of W^{p_2}-functions in the p_1-descendant, if such functions exist, divides into a set of W^{p_2}-points which are called the p_2-descendants of the p_1-descendant. They are at the same time p_2-descendants of the W-point. We will prove *that only one of the p_1-descendants of the W-point can have p_2-descendants for any $p_2 > p_1$*, so that all the p_2-descendants of the W-point (if existing) are p_2-descendants of one and the same p_1-descendant. In fact, the difference of two functions, each taken from its p_2-descendant, is a W-zero function and a W^{p_2}-function, and hence, in consequence of Theorem 2, Chapter III, a W^p-zero function for $p < p_2$, in particular for $p = p_1$. This »generating« W^{p_1}-point is called the p_1-generator; all the other p_1-descendants die at the time p_1 at the moment they are »born« (i. e. come into existence as points) and are therefore called the stillborn brothers of the p_1-descendant. The p_1-generator is defined for $1 < p_1 < P$.

If the W-point from which we are starting is a W-a. p. point the p_1-generator ($1 < p_1 < P$) *will be W^{p_1}-a. p.* In fact, the p_2-descendants of the p_1-generator ($p_2 > p_1$) consist of W^{p_2}-functions which are simultaneously W-a. p. functions; thus, in consequence of Theorem 1, Chapter III these functions are W^{p_1}-a. p. and, lying in the p_2-descendants of the p_1-generator, they also lie in the p_1-generator itself, which is therefore W^{p_1}-a. p. *In a W-a. p. point at most one of the p-descendants can be W^p-a. p.* (so that for $p < P$ the p-generator is the only W^p-a. p. p-descendant). For a W^p-a. p. point in the W-a. p. point has the same Fourier series as the W-a. p. point itself, and, in consequence of the uniqueness theorem, there exists only one W^p-a. p. point with a given Fourier series.

In the preceding we have used »biological« phrases. We will also give another methaphor of the situation. We speak of a W-rocket, the »components« of which at the time p are the p-descendants of the W-point; the p-generator is called the p-nucleus and the still-born brothers of the p-generator are called the p-sparks of the rocket (see Fig. 5, which suggests the »evolution« of the W-point »in the course of time«, i. e. for increasing p).

In connection with the figure we remind of certain facts given above: If the W-point is W-a. p., the p-generator is W^p-a. p. for every p, $1 < p < P$, whereas no one of its still-born brothers is. Further, if the W-a. p. point is alive at the moment of death P as to the W^p-spaces, so that there exist P-descendants, at most one of them is W^P-a. p. As we shall see in § 5, some of those W-a. p. points have a W^P-a. p. P-descendant, whereas others have not.

Next we consider the single functions in the W-point. A function $f(x)$ is called alive at the time p_1 as to the W^p-spaces, if $f(x)$ is a W^{p_1}-function; otherwise $f(x)$ is said to be dead at the time p_1. If the W-point is W-a. p., we define in an analogous way what is to be understood by $f(x)$ being alive or dead at the time p_1 as to the W^p-a. p. spaces. The upper bound of the p for

6

which $f(x)$ is alive as to the W^p-spaces is called the **lifetime** of $f(x)$ as to the W^p-spaces. If the W-point is W-a. p., the lifetime of $f(x)$ as to the W^p-a. p. spaces is defined in the analogous way. In this case, in consequence of Theorem 1, Chapter III, *the function $f(x)$ has the same lifetime as to the W^p-spaces and as to the W^p-a. p. spaces.*

We start our investigations about the functions in a given W-point by mentioning, without proof, that in every W-point there exists a **through**

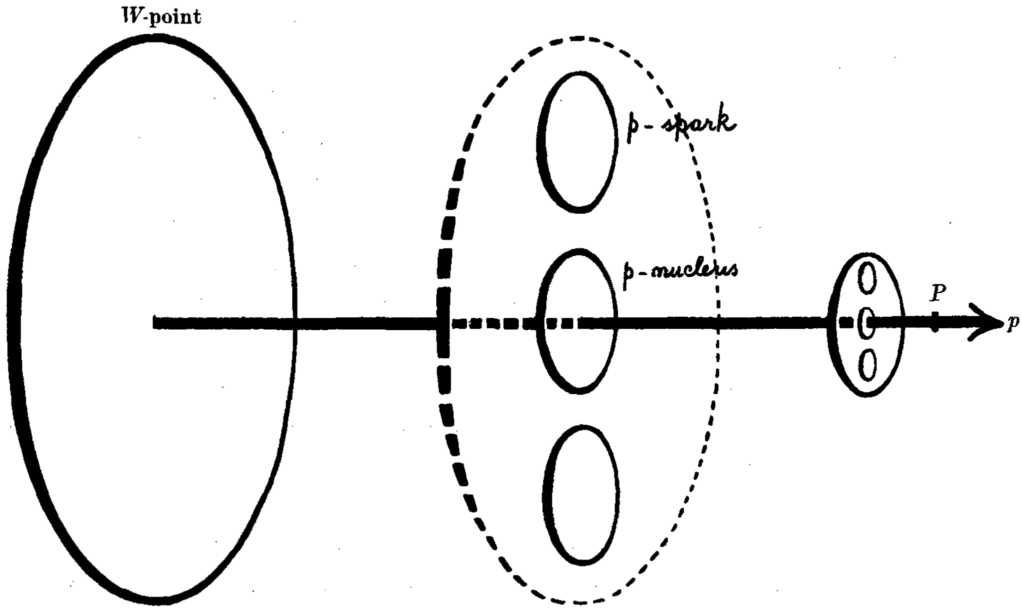

Fig. 5.

function as to the W^p- and the W^p-a. p. spaces, i. e. a function which is a W^p-function for just those p for which the W-point contains W^p-functions and a W^p-a. p. function for just those p for which the W-point contains W^p-a. p. functions. If the W-point is alive at the time P as to the W^p-a. p. spaces, or is not alive at the time P as to the W^p-a. p. spaces but is alive as to the W^p-spaces, it is obvious that there exists a through function; in fact an arbitrary one of the W^p-a. p., respectively W^p-functions contained in the W-point will be a through function. The problem is to show that there exists a through function also in the case where (if $P < \infty$) the W-point is dead at the time P as to the W^p-spaces. In order to prove this, it is of course sufficient to show that there exists a through function as to the W^p-spaces, since such a function, if the

W-point be a W-a. p. point, will at the same time be a through function as to the W^p-a. p. spaces. We postpone the proof to § 6. Taking the existence of a through function for granted, we shall now give a complete account of the functions lying in our arbitrarily given W-point, whose lifetime P ($1 \leqq P \leqq \infty$) and behaviour at the moment of death P (if $P < \infty$) as to the W^p-spaces and the W^p-a. p. spaces are assumed to be known. (What possibilities may occur for a W-point in this respect will, as mentioned in § 1, be discussed in § 4 and § 5). By the investigation of the functions in our W-point we distinguish between the W-point not being W-a. p. or being W-a. p.

I. The W-point is not W-a. p. Denoting the lifetime of a function $f(x)$ in the W-point as to the W^p-spaces by P_1, there are the following possibilities. *The lifetime* P_1 *may be an arbitrary number*, $1 \leqq P_1 \leqq P$, and for any fixed choice of P_1 there are, if $P_1 < \infty$, the *two possibilities:*

1. $f(x)$ is dead as to the W^p-spaces at the time P_1,
2. $f(x)$ is alive as to the W^p-spaces at the time P_1,

with exception, however, of the case $P_1 = 1$ where of course only 2. can occur, and the case $P_1 = P$ where 2. can only occur if the W-point is alive as to the W^p-spaces at the time P.

Proof. Let $g(x)$ be a through function in the W-point as to the W^p-spaces. In the special cases where $P_1 = P = \infty$, or $P_1 = P < \infty$ and moreover the given W-point and the desired function $f(x)$ have the same behaviour as to the W^p-spaces at their common moment of death $P = P_1$, we may as $f(x)$ simply use the function $g(x)$ itself. In all other cases we obtain, on account of the linearity of the W^p-sets, a function $f(x)$ of the type wanted by adding to $g(x)$ a W-zero function of lifetime P_1 which in case of 1. is not a W^{P_1}-function, and in case of 2. is a W^{P_1}-function.

We observe that for $1 < P_1 < P$ all the functions in the P_1-sparks are of the type 2.

II. The W-point is W-a. p. *The lifetime* P_1 *of a function* $f(x)$ *in the* W-*a. p. point as to the* W^p- *and the* W^p-*a. p. spaces may be an arbitrary number in the interval* $1 \leqq P_1 \leqq P$, and for every fixed choice of P_1 there are, if $P_1 < \infty$, the following *three possibilities:*

1. $f(x)$ is dead as to the W^p-spaces at the time P_1,
2. $f(x)$ is alive as to the W^p-spaces, but dead as to the W^p-a. p. spaces at the time P_1,
3. $f(x)$ is alive as to the W^p-a. p. spaces at the time P_1,

with exception, however, of the case $P_1 = 1$ where of course only 3. can occur, and the case $P_1 = P$ where 3. can only occur if the W-a. p. point is alive as to the W^p-a. p. spaces at the time P, and 2. can only occur if the W-a. p. point is alive as to the W^p-spaces at the time P.

Proof. Let $g(x)$ be a through function in the W-a. p. point as to the W^p- and the W^p-a. p. spaces. If $P_1 = P = \infty$, or $P_1 = P < \infty$ and moreover the given W-a. p. point and the desired function $f(x)$ have the same behaviour as to the W^p-spaces and the W^p-a. p. spaces at their common moment of death $P = P_1$, we may as $f(x)$ simply use the through function $g(x)$ itself. In all other cases we obtain, on account of the linearity of the W^p-sets, a function $f(x)$ of the type wanted by adding to $g(x)$ a suitable W-zero function: We get a function $f(x)$ of the type 1., by adding to $g(x)$ a W-zero function with the lifetime P_1 as to the W^p-spaces which is not a W^{P_1}-function. Similarly we get a function $f(x)$ of the type 2. by adding to $g(x)$ a W-zero function which is a W^{P_1}-function but not a W^{P_1}-zero function (since, in consequence of the uniqueness theorem, two W^{P_1}-a. p. functions in our W-a. p. point must differ by a W^{P_1}-zero function). Finally we get a function of the type 3. by adding to $g(x)$ a W-zero function which has the lifetime P_1 as to the W^p-spaces and is a W^{P_1}-zero function.

We observe that for $1 < P_1 < P$ all the functions in the P_1-sparks are of type 2.

§ 4.

W-Points which are not W-a. p. Points.

In this paragraph we shall consider the W-points which are not W-a. p. points, and we shall investigate what possibilities may occur for such points concerning as well the lifetime P as the behaviour at the moment of death as to the W^p-spaces. We shall show *that all possibilities which are imaginable beforehand may occur*, viz.

1. $P = \infty$.
2. P arbitrary finite, $1 \leqq P < \infty$.
   2 a. The point is dead as to the W^p-spaces at the time P $(P > 1)$.
   2 b. The point is alive as to the W^p-spaces at the time P $(P \geqq 1)$.

A (trivial) example to 1. with the lifetime ∞ is first given. Next, in order to get examples of W-points which are not W-a. p. and have an arbitrarily given finite lifetime P and a given behaviour at the moment of death as to the W^p-spaces, we add the W-point of the first example to a periodic W-point with the lifetime P and the desired behaviour at the moment of death as to the W^p-spaces. (In consequence of the theorem of Chapter III on the periodic G-points, the point »behaves« entirely as the periodic function contained in it). By this addition, the almost periodicity of the periodic W-point is destroyed, whereas its lifetime and behaviour at the moment of death as to the W^p-spaces are preserved.

Example to 1.

Let

$$f(x) = \begin{cases} 1 \text{ for } & 0 \leqq x < \infty \\ -1 \text{ for } & -\infty < x < 0. \end{cases}$$

The function $f(x)$ being bounded is obviously a W^p-function for every p; further $f(x)$ is not W-a. p., as

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)\,dx = 1 \quad \text{while} \quad \lim_{T \to \infty} \frac{1}{T} \int_{-T}^0 f(x)\,dx = -1 \;(\neq 1).$$

The W-point around $f(x)$ is thus not W-a. p. and has the lifetime $P = \infty$ as to the W^p-spaces.

Example to 2 a.

Let P be an arbitrary number, $1 < P < \infty$. Let $f(x)$ be the function of example 1, and $h(x)$ a periodic function which is p-integrable for $p < P$ but not P-integrable. Denote by $\mathfrak{A}$ the W-point around $f(x)$ and by $\mathfrak{B}$ the W-point around $h(x)$. Then the point $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$ will not be W-a. p., will have the lifetime P as to the W^p-spaces and be dead at the time P. That $\mathfrak{C}$ is not W-a. p. results from the linearity of the W-a. p. space, $\mathfrak{B}$ being W-a. p. and $\mathfrak{A}$ not being W-a. p. Further the point $\mathfrak{C}$ contains the function $f(x) + h(x)$ which is a

W^p-function for $p<P$. Finally no W^p-function lies in the point $\mathfrak{C}$; in fact the functions in $\mathfrak{C}$ can be obtained by adding to $f(x)$ all the functions in $\mathfrak{B}$, and $f(x)$ is a W^p-function, whereas, in consequence of the theorem on the periodic points, no function in $\mathfrak{B}$ is a W^p-function.

Example to 2 b.

Let P be an arbitrary number, $1 \leqq P < \infty$. Let $f(x)$ be the function of example 1, and $h(x)$ a periodic function which is P-integrable but not p-integrable for $p>P$. Denote by $\mathfrak{A}$ the W-point around $f(x)$ and by $\mathfrak{B}$ the W-point around $h(x)$. Then the W-point $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$ will not be W-a. p., will have the lifetime P as to the W^p-spaces and be alive at the time P. The proof is quite analogous to that of example 2 a: From the linearity of the W-a. p. space it results that $\mathfrak{C}$ is not W-a. p., the point $\mathfrak{B}$ being W-a. p. and $\mathfrak{A}$ not being W-a. p. Further the point $\mathfrak{C}$ contains the function $f(x) + h(x)$ which is a W^P-function. Finally for $p>P$ no W^p-function lies in the point $\mathfrak{C}$, as this latter point consists just of the functions obtained by adding $f(x)$ to all the functions in $\mathfrak{B}$, and $f(x)$ is a W^p-function, whereas, by the theorem on the periodic points, no function in $\mathfrak{B}$ is a W^p-function.

§ 5.

W-a. p. Points. Main Example 3.

In this paragraph we consider an arbitrary W-a. p. point, whose lifetime as to the W^p- and the W^p-a. p. spaces is denoted by P, and shall show *that also here all possibilities which are imaginable beforehand may occur*, viz.

1. $P = \infty$.
2. P arbitrary finite, $1 \leqq P < \infty$.
   - 2 a. The point is dead as to the W^p-spaces at the time P $(P > 1)$.
   - 2 b. The point is alive as to the W^p-spaces at the time P.
     - 2 b α. The point is alive as to the W^p-a. p. spaces at the time P $(P \geqq 1)$.
     - 2 b β. The point is dead as to the W^p-a. p. spaces at the time P $(P > 1)$.

Example to 1.

The W-point around a bounded periodic function.

Example to 2 a.

Let P be an arbitrary number, $1 < P < \infty$, and $h(x)$ a periodic function which is p-integrable for $p < P$ but not P-integrable. Then the W-point around $h(x)$ has the desired properties. Firstly, it is obviously W-a. p., $h(x)$ being W-a. p. Secondly, it contains a W^p-a. p. function for $p < P$, viz. $h(x)$. And thirdly, by the theorem on the periodic points, it does not contain any W^P-function.

Example to 2 b α.

Let P be an arbitrary number, $1 \leqq P < \infty$, and $h(x)$ a periodic function which is P-integrable, but not p-integrable for $p > P$. Then the W-point around $h(x)$ has the wanted properties. Firstly, the W-point is W-a. p., $h(x)$ being W-a. p. Secondly, it contains a W^P-a. p. function, viz. $h(x)$. And thirdly, by the theorem on the periodic points, it does not contain any W^p-function for $p > P$.

The case 2 b β remains. A rather complicated construction is necessary in order to show that this case can be realized.

Example to 2 bβ. **Main example 3.**

Let P be arbitrarily given, $1 < P < \infty$. We wish to construct *a function $F(x)$ which is a W^p-a. p. function for $p < P$ and a W^P-function, but such that the W-point around $F(x)$ does not contain any W^P-a. p. function.* Then the W-point around $F(x)$ will be an example to 2 bβ.

Let $m_1, m_2, \ldots$ be a sequence of odd numbers $\geqq 3$, increasing so strongly that the product

$$\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\left(1 - \frac{1}{m_3}\right)\cdots$$

is convergent (> 0). As usually we put

$$h_1 = m_1, \quad h_2 = m_1 m_2, \quad h_3 = m_1 m_2 m_3, \ldots .$$

Further let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of numbers such that $1 > \varepsilon_1 > \varepsilon_2 \cdots \to 0$. By a tower of type n we understand a tower with the 1-integral ε_n and the P-integral 1. Since $\varepsilon_n < 1$, the breadth of a tower of type n is less than 1. In the following, the letters ν, μ, η denote integers. We construct a sequence of functions $F_1(x), F_2(x), \ldots$ in the following way:

$F_1(x)$: In every interval $\nu h_1 \leqq x < (\nu + 1) h_1$ a tower of type 1 is placed on the central one of the subintervals $\mu \leqq x < \mu + 1$.

$F_2(x)$: Is obtained from $F_1(x)$ by »filling up« the central[1] one of the subintervals $\mu h_1 \leqq x < (\mu + 1) h_1$ of every interval $\nu h_2 \leqq x < (\nu + 1) h_2$ by towers of type 2, i. e. by placing a tower of type 2 on every subinterval $\eta \leqq x < \eta + 1$ of the mentioned central interval where no tower of $F_1(x)$ is standing.

$F_3(x)$: Is obtained from $F_2(x)$ by »filling up« the central one of the subintervals $\mu h_2 \leqq x < (\mu + 1) h_2$ of every interval $\nu h_3 \leqq x < (\nu + 1) h_3$ by towers of type 3, i. e. by placing a tower of type 3 on every subinterval $\eta \leqq x < \eta + 1$ of the mentioned central interval where no tower of $F_2(x)$ is standing.

⋮

$F_{n+1}(x)$: Is obtained from $F_n(x)$ by »filling up« the central one of the subintervals $\mu h_n \leqq x < (\mu + 1) h_n$ of every interval $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$ by towers of type $n + 1$, i. e. by placing a tower of type $n + 1$ on every subinterval $\eta \leqq x < \eta + 1$ of the mentioned central interval where no tower of $F_n(x)$ is standing.

⋮

(see Fig. 6, where $m_1 = m_2 = 3$ and $n = 2$).

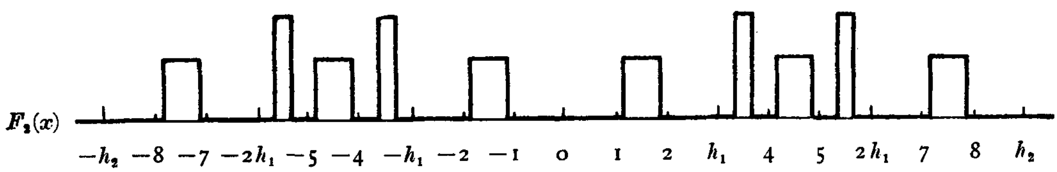

Fig. 6.

$F_n(x)$ is a bounded periodic function with the period h_n. Further $F_n(x) = F_{n+1}(x)$ for $-h_n \leqq x < h_n$ and moreover $F_n(x) = F_{n+1}(x) = F_{n+2}(x) = \cdots$ for the same x, since $h_n < h_{n+1} < h_{n+2} < \cdots$; consequently, as $h_n \to \infty$, a limit function

$$F(x) = \lim_{n \to \infty} F_n(x)$$

exists for $-\infty < x < \infty$, and

$$F(x) = F_n(x) \quad \text{for} \quad -h_n \leqq x < h_n.$$

[1] Here and in the following instead of the central one of the subintervals we could have chosen any one of the subintervals with exception of the first and the last, but in order to be able to use main example 3 to further purposes in the appendix we have made the specialisation mentioned.

The function $F_n(x)$, differing from $F_{n+q}(x)$ only by towers of the types $n+1$, $n+2, \ldots, n+q$, differs from $F(x)$ only by towers of types $n+1$, $n+2, \ldots$; hence (as $\varepsilon_{n+1} > \varepsilon_{n+2} > \cdots$)

$$\int_m^{m+1} (F(x) - F_n(x))\,dx \leqq \varepsilon_{n+1} \quad \text{for each integer } m$$

and thus

$$\int_x^{x+1} (F(t) - F_n(t))\,dt \leqq 2\,\varepsilon_{n+1} \quad \text{for all } x,$$

i. e.

$$D_S[F(x), F_n(x)] \leqq 2\,\varepsilon_{n+1}.$$

Since $\varepsilon_n \to 0$, we have $F_n(x) \xrightarrow{S} F(x)$; thus the function $F(x)$ is an S-limit periodic function, in particular an S-a. p. function. All our towers having the P-integral 1, obviously $F(x)$ is simultaneously an S^P-function (or, what is equivalent, a W^P-function) and therefore, in consequence of Theorem 1, Chapter III, also an S^p-a. p. function for $p<P$. Hence our function is not only a W^P-function and a W^p-a. p. function for $p<P$, as desired, but *moreover an S^p-a. p. function for $p<P$.*

We have to prove that the W-point around $F(x)$ does not contain any W^P-a. p. function. As a preparation we prove that the function $F(x)$ itself is not a W^P-a. p. function.

We begin by some preliminary remarks:

By d_i ($i = 1, 2, \ldots$) we denote the relative density of all the »places« $\eta \leqq x < \eta + 1$ on which there are standing no towers in $F_i(x)$, exactly speaking, the ratio $d_i = \frac{e_i}{h_i}$ between the number e_i of the »empty« places in a period interval $\mu h_i \leqq x < (\mu + 1) h_i$ of $F_i(x)$ and the total number h_i of places in such an interval. It is easy to see that

$$d_{i+1} = \left(1 - \frac{1}{m_{i+1}}\right) d_i.$$

In fact, when passing over from $F_i(x)$ to $F_{i+1}(x)$ we fill out just one of the m_{i+1} period intervals $\mu h_i \leqq x < (\mu + 1) h_i$ of $F_i(x)$ of which a period interval $\nu h_{i+1} \leqq x < (\nu + 1) h_{i+1}$ of $F_{i+1}(x)$ consists, so that $e_{i+1} = (m_{i+1} - 1) e_i$ and hence

$$d_{i+1} = \frac{e_{i+1}}{h_{i+1}} = \frac{(m_{i+1} - 1) e_i}{m_{i+1} h_i} = \left(1 - \frac{1}{m_{i+1}}\right) \frac{e_i}{h_i} = \left(1 - \frac{1}{m_{i+1}}\right) d_i.$$

Especially we get (by induction), as $d_1 = \frac{e_1}{h_1} = \left(1 - \frac{1}{m_1}\right)$,

$$d_n = \left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots\left(1 - \frac{1}{m_n}\right) \qquad (n = 1, 2, \ldots).$$

We emphasise that, on account of the convergence of the infinite product

$$\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots,$$

the relative density d_n of the empty places in the function $F_n(x)$ keeps greater than a positive constant when $n \to \infty$, so that after each construction of one of our functions $F_n(x)$ an »essential« part of the x-axis is kept free from towers.

We can now easily show that our function $F(x)$ is not W^P-a. p. To this purpose we consider, for an arbitrary fixed n, among the m_{n+1} intervals $\mu h_n \leqq x < (\mu + 1) h_n$ of the interval $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$, the central one which we denote by $\alpha_\nu \leqq x < \alpha_\nu + h_n$. Then we have

$$(1) \quad \sqrt[P]{\frac{1}{h_n}\int_{\alpha_\nu}^{\alpha_\nu + h_n} (F_{n+1}(x) - F_{n+1}(x + h_n))^P \, dx} = \sqrt[P]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots\left(1 - \frac{1}{m_n}\right)}.$$

For, in the interval $\alpha_\nu + h_n \leqq x < \alpha_\nu + 2h_n$ (to the right of the central interval), $F_{n+1}(x)$ has the same towers as $F_n(x)$, whereas, in consequence of the above, in the central interval $\alpha_\nu \leqq x < \alpha_\nu + h_n$ itself $F_{n+1}(x)$ has the same towers as $F_n(x)$ plus $h_n d_n = h_n \left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots\left(1 - \frac{1}{m_n}\right)$ towers of type $n + 1$ (see Fig. 7),

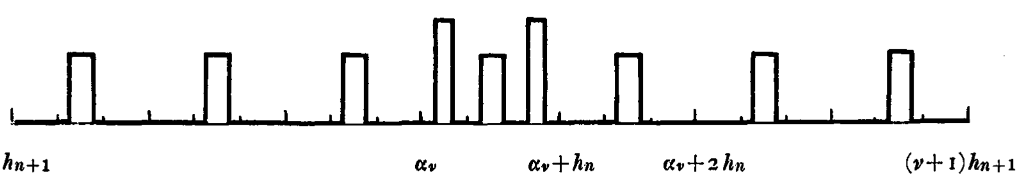

Fig. 7.

and all our towers have the P-integral 1. Now, however, $F(x) = F_{n+1}(x)$ for $-h_{n+1} \leqq x < h_{n+1}$. Hence we have ($\alpha_0 \leqq x < \alpha_0 + h_n$ denoting the central one of the intervals $\mu h_n \leqq x < (\mu + 1) h_n$ in $0 \leqq x < h_{n+1}$)

$$(2) \qquad \sqrt[P]{\frac{1}{h_n}\int_{\alpha_0}^{\alpha_0+h_n} (F(x)-F(x+h_n))^P\,dx} = \sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_n}\right)}.$$

By help of (2) we conclude that $F(x)$ is not W^P-a. p. Otherwise, in consequence of Theorem 2 a, Chapter I, since $F_n(x)$ is a sequence of periodic functions with the periods h_n which S-converges to $F(x)$, we should have

$$D_{S_L^P}[F(x+h_n),\ F(x)] \leqq \text{say } \frac{1}{2}\sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots}$$

for $L \geqq$ some L_0 and $n \geqq$ some N_0, and therefore, choosing n so large that $h_n \geqq L_0$ and $n \geqq N_0$,

$$D_{S_{h_n}^P}[F(x+h_n),\ F(x)] \leqq \frac{1}{2}\sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots}$$

which contradicts the relation (2).

More generally, however, we have to show that in the whole W-point around $F(x)$ there lies no W^P-a. p. function, i. e. that a function $G(x) = F(x)+J(x)$ where $J(x)$ is a W-zero function can never be a W^P-a. p. function. Assuming, indirectly, that $G(x) = F(x) + J(x)$ be a W^P-a. p. function, the function $J(x)$, being a W-zero function and a W^P-function, would (on account of Theorem 2, Chapter III) be a W^p-zero function for $p < P$. It might be supposed that, in a similar way as above, we could arrive at a contradiction by considering, for sufficiently large n, only the »first« of the central intervals, $\alpha_0 \leqq x < \alpha_0 + h_n$, and by concluding from the fact that

$$\frac{1}{h_n}\int_{\alpha_0}^{\alpha_0+h_n} (F(x)-F(x+h_n))^P\,dx$$

is »large« (i. e. not vanishing) that

$$\frac{1}{h_n}\int_{\alpha_0}^{\alpha_0+h_n} (F(x)-F(x+h_n))^p\,dx$$

for $p < P$ and near to P would also be »large«. This would namely involve, as $D_{S_{h_n}^P}[G(x+h_n),\ G(x)]$ and therefore $D_{S_{h_n}^p}[G(x+h_n),\ G(x)]$ for $p < P$ should be

»small«, that $D_{S^p_{h_n}}[J(x+h_n), J(x)]$ and hence $2\,D_{S^p_{h_n}}[J(x)]$ would be »large«, in contradiction to $J(x)$ being a W^p-zero function for every $p < P$. However, this attempt of argumentation fails for the following reason: The larger n is chosen, the nearer to P we have to choose the number p, so that we must operate with a variable p, while on the other hand the carrying out of the idea indicated would claim a fixed $p < P$. In the real proof we are forced to consider all the central intervals $\alpha_\nu \leqq x < \alpha_\nu + h_n$ (for sufficiently large n), and not only the first one; this means a certain complication, as $F(x)$ is not equal to $F_{n+1}(x)$ in all the intervals $\alpha_\nu \leqq x < \alpha_\nu + 2\,h_n$ (as in the first one $\alpha_0 \leqq x < \alpha_0 + 2\,h_n$). But except that, the reasoning is still the same as in the above attempt. Besides, in this way it is as easy to prove *that even in the B-point and not only in the W-point around $F(x)$ there is no W^p-a. p. function*, and this we are therefore going to do.

Denoting, as before, the central one of the intervals $\mu h_n \leqq x < (\mu+1)\,h_n$ in $\nu h_{n+1} \leqq x < (\nu+1)\,h_{n+1}$ by $\alpha_\nu \leqq x < \alpha_\nu + h_n$ we can, to

$$\varepsilon = \text{say } \frac{1}{4}\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots$$

and an arbitrarily fixed n, determine $P_0(n) < P$ such that the inequality

$$(3)\qquad \sqrt[p]{\frac{1}{h_n}\int_{\alpha_\nu}^{\alpha_\nu+h_n}(F_{n+1}(x)-F_{n+1}(x+h_n))^p\,dx} > \sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_n}\right)} - \varepsilon$$

is valid for every ν and for $P_0(n) < p < P$. This results, through continuity reasons, from the relation (1) using that for $p \to P$ the p-integral of a tower of type $n+1$ tends to its P-integral ($=1$).

The problem is to pass from $F_{n+1}(x)$ to $F(x)$, or more conveniently to $F_{n+q}(x)$ (which for a large q is identical with $F(x)$ in the large interval $-h_{n+q} \leqq x < h_{n+q}$). To this purpose, for an arbitrary $q > 0$, we consider a period interval (of length h_{n+q}) of $F_{n+q}(x)$ consisting of $\frac{h_{n+q}}{h_{n+1}}$ $(= m_{n+2}\,m_{n+3}\ldots m_{n+q})$ period intervals of $F_{n+1}(x)$ and ask: What is the relative density $d^{(n+1)}_{n+q}$ of those of the latter intervals in which $F_{n+q}(x)$ is identical with $F_{n+1}(x)$, or exactly speaking, what is the ratio $d^{(n+1)}_{n+q}$ between the number of those of the intervals in which $F_{n+q}(x)$ is identical with $F_{n+1}(x)$ and the total number $\frac{h_{n+q}}{h_{n+1}}$ of these intervals?

Above, we have met a similar question, viz. that of determining the relative density d_n of the empty places in the function $F_n(x)$. By a similar reasoning we find that the answer to our present question is

$$d_{n+q}^{(n+1)} = \left(1 - \frac{1}{m_{n+2}}\right)\left(1 - \frac{1}{m_{n+3}}\right) \cdots \left(1 - \frac{1}{m_{n+q}}\right).$$

[It may however be observed that in order to get this expression for $d_{n+q}^{(n+1)}$ we need not carry out this similar reasoning as we can directly establish the relation $d_{n+q}^{(n+1)} = \frac{d_{n+q}}{d_{n+1}}$ from which (using the expression for d_i) we just get the expression for $d_{n+q}^{(n+1)}$ given above. The relation in question may be obtained in the following way: We consider the empty places of the function $F_{n+q}(x)$ in one of its period intervals. On the one hand, the number of those places is, per definition, e_{n+q}. On the other hand however, as such empty places only occur in the $d_{n+q}^{(n+1)} \cdot \frac{h_{n+q}}{h_{n+1}}$ intervals $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$ in which $F_{n+q}(x)$ is identical with $F_{n+1}(x)$, the number in question may also be expressed by $d_{n+q}^{(n+1)} \frac{h_{n+q}}{h_{n+1}} \cdot e_{n+1}$. Putting this last expression equal to e_{n+q} $\left(\text{and using that } \frac{e_i}{h_i} = d_i\right)$ we just get the relation $d_{n+q}^{(n+1)} = \frac{d_{n+q}}{d_{n+1}}$].

Since

$$F(x) = F_{n+q}(x) \quad \text{for} \quad -h_{n+q} \leqq x < h_{n+q}$$

the function $F_{n+1}(x)$ is identical with $F(x)$ in $2\, d_{n+q}^{(n+1)} \frac{h_{n+q}}{h_{n+1}}$ of the $2 \frac{h_{n+q}}{h_{n+1}}$ intervals $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$ of which the interval $-h_{n+q} \leqq x < h_{n+q}$ consists. By (3) in every one of these $2\, d_{n+q}^{(n+1)} \frac{h_{n+q}}{h_{n+1}}$ intervals $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$ there is lying an interval $\alpha_\nu \leqq x < \alpha_\nu + h_n$ such that

$$(4) \quad \sqrt[p]{\frac{1}{h_n} \int_{\alpha_\nu}^{\alpha_\nu + h_n} (F(x) - F(x + h_n))^p \, dx} > \sqrt[p]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right) \cdots \left(1 - \frac{1}{m_n}\right)} - \varepsilon$$

holds for $P_0(n) < p < P$, where $P_0(n)$ is independent of q.

We shall show that this involves that no W^P-a. p. function can lie in the B-point around $F(x)$. Indirectly, we assume that such a function $G(x)$ exists. Then $F(x) - G(x) = J(x)$ is a B-zero function and a B^P-function and therefore, in consequence of Theorem 2, Chapter III, a B^p-zero function for $p < P$. Further, $F_n(x)$ being a sequence of periodic functions, with the periods h_n, which S-converges to $F(x)$ and hence B-converges to $G(x)$, the number h_n, in consequence of Theorem 2 a, Chapter I, is an S_L^P-translation number of our W^P-a. p. function $G(x)$ belonging to our

$$\varepsilon = \frac{1}{4}\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots$$

for $L \geqq$ some L_0, $n \geqq$ some N_0, i. e.

$$D_{S_L^P}[G(x + h_n), G(x)] \leqq \varepsilon \quad \text{for} \quad n \geqq N_0,\ L \geqq L_0.$$

We choose N so large that $h_N \geqq L_0$ and $N \geqq N_0$. Then we have

$$D_{S_{h_N}^P}[G(x + h_N), G(x)] \leqq \varepsilon$$

and therefore a fortiori

$$(5) \qquad D_{S_{h_N}^p}[G(x + h_N), G(x)] \leqq \varepsilon \quad \text{for} \quad p < P.$$

Now we consider $F(x)$ in the interval $-h_{N+q} \leqq x < h_{N+q}$. In $2\, d_{N+q}^{(N+1)} \cdot \dfrac{h_{N+q}}{h_{N+1}}$ of the $2\,\dfrac{h_{N+q}}{h_{N+1}}$ intervals $\nu\, h_{N+1} \leqq x < (\nu + 1)\, h_{N+1}$ of which the interval $-h_{N+q} \leqq x < h_{N+q}$ consists, there is, as we have seen above (at (4)), an interval $\alpha_\nu \leqq x < \alpha_\nu + h_N$ such that

$$(6) \qquad \sqrt[p]{\frac{1}{h_N}\int_{\alpha_\nu}^{\alpha_\nu + h_N} (F(x) - F(x + h_N))^p\, dx} > \sqrt[p]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots\left(1 - \frac{1}{m_N}\right)} - \varepsilon$$

for $P_0(N) < p < P$, where $P_0(N)$ is independent of q.

For every p satisfying $P_0(N) < p < P$ the inequalities (5) and (6) involve by help of Minkowski's inequality

$$\sqrt[p]{\frac{1}{h_N}\int_{\alpha_\nu}^{\alpha_\nu+h_N} |J(x) - J(x+h_N)|^p\,dx} =$$

$$\sqrt[p]{\frac{1}{h_N}\int_{\alpha_\nu}^{\alpha_\nu+h_N} |(F(x) - F(x+h_N)) - (G(x) - G(x+h_N))|^p\,dx} \geqq$$

$$\sqrt[p]{\frac{1}{h_N}\int_{\alpha_\nu}^{\alpha_\nu+h_N} (F(x) - F(x+h_N))^p\,dx} - \sqrt[p]{\frac{1}{h_N}\int_{\alpha_\nu}^{\alpha_\nu+h_N} |G(x) - G(x+h_N)|^p\,dx} >$$

$$\sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_N}\right)} - 2\varepsilon >$$

$$\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots - 2\varepsilon = \frac{1}{2}\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots = k$$

where k is a constant (independent of q).

Finally, p being a fixed number, $P_0(N)<p<P$, we get (using the expression for the relative density $d_{n+q}^{(n+1)}$)

$$\frac{1}{2h_{N+q}}\int_{-h_{N+q}}^{h_{N+q}} |J(x) - J(x+h_N)|^p\,dx \geqq \frac{1}{2h_{N+q}}\cdot 2\,d_{N+q}^{(N+1)}\,\frac{h_{N+q}}{h_{N+1}}\cdot k^p h_N =$$

$$\frac{h_N}{h_{N+1}}\,k^p\,d_{N+q}^{(N+1)} = \frac{k^p}{m_{N+1}}\left(1-\frac{1}{m_{N+2}}\right)\left(1-\frac{1}{m_{N+3}}\right)\cdots\left(1-\frac{1}{m_{N+q}}\right) >$$

$$\frac{k^p}{m_{N+1}}\left(1-\frac{1}{m_{N+2}}\right)\left(1-\frac{1}{m_{N+3}}\right)\cdots = k' > 0$$

where $k' = k'(N)$ is independent of q. Hence

$$(7)\qquad D_{B^p}[J(x), J(x+h_N)] = \overline{\lim_{T\to\infty}}\sqrt[p]{\frac{1}{2T}\int_{-T}^{T} |J(x) - J(x+h_N)|^p\,dx} \geqq \sqrt[p]{k'} > 0.$$

On the other hand, $J(x)$ being a B^p-zero function, we have $D_{B^p}[J(x), J(x+h_N)] \leqq 2\,D_{B^p}[J(x)] = 0$, i. e.

$$D_{B^p}[J(x), J(x+h_N)] = 0,$$

in contradiction to (7).

§ 6.

Through Functions.

Already in § 3 we used the following

Theorem. *Let $\mathfrak{A}$ be a W-point with the lifetime P, $1 < P \leqq \infty$, as to the W^p-spaces which is dead at the time P (if $P < \infty$). Then there exists a through function $f^*(x)$ in $\mathfrak{A}$ as to the W^p-spaces, i. e. a function which is a W^p-function for every $p < P$.*

We shall now prove this theorem. Roughly speaking, the method is as follows: In our W-point we choose functions $f_1(x), f_2(x), \ldots$ whose lifetimes approach P more and more. Starting from these functions we shall arrive at a through function. The first idea might perhaps be to consider the function which is equal to $f_1(x)$ say for $|x| < 1$, to $f_2(x)$ for $1 \leqq |x| < 2$, to $f_3(x)$ for $2 \leqq |x| < 3$ etc. This function, however, is far from being a through function; it needs not even to be p-integrable for larger p than is $f_1(x)$. However, it proves possible to modify the functions $f_1(x), f_2(x), \ldots$ in such a way that, applying the above method on the modified functions $f_1^*(x), f_2^*(x), \ldots$, we really get a through function $f^*(x)$. By our modification of the functions $f_1(x), f_2(x), \ldots$ we wish to obtain, firstly, that the modified functions $f_1^*(x), f_2^*(x), \ldots$ be p-integrable for all $p < P$, and secondly, that each function of the modified sequence $f_1^*(x), f_2^*(x), \ldots$ differs »so little« from its successor that the »composed« function $f^*(x)$ differs »little« from each of the functions $f_1^*(x), f_2^*(x), \ldots$ in the following sense: $f^*(x)$ is a W^p-function »almost as far« as each of these functions $f_n^*(x)$ (and is therefore a W^p-function for all $p < P$), and like these functions it belongs to the W-point.

We begin with the two following remarks.

Remark 1. *Let $f(x)$ be a 1-integrable function. Then we can always by adding a W-zero function obtain a function $g(x)$ which is p-integrable for every p, and such that $|g(x)| \leqq |f(x)|$ for all x.*

This may be done in the following way. For $n = 0, \pm 1, \pm 2, \ldots$ we determine N_n so large that

$$\int_n^{n+1} |f(x) - (f(x))_{N_n}| \, dx < \frac{1}{|n| + 1},$$

$(f(x))_{N_n}$ denoting, as usually, the function cut off at N_n, and define

$$g(x) = (f(x))_{N_n} \quad \text{for} \quad n \leqq x < n+1 \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Then we have

$$D_W[f(x), g(x)] = 0,$$

since the mean value of $|f(x) - g(x)|$ over a sufficiently large interval is arbitrarily small wherever on the x-axis the integration starts. Further, the function $g(x)$, being bounded in every interval $n \leqq x < n+1$, is p-integrable for every p. Finally $|g(x)| \leqq |f(x)|$ since $|(f(x))_{N_n}| \leqq |f(x)|$.

Remark 2. As easily seen, for an S^p-function it does not always hold (as in the case of a p-integrable periodic function) that $D_{S^p}[f(x), (f(x))_N] \to 0$ for $N \to \infty$. *But, if* $1 \leqq p_1 < p_2$ *and if* $f(x)$ *is an* S^{p_2}*-function, it does hold that*

$$D_{S^{p_1}}[f(x), (f(x))_N] \to 0 \quad \text{for} \quad N \to \infty,$$

i. e. to a given $\varepsilon > 0$ the inequality

$$\sqrt[p_1]{\int_x^{x+1} |f(t) - (f(t))_N|^{p_1}\, dt} < \varepsilon$$

is valid for all x provided N is chosen sufficiently large.

This may be seen in the following way. We have for all t

$$|f(t) - (f(t))_N|^{p_1} \leqq \frac{1}{N^{p_2 - p_1}} \cdot |f(t)|^{p_2};$$

in fact, the inequality is obvious for those t for which $|f(t)| \leqq N$ as the left-hand side is 0, and for those t for which $|f(t)| \geqq N$ we have

$$|f(t) - (f(t))_N|^{p_1} \leqq |f(t)|^{p_1} \leqq \frac{1}{N^{p_2 - p_1}} \cdot |f(t)|^{p_2}.$$

Hence for all x

$$\int_x^{x+1} |f(t) - (f(t))_N|^{p_1}\, dt \leqq \frac{1}{N^{p_2 - p_1}} \int_x^{x+1} |f(t)|^{p_2}\, dt \leqq \frac{1}{N^{p_2 - p_1}} (D_{S^{p_2}}[f(x)])^{p_2}$$

which tends to 0 for $N \to \infty$.

Now, we pass to the proof of the theorem. Let $1 < p_1 < p_2 < \cdots \to P$. Then in the W-point $\mathfrak{A}$ there exists a W^{p_1}-function $f_1(x)$, a W^{p_2}-function $f_2(x), \ldots$, and, in consequence of Remark 1, we may assume that all the functions $f_n(x)$ are p-integrable for every p. Let

$$f_2(x) - f_1(x) = j_1(x)$$
$$f_3(x) - f_2(x) = j_2(x)$$
$$f_4(x) - f_3(x) = j_3(x)$$
$$\vdots$$

Then $j_1(x), j_2(x), \ldots$ are all W-zero functions, and $j_1(x)$ is a W^{p_1}-function i. e. an S^{p_1}-function, $j_2(x)$ an S^{p_2}-function, ... Let further

$$j_1^*(x) = j_1(x) - (j_1(x))_{N_1}$$
$$j_2^*(x) = j_2(x) - (j_2(x))_{N_2}$$
$$j_3^*(x) = j_3(x) - (j_3(x))_{N_3}$$
$$\vdots$$

where $N_1, N_2, \ldots$ will be chosen below. All the functions $j_1^*(x), j_2^*(x), \ldots$ are W-zero functions, since $|j_1^*(x)| \leqq |j_1(x)|$, $|j_2^*(x)| \leqq |j_2(x)|$, ... For the same reason the function $j_1^*(x)$ is an S^{p_1}-function, $j_2^*(x)$ an S^{p_2}-function, ... Let $\sum\limits_1^\infty \varepsilon_n$ be a convergent series of positive terms. In consequence of Remark 2 it is possible to choose $N_1, N_2, \ldots$ such that for all x

$$\int_x^{x+1} |j_1^*(t)|\, dt < \varepsilon_1$$
$$\sqrt[p_1]{\int_x^{x+1} |j_2^*(t)|^{p_1}\, dt} < \varepsilon_2$$
$$\sqrt[p_2]{\int_x^{x+1} |j_3^*(t)|^{p_2}\, dt} < \varepsilon_3$$
$$\vdots$$

We can now indicate the »modified« functions $f_1^*(x), f_2^*(x), \ldots$. They are successively determined by

$$f_1^*(x) = f_1(x)$$
$$f_2^*(x) = f_1^*(x) + j_1^*(x)$$
$$f_3^*(x) = f_2^*(x) + j_2^*(x)$$

It may be observed that these functions $f_1^*(x), f_2^*(x), \ldots$ are constructed starting from $f_1(x)$ by help of the functions $j_1^*(x), j_2^*(x), \ldots$ in exactly the same way as $f_1(x), f_2(x), \ldots$ may be obtained starting from $f_1(x)$ by help of the functions $j_1(x), j_2(x), \ldots$. We find

$$(1)\qquad \begin{aligned} f_1^*(x) &= f_1(x) \\ f_2^*(x) &= f_2(x) - (j_1(x))_{N_1} \\ f_3^*(x) &= f_3(x) - (j_1(x))_{N_1} - (j_2(x))_{N_2} \\ &\;\;\vdots \end{aligned}$$

All the functions $j_1^*(x), j_2^*(x), \ldots$ being W-zero functions, the functions $f_1^*(x)$, $f_2^*(x), \ldots$ will (on account of their definitions) belong to $\mathfrak{A}$. Further, all the functions $(j_1(x))_{N_1}$, $(j_2(x))_{N_2}, \ldots$ being bounded, it results from the equations (1) that $f_1^*(x)$ is a W^{p_1}-function, $f_2^*(x)$ is a W^{p_2}-function, ..., and that they are all p-integrable for every p. From the way in which $N_1, N_2, \ldots$ are determined, we conclude that for all x and all $n = 1, 2, \ldots$ (putting $p_0 = 1$)

$$(2)\qquad \sqrt[p_{n-1}]{\int_x^{x+1} |f_m^*(t) - f_n^*(t)|^{p_{n-1}}\,dt} < \varepsilon_n + \varepsilon_{n+1} + \cdots \quad \text{for} \quad m > n,$$

since

$$\sqrt[p_{n-1}]{\int_x^{x+1} |f_m^*(t) - f_n^*(t)|^{p_{n-1}}\,dt} \leqq$$

$$\sqrt[p_{n-1}]{\int_x^{x+1} |f_{n+1}^*(t) - f_n^*(t)|^{p_{n-1}}\,dt} + \sqrt[p_{n-1}]{\int_x^{x+1} |f_{n+2}^*(t) - f_{n+1}^*(t)|^{p_{n-1}}\,dt} + \cdots +$$

$$+ \sqrt[p_{n-1}]{\int_x^{x+1} |f_m^*(t) - f_{m-1}^*(t)|^{p_{n-1}}\,dt} =$$

$$\sqrt[p_{n-1}]{\int_x^{x+1} |j_n^*(t)|^{p_{n-1}}\,dt} + \sqrt[p_{n-1}]{\int_x^{x+1} |j_{n+1}^*(t)|^{p_{n-1}}\,dt} + \cdots + \sqrt[p_{n-1}]{\int_x^{x+1} |j_{m-1}^*(t)|^{p_{n-1}}\,dt} \leqq$$

$$\sqrt[p_{n-1}]{\int_x^{x+1} |j_n^*(t)|^{p_{n-1}}\,dt} + \sqrt[p_n]{\int_x^{x+1} |j_{n+1}^*(t)|^{p_n}\,dt} + \cdots + \sqrt[p_{m-2}]{\int_x^{x+1} |j_{m-1}^*(t)|^{p_{m-2}}\,dt} <$$

$$\varepsilon_n + \varepsilon_{n+1} + \cdots + \varepsilon_{m-1} < \varepsilon_n + \varepsilon_{n+1} + \cdots.$$

Finally we define $f^*(x)$ by (see Fig. 8)

$$f^*(x) = f^*_{n+1}(x)$$

for

$$n \leqq |x| < n + 1 \qquad (n = 0, 1, 2, \ldots).$$

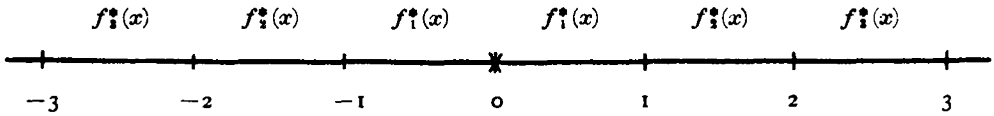

Fig. 8.

The function $f^*(x)$ is p-integrable for each p, all $f^*_n(x)$ being p-integrable. We consider the difference $f^*(x) - f^*_n(x)$ for an arbitrarily fixed n, and shall estimate

$$D_{W^{p_{n-1}}}[f^*(x), f^*_n(x)].$$

The inequality (2) tells us, how much $f^*(x)$ differs from $f^*_n(x)$ for all x outside the finite interval $-n \leqq x < n$. Since for the determination of $D_{W^{p_{n-1}}}$ the values of the function in an arbitrary finite interval are irrelevant if only the function is p_{n-1}-integrable in this interval, we get from (2)

$$D_{W^{p_{n-1}}}[f^*(x), f^*_n(x)] \leqq \varepsilon_n + \varepsilon_{n+1} + \cdots \tag{3}$$

which tends to 0 for $n \to \infty$. From (3) it results in particular that

$$D_{W}[f^*(x), f^*_n(x)] \to 0 \quad \text{for} \quad n \to \infty;$$

hence $f^*(x)$ belongs to $\mathfrak{A}$, a G-point considered as a set of functions being G-closed. Further we get from (3) that

$$D_{W^{p_{n-1}}}[f^*(x)] \leqq D_{W^{p_{n-1}}}[f^*_n(x)] + \varepsilon_n + \varepsilon_{n+1} + \cdots,$$

so that $f^*(x)$ is a $W^{p_{n-1}}$-function for every $n = 1, 2, \ldots$ and therefore a W^p-function for $p < P$.

CHAPTER VI.

The Mutual Relations of the B^p- and the B^p-a. p. Spaces.

§ 1.

Introduction.

In this Chapter we shall consider an arbitrary B-point as to the B^p-spaces and the B^p-a. p. spaces. We proceed in quite the same way as by the corresponding investigation in Chapter V of the behaviour of a W-point as to the W^p- and the W^p-a. p. spaces. On the one side we investigate what B^p-points belong to our B-point, and on the other side we consider the single functions in the B-point. Both the B^p-points and the functions are characterised by means of the B^p- and the B^p-a. p. spaces. In many respects also the results of our investigations will prove to be analogous to those of Chapter V. The results of Chapter V, § 2 on W-zero functions may even be transferred verbally to the B-zero functions, for by a retrospective glance we see immediately that we may replace »W« by »B« everywhere in the text without changing the examples. Also the general investigation in Chapter V, § 3 of the W^p-points in a given W-point can be transferred word for word; here too we may replace »W« by »B» everywhere. Whether the investigation of the single functions in Chapter V, § 3 can also be transferred, obviously depends on the question whether (analogously to the W-case) in a B-point there is always a through function as to the B^p-spaces and the B^p-a. p. spaces, i. e. a function which is a B^p-function for those p for which the B-point contains B^p-functions, and a B^p-a. p. function for those p for which the B-point contains B^p-a. p. functions. As we shall see, such a general theorem is really valid. Evidently, to establish this theorem it is, just as in the W-case, sufficient to prove that every B-point with the lifetime P which (if $P < \infty$) is dead at the time P contains a through function as to the B^p-spaces. By means of this theorem the investigation of the single functions in a given W-point can be transferred word for word to the given B-point. The proof of the theorem on the existence of a through function, however, is not analogous to that in the W-case, and it will be postponed to § 6.

But there is an interesting difference between the W-a. p. points and the B-a. p. points. In the W-case we gave an example of a W-a. p. point which is alive at the time P as to the W^p-spaces but is dead as to the W^p-a. p. spaces (Main example 3, Chapter V, § 5); in the B-case, however, such an example does not exist, the theorem being valid that a B-a. p. point which is alive at the time P as to the B^p-spaces is also alive at the time P as to the B^p-a. p. spaces. Hence, if a B-a. p. point with the lifetime P possesses P-descendants at all, one (and of course only one) of them will always be a B^p-a. p. point.

As to the B-zero functions, we simply refer to the treatment in Chapter V, § 2 of the W-zero functions where, as mentioned above, the letter »W« may right away be changed to »B«. From systematical reasons, however, we shall (in spite of the complete analogy with the W-case) in § 2 give a brief account of the behaviour of the B^p-points and the functions in a given B-point as to the B^p- and the B^p-a. p. spaces. In § 3 we give the proof of the theorem on B-a. p. points indicated above. Next, in § 4 and § 5 we state all the possibilities for a B-point which is not a B-a. p. point, respectively is a B-a. p. point. Finally in § 6 we prove the theorem on the through function.

§ 2.

General Remarks on B-Points.

Already in § 1 we spoke about a lifetime P of a B-point as to the B^p-spaces and eventually the B^p-a. p. spaces, and used the fact *that a B-a. p. point has the same lifetime as to these spaces.* As in the W-case, we say that we know the behaviour of a B-point at the time p_1 as to the B^p-spaces and (eventually) the B^p-a. p. spaces, if we know whether the point is alive or dead at the time p_1 as to the B^p-spaces and the B^p-a. p. spaces. Further we speak of the p-descendants of our B-point (or of the »components« of our B-rocket) and for every p, $1<p<P$, of a p-generator (p-nucleus). *The p-generator is the only one of the p-descendants which has descendants itself,* all the other p-descendants (the still-born brothers, or p-sparks) dying at the time p in the moment they are born. *If the B-point is B-a. p., the p-generator is B^p-a. p.* As to the general situation we may refer to the Fig. 5 (with »W« replaced by »B«).

We now pass to the single functions in a B-point. We speak about a function being alive or dead as to the B^p-spaces and the B^p-a. p. spaces at a definite date, and we speak of its lifetime P_1. *If the B-point is B-a. p., a function in this point has the same lifetime as to the B^p- and the B^p-a. p. spaces.*

If the B-point (with the lifetime P) is not B-a. p., there are the following possibilities for a function $f(x)$ with the lifetime P_1 contained in the B-point. *The lifetime P_1 may be an arbitrary number*, $1 \leqq P_1 \leqq P$, and for each fixed P_1 there are, if $P_1 < \infty$, the *two possibilities:*

1. $f(x)$ is dead as to the B^p-spaces at the time P_1,
2. $f(x)$ is alive as to the B^p-spaces at the time P_1,

with exception, however, of the case $P_1 = 1$ where of course only 2. can occur, and the case $P_1 = P$ where 2. can only occur if the B-point is alive as to the B^p-spaces at the time P.

If the B-point is B-a. p., *the lifetime P_1 of a function in the point may be an arbitrary number in the interval* $1 \leqq P_1 \leqq P$, and for each fixed P_1 there are, if $P_1 < \infty$, the *three possibilities:*

1. $f(x)$ is dead as to the B^p-spaces at the time P_1,
2. $f(x)$ is alive as to the B^p-spaces, but dead as to the B^p-a. p. spaces at the time P_1,
3. $f(x)$ is alive as to the B^p-a. p. spaces at the time P_1,

with exception, however, of the case $P_1 = 1$ where of course only 3. can occur, and the case $P_1 = P$ where 3. can only occur if the B-a. p. point is alive as to the B^p-a. p. spaces at the time P, and 2. can only occur if the B-a. p. point is alive as to the B^p-spaces at the time P.

§ 3.

A Theorem on the Behaviour of B-a. p. Points in their Moments of Death.

In this paragraph we prove the following theorem concerning the B-a. p. points which has no analogue in the W-case.

Theorem. *A B-a. p. point which contains a B^p-function $f(x)$ contains also a B^p-a. p. function $g(x)$.*

The general proof of this theorem uses the notion of asymptotic distribution function of a real B-a. p. function. In the special case $P = 2$, however, the theorem can be proved in another and more simple way, namely by help of Besicovitch's Theorem on Fourier series of B^2-a. p. functions. We shall begin by giving this proof which is only applicable in the case $P = 2$.

The special case $P = 2$. Let our B-a. p. and B^2-function $f(x)$ have the Fourier series $\Sigma A_n e^{i\Lambda_n x}$. We first show that

$$\Sigma |A_n|^2 \leqq (D_{B^2}[f(x)])^2. \tag{1}$$

Let

$$\sigma_q(x) = \sum_{n=1}^{N(q)} k_n^{(q)} A_n e^{i\Lambda_n x}$$

be a Bochner-Fejér sequence of $f(x)$. Then

$$\sum_{n=1}^{N(q)} \{k_n^{(q)}\}^2 |A_n|^2 = M\{|\sigma_q(x)|^2\} = (D_{B^2}[\sigma_q(x)])^2$$

Hence on account of the inequality (Chapter I)

$$D_{B^2}[\sigma_q(x)] \leqq D_{B^2}[f(x)]$$

we get

$$\sum_{n=1}^{N(q)} \{k_n^{(q)}\}^2 |A_n|^2 \leqq (D_{B^2}[f(x)])^2.$$

As $0 \leqq k_n^{(q)} \leqq 1$, and $k_n^{(q)} \to 1$ for fixed n and $q \to \infty$, we immediately get for $q \to \infty$ the desired inequality (1). In particular $\Sigma |A_n|^2$ is convergent and thus, in consequence of Besicovitch's Theorem, $\Sigma A_n e^{i\Lambda_n x}$ is the Fourier series of a B^2-a. p. function $g(x)$. As the two functions $g(x)$ and $f(x)$ (considered as B-a. p. functions) have the same Fourier series, the function $g(x)$ lies in our B-a. p. point around $f(x)$.

As mentioned above the proof of the theorem in the general case uses the notion of asymptotic distribution function of a real B-a. p. function. The asymptotic distribution functions for the different types of almost periodic functions are dealt with by Jessen and Wintner in their paper: Distribution Functions and the Riemann Zeta Function, Trans. of the Amer. Math. Soc., vol. 38. We shall only apply a single theorem of this paper, and as we shall not assume the knowledge of the paper we shall not merely state the theorem but also give a direct proof of it (communicated to us by Jessen).

To begin with we remind of two well known and elementary facts concerning real monotonic functions (in the wide sense) defined on the whole axis.

1°. A monotonic function has at most an enumerable number of discontinuity points.

2°. Let $\psi(\alpha)$ and $\psi_1(\alpha)$ be increasing functions with the following two properties:

$$\psi_1(\alpha) \leqq \psi(\alpha) \text{ for all } \alpha, \quad \text{and} \quad \psi_1(\beta) \geqq \psi(\alpha) \text{ for } \beta > \alpha.$$

Then $\psi(\alpha)$ and $\psi_1(\alpha)$ have the same discontinuity points, and $\psi_1(\alpha) = \psi(\alpha)$ in all the continuity points. (For, if α is a continuity point of $\psi_1(\alpha)$, it results from $\psi_1(\beta) \geqq \psi(\alpha)$ for $\beta > \alpha$ that $\psi_1(\alpha) = \psi_1(\alpha +) \geqq \psi(\alpha)$ which together with $\psi_1(\alpha) \leqq \psi(\alpha)$ gives $\psi_1(\alpha) = \psi(\alpha)$).

We say that a real function $f(x)$ defined on the whole x-axis has an asymptotic distribution function, if there exists an increasing function $\psi(\alpha)$ (in the wide sense) defined on the whole α-axis so that:

1) In a continuity point α of $\psi(\alpha)$ the two »relative measures«

$$m_{\text{rel}}\{[f(x) \leqq \alpha]\} = \lim_{T \to \infty} \frac{1}{2T} m\{[f(x) \leqq \alpha] \times [-T \leqq x \leqq T]\}$$

and

$$m_{\text{rel}}\{[f(x) < \alpha]\} = \lim_{T \to \infty} \frac{1}{2T} m\{[f(x) < \alpha] \times [-T \leqq x \leqq T]\}$$

both exist and are equal to $\psi(\alpha)$ (then obviously $0 \leqq \psi(\alpha) \leqq 1$).

2) $\psi(\alpha) \to 1$ for $\alpha \to \infty$, and $\psi(\alpha) \to 0$ for $\alpha \to -\infty$.

By the distribution function of $f(x)$ we then understand the function $\psi(\alpha)$ in its continuity points.

We can now state the theorem of Jessen and Wintner:

Auxiliary theorem. *Every real B-a. p. function $f(x)$ possesses an asymptotic distribution function.*

Proof. Let

$$\psi(\alpha) = \overline{m}_{\text{rel}}\{[f(x) \leqq \alpha]\} = \overline{\lim_{T \to \infty}} \frac{1}{2T} m\{[f(x) \leqq \alpha] \times [-T \leqq x \leqq T].$$

Obviously the function $\psi(\alpha)$ is an increasing function of α (in the wide sense) defined on the whole α-axis. We shall show that the two relative measures $m_{\text{rel}}\{[f(x) \leqq \alpha]\}$ and $m_{\text{rel}}\{[f(x) < \alpha]\}$ exist in every continuity point of $\psi(\alpha)$ and are both equal to $\psi(\alpha)$, and that $\psi(\alpha) \to 1$ for $\alpha \to \infty$ and $\psi(\alpha) \to 0$ for $\alpha \to -\infty$. Then, according to our definition, the function $\psi(\alpha)$ considered in its continuity points is an asymptotic distribution function of $f(x)$.

Together with $\psi(\alpha)$ we consider the other increasing function

$$\psi_1(\alpha) = \underline{m}_{\text{rel}}\{[f(x) < \alpha]\} = \varliminf_{T\to\infty} \frac{1}{2T} m\{[f(x) < \alpha] \times [-T \leqq x \leqq T]\}.$$

First we shall show by help of 2° that $\psi(\alpha)$ and $\psi_1(\alpha)$ have the same discontinuity points and are equal in their continuity points. Obviously $\psi_1(\alpha) \leqq \psi(\alpha)$; hence it is sufficient to show that $\psi_1(\beta) \geqq \psi(\alpha)$ for $\beta > \alpha$. In order to do that we introduce the auxiliary function (see Fig. 9):

$$\Phi(z) = \begin{cases} 1 & \text{for } z \leqq \alpha \\ \dfrac{\beta - z}{\beta - \alpha} & \text{for } \alpha \leqq z \leqq \beta \\ 0 & \text{for } z \geqq \beta. \end{cases}$$

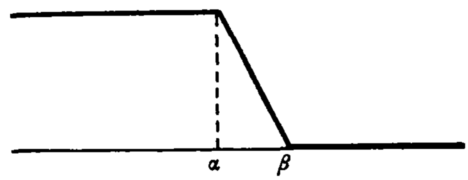

Fig. 9.

This continuous function $\Phi(z)$ (which for β »near to« α differs unessentially from the function which is 1 for $z \leqq \alpha$ and 0 for $z > \alpha$) has a bounded difference quotient. Hence $\Phi(f(x))$ is a B-a. p. function (Chapter I). In particular, what is of decisive importance in the proof, $\Phi(f(x))$ has a mean value $M\{\Phi(f(x))\}$. As $\Phi(f(x)) \leqq 1$ for $f(x) < \beta$ and $\Phi(f(x)) = 0$ for $f(x) \geqq \beta$, we have

$$\psi_1(\beta) = \underline{m}_{\text{rel}}\{[f(x) < \beta]\} \geqq M\{\Phi(f(x))\},$$

and as $\Phi(f(x)) = 1$ for $f(x) \leqq \alpha$ and $\Phi(f(x)) \geqq 0$ for $f(x) > \alpha$, we have

$$\psi(\alpha) = \overline{m}_{\text{rel}}\{[f(x) \leqq \alpha]\} \leqq M\{\Phi(f(x))].$$

From the two latter inequalities the desired inequality $\psi_1(\beta) \geqq \psi(\alpha)$ results.

Further, in an arbitrary one of the (common) continuity points for $\psi(\alpha)$ and $\psi_1(\alpha)$ we have

$$\psi_1(\alpha) = \underline{m}_{\text{rel}}\{[f(x) < \alpha]\} \leqq \left\{\begin{matrix} \underline{m}_{\text{rel}}\{[f(x) \leqq \alpha]\} \\ \overline{m}_{\text{rel}}\{[f(x) < \alpha]\} \end{matrix}\right\} \leqq \overline{m}_{\text{rel}}\{[f(x) \leqq \alpha]\} = \psi(\alpha),$$

and as the first and the last term in this chain of inequalities are equal, all the terms must be equal. Consequently $m_{\text{rel}}\{[f(x) \leqq \alpha]\}$ and $m_{\text{rel}}\{[f(x) < \alpha]\}$ both exist and are equal to $\psi(\alpha)$.

It remains to prove that

$$\psi(\alpha) \to 1 \text{ for } \alpha \to \infty \quad \text{and} \quad \psi(\alpha) \to 0 \text{ for } \alpha \to -\infty.$$

We begin by proving the first of these limit relations. As $\psi(\alpha)$ is increasing and $\leqq 1$, the limit $\lim_{\alpha\to\infty}\psi(\alpha)$ exists and is $\leqq 1$. Proceeding indirectly we assume that $\lim_{\alpha\to\infty}\psi(\alpha)=g<1$ and hence $\psi(\alpha)\leqq g<1$ for every α. In the following we may let α avoid the discontinuity points of $\psi(\alpha)$. Obviously

$$m_{\rm rel}\{[f(x)\geqq\alpha]\}=1-\psi(\alpha).$$

From our assumption it would follow that for every α

$$m_{\rm rel}\{[f(x)\geqq\alpha]\}\geqq 1-g>0,$$

and hence for arbitrary large α (indeed for every $\alpha>0$)

$$D_B[f(x)]=\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}|f(x)|\,dx\geqq\overline{\lim_{T\to\infty}}\frac{1}{2T}\int\limits_{[f(x)\geqq\alpha]\times[-T\leqq x\leqq T]}f(x)\,dx\geqq$$

$$\lim_{T\to\infty}\alpha\cdot\frac{1}{2T}m\{[f(x)\geqq\alpha]\times[-T\leqq x\leqq T]\}=\alpha\, m_{\rm rel}\{[f(x)\geqq\alpha]\}\geqq\alpha(1-g)$$

in contradiction to $D_B[f(x)]$ being finite. The other limit relation $\psi(\alpha)\to 0$ for $\alpha\to-\infty$ follows immediately from the first limit relation $\psi(\alpha)\to 1$ for $\alpha\to\infty$ by applying the latter to the function $-f(x)$ and using that

$$m_{\rm rel}\{[f(x)<\alpha]\}=1-m_{\rm rel}\{[f(x)\geqq\alpha]\}=1-m_{\rm rel}\{[-f(x)\leqq-\alpha]\}.$$

Having proved the auxiliary theorem, we now pass to the p r o o f o f o u r t h e o r e m i n t h e g e n e r a l c a s e i. e. f o r a n a r b i t r a r y $P>1$. This proof may be formulated in the shortest way by help of Stieltjes' integrals, but not having to use Stieltjes' integrals elsewhere in our paper we prefer to accomplish the proof in a more elementary manner.

Let $f(x)$ be the B-a. p. and B^P-function of the theorem. Then $|f(x)|$ is a real B-a. p. function and hence possesses an asymptotic distribution function $\psi(\alpha)$. For the sake of convenience we will assume that no point of the, at most enumerable, set of discontinuity points of $\psi(\alpha)$ is a positive integer; otherwise we might consider the function $kf(x)$, instead of $f(x)$, where k is a suitably chosen positive constant. $\Bigl($If $\psi(\alpha)$ has the discontinuity points d_n, the function $|kf(x)|$ has the distribution function $\psi\left(\frac{\alpha}{k}\right)$ with the discontinuity points kd_n, and disposing of k in a suitable way we can of course provide for none of these latter numbers being a positive integer$\Bigr)$.

For $n = 1, 2, \ldots$ we put

$$\mu_n = m_{\text{rel}}\{[n \leqq |f(x)| < n + 1]\};$$

then

$$\mu_n = \psi(n + 1) - \psi(n).$$

We begin with two remarks which easily result from the fact that $|f(x)|$ has the distribution function $\psi(\alpha)$.

1°. It is evident that

$$m_{\text{rel}}\{[n \leqq |f(x)| \leqq \infty]\} = \mu_n + \mu_{n+1} + \cdots;$$

for on the one hand

$$m_{\text{rel}}\{[n \leqq |f(x)| \leqq \infty]\} = 1 - \psi(n)$$

and on the other hand

$$\mu_n + \mu_{n+1} + \cdots = (\psi(n+1) - \psi(n)) + (\psi(n+2) - \psi(n+1)) + \cdots = \lim_{\nu \to \infty} \psi(\nu) - \psi(n) = 1 - \psi(n).$$

2°. Further, *the series*

$$\mu_1 \cdot 1^p + \mu_2 \cdot 2^p + \cdots + \mu_n \cdot n^p + \cdots$$

is convergent with a sum $\leqq (D_{B^p}[f(x)])^p$, in other words, the inequality

$$\mu_1 \cdot 1^p + \mu_2 \cdot 2^p + \cdots + \mu_n \cdot n^p \leqq (D_{B^p}[f(x)])^p$$

holds for an arbitrary fixed n. In order to prove this latter inequality we estimate $(D_{B^p}[f(x)])^p$ from below in the following way: Taking only those x for which $1 \leqq |f(x)| < n + 1$ into consideration, we get

$$(D_{B^p}[f(x)])^p = \overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{-T}^{T} |f(x)|^p \, dx \geqq \overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{[-T \leqq x \leqq T] \times [1 \leqq |f(x)| < n+1]} |f(x)|^p \, dx.$$

Therefore we consider, for a fixed T, the integral

$$\frac{1}{2T} \int_{[-T \leqq x \leqq T] \times [1 \leqq |f(x)| < n+1]} |f(x)|^p \, dx.$$

We divide the range of integration $[-T \leqq x \leqq T] \times [1 \leqq |f(x)| < n + 1]$ into the n subsets $[-T \leqq x \leqq T] \times [\nu \leqq |f(x)| < \nu + 1]$ $(\nu = 1, 2, \ldots n)$, and correspondingly the integral into the n integrals

$$\frac{1}{2T} \int_{[-T \leqq x \leqq T] \times [\nu \leqq |f(x)| < \nu+1]} |f(x)|^p \, dx \qquad (\nu = 1, 2, \ldots n).$$

For each of these integral we have

$$\frac{1}{2T}\int\limits_{[-T\leqq x\leqq T]\times[\nu\leqq|f(x)|<\nu+1]} |f(x)|^p\,dx \quad \geqq \nu^p\cdot m\{[-T\leqq x\leqq T]\times[\nu\leqq|f(x)|<\nu+1]\},$$

where the left-hand side tends to $\nu^p\cdot\mu_\nu$ for $T\to\infty$. Thus we get

$$\overline{\lim_{T\to\infty}}\,\frac{1}{2T}\int\limits_{[-T\leqq x\leqq T]\times[1\leqq|f(x)|<n+1]} |f(x)|^p\,dx \quad \geqq \mu_1\cdot 1^p+\mu_2\cdot 2^p+\cdots+\mu_n\cdot n^p$$

and hence the wanted inequality

$$\mu_1\cdot 1^p+\mu_2\cdot 2^p+\cdots+\mu_n\cdot n^p\leqq(D_{B^p}[f(x)])^p.$$

Now we pass to the proper proof. The salient point is to demonstrate that the sequence $(f(x))_n$ is a B^p-fundamental sequence. To this purpose we have to estimate

$$\left(D_{B^p}[(f(x))_n,\ (f(x))_m]\right)^p=\overline{\lim_{T\to\infty}}\,\frac{1}{2T}\int_{-T}^{T}|(f(x))_m-(f(x))_n|^p\,dx$$

for $n<m$. For those x for which $|f(x)|<n$ we have $(f(x))_m-(f(x))_n=0$, for those x for which $\nu\leqq|f(x)|<\nu+1$ $(\nu=n,n+1,\ldots,m-1)$ we have $|(f(x))_m-(f(x))_n|<\nu+1-n$, and for those x for which $|f(x)|\geqq m$ we have $|(f(x))_m-(f(x))_n|=m-n$. Thus we get

$$\left(D_{B^p}[(f(x))_n,\ (f(x))_m]\right)^p=\overline{\lim_{T\to\infty}}\,\frac{1}{2T}\int_{-T}^{T}|(f(x))_m-(f(x))_n|^p\,dx\leqq$$

$$\sum_{\nu=n}^{m-1}\mu_\nu(\nu+1-n)^p+(m-n)^p\,m_{\mathrm{rel}}\{[m\leqq|f(x)|\leqq\infty]\}=$$

$$\sum_{\nu=n}^{m-1}\mu_\nu(\nu+1-n)^p+(m-n)^p\sum_{\nu=m}^{\infty}\mu_\nu.$$

Enlarging $\nu+1-n$ to ν in the first sum, and $(m-n)^p\mu_\nu$ to $\nu^p\mu_\nu$ in the last term, we get

$$\left(D_{B^p}[(f(x))_n,\ (f(x))_m]\right)^p\leqq\sum_{\nu=n}^{\infty}\mu_\nu\cdot\nu^p$$

where the right-hand side is independent of m and tends to 0 for $n\to\infty$ since, according to $2°$, the series $\Sigma\mu_\nu\cdot\nu^p$ is convergent. Consequently $(f(x))_n$ is a B^p-fundamental sequence.

As $f(x)$ is B-a. p., the function $(f(x))_n$ is also B-a. p. (Chapter I) and, being bounded, it is therefore B^p-a. p. for all p, in particular it is B^P-a. p. Hence the sequence $(f(x))_n$ is a B^P-fundamental sequence of B^P-a. p. functions. The B^P-a. p. space being complete, the sequence $(f(x))_n$ thus B^P-converges to a B^P-a. p. function $g(x)$. This function $g(x)$ must lie in our B-a. p. point around $f(x)$ as the sequence $(f(x))_n$ B-converges to $g(x)$ and B-converges to $f(x)$, the latter because $f(x)$ is B-a. p. (Chapter I).

We observe that the »reason« why no corresponding theorem holds for the W-a. p. points is the incompleteness of the W^p-a. p. spaces; for as regards the distribution functions a wholly analogous notion exists for W-a. p. functions, only a relative measure in the W-sense being used instead of a relative measure in the B-sense. In the S-case we have completeness of the S^p-a. p. spaces but the notion asymptotic distribution function has no meaning in the S-case (and as we have seen in Chapter IV a function $f(x)$ may very well be an S-a. p. and S^P-function without being S^P-a. p.).

§ 4.

B-Points which are not B-a. p. Points.

In this paragraph we shall consider the B-points which are not B-a. p. points, and we shall investigate what possibilities may occur for such points concerning as well the lifetime P as the behaviour in the moment of death as to the B^p-spaces. We shall show that (as in the W-case) *all possibilities which are imaginable beforehand may occur*, viz.

1. $P = \infty$.
2. P arbitrarily finite, $1 \leqq P < \infty$.
   - 2 a. The point is dead as to the B^p-spaces at the time P $(P > 1)$.
   - 2 b. The point is alive as to the B^p-spaces at the time P $(P \geqq 1)$.

The examples which we shall give are quite similar to those used in the corresponding investigation in Chapter V, § 4 on W-points which are not W-a. p.

Example to 1.

Let

$$f(x)=\begin{cases} 1 & \text{for } x\geqq 0 \\ -1 & \text{for } x<0. \end{cases}$$

The function $f(x)$ being bounded is a B^p-function for all p. Further $f(x)$ is no B-a. p. function as

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\,dx=1 \quad \text{while} \quad \lim_{T\to\infty}\frac{1}{T}\int_{-T}^0 f(x)\,dx=-1\ (\neq 1).$$

Thus the B-point around $f(x)$ is not B-a. p. and has the lifetime $P=\infty$.

Example to 2 a.

In order to get a B-point (not B-a. p.) with an arbitrary finite lifetime $P\,(>1)$ which is dead at the time P we add to the B-point of the first example a B-point around a periodic function $h(x)$ which is p-integrable for $p<P$ but not P-integrable. The B-point thus constructed is not B-a. p. as the B-point of the first example is not, while the B-point around $h(x)$ is. Further the point contains the function $f(x)+h(x)$ which is a B^p-function for $p<P$, but it does not contain any B^P-function, as the functions of the point can be obtained by adding to $f(x)$ all the functions in the B-point around $h(x)$, and $f(x)$ is a B^P-function whereas, in consequence of the theorem on the periodic points, the B-point around $h(x)$ does not contain any B^P-function.

Example to 2 b.

In order to get a B-point (not B-a. p.) with an arbitrary finite lifetime $P\,(\geqq 1)$ which is alive at the time P we add in an analogous way to the B-point from the first example a B-point around a periodic function $h(x)$ which is P-integrable but not p-integrable for $p>P$.

§ 5.

B-a. p. Points.

In this paragraph we consider an arbitrary B-a. p. point whose lifetime as to the B^p- and the B^p-a. p. spaces is denoted by P. In consequence of the theorem in § 3 it holds (in contrast to the W-case) that every B-a. p. point »behaves in the same way« as to the B^p- and the B^p-a. p. spaces in the following sense: *If a B-a. p. point contains a B^p-function, it contains also a B^p-a. p. function.* We shall prove that there are *the following possibilities* for a B-a. p. point as regards its lifetime P and behaviour in the moment of death.

1. $P = \infty$.
2. P arbitrary finite, $1 \leqq P < \infty$.
   - 2 a. The point is dead as to the B^p- and the B^p-a. p. spaces at the time P $(P > 1)$.
   - 2 b. The point is alive as to the B^p- and the B^p-a. p. spaces at the time P $(P \geqq 1)$.

Example to 1.

The B-point around a bounded periodic function.

Example to 2 a.

The B-point around a periodic function which is p-integrable for $p < P$ but not P-integrable.

Example to 2 b.

The B-point around a periodic function which is P-integrable but not p-integrable for $p > P$.

§ 6.

Through Functions.

Finally in this paragraph we prove the following theorem which has already been used in § 2.

Theorem. *Let $\mathfrak{A}$ be a B-point with the lifetime P, $1 \leqq P \leqq \infty$, which (if $P < \infty$) is dead at the time P. Then there exists in $\mathfrak{A}$ a through function $f^*(x)$ as to the B^p-spaces, i. e. a function in $\mathfrak{A}$ which is a B^p-function for every $p < P$.*

In the proof of this theorem we use a remark made in the proof of the corresponding theorem on W-points, viz. that a 1-integrable function can always be modified by a W-zero function, and hence still more by a B-zero function, so that it becomes p-integrable for all p, and so that its modulus is not enlarged for any x. Further we shall use the operation of forming the minimum of two functions, in the sense indicated in the introduction.

Let $1 \leqq p_1 < p_2 < \cdots \to P$. We choose in $\mathfrak{A}$ a B^{p_1}-function $f_1(x)$, a B^{p_2}-function $f_2(x), \ldots$ and in consequence of the remark above we may assume these functions to be p-integrable for all p. We replace $f_1(x), f_2(x), \ldots$ by other functions $f_1^*(x), f_2^*(x), \ldots$ in $\mathfrak{A}$ where $f_n^*(x)$ like $f_n(x)$ is a B^{p_n}-function and p-integrable for all p and so that moreover the chain of inequalities

$$|f_1^*(x)| \geqq |f_2^*(x)| \geqq \cdots$$

holds for every x. As such functions $f_1^*(x), f_2^*(x), \ldots$ we may use

$$f_1^*(x) = f_1(x), \quad f_2^*(x) = \min\,[f_1^*(x), f_2(x)], \quad f_3^*(x) = \min\,[f_2^*(x), f_3(x)], \; \ldots.$$

In fact, firstly $|f_n^*(x)| \leqq |f_n(x)|$ for every x which involves that $f_n^*(x)$ like $f_n(x)$ is a B^{p_n}-function and p-integrable for all p, secondly $|f_1^*(x)| \geqq |f_2^*(x)| \geqq \cdots$ for every x, and thirdly $f_1^*(x), f_2^*(x), \ldots$ are all contained in $\mathfrak{A}$, as a G-point considered as a set of functions is closed with respect to the minimum-operation.

The functions $f_1^*(x), f_2^*(x), \ldots$ lying in $\mathfrak{A}$ form in particular a B-fundamental sequence. Now we make use of the special method of constructing a B-limit function of a B-fundamental sequence indicated in Chapter II in the proof of the completeness of the B^p-spaces. Constructing by this method (see Fig. 2) a B-limit function of our B-fundamental sequence $f_1^*(x), f_2^*(x), \ldots$ we get a function $f^*(x)$ which is a through function for our point $\mathfrak{A}$. On the one hand, this function $f^*(x)$ lies in $\mathfrak{A}$, as a G-point considered as a set of G-functions is G-closed. On the other hand, as $|f_n^*(x)| \geqq |f_{n+1}^*(x)| \geqq \cdots$ we have $|f^*(x)| \leqq |f_n^*(x)|$ for $x > T_{n-1}$ (and analogously for negative x with a large modulus) which, together with the fact that $f_1^*(x), f_2^*(x), \ldots$ are p-integrable for all p, shows that

$$D_{B^{p_n}}[f^*(x)] \leqq D_{B^{p_n}}[f_n^*(x)];$$

hence $f^*(x)$ is a B^{p_n}-function for every n and consequently a B^p-function for every $p < P$.

APPENDIX.

BY

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In the proper paper the reciprocal interaction between the G^p- and the G^p-a. p. spaces was treated in every one of the three cases $G = S$, $G = W$ and $G = B$. As mentioned in the preface the reciprocal interaction between all the spaces will be investigated in a later paper. For this investigation a new series of main examples will be needed. In every one of these main examples the problem is to construct a B-a. p. point (represented by a B-a. p. function $F(x)$) with certain particular properties, and each example deals with an »extreme case«. The main examples serve as bricks in the construction of all the types of B-a. p. points, as the »medium cases« can be obtained by addition of different extreme cases. Naturally these main examples are more varied and complicated than our former main examples 1, 2 and 3, but on the other hand they are more or less analogous to them. Therefore we have preferred to indicate them — with exception of a single especially complicated one — in an appendix to the present paper. The examples in this appendix are numbered by Roman numerals I, II, . . . with subsequent letters a, b, ... Every main example numbered by one of the Roman numerals I, II or III is nearly associated with the main example with the corresponding Arabian numeral in the paper itself. In connexion with main example II some lemmas concerning integral-estimations are proved which also will be used in the later paper. We shall not here try to give a comprehensive view of the examples, as such a view can first be properly gained in the course of the later paper where the examples are put in their natural places as counter examples to the general theorems.

Contents.

Main Example I.

In the main example I we constructed a sequence $F_1(x)$, $F_2(x)$, ... of bounded periodic functions with periods $h_1 = m_1$, $h_2 = m_1 m_2$, ... which is a W^p-fundamental sequence for every p, but not W-convergent. We shall now prove that there exists a function $F(x)$ such that the quoted sequence $F_n(x)$ is B^p-convergent to $F(x)$ for every p ($\geqq 1$) and such that the B-point around $F(x)$ does not contain any W-function. We remark that these latter properties involve in particular that the sequence $F_n(x)$ cannot W-converge to any W-function, as an eventual W-limit function of $F_n(x)$, being a B-limit function of $F_n(x)$, would lie in the B-point around $F(x)$. We emphasise, and this is the real content of the example, that we hereby get *a function $F(x)$ which is B^p-a. p. for all p, whereas the B-point around $F(x)$ does not contain W-functions.*

For $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$ our sequence $F_n(x) = n$ tends to ∞, while for $-\infty < x < -\frac{1}{2}$ and $\frac{1}{2} < x < \infty$ the limit $\lim\limits_{n \to \infty} F_n(x)$ exists and is finite. In fact for $-h_{n+1} + \frac{1}{2} < x < -\frac{1}{2}$ and $\frac{1}{2} < x < h_{n+1} - \frac{1}{2}$ we have $F_n(x) = F_{n+1}(x)$ and hence for the same x (as $h_1 < h_2 < \cdots$)

$$F_n(x) = F_{n+1}(x) = F_{n+2}(x) = \cdots$$

and for n sufficiently large every $x < -\frac{1}{2}$ and $x > \frac{1}{2}$ is caught in the quoted intervals. For $-h_{n+1} + \frac{1}{2} < x < -\frac{1}{2}$ and $\frac{1}{2} < x < h_{n+1} - \frac{1}{2}$ we get

$$\lim_{\nu \to \infty} F_\nu(x) = F_n(x).$$

We shall see that as our $F(x)$ we can use the function

$$F(x)=\begin{cases}\text{say } 0 \text{ for } -\frac{1}{2}\leqq x\leqq\frac{1}{2}\\ \lim\limits_{n\to\infty}F_n(x) \text{ for } -\infty<x<-\frac{1}{2} \text{ and } \frac{1}{2}<x<\infty.\end{cases}$$

Thus $F(x)$ consists of:

Towers of the breadth 1 and the height 1 placed on all the numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$,

towers of the breadth 1 and the height 2 placed on all the numbers $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$,

towers of the breadth 1 and the height 3 placed on all the numbers $\equiv 0 \pmod{h_3}$ but $\not\equiv 0 \pmod{h_4}$,

$\vdots$

We shall first show that $F_n(x)\xrightarrow{B^p}F(x)$ for every $p\geqq 1$, so that $F(x)$ is B^p-a. p. for all p.

We have

$$(D_{B^p}[F(x),\,F_n(x)])^p=\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}|F(x)-F_n(x)|^p\,dx.$$

Thus we shall estimate

$$\frac{1}{2T}\int_{-T}^{T}|F(x)-F_n(x)|^p\,dx$$

for fixed n and large T, say $T\geqq h_n$. For a given $T\geqq h_n$ we determine first $q\geqq 0$ such that $h_{n+q}\leqq T<h_{n+q+1}$ and then ν among the numbers $1, 2, \ldots, m_{n+q+1}-1$ such that $\nu h_{n+q}\leqq T<(\nu+1)h_{n+q}$.

To begin with we distinguish between the case $\nu\leqq m_{n+q+1}-2$ and the case $\nu=m_{n+q+1}-1$.

In the first case we get

$$\frac{1}{2T}\int_{-T}^{T}|F(x)-F_n(x)|^p\,dx\leqq\frac{1}{2\nu h_{n+q}}\int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}}|F(x)-F_n(x)|^p\,dx\leqq$$

$$\frac{1}{2\nu h_{n+q}}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}|0-n|^p\,dx+\int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}}(F_{n+q}(x)-F_n(x))^p\,dx\right]$$

(as $F(x) = F_{n+q}(x)$ for $-(\nu+1)h_{n+q} \leqq x < -\frac{1}{2}$ and for $\frac{1}{2} < x \leqq (\nu+1)h_{n+q}$, $\nu = 1, 2, \ldots, m_{n+q+1} - 2$). Since $(F_{n+q}(x) - F_n(x))^p$ is periodic with the period h_{n+q}, the last term of the inequality is equal to

$$\frac{n^p}{2\nu h_{n+q}} + \frac{\nu+1}{\nu} M\{(F_{n+q}(x) - F_n(x))^p\}.$$

In the second case ($\nu = m_{n+q+1} - 1$) we get, as $F(x) = F_{n+q+1}(x)$ for $-h_{n+q+1} \leqq x < -\frac{1}{2}$ and $\frac{1}{2} < x \leqq h_{n+q+1}$,

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^p\, dx \leqq \frac{1}{2\nu h_{n+q}} \int_{-h_{n+q+1}}^{h_{n+q+1}} |F(x) - F_n(x)|^p\, dx \leqq$$

$$\frac{1}{2\nu h_{n+q}}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} |0 - n|^p\, dx + \int_{-h_{n+q+1}}^{h_{n+q+1}} (F_{n+q+1}(x) - F_n(x))^p\, dx\right].$$

As $(F_{n+q+1}(x) - F_n(x))^p$ is periodic with the period h_{n+q+1}, the last term of the inequality is equal to

$$\frac{n^p}{2\nu h_{n+q}} + \frac{\nu+1}{\nu} M\{(F_{n+q+1}(x) - F_n(x))^p\}.$$

Using the estimation of $M\{(F_{n+q}(x) - F_n(x))^p\}$ from main example 1 (see pages 59—60) we get in both cases, as $\nu \geqq 1$ and $\frac{\nu+1}{\nu} \leqq 2$,

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^p\, dx \leqq \frac{n^p}{2h_{n+q}} + 2R_n^p$$

where R_n is the remainder after the n-th term in the geometrical series $\sum_1^\infty \frac{1}{2^{\frac{n}{p}}}$. We let now T and consequently $q \to \infty$. Then we get

$$\overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^p\, dx \leqq 2R_n^p,$$

i. e.

$$D_{B^p}[F(x), F_n(x)] \leqq \sqrt[p]{2}\, R_n.$$

The last inequality shows that $F_n(x) \xrightarrow{B^p} F(x)$ for every $p \geqq 1$.

Next we shall show that the B-point around $F(x)$ does not contain any W-function. The proof runs in a similar way as the proof (given in main example 1) of the more special fact that the sequence $F_n(x)$ is not W-convergent. Proceeding indirectly we assume that there exists a B-zero function $J(x)$ such that the function

$$F(x) + J(x) = G(x)$$

is a W-function (i. e. an S-function). Let

$$D_S[G(x)] = K < \infty.$$

We choose a fixed integer $N > K$. In $F(x)$ on all the numbers $m \equiv 0 \pmod{h_N}$ except the number 0 there are standing towers of the breadth 1 and a height $\geqq N$. As $\int_x^{x+1} |G(t)|\, dt \leqq K$ for all x, the inequality

$$\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x) - G(x)|\, dx \geqq \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} F(x)\, dx - \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|\, dx \geqq N - K$$

is valid for every one of the quoted m. Hence

$$D_B[J(x)] = D_B[G(x) - F(x)] = \overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{-T}^{T} |F(x) - G(x)|\, dx \geqq \frac{N-K}{h_N} > 0,$$

and consequently $D_B[J(x)] > 0$, in contradiction to $J(x)$ being a B-zero function.

Finally we observe that as in main example 2 we might have chosen all the numbers $m_1, m_2, \ldots$ equal to 2.

Main Examples II a, II b and II c.

In main example 2 we constructed a function $F(x)$ which is an S^p-a. p. function for $p < P$, an S^P-function, but not an S^P-a. p. function. The following three main examples II a, II b and II c are generalisations of this main example.

In main example II a the B-point around the function $F(x)$ of main example 2 is considered, the numbers $m_1, m_2, \ldots$ occurring in it being only assumed to increase suitably strongly to ∞. In this way we get, as we shall see, *a function $F(x)$ which is S^p-a. p. for $p < P$, W^P-a. p., B^p-a. p. for all p, and such that the B-point around $F(x)$ does not contain any S^P-a. p. function.* Here P is an arbitrarily given number, $1 < P < \infty$.

In the main examples II b and II c other types of towers are used than in main example 2, but apart from this the construction is quite the same.

In main example II b we construct *a function $F(x)$ which is S^p-a. p. for $p<P$, B^p-a. p. for all p and such that the B-point around $F(x)$ does not contain any S^P-function.* Here P is an arbitrarily given number, $1 < P < \infty$.

Finally in main example II c we construct *a function $F(x)$ which is S^P-a. p., B^p-a. p. for all p and such that the B-point around $F(x)$ does not contain any S^p-function for any $p > P$.* Here P is an arbitrarily given number, $1 \leqq P < \infty$.

Main Example II a.

As mentioned, in main example II a we consider the B-a. p. point around the function $F(x)$ of main example 2. In this latter example we saw that $F(x)$ is an S^p-a. p. function for $p<P$ and an S^P-function, but not an S^P-a. p. function. From this it can easily be concluded that the B-a. p. point around $F(x)$ does not contain any S^P-a. p. function. Indirectly we assume that the point contains an S^P-a. p. function $G(x)$. Then $G(x)$ has the same Fourier series as $F(x)$, and both $F(x)$ and $G(x)$ are S-a. p. functions; in consequence of the uniqueness theorem they can therefore only differ by an S-zero function, which is also an S^P-zero function, and $G(x)$ being S^P-a. p., $F(x)$ would also be S^P-a. p., which is not the case.

Next we show that $F(x)$ is a W^P-a. p. function. As mentioned in main example 2, the function $F(x)$ differs from $F_n(x)$ only at the numbers $m \equiv 0$ (mod h_{n+1}). Therefore for all $m \equiv 0$ (mod h_1) but $\not\equiv 0$ (mod h_{n+1}) we have

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|^P \, dx} = 0;$$

and further, as all our towers have the P-integral 1, we get in consequence of Minkowski's inequality for all $m \equiv 0$ (mod h_{n+1})

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|^P\,dx} \leqq$$

$$\sqrt[P]{\int_{m-\frac{h^1}{2}}^{m+\frac{h_1}{2}} |F(x)|^P\,dx} + \sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F_n(x)|^P\,dx} \leqq 2.$$

Hence

$$D_{W^P}[F(x), F_n(x)] \leqq \frac{2}{\sqrt[P]{h_{n+1}}},$$

where the right-hand side tends to 0 for $n \to \infty$.

Finally we shall show that by letting the numbers $m_1, m_2, \ldots$ increase sufficiently strongly to infinity we can obtain that $F(x)$ becomes B^p-a. p. for all p. However, as it will be convenient to prove this property of $F(x)$, which is common for the main examples II a, II b and II c, simultaneously for all three main examples, we postpone the proof.

In the main examples II b and II c we shall use the following

Lemma 1 a. *Let $f(x)$ be a function, defined in a finite interval $a \leqq x \leqq a+L$, which consists of a number of congruent towers placed in some way or other in the interval. Then every function $t(x)$, satisfying the inequality*

$$\sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x) + t(x)|^P\,dx} \leqq k \sqrt[P]{\frac{1}{L}\int_a^{a+L} (f(x))^P\,dx}$$

where $0 < k < 1$ and $1 \leqq P < \infty$, will satisfy the inequality

$$\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha\,dx \geqq (1-k)^P \frac{1}{L}\int_a^{a+L} (f(x))^\alpha\,dx$$

for an arbitrary α, $1 \leqq \alpha \leqq P$.

In the main examples II b and II c, however, the lemma will only be applied in the case where $\alpha = 1$ and $f(x)$ consists of only one tower.

Proof. Let the towers of $f(x)$ have the breadth b and the height h and let their number be ν. We may assume that $|t(x)| \leqq h$; otherwise we consider $(t(x))_h$ which in consequence of the inequality

$$|f(x) + (t(x))_h| = |(f(x))_h + (t(x))_h| \leqq |f(x) + t(x)|$$

satisfies the same assumption as $t(x)$; on account of $|t(x)| \geqq |(t(x))_h|$ the conclusion for $t(x)$ results from the conclusion for $(t(x))_h$. Then (i. e. for $|t(x)| \leqq h$) we have for every α, $1 \leqq \alpha \leqq P$,

$$\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha dx \geqq \frac{1}{h^{P-\alpha}} \frac{1}{L}\int_a^{a+L} |t(x)|^P dx = \frac{1}{h^{P-\alpha}} \left\{ \sqrt[P]{\frac{1}{L}\int_a^{a+L} |t(x)|^P dx} \right\}^P \geqq$$

$$\frac{1}{h^{P-\alpha}} \left\{ \sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x)|^P dx} - \sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x) + t(x)|^P dx} \right\}^P \geqq$$

$$\frac{1}{h^{P-\alpha}} (1-k)^P \frac{1}{L}\int_a^{a+L} |f(x)|^P dx = \frac{1}{h^{P-\alpha}} (1-k)^P \frac{\nu}{L} b h^P = (1-k)^P \frac{\nu}{L} b h^\alpha =$$

$$(1-k)^P \frac{1}{L}\int_a^{a+L} (f(x))^\alpha dx.$$

Main Example II b.

We construct a sequence $F_1(x), F_2(x), \ldots$ and a function $F(x)$ in exactly the same way as in main example II a. Only by »a tower of type n« we shall now understand a tower with the p_n-integral ε_n and the P-integral n where the sequence $p_1, p_2, \ldots$ is chosen such that $1 \leqq p_1 < p_2 < \cdots \to P$. As before $1 > \varepsilon_1 > \varepsilon_2 > \cdots \to 0$. The n-th function $F_n(x)$ is a bounded periodic function with the period h_n.

First, we shall show that the sequence $F_n(x)$ is S^p-convergent to $F(x)$ for every $p < P$, so that $F(x)$ is an S^p-limit periodic function for $p < P$. For every $m \equiv 0 \pmod{h_1}$ and $\not\equiv 0 \pmod{h_{n+1}}$ the quantity

$$\sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|^{p_n} dx}$$

is equal to 0 (cp. page 115) and for $m \equiv 0 \pmod{h_{n+1}}$ we have

$$\sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)-F_n(x)|^{p_n}\, dx} \leqq \sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)|^{p_n}\, dx} + \sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F_n(x)|^{p_n}\, dx},$$

which for $m=0$ is equal to $\sqrt[p_n]{\varepsilon_n}$, while for $m \neq 0$, denoting by $n+q(m)$, $q(m) \geqq 1$, the type of the tower placed on the number m in $F(x)$, it is

$$\leqq \sqrt[p_{n+q(m)}]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)|^{p_{n+q(m)}}\, dx} + \sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F_n(x)|^{p_n}\, dx} = \sqrt[p_{n+q(m)}]{\varepsilon_{n+q(m)}} + \sqrt[p_n]{\varepsilon_n}.$$

As $1 > \varepsilon_1 > \varepsilon_2 > \cdots$, we have consequently for every $m \equiv 0 \pmod{h_1}$

$$\sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)-F_n(x)|^{p_n}\, dx} \leqq 2\sqrt[P]{\varepsilon_n},$$

and hence for every x

$$\sqrt[p_n]{\int_{x-\frac{h_1}{2}}^{x+\frac{h_1}{2}} |F(t)-F_n(t)|^{p_n}\, dt} \leqq \sqrt[p_n]{2} \cdot 2\sqrt[P]{\varepsilon_n}.$$

Thus

$$D_{S_{h_1}^{p_n}}[F(x), F_n(x)] \leqq \sqrt[p_n]{\frac{2}{h_1}} \cdot 2\sqrt[P]{\varepsilon_n} \leqq 2\sqrt[P]{\varepsilon_n},$$

which tends to 0 for $n \to \infty$. From this it results that

$$D_{S_{h_1}^{p}}[F(x), F_n(x)] \to 0 \quad \text{for} \quad p < P,$$

as for sufficiently large n we have $p_n > p$ and therefore

$$D_{S_{h_1}^{p}}[F(x), F_n(x)] \leqq D_{S_{h_1}^{p_n}}[F(x), F_n(x)] \to 0.$$

Next we show that the B-point around $F(x)$ does not contain any S^P-function. Proceeding indirectly we assume that the B-point around $F(x)$ contains an S^P-function $G(x)$. Then $D_{S^P}[G(x)] = K < \infty$. Let N be a fixed number so that $\sqrt[P]{N} \geqq 2K$. In $F(x)$ on all numbers $m \equiv 0 \pmod{h_N}$ but $\not\equiv 0 \pmod{h_{N+1}}$ towers of type N are standing and these towers have the P-integral N. Thus we have for the m in question

$$\sqrt[P]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x)|^P\, dx} = \sqrt[P]{N} \geqq 2K.$$

At the same time the inequality

$$\sqrt[P]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|^P\, dx} \leqq K$$

is valid so that for the quoted m we get the inequality

$$\sqrt[P]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|^P\, dx} \leqq \frac{1}{2} \sqrt[P]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x)|^P\, dx}.$$

Thus we have, on account of the lemma 1 a above, for these m

$$\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x) - F(x)|\, dx \geqq \left(\frac{1}{2}\right)^P \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x)|\, dx = \left(\frac{1}{2}\right)^P k' > 0,$$

where k' denotes the 1-integral of a tower of type N. Hence

$$D_B[G(x), F(x)] = \overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{-T}^{T} |G(x) - F(x)|\, dx \geqq \frac{\left(\frac{1}{2}\right)^P k'(m_{N+1} - 1)}{h_{N+1}} > 0,$$

which contradicts the fact that $F(x)$ and $G(x)$ lie in the same B-point.

Main Example II c.

In just the same way as in the main examples II a and II b a sequence $F_1(x)$, $F_2(x)$, ... and a function $F(x)$ are constructed. Only by a »tower of type n« we shall now understand a tower with the P-integral ε_n and the p_n-integral n where the sequence of numbers $p_1, p_2, \ldots$ is chosen such that $p_1 > p_2 > \cdots \to P$. As before $1 > \varepsilon_1 > \varepsilon_2 > \cdots \to 0$. The n-th function $F_n(x)$ is a bounded periodic function with the period h_n.

First, we shall show that the sequence $F_n(x)$ is S^P-convergent to $F(x)$, so that $F(x)$ is an S^P-limit periodic function. The quantity

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)-F_n(x)|^P\,dx}$$

is equal to 0 for every $m \equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_{n+1}}$, whereas for $m \equiv 0 \pmod{h_{n+1}}$ we have, denoting for $m \neq 0$ by $n + q(m)$, $q(m) \geqq 1$, the type of the tower which stands in $F(x)$ on the number m,

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)-F_n(x)|^P\,dx} \leqq \sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)|^P\,dx} + \sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F_n(x)|^P\,dx} = \begin{cases} \sqrt[P]{\varepsilon_n} & \text{for } m = 0 \\ \sqrt[P]{\varepsilon_{n+q(m)}} + \sqrt[P]{\varepsilon_n} & \text{for } m \neq 0 \end{cases}$$

which is

$$\leqq 2\sqrt[P]{\varepsilon_n}.$$

Hence for every $m \equiv 0 \pmod{h_1}$

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)-F_n(x)|^P\,dx} \leqq 2\sqrt[P]{\varepsilon_n},$$

and hence for every x

$$\sqrt[P]{\int_{x-\frac{h_1}{2}}^{x+\frac{h_1}{2}} |F(t)-F_n(t)|^P\, dt} \leqq \sqrt[P]{2}\cdot 2\sqrt[P]{\varepsilon_n}.$$

Consequently

$$D_{S^P_{h_1}}[F(x), F_n(x)] \leqq \sqrt[P]{\frac{2}{h_1}}\cdot 2\sqrt[P]{\varepsilon_n},$$

which tends to o for $n \to \infty$.

Next we shall prove that the B-point around $F(x)$ does not contain S^p-functions for any $p > P$. Proceeding indirectly we assume that the B-point around $F(x)$ contains an S^p-function $G(x)$ for a $p > P$. Then

$$D_{S^p}[G(x)] = K < \infty.$$

Let N be a fixed number so that $\sqrt[p]{N} \geqq 2K$ and $p_N \leqq p$. In $F(x)$ on all the numbers $m \equiv$ o (mod h_N) but $\not\equiv$ o (mod h_{N+1}) there are standing towers of type N and these towers have the p_N-integral N and therefore a p-integral which is $\geqq N$. Hence for the m in question

$$\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} (F(x))^p\, dx} \geqq \sqrt[p]{N} \geqq 2K.$$

At the same time the inequality

$$\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|^p\, dx} \leqq K$$

holds, so that for the quoted m

$$\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|^p\, dx} \leqq \frac{1}{2}\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} (F(x))^p\, dx}.$$

Consequently, by the lemma, we have for these m

$$\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x) - F(x)|\, dx \geqq \left(\frac{1}{2}\right)^p \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} F(x)\, dx = \left(\frac{1}{2}\right)^p k' > 0,$$

where k' denotes the 1-integral of a tower of type N. Hence

$$D_B[G(x), F(x)] = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |G(x) - F(x)|\, dx \geqq \frac{\left(\frac{1}{2}\right)^p k'(m_{N+1} - 1)}{h_{N+1}} > 0$$

which contradicts the fact that $F(x)$ and $G(x)$ lie in the same B-point.

»Spreading» of the Towers in the Main Examples II a, II b and II c.

Finally we show, simultaneously for all three main examples, that by letting $m_1, m_2, \ldots$ increase sufficiently strongly we can obtain that the sequence $F_1(x), F_2(x), \ldots$ is B^p-convergent to $F(x)$ for every p, so that $F(x)$ is B^p-limit periodic for every p.

We put (cp. main example 1)

$$f_1(x) = F_1(x),\quad f_2(x) = F_2(x) - F_1(x),\quad f_3(x) = F_3(x) - F_2(x), \ldots.$$

Let $\delta_1, \delta_2, \ldots$ be a sequence of positive numbers so that $\sum_1^\infty \delta_n$ is convergent, and let $P_1, P_2, \ldots$ be a sequence of numbers, $1 \leqq P_1 < P_2 < \cdots$, tending to ∞. Successively we may choose $m_1, m_2, \ldots$ so large that

$$D_{B^{P_n}}[f_n(x)] < \delta_n \quad \text{for} \quad n = 1, 2, \ldots.$$

In fact $f_n(x) = F_n(x) - F_{n-1}(x)$ differs from 0 only at the numbers $\equiv 0 \pmod{h_n}$, and on these numbers in $F_{n-1}(x)$ there stand towers of type $n-1$, whereas in $F_n(x)$ towers of type n are standing, so that

$$D_{B^{P_n}}[f_n(x)] = D_{B^{P_n}}[F_n(x) - F_{n-1}(x)] \leqq \frac{\sqrt[P_n]{I_n'} + \sqrt[P_n]{I_n}}{h_n},$$

where I_n' denotes the P_n-integral of a tower of type $n-1$ and I_n denotes the P_n-integral of a tower of type n; assuming the numbers $m_1, m_2, \ldots, m_{n-1}$ already fixed, the number m_n and therewith $h_n = m_1 m_2 \ldots m_n$ may evidently be chosen so large that the right-hand side of the inequality and therefore $D_{B^{P_n}}[f_n(x)]$ becomes $< \delta_n$.

After this choice of $m_1, m_2, \ldots$ we can prove that

$$D_{B^{P_n}}[F(x), F_n(x)] \to 0 \quad \text{for} \quad n \to \infty.$$

From this we get immediately the desired relation $D_{B^p}[F(x), F_n(x)] \to 0$ for every fixed p, as for n sufficiently large $P_n > p$ and therefore $D_{B^p}[F(x), F_n(x)] \leqq D_{B^{P_n}}[F(x), F_n(x)] \to 0$.

In order to prove that

$$D_{B^{P_n}}[F(x), F_n(x)] = \overline{\lim_{T \to \infty}} \sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{P_n} dx} \to 0 \quad \text{for} \quad n \to \infty$$

we estimate (cp. main example I)

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{P_n} dx$$

for fixed n and large T, say $T \geqq h_n$. First we determine $q \geqq 0$ so that $h_{n+q} \leqq T < h_{n+q+1}$, and then ν among the numbers $1, 2, \ldots, m_{n+q+1} - 1$ so that $\nu h_{n+q} \leqq T < (\nu + 1) h_{n+q}$.

To begin with we distinguish between the two cases $\nu \leqq m_{n+q+1} - 2$ and $\nu = m_{n+q+1} - 1$.

In the first case we have

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{P_n} dx \leqq \frac{1}{2\nu h_{n+q}} \int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} |F(x) - F_n(x)|^{P_n} dx \leqq$$

$$\frac{1}{2\nu h_{n+q}}\left[I_n + \int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} |F_{n+q}(x) - F_n(x)|^{P_n} dx\right],$$

where, as before, I_n denotes the P_n-integral of a tower of type n; for on 0 there is standing no tower in $F(x)$, whereas a tower of type n is standing in $F_n(x)$, and $F(x) = F_{n+q}(x)$ for $-(\nu + 1) h_{n+q} < x < -\frac{h_1}{2}$ and for $\frac{h_1}{2} < x < (\nu + 1) h_{n+q}$,

$\nu = 1, 2, \ldots, m_{n+q+1}-2$. As $|F_{n+q}(x)-F_n(x)|^{p_n}$ is periodic with the period h_{n+q}, the right-hand side is equal to

$$\frac{I_n}{2\nu h_{n+q}} + \frac{\nu+1}{\nu} M\{|F_{n+q}(x) - F_n(x)|^{p_n}\}.$$

In the second case ($\nu = m_{n+q+1} - 1$) we get

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{p_n}\, dx \leqq \frac{1}{2\nu h_{n+q}} \int_{-h_{n+q+1}}^{h_{n+q+1}} |F(x) - F_n(x)|^{p_n}\, dx \leqq$$

$$\frac{1}{2\nu h_{n+q}}\left[I_n + \int_{-h_{n+q+1}}^{h_{n+q+1}} |F_{n+q+1}(x) - F_n(x)|^{p_n}\, dx\right],$$

as $F(x) = F_{n+q+1}(x)$ for $-h_{n+q+1} < x < -\frac{h_1}{2}$ and for $\frac{h_1}{2} < x < h_{n+q+1}$. Since $|F_{n+q+1}(x)-F_n(x)|^{p_n}$ is periodic with the period h_{n+q+1}, the last term is equal to

$$\frac{I_n}{2\nu h_{n+q}} + \frac{\nu+1}{\nu} M\{|F_{n+q+1}(x) - F_n(x)|^{p_n}\}.$$

An estimation of $M\{|F_{n+q}(x) - F_n(x)|^{p_n}\}$ is got in the following way:

$$\sqrt[p_n]{M\{|F_{n+q}(x) - F_n(x)|^{p_n}\}} = \sqrt[p_n]{M\{|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+q}(x)|^{p_n}\}} \leqq$$

$$\sqrt[p_n]{M\{|f_{n+1}(x)|^{p_n}\}} + \sqrt[p_n]{M\{|f_{n+2}(x)|^{p_n}\}} + \cdots + \sqrt[p_n]{M\{|f_{n+q}(x)|^{p_n}\}} \leqq$$

$$\sqrt[p_{n+1}]{M\{|f_{n+1}(x)|^{p_{n+1}}\}} + \sqrt[p_{n+2}]{M\{|f_{n+2}(x)|^{p_{n+2}}\}} + \cdots + \sqrt[p_{n+q}]{M\{|f_{n+q}(x)|^{p_{n+q}}\}} \leqq$$

$$\delta_{n+1} + \delta_{n+2} + \cdots + \delta_{n+q} \leqq \delta_{n+1} + \delta_{n+2} + \cdots.$$

Thus we have

$$M\{|F_{n+q}(x) - F_n(x)|^{p_n}\} \leqq (\delta_{n+1} + \delta_{n+2} + \cdots)^{p_n}.$$

Using this estimation in each of the two above cases, we get

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{p_n}\, dx \leqq \frac{I_n}{2 h_{n+q}} + 2(\delta_{n+1} + \delta_{n+2} + \cdots)^{p_n}.$$

For $T \to \infty$ and therefore $q \to \infty$, we have $\frac{I_n}{2 h_{n+q}} \to 0$; hence

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |F(x) - F_n(x)|^{p_n} dx \leqq 2(\delta_{n+1} + \delta_{n+2} + \cdots)^{p_n}$$

or

$$D_{B^{p_n}}[F(x), F_n(x)] \leqq \sqrt[p_n]{2}(\delta_{n+1} + \delta_{n+2} + \cdots),$$

which shows that

$$D_{B^{p_n}}[F(x), F_n(x)] \to 0 \quad \text{for} \quad n \to \infty.$$

In connexion with the lemma 1 a we insert here two lemmas of similar character which will be applied in the later paper.

Lemma 1 b. *Let $f(x)$ be a function, defined in a finite interval $a \leqq x \leqq a + L$, which consists of a number of congruent towers placed in some way or other in the interval. Then every function $t(x)$ satisfying the inequality*

$$\sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x) + t(x)|^P dx} \leqq k \sqrt[P]{\frac{1}{L}\int_a^{a+L} (f(x))^P dx},$$

where $0 < k < 1$ and $1 \leqq P < \infty$, will satisfy the inequality

$$\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha dx \geqq (1 - k)^\alpha \frac{1}{L}\int_a^{a+L} (f(x))^\alpha dx$$

for an arbitrary α, $P \leqq \alpha < \infty$.

Proof. We may assume that $t(x) = 0$ where $f(x) = 0$. Otherwise we may consider the function $t^*(x)$ which is equal to 0 where $f(x) = 0$ and equal to $t(x)$ where $f(x) \neq 0$; like $t(x)$ this function $t^*(x)$ satisfies the assumption of the lemma 1 b, since $|f(x) + t^*(x)| \leqq |f(x) + t(x)|$, and the conclusion for $t(x)$ results from the conclusion for $t^*(x)$, as $|t(x)| \geqq |t^*(x)|$. Let the towers of $f(x)$ have the breadth b and the height h, let the number of the towers be ν, and denote by $e(x)$ the function which has towers in the same places and with the same breadth as the towers of $f(x)$ but with the height 1. In consequence of HÖLDER's inequality we have for every $\alpha \geqq P$

$$\frac{1}{L}\int_a^{a+L} |t(x)|^P\,dx = \frac{1}{L}\int_a^{a+L} |t(x)|^P e(x)\,dx \leqq$$

$$\left(\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha\,dx\right)^{\frac{P}{\alpha}} \left(\frac{1}{L}\int_a^{a+L} (e(x))^{\frac{1}{1-\frac{P}{\alpha}}}\,dx\right)^{1-\frac{P}{\alpha}} = \left(\frac{\nu}{L}b\right)^{1-\frac{P}{\alpha}} \left(\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha\,dx\right)^{\frac{P}{\alpha}}.$$

Hence

$$\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha\,dx \geqq \frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} \left(\frac{1}{L}\int_a^{a+L} |t(x)|^P\,dx\right)^{\frac{\alpha}{P}} =$$

$$\frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} \left[\sqrt[P]{\frac{1}{L}\int_a^{a+L} |t(x)|^P\,dx}\right]^\alpha \geqq$$

$$\frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} \left[\sqrt[P]{\frac{1}{L}\int_a^{a+L} (f(x))^P\,dx} - \sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x)+t(x)|^P\,dx}\right]^\alpha \geqq$$

$$\frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} \left[(1-k)\sqrt[P]{\frac{1}{L}\int_a^{a+L} (f(x))^P\,dx}\right]^\alpha = \frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} (1-k)^\alpha \left(\frac{\nu}{L}b h^P\right)^{\frac{\alpha}{P}} =$$

$$(1-k)^\alpha\left(\frac{\nu}{L}b\right)h^\alpha = (1-k)^\alpha \frac{1}{L}\int_a^{a+L} (f(x))^\alpha\,dx.$$

Lemma 2. *Let the function $f(x)$ be defined in a finite interval $a \leqq x \leqq a+L$ and let P be a number, $1 \leqq P < \infty$. Let further $A \geqq 0$ be given so that*

$$A < \sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x)|^P\,dx}.$$

Then there exists a constant $c > 0$ such that every function $t(x)$ which satisfies the inequality

$$\sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x) + t(x)|^P\,dx} \leqq A$$

also satisfies the inequality

$$\frac{1}{L}\int_a^{a+L} |t(x)|\,dx \geqq c$$

Proof. We determine N so large that

$$\sqrt[P]{\frac{1}{L}\int_a^{a+L} |(f(x))_N|^P\,dx} = A_1 > A.$$

Then

$$\frac{1}{L}\int_a^{a+L} |t(x)|\,dx \geqq \frac{1}{L}\int_a^{a+L} |(t(x))_N|\,dx \geqq$$

$$\frac{1}{N^{P-1}}\,\frac{1}{L}\int_a^{a+L} |(t(x))_N|^P\,dx = \frac{1}{N^{P-1}}\left\{\sqrt[P]{\frac{1}{L}\int_a^{a+L} |(t(x))_N|^P\,dx}\right\}^P \geqq$$

$$\frac{1}{N^{P-1}}\left\{\sqrt[P]{\frac{1}{L}\int_a^{a+L} |(f(x))_N|^P\,dx} - \sqrt[P]{\frac{1}{L}\int_a^{a+L} |(f(x))_N + (t(x))_N|^P\,dx}\right\}^P \geqq$$

$$\frac{1}{N^{P-1}}\left\{\sqrt[P]{\frac{1}{L}\int_a^{a+L} |(f(x))_N|^P\,dx} - \sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x) + t(x)|^P\,dx}\right\}^P \geqq$$

$$\frac{1}{N^{P-1}}\,\{A_1 - A\}^P = c > 0.$$

It may be observed that lemma 2 is of a somewhat other character than the two lemmas 1 a and 1 b. In fact the lower bound c indicated for $\frac{1}{L}\int_a^{a+L} |t(x)|\,dx$ in lemma 2 depends on the function $f(x)$; it is easily seen that there exists no form of lemma 2 corresponding to the lemmas 1 a and 1 b (where the indicated lower bounds are independent of $f(x)$).

Main Examples III a and III b.

Main Example III a.

In main example 3 we constructed *a function* $F(x)$ *which is an* S^p*-a. p. function for* $p < P$, *an* S^P*-function and such that the B-point around* $F(x)$ *does not contain* W^P*-a. p. functions.* Now we shall show *that* $F(x)$ *becomes* B^p*-a. p. for all* p, if the numbers $m_1, m_2, \ldots$ increase sufficiently rapidly to ∞. We remark, that it was in order to be able to obtain this property that already in main example 3 we chose to fill out just the central subintervals.

Let $1 \leqq P_1 < P_2 < \cdots \to \infty$ and let $\sum_1^\infty \delta_n$ be a convergent series of positive numbers. We put (cp. main examples I and II)

$$f_1(x) = F_1(x),\; f_2(x) = F_2(x) - F_1(x),\; f_3(x) = F_3(x) - F_2(x), \ldots .$$

The function $f_n(x)$ is periodic with the period h_n and consists in a period interval $\nu h_n \leqq x < (\nu + 1) h_n$ of the towers of type n which by the transition from $F_{n-1}(x)$ to $F_n(x)$ we filled into the central one of the subintervals $\mu h_{n-1} \leqq x < (\mu + 1) h_{n-1}$. We have calculated the number of these towers exactly, but here we need only observe that it is (of course) at most equal to the total number of subintervals $\eta \leqq x < \eta + 1$ in the mentioned central interval, viz. $\leqq h_{n-1}$. The P_n-integral of a tower of type n being denoted by I_n we have the estimation

$$\sqrt[P_n]{M\{(f_n(x))^{P_n}\}} \leqq \sqrt[P_n]{\frac{h_{n-1} I_n}{h_n}} = \sqrt[P_n]{\frac{I_n}{m_n}}.$$

Now we choose m_n so large that

$$\sqrt[P_n]{\frac{I_n}{m_n}} < \delta_n, \quad n = 1, 2, \ldots .$$

In particular we have

$$\sqrt[P_n]{M\{(f_n(x))^{P_n}\}} < \delta_n, \quad n = 1, 2, \ldots .$$

We shall prove that

$$D_{B^{P_n}}[F(x),\, F_n(x)] \to 0$$

for such a choice of $m_1, m_2, \ldots$, which involves (on account of $P_n \to \infty$, cp. main example II) that $D_{B^p}[F(x),\, F_n(x)] \to 0$ for every fixed p, so that $F(x)$ is B^p-a. p. for all p.

We have

$$D_{B^{P_n}}[F(x),\, F_n(x)] = \overline{\lim_{T\to\infty}} \sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F(x) - F_n(x))^{P_n}\, dx}.$$

Thus we shall estimate

$$\frac{1}{2T}\int_{-T}^{T} (F(x) - F_n(x))^{P_n}\, dx$$

for fixed n and large T, say $T \geqq h_n$ (cp. the main examples I and II). We choose first $q \geqq 0$ such that $h_{n+q} \leqq T < h_{n+q+1}$ and then ν among the numbers $1, 2, \ldots, m_{n+q+1} - 1$ such that $\nu h_{n+q} \leqq T < (\nu+1)\, h_{n+q}$. As $F(x) = F_{n+q+1}(x)$ for $-h_{n+q+1} \leqq x < h_{n+q+1}$, we have

$$\sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F(x) - F_n(x))^{P_n}\, dx} = \sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F_{n+q+1}(x) - F_n(x))^{P_n}\, dx} =$$

$$\sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F_{n+q}(x) - F_n(x) + f_{n+q+1}(x))^{P_n}\, dx} \leqq$$

$$\sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F_{n+q}(x) - F_n(x))^{P_n}\, dx} + \sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (f_{n+q+1}(x))^{P_n}\, dx};$$

and as $f_{n+q+1}(x) = 0$ for

$$-\frac{m_{n+q+1} - 1}{2}\, h_{n+q} \leqq x < \frac{m_{n+q+1} - 1}{2}\, h_{n+q},$$

we plainly have

$$\sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (f_{n+q+1}(x))^{P_n}\, dx} \leqq \sqrt[P_n]{\frac{1}{2\,\frac{m_{n+q+1} - 1}{2}\, h_{n+q}}\int_{-h_{n+q+1}}^{h_{n+q+1}} (f_{n+q+1}(x))^{P_n}\, dx},$$

and the above quantity is thus

$$\leqq \sqrt[p_n]{\frac{1}{2\nu h_{n+q}} \int\limits_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (F_{n+q}(x) - F_n(x))^{p_n}\, dx} +$$

$$\sqrt[p_n]{\frac{1}{2\,\dfrac{m_{n+q+1}-1}{2}\, h_{n+q}} \int\limits_{-h_{n+q+1}}^{h_{n+q+1}} (f_{n+q+1}(x))^{p_n}\, dx}.$$

As $(F_{n+q}(x) - F_n(x))^{p_n}$ is periodic with the period h_{n+q} and $(f_{n+q+1}(x))^{p_n}$ is periodic with the period h_{n+q+1}, the latter quantity is

$$= \sqrt[p_n]{\frac{\nu+1}{\nu} M\{(F_{n+q}(x) - F_n(x))^{p_n}\}} + \sqrt[p_n]{\frac{m_{n+q+1}}{\dfrac{m_{n+q+1}-1}{2}} M\{(f_{n+q+1}(x))^{p_n}\}} \leqq$$

$$\sqrt[p_n]{2\, M\{(F_{n+q}(x) - F_n(x))^{p_n}\}} + \sqrt[p_n]{4\, M\{(f_{n+q+1}(x))^{p_n}\}} \leqq$$

$$\sqrt[p_n]{4}\left(\sqrt[p_n]{M\{(F_{n+q}(x) - F_n(x))^{p_n}\}} + \sqrt[p_n]{M\{(f_{n+q+1}(x))^{p_n}\}}\right).$$

By an estimation of

$$\sqrt[p_n]{M\{(F_{n+q}(x) - F_n(x))^{p_n}\}}$$

in a way quite analogous to that on page 124 we see that the right-hand side is

$$\leqq \sqrt[p_n]{4}\left(\sqrt[p_{n+1}]{M\{(f_{n+1}(x))^{p_{n+1}}\}} + \sqrt[p_{n+2}]{M\{(f_{n+2}(x))^{p_{n+2}}\}} + \cdots + \sqrt[p_{n+q+1}]{M\{(f_{n+q+1}(x))^{p_{n+q+1}}\}}\right) \leqq$$

$$\sqrt[p_n]{4}\,(\delta_{n+1} + \delta_{n+2} + \cdots + \delta_{n+q+1}) \leqq \sqrt[p_n]{4}\,(\delta_{n+1} + \delta_{n+2} + \cdots).$$

Hence

$$\sqrt[p_n]{\frac{1}{2T} \int\limits_{-T}^{T} (F(x) - F_n(x))^{p_n}\, dx} \leqq \sqrt[p_n]{4}\,(\delta_{n+1} + \delta_{n+2} + \cdots) \quad \text{for} \quad T \geqq h_n.$$

It results for $T \to \infty$ that $D_B{}^{p_n}[F(x), F_n(x)] \leqq \sqrt[p_n]{4}\,(\delta_{n+1} + \delta_{n+2} + \cdots)$ so that $D_B{}^{p_n}[F(x), F_n(x)] \to 0$ for $n \to \infty$.

Main Example III b.

This main example is formed in a manner quite analogous to main example III a, but is somewhat simpler in so far as all the towers are taken to be congruent, viz. with the breadth 1 and the height 1. The function $F(x)$ thus constructed will have the same properties corresponding to $P = 1$ as the function $F(x)$ of main example III a (where P was > 1) — with exception of course of the function being S^p-a. p. for $p < P$ — and the proofs can directly be transferred. It may be remarked, however, that, on account of $F(x)$ here being bounded, in order to prove that $F(x)$ is B^p-a. p. for all p, we need only to show that $F(x)$ is B-a. p.; hence in the proof we may put $P_1 = P_2 = \cdots = 1$. In this way we get *a function $F(x)$ which is an S^p-function (even bounded) and a B^p-a. p. function for all p and such that the B-point around $F(x)$ does not contain any W-a. p. function.*

Main Example IV.

We construct *a function $F(x)$ which is W^p-a. p. for all p and such that the B-point around $F(x)$ does not contain any S-a. p. function.*

Throughout this main example by a »tower« we shall always understand a tower with the height 1 and the breadth 1. Let $m_1, m_2, \ldots$ be arbitrary integers $\geqq 2$. As usual, we put $h_1 = m_1$, $h_2 = m_1 m_2$, $h_3 = m_1 m_2 m_3, \ldots$, and construct (cp. main examples I and II) a sequence of functions $F_1(x), F_2(x), \ldots$ in the following way:

$F_1(x)$:	On all numbers	$\equiv 0 \pmod{h_1}$		a tower is placed.
$F_2(x)$:	On all numbers	$\equiv 0 \pmod{h_1}$	but $\not\equiv 0 \pmod{h_2}$	a tower is placed.
	» » »	$\equiv 0 \pmod{h_2}$		no » » » .
⋮				
$F_{2n}(x)$:	On all numbers	$\equiv 0 \pmod{h_1}$	but $\not\equiv 0 \pmod{h_2}$	a tower is placed.
	» » »	$\equiv 0 \pmod{h_2}$	» $\not\equiv 0 \pmod{h_3}$	no » » » .
	⋮			
	» » »	$\equiv 0 \pmod{h_{2n-1}}$	» $\not\equiv 0 \pmod{h_{2n}}$	a » » » .
	» » »	$\equiv 0 \pmod{h_{2n}}$		no » » » .
$F_{2n+1}(x)$:	On all numbers	$\equiv 0 \pmod{h_1}$	but $\not\equiv 0 \pmod{h_2}$	a tower is placed.
	» » »	$\equiv 0 \pmod{h_2}$	» $\not\equiv 0 \pmod{h_3}$	no » » » .
	⋮			
	» » »	$\equiv 0 \pmod{h_{2n}}$	» $\not\equiv 0 \pmod{h_{2n+1}}$	no » » » .
	» » »	$\equiv 0 \pmod{h_{2n+1}}$		a » » » .
⋮				

(see Fig. 10 which represents $F_3(x)$ for $m_1 = m_2 = m_3 = 2$).

Fig. 10.

Obviously $F_n(x)$ is a bounded periodic function with the period h_n.

We begin by proving that $F_n(x)$ is W-convergent to the following function:

$F(x)$: On all numbers	$\equiv 0 \pmod{h_1}$	but $\not\equiv 0 \pmod{h_2}$	a tower is placed.
» » »	$\equiv 0 \pmod{h_2}$	» $\not\equiv 0 \pmod{h_3}$	no » » » .
⋮			
» » »	$\equiv 0 \pmod{h_{2n-1}}$	» $\not\equiv 0 \pmod{h_{2n}}$	a » » » .
» » »	$\equiv 0 \pmod{h_{2n}}$	» $\not\equiv 0 \pmod{h_{2n+1}}$	no » » » .
⋮			

If we leave the interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$ out of account, obviously $F(x)$ can also be defined as $\lim_{n \to \infty} F_n(x)$ (cp. the main examples I and II). $F(x)$ is a bounded function and differs from $F_n(x)$ at most on the numbers $m \equiv 0 \pmod{h_{n+1}}$, and we have for each such m

$$\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x) - F_n(x)|\, dx \leqq 1,$$

viz. either 0 or 1. Hence

$$D_W[F(x), F_n(x)] \leqq \frac{1}{h_{n+1}}$$

which tends to 0 for $n \to \infty$, so that $F_n(x) \overset{W}{\to} F(x)$ for $n \to \infty$. Thus the function $F(x)$ is a W-limit periodic function, and $F(x)$ being bounded is therefore W^p-a. p. for all p.

Next we show that the B-point around $F(x)$ does not contain any S-a. p. function. Indirectly, we assume that $G(x)$ is such a function. Then we have

$$G(x) = F(x) + J(x)$$

where $J(x)$ is a B-zero function. Further, $F_n(x)$ being a sequence of periodic functions with the periods h_n which B-converge to $G(x)$, the period h_n is, in consequence of Theorem 1 a of Chapter I, for n sufficiently large, an S-translation number of $G(x)$ belonging to an arbitrary given $\varepsilon > 0$. We choose $\varepsilon = \frac{1}{2}$ and determine a fixed N so large that

$$(1) \qquad D_S[G(x+h_N), G(x)] \leqq \tfrac{1}{2}.$$

Let now m' denote numbers $\equiv 0 \pmod{h_N}$ but $\not\equiv 0 \pmod{h_{N+1}}$, and let m'' denote numbers $\equiv 0 \pmod{h_{N+1}}$ but $\not\equiv 0 \pmod{h_{N+2}}$. Either, in $F(x)$, there are towers on all numbers m' and none on the numbers m'', or conversely. By a translation h_N all m''-points are translated into certain of the m'-points, as

$$m'' + h_N \equiv 0 + h_N \not\equiv 0 \pmod{h_{N+1}}$$

and

$$m'' + h_N \equiv 0 + 0 = 0 \pmod{h_N}.$$

Hence

$$(2) \qquad \int_{m''-\frac{1}{2}}^{m''+\frac{1}{2}} |F(x+h_N) - F(x)|\, dx = 1$$

for all the numbers m''. In consequence of (1) we have in particular

$$(3) \qquad \int_{m''-\frac{1}{2}}^{m''+\frac{1}{2}} |G(x+h_N) - G(x)|\, dx \leqq \tfrac{1}{2}.$$

By (2) and (3) we get for the function $J(x) = G(x) - F(x)$

$$\int_{m''-\frac{1}{2}}^{m''+\frac{1}{2}} |J(x+h_N) - J(x)|\, dx \geqq \tfrac{1}{2}.$$

Hence

$$D_B[J(x+h_N), J(x)] = \overline{\lim_{T\to\infty}}\, \frac{1}{2T} \int_{-T}^{T} |J(x+h_N) - J(x)|\, dx \geqq \tfrac{1}{2}\, \frac{m_{N+2}-1}{h_{N+2}} > 0.$$

Consequently $D_B[J(x+h_N), J(x)] > 0$ which contradicts the fact that $J(x)$ is a B-zero function.

We observe that in this main example we were not forced to impose additional conditions on the sequence $m_1, m_2, \ldots$.

Main Examples V a, V b and VI.

In main example V a *a function* $F(x)$ *is constructed which is an* S^p*-a. p. function for* $p < P$, *not a* B^P*-function and such that the B-point around* $F(x)$ *contains a function* $G(x)$ *which is* B^p*-a. p. for all* p. Here P is an arbitrarily given number, $1 < P < \infty$.

In main example VI *a function* $F(x)$ *is constructed which is an* S^p*-a. p. function for* $p < P$, *an* S^P*-function, but no* B^P*-a. p. function and such that the B-point around* $F(x)$ *contains a function* $G(x)$ *which is* B^p*-a. p. for all* p. Here P is an arbitrarily given number, $1 < P < \infty$.

In main example V b *a function* $F(x)$ *is constructed which is an* S^P*-a. p. function, but not a* B^p*-function for any* $p > P$ *and such that the B-point around* $F(x)$ *contains a function* $G(x)$ *which is* B^p*-a. p. for all* p. Here P is an arbitrarily given number, $1 \leqq P < \infty$.

Main Examples V a and V b.

The two main examples V a and V b are constructed in an analogous way. In both cases we start from a positive function $t(x)$ defined for $-\frac{1}{2} \leqq x < \frac{1}{2}$ which is bounded in every interval $-\frac{1}{2} \leqq x \leqq \alpha < \frac{1}{2}$; in main example V a this function $t(x)$ is p-integrable for $p < P$ but not for $p = P$, while in main example V b the function $t(x)$ is P-integrable, but not p-integrable for $p > P$.

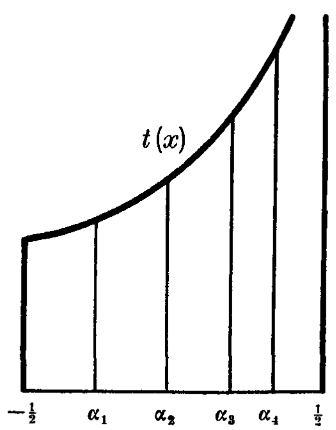

Fig. 11.

Let $-\frac{1}{2} < \alpha_1 < \alpha_2 < \cdots \to \frac{1}{2}$ (see Fig. 11). For $-\frac{1}{2} \leqq x < \frac{1}{2}$ we define

$$t_1(x) = \begin{cases} t(x) & \text{for } -\frac{1}{2} \leqq x < \alpha_1 \\ 0 & \text{elsewhere,} \end{cases}$$

$$t_2(x) = \begin{cases} t(x) & \text{for } \alpha_1 \leqq x < \alpha_2 \\ 0 & \text{elsewhere,} \end{cases}$$

$$t_3(x) = \begin{cases} t(x) & \text{for } \alpha_2 \leqq x < \alpha_3 \\ 0 & \text{elsewhere.} \end{cases}$$

$$\vdots$$

Let $m_1, m_2, \ldots$ be a sequence of integers $\geqq 2$ and let $h_1 = m_1$, $h_2 = m_1 m_2$, $h_3 = m_1 m_2 m_3, \ldots$. By $f_n(x)$ we denote the function arising from the function $t_n(x)$ by repeating it periodically with the period h_n, $n = 1, 2, \ldots$. We put (cp. main example I)

$$F_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

(see Fig. 12 where $m_1 = m_2 = m_3 = 2$ and $n = 3$).

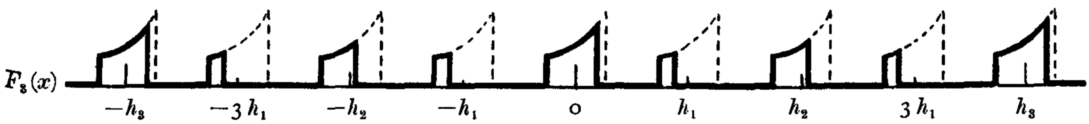

Fig. 12.

Further we put

$$F(x) = f_1(x) + f_2(x) + \cdots;$$

this last series is convergent for every x, since at most one of the terms is different from o for a given x. The function $F_n(x)$ is bounded and periodic with the period h_n.

It is easily seen that

$$F_n(x) \xrightarrow{S^p} F(x)$$

for $p<P$, respectively for $p=P$, so that $F(x)$ is S^p-a. p. for $p<P$, respectively for $p=P$; in fact for an arbitrary $\varepsilon > 0$ we have

$$D_{S^p}[f_n(x) + f_{n+1}(x) + \cdots] < \varepsilon$$

for $p<P$, respectively for $p=P$, when $n >$ some $N = N(\varepsilon, p)$, as for $n \to \infty$

$$\int_{a_n}^{\frac{1}{2}} (t(x))^p \, dx \to 0$$

for $p<P$, respectively $p=P$.

The function $F(x)$ is no B^p-function for $p=P$, respectively for $p>P$, since $F(x) = t(x)$ for $-\frac{1}{2} \leqq x < \frac{1}{2}$ and $t(x)$ is not p-integrable for $p=P$, respectively for $p>P$.

Finally we shall show that the B-point (even the W-point) around $F(x)$ contains a function $G(x)$ which is B^p-a. p. for all p, if we only let the numbers $m_1, m_2, \ldots$ increase sufficiently rapidly to ∞.

Let $1 \leqq P_1 < P_2 < \cdots \to \infty$ and let $\sum_1^\infty \delta_n$ be a convergent series of positive numbers. We choose the number m_1 so large that $\sqrt[P_1]{M\{(f_1(x))^{P_1}\}} < \delta_1$, the number m_2 so large that $\sqrt[P_2]{M\{(f_2(x))^{P_2}\}} < \delta_2, \ldots$.

Subtracting from $F(x)$ the W-zero function

$$j(x) = \begin{cases} t(x) & \text{for } -\frac{1}{2} \leqq x < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

we get a function $G(x) = F(x) - j(x)$ which will prove to be B^p-a. p. for all p.

Putting

$$j_n(x) = \begin{cases} t(x) & \text{for } -\frac{1}{2} \leqq x < \alpha_n \\ 0 & \text{elsewhere,} \end{cases}$$

and $G_n(x) = F_n(x) - j_n(x)$, we have

$$D_{B^{P_n}}[G(x), F_n(x)] = D_{B^{P_n}}[G(x), G_n(x)],$$

since $j_n(x)$ is a W^p-zero function for all p. Further

$$D_{B^{P_n}}[G(x), G_n(x)] = \overline{\lim_{T\to\infty}} \sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (G(x) - G_n(x))^{P_n}\, dx},$$

and we shall therefore estimate

$$\frac{1}{2T}\int_{-T}^{T} (G(x) - G_n(x))^{P_n}\, dx$$

for fixed n and large T, say $T \geqq h_n$, proceeding in a similar way as in the main examples I, II and III. We determine first $q \geqq 0$ so that $h_{n+q} \leqq T < h_{n+q+1}$ and then ν among the numbers $1, 2, \ldots, m_{n+q+1} - 1$ so that $\nu h_{n+q} \leqq T < (\nu+1) h_{n+q}$.

To begin with we distinguish between the two cases $\nu \leqq m_{n+q+1} - 2$ and $\nu = m_{n+q+1} - 1$.

In the first case we get

$$\frac{1}{2T}\int_{-T}^{T} (G(x) - G_n(x))^{P_n}\, dx \leqq \frac{1}{2\nu h_{n+q}} \int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (G(x) - G_n(x))^{P_n}\, dx =$$

$$\frac{1}{2\nu h_{n+q}} \int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (G_{n+q}(x) - G_n(x))^{P_n}\, dx,$$

as $G(x) = G_{n+q}(x)$ for

$$-(\nu + 1) h_{n+q} \leqq x < (\nu + 1) h_{n+q}, \quad \nu = 1, 2, \ldots, m_{n+q+1} - 2.$$

Here the right-hand side is

$$\leqq \frac{1}{2 \nu h_{n+q}} \int\limits_{-(\nu+1) h_{n+q}}^{(\nu+1) h_{n+q}} (F_{n+q}(x) - F_n(x))^{P_n} dx,$$

as

$$0 \leqq G_{n+q}(x) - G_n(x) = F_{n+q}(x) - F_n(x) - (j_{n+q}(x) - j_n(x)) \leqq F_{n+q}(x) - F_n(x),$$

and this quantity is

$$= \frac{\nu + 1}{\nu} M\{(F_{n+q}(x) - F_n(x))^{P_n}\},$$

as $(F_{n+q}(x) - F_n(x))^{P_n}$ is periodic with the period h_{n+q}.

In the other case we get

$$\frac{1}{2T} \int\limits_{-T}^{T} (G(x) - G_n(x))^{P_n} dx \leqq \frac{1}{2 \nu h_{n+q}} \int\limits_{-h_{n+q+1}}^{h_{n+q+1}} (G(x) - G_n(x))^{P_n} dx =$$

$$\frac{1}{2 \nu h_{n+q}} \int\limits_{-h_{n+q+1}}^{h_{n+q+1}} (G_{n+q+1}(x) - G_n(x))^{P_n} dx,$$

as $G(x) = G_{n+q+1}(x)$ for $-h_{n+q+1} \leqq x < h_{n+q+1}$. Here the right-hand side is

$$\leqq \frac{1}{2 \nu h_{n+q}} \int\limits_{-h_{n+q+1}}^{h_{n+q+1}} (F_{n+q+1}(x) - F_n(x))^{P_n} dx,$$

as $0 \leqq G_{n+q+1}(x) - G_n(x) \leqq F_{n+q+1}(x) - F_n(x)$, and this is further

$$= \frac{\nu + 1}{\nu} M\{(F_{n+q+1}(x) - F_n(x))^{P_n}\},$$

as $(F_{n+q+1}(x) - F_n(x))^{P_n}$ is periodic with the period h_{n+q+1}.

Estimating $M\{(F_{n+q}(x) - F_n(x))^{P_n}\}$ in the same manner as on page 124 we get

$$\sqrt[P_n]{M\{(F_{n+q}(x) - F_n(x))^{P_n}\}} \leqq \sqrt[P_{n+1}]{M\{(f_{n+1}(x))^{P_{n+1}}\}} + \sqrt[P_{n+2}]{M\{(f_{n+2}(x))^{P_{n+2}}\}} + \cdots$$

$$+ \sqrt[P_{n+q}]{M\{(f_{n+q}(x))^{P_{n+q}}\}} \leqq \delta_{n+1} + \delta_{n+2} + \cdots + \delta_{n+q} \leqq \delta_{n+1} + \delta_{n+2} + \cdots.$$

Thus we get in both cases, as $\frac{\nu+1}{\nu} \leqq 2$,

$$\sqrt[p_n]{\frac{1}{2T}\int_{-T}^{T}(G(x)-G_n(x))^{p_n}\,dx} \leqq \sqrt[p_n]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots)$$

for $T \geqq h_n$. Letting $T\to\infty$ we get

$$D_{B^{p_n}}[G(x),\, G_n(x)] \leqq \sqrt[p_n]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots).$$

Since $D_{B^{p_n}}[G(x), F_n(x)] = D_{B^{p_n}}[G(x), G_n(x)]$, it results that $D_{B^{p_n}}[G(x), F_n(x)]\to 0$ for $n\to\infty$. Hence

$$F_n(x) \xrightarrow{B^p} G(x) \text{ for all } p,$$

and consequently $G(x)$ is a B^p-a. p. function for all p.

Main Example VI.

This main example is constructed in a similar way as main example III a. As in that example we construct a sequence $F_1(x)$, $F_2(x)$, ... of bounded periodic functions with the periods $h_1 = m_1$, $h_2 = m_1 m_2$, ... and consider $F(x) = \lim_{n\to\infty} F_n(x)$. In main example III a we obtained $F_{n+1}(x)$ from $F_n(x)$ by filling out the central one of the subintervals $\eta h_n \leqq x < (\eta+1)h_n$ of every interval $\nu h_{n+1} \leqq x < (\nu+1)h_{n+1}$ by towers of type $n+1$, i. e. by towers with the 1-integral ε_{n+1} and the P-integral 1. Now, however, instead of filling out the central one of these subintervals we fill out the first, i. e. that farthest to the left (see Fig. 13 where $m_1 = m_2 = 3$ and $n = 2$).

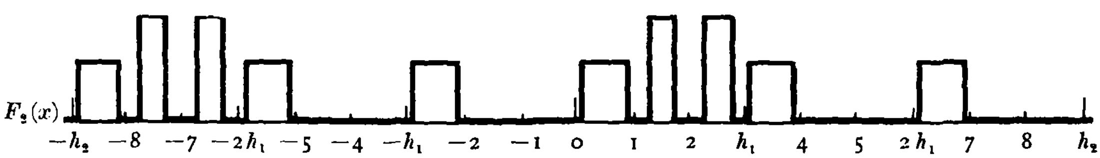

Fig. 13.

As in main example III a we denote the added function $F_{n+1}(x) - F_n(x)$ by $f_{n+1}(x)$ $(f_1(x) = F_1(x))$ and assume that

$$\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots$$

is convergent. Since this time we fill out the first interval instead of the central one, it does not hold that $F(x) = F_n(x)$ for $-h_n \leqq x < h_n$, but only that $F(x) = F_n(x)$ for $-h_n \leqq x < h_{n-1}$; but obviously it is still valid that $F_n(x) \xrightarrow{S} F(x)$ so that $F(x)$ is S-a. p., and that $F(x)$ is an S^p-function.

While $F(x)$ of main example III a is B^p-a. p. for all p for a suitable choice of $m_1, m_2, \ldots$ we shall now show that in the present case $F(x)$ is not B^p-a. p. We prove this by showing that

$$D^*_{B^p}\,[F(x), (F(x))_N] \geqq \frac{1}{2}\sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots} > 0$$

for every $N > 0$; this involves that $F(x)$ is not B^p-a. p., as otherwise (see Chapter I) $D^*_{B^p}[F(x), (F(x))_N] \to 0$ for $N \to \infty$.

Let, then, N be an arbitrary number > 0. The height k_n of a tower of type n being equal to $\left(\frac{1}{\varepsilon_n}\right)^{\frac{1}{p-1}}$ (see page 43) tends to ∞ for $n \to \infty$. We choose N_1 so large that $k_n \geqq 2N$ for $n \geqq N_1$. If $t(x)$ denotes a tower of type n for $n \geqq N_1$ standing on the interval $\eta \leqq x < \eta + 1$ we have

$$\sqrt[p]{\int_{\eta}^{\eta+1} (t(x) - (t(x))_N)^p\, dx} \geqq \tfrac{1}{2}.$$

We consider $F(x)$ in the interval $0 \leqq x < h_{n-1}$. In this interval $F(x) = F_n(x)$, and $F_n(x)$ contains $h_{n-1}\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_{n-1}}\right)$ towers of type n, namely all the towers of type n which were filled into the first of the subintervals $\nu h_{n-1} \leqq x < (\nu+1) h_{n-1}$ of the interval $0 \leqq x < h_n$ when passing from $F_{n-1}(x)$ to $F_n(x)$ (cp. page 86). Therefore for all $n \geqq N_1$ we have

$$\sqrt[p]{\frac{1}{h_{n-1}}\int_0^{h_{n-1}} (F(x) - (F(x))_N)^p\, dx} \geqq$$

$$\sqrt[p]{\frac{1}{h_{n-1}}\, h_{n-1}\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_{n-1}}\right)\left(\frac{1}{2}\right)^p} =$$

$$\frac{1}{2}\sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_{n-1}}\right)} \geqq \frac{1}{2}\sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots};$$

and thus

$$\overline{\lim_{T\to\infty}} \sqrt[P]{\frac{1}{T}\int_0^T (F(x)-(F(x))_N)^P\,dx} \geqq \frac{1}{2}\sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots}.$$

Hence

$$D^*_{B^P}[F(x),\ (F(x))_N] \geqq \frac{1}{2}\sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots}.$$

Finally we shall prove that, by letting $m_1, m_2, \ldots$ increase sufficiently rapidly to ∞, we can obtain, that the B-point (even the W-point) around $F(x)$ contains a function $G(x)$ which is B^p-a. p. for all p.

Let $1 \leqq P_1 < P_2 < \cdots \to \infty$ and let $\sum_1^\infty \delta_n$ be a convergent series of positive numbers. We denote the P_n-integral of a tower of type n by I_n and choose m_n, $n = 1, 2, \ldots$, so large that

$$\sqrt[P_n]{\frac{I_n}{m_n}} < \delta_n$$

and thus (see page 128)

$$\sqrt[P_n]{M\{(f_n(x))^{P_n}\}} < \delta_n.$$

Let

$$f_n^*(x) = \begin{cases} f_n(x) & \text{for } -h_n \leqq x < h_n \\ 0 & \text{elsewhere} \end{cases}, \quad n = 1, 2, \ldots.$$

Corresponding to $F(x) = f_1(x) + f_2(x) + \cdots$ we form the function

$$j(x) = f_1^*(x) + f_2^*(x) + \cdots$$

(the series is convergent, since for a given x at most one of the terms is $\neq 0$).

We shall prove that $j(x)$ is a W-zero function and that the difference $G(x) = F(x) - j(x)$ is B^p-a. p. for all p.

It is easily seen that $j(x)$ is a W-zero function; for outside the interval $-h_n \leqq x < h_n$ the towers of $j(x)$ are all of types $\geqq n+1$ and such towers have 1-integrals $\leqq \varepsilon_{n+1}$.

Next we prove that $G(x) = F(x) - j(x)$ is B^p-a. p. for all p by showing that $F_n(x) \xrightarrow{B^p} G(x)$ for all p. To this purpose, corresponding to

$$F_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) \quad \text{we put} \quad j_n(x) = f_1^*(x) + f_2^*(x) + \cdots + f_n^*(x)$$

and consider the function $G_n(x) = F_n(x) - j_n(x)$, $n = 1, 2, \ldots$. Obviously $j_n(x)$ is a W^p-zero function for all p and $G(x) = \lim_{n \to \infty} G_n(x)$. Further

$$G_{n+1}(x) = G_n(x) + f_{n+1}(x) - f_{n+1}^*(x),$$

since

$$G_{n+1}(x) = F_{n+1}(x) - j_{n+1}(x) = F_n(x) + f_{n+1}(x) - j_n(x) - f_{n+1}^*(x) = G_n(x) + f_{n+1}(x) - f_{n+1}^*(x).$$

Hence we have, on account of the definition of $f_{n+1}^*(x)$,

$$G_{n+1}(x) = G_n(x) \quad \text{for} \quad -h_{n+1} \leqq x < h_{n+1};$$

uccessively applying this equation, we get

$$G(x) = G_n(x) \quad \text{for} \quad -h_{n+1} \leqq x < h_{n+1}.$$

As $j_n(x)$ is a W^p-zero function (and hence a fortiori a B^p-zero function) for all p, we have

$$D_{B^{p_n}}[G(x), F_n(x)] = D_{B^{p_n}}[(G(x), G_n(x)] = \overline{\lim_{T \to \infty}} \sqrt[p_n]{\frac{1}{2T} \int_{-T}^{T} (G(x) - G_n(x))^{p_n} \, dx}.$$

Thus we shall estimate

$$\frac{1}{2T} \int_{-T}^{T} (G(x) - G_n(x))^{p_n} \, dx$$

for fixed n and large T, say $T \geqq h_n$ (cp. the main examples I, II, III, V). First $q \geqq 0$ is determined so that $h_{n+q} \leqq T < h_{n+q+1}$, and next ν among the numbers $1, 2, \ldots, m_{n+q+1} - 1$ so that $\nu h_{n+q} \leqq T < (\nu + 1) h_{n+q}$. Then we have

$$\frac{1}{2T} \int_{-T}^{T} (G(x) - G_n(x))^{p_n} \, dx \leqq \frac{1}{2 \nu h_{n+q}} \int_{-(\nu+1) h_{n+q}}^{(\nu+1) h_{n+q}} (G(x) - G_n(x))^{p_n} \, dx =$$

$$\frac{1}{2 \nu h_{n+q}} \int_{-(\nu+1) h_{n+q}}^{(\nu+1) h_{n+q}} (G_{n+q}(x) - G_n(x))^{p_n} \, dx,$$

since

$$G(x) = G_{n+q}(x) \quad \text{for} \quad -h_{n+q+1} \leqq x < h_{n+q+1}.$$

As $0 \leqq G_{n+q}(x) - G_n(x) \leqq F_{n+q}(x) - F_n(x)$, the right-hand side is

$$\leqq \frac{1}{2\nu h_{n+q}} \int\limits_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (F_{n+q}(x) - F_n(x))^{P_n}\, dx,$$

and this is

$$= \frac{\nu+1}{\nu} M\{(F_{n+q}(x) - F_n(x))^{P_n}\},$$

as $(F_{n+q}(x) - F_n(x))^{P_n}$ is periodic with the period h_{n+q}. Further, in consequence of the estimation on page 124 the last quantity is

$$\leqq 2\left(\sqrt[P_{n+1}]{M\{(f_{n+1}(x))^{P_{n+1}}\}} + \sqrt[P_{n+2}]{M\{(f_{n+2}(x))^{P_{n+2}}\}} + \cdots + \sqrt[P_{n+q}]{M\{(f_{n+q}(x))^{P_{n+q}}\}}\right)^{P_n} \leqq$$

$$2\,(\delta_{n+1} + \delta_{n+2} + \cdots + \delta_{n+q})^{P_n} \leqq 2\,(\delta_{n+1} + \delta_{n+2} + \cdots)^{P_n}.$$

Hence for $T \geqq h_n$ we have

$$\sqrt[P_n]{\frac{1}{2T}\int\limits_{-T}^{T} (G(x) - G_n(x))^{P_n}\, dx} \leqq \sqrt[P_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots);$$

letting $T \to \infty$, we get

$$D_{B^{P_n}}[G(x), G_n(x)] \leqq \sqrt[P_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots).$$

Since

$$D_{B^{P_n}}[G(x), F_n(x)] = D_{B^{P_n}}[G(x), G_n(x)],$$

we conclude that $D_{B^{P_n}}[G(x), F_n(x)] \to 0$ for $n \to \infty$ and consequently that $F_n(x) \xrightarrow{B^p} G(x)$ for all p.

Main Examples VII a, VII b and VII c.

The number α, $1 < \alpha < \infty$, being arbitrarily given, in all the three main examples we construct *a function* $F(x)$ *which is an* S^p*-a. p. function for* $p<\alpha$, *an* S^α*-function and such that the B-point around* $F(x)$ *contains a function* $G(x)$ *which is* B^p*-a. p. for all* p.

In main example VII a the number P being arbitrarily given such that $\alpha < P < \infty$, and in the main examples VII b and VII c the number P being arbitrarily given such that $\alpha \leqq P < \infty$, in the different examples the function $F(x)$ has further the following properties.

In main example VII a: $F(x)$ *is* B^p*-a. p. for* $p<P$, *and the* W^α*-point around* $F(x)$ *contains no* B^P*-functions.*

In main example VII b: $F(x)$ *is a* B^P*-function, and the* W^α*-point around* $F(x)$ *contains no* B^P*-a. p. functions.* In the case $P=\alpha$ it results already from the above that $F(x)$ is a B^α-function; in fact it is even an S^α-function.

In main example VII c: $F(x)$ *is a* B^P*-a. p. function, and the* W^α*-point around* $F(x)$ *contains no* B^p*-functions for* $p > P$.

We remark that in the later paper, where the examples of the appendix will be used, we shall see that, on account of general theorems, the B-points around the functions $F(x)$ of the three main examples cannot contain W^α-a. p. functions.

The main examples VII a, VII b and VII c are constructed in an analogous way, and they are of a similar type as the main examples III a and VI. Just as in these examples, we construct a sequence $F_1(x)$, $F_2(x)$, ... of bounded periodic functions with the periods $h_1 = m_1$, $h_2 = m_1 m_2$, ... where

$$\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots$$

is convergent and consider $F(x) = \lim_{n\to\infty} F_n(x)$. In main example III a, respectively VI, we passed from $F_n(x)$ to $F_{n+1}(x)$ by filling out the central, respectively the first, of the subintervals $\mu h_n \leqq x < (\mu+1) h_n$ of every interval $\nu h_{n+1} \leqq x < (\nu+1) h_{n+1}$ by towers of type $n+1$, i. e. towers with the 1-integral ε_{n+1} and the P-integral 1. In the present construction, however, α takes the place of P, so that a tower of type n means a tower with the 1-integral ε_n and the α-integral 1. Further, by the transition from $F_n(x)$ to $F_{n+1}(x)$ we do not fill out just the central or the first of the subintervals $\mu h_n \leqq x < (\mu + 1) h_n$ by towers of type $n+1$, but another of the subintervals, later precisely indicated. The subintervals to be filled out shall of course as usual lie periodically with the period h_{n+1}. As we shall see, by a suitable choice of these intervals, we can obtain that $F(x)$ gets the desired »B-properties«. Let the subinterval $\nu h_n \leqq x < (\nu + 1) h_n$ of the interval $0 \leqq x < h_{n+1}$ which is filled out at the transition from $F_n(x)$ to $F_{n+1}(x)$ be denoted by $\nu_{n+1} h_n \leqq x < (\nu_{n+1} + 1) h_n$. For the sake of convenience we shall choose

$$\nu_{n+1} < \frac{m_{n+1}}{2}$$

so that the interval $\nu_{n+1} h_n \leqq x < (\nu_{n+1} + 1) h_n$ is that (or eventually one of the two) of the subintervals filled out at the mentioned transition which lies nearest to 0.

It is plain that $F_n(x) \xrightarrow{S} F(x)$ for $n \to \infty$, as the 1-integral ε_n of a tower of type n tends to 0 for $n \to \infty$, and thus the function $F(x)$ is an S-a. p. function. Further all towers of $F(x)$ having the α-integral 1, the function $F(x)$ is an S^α-function.

We introduce similar notions as in main example VI. We put $f_1(x) = F_1(x)$, $f_2(x) = F_2(x) - F_1(x)$, $f_3(x) = F_3(x) - F_2(x), \ldots$, so that $F_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$ and $F(x) = f_1(x) + f_2(x) + \cdots$. Further we put

$$f_n^*(x) = \begin{cases} f_n(x) & \text{for } -h_n \leqq x < h_n \\ 0 & \text{elsewhere,} \end{cases}$$

$$j_n(x) = f_1^*(x) + f_2^*(x) + \cdots + f_n^*(x), \quad j(x) = f_1^*(x) + f_2^*(x) + \cdots,$$

$$G_n(x) = F_n(x) - j_n(x), \quad G(x) = F(x) - j(x).$$

For $h_{n+q} \leqq T < h_{n+q+1}$ we get the estimation (cp. the analogous estimation on pages 129—130)

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T} (F(x) - F_n(x))^p \, dx} \leqq \sqrt[p]{2}\left(\sqrt[p]{M\{(f_{n+1}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}}\right) + \sqrt[p]{\frac{1}{2\,\nu_{n+q+1} h_{n+q}} \int_{-h_{n+q+1}}^{h_{n+q+1}} (f_{n+q+1}^*(x))^p \, dx}$$

$\left(\text{if } \nu_{n+q+1} = 0 \text{ we put } \frac{1}{0} = \infty\right)$, or, introducing the notion

$$A_p(n) = \sqrt[p]{\frac{1}{2\,\nu_n h_{n-1}} \int_{-h_n}^{h_n} (f_n^*(x))^p\, dx},$$

the estimation

$$(1) \quad \sqrt[p]{\frac{1}{2\,T} \int_{-T}^{T} (F(x) - F_n(x))^p\, dx} \leqq$$
$$\sqrt[p]{2}\left(\sqrt[p]{M\{(f_{n+1}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}}\right) + A_p(n+q+1).$$

Further we have (cp. page 137)

$$(2) \quad \sqrt[p]{\frac{1}{2\,T} \int_{-T}^{T} (G(x) - G_n(x))^p\, dx} \leqq \sqrt[p]{2}\left(\sqrt[p]{M\{(f_{n+1}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}}\right).$$

Let $1 \leqq P_1 < P_2 < \cdots \to \infty$ and let $\sum_1^\infty \delta_n$ be a convergent series of positive numbers. We let $m_1, m_2, \ldots$ increase so strongly that (cp. page 128)

$$(3) \qquad \sqrt[P_n]{M\{(f_n(x))^{P_n}\}} < \delta_n.$$

For $h_{n+q} \leqq T < h_{n+q+1}$ we get from (2) and (3)

$$\sqrt[P_n]{\frac{1}{2\,T} \int_{-T}^{T} (G(x) - G_n(x))^{P_n}\, dx} \leqq \sqrt[P_n]{2}\left(\sqrt[P_n]{M\{(f_{n+1}(x))^{P_n}\}} + \cdots + \sqrt[P_n]{M\{(f_{n+q}(x))^{P_n}\}}\right) \leqq$$
$$\sqrt[P_n]{2}\left(\sqrt[P_{n+1}]{M\{(f_{n+1}(x))^{P_{n+1}}\}} + \cdots + \sqrt[P_{n+q}]{M\{(f_{n+q}(x))^{P_{n+q}}\}}\right) \leqq$$
$$\sqrt[P_n]{2}\,(\delta_{n+1} + \cdots + \delta_{n+q}) \leqq \sqrt[P_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots).$$

Hence

$$D_B^{P_n}[G(x), G_n(x)] \leqq \sqrt[P_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots),$$

and, $j_n(x)$ being a W^p-zero function for all p, we have

$$D_{B^{P_n}}[G(x), F_n(x)] = D_{B^{P_n}}[G(x), G_n(x)] \leqq \sqrt[P_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots)$$

so that $D_{B^{P_n}}[G(x), F_n(x)] \to 0$ for $n \to \infty$. Thus $G(x)$ is a B^p-a. p. function for all p. Since $j(x)$ is a W-zero function (cp. main example VI), $G(x)$ lies in the W-point around $F(x)$ and in particular in the B-point around $F(x)$.

Having discussed the common properties of the main examples VII a, VII b and VII c we now pass to consider these examples separately, as regards their mutual differences.

Main Example VII a.

We wish to choose the numbers $\nu_1, \nu_2, \ldots$ defined above so that $F(x)$ becomes B^p-a. p. for $p < P$ and so that the W^α-point around $F(x)$ does not contain any B^P-function.

We shall first show that we can choose $\nu_1, \nu_2, \ldots$ so that

$$A_p(n) \to 0 \quad \text{for} \quad p < P \quad \text{and} \quad n \to \infty$$

and

$$A_P(n) \to \infty \quad \text{for} \quad n \to \infty;$$

later we shall show that $F(x)$ then gets the desired properties.

A necessary condition for $A_P(n) \to \infty$ is that $\frac{\nu_n}{m_n} \to 0$. For, as

$$A_P(n) = \sqrt[P]{\frac{1}{2\,\nu_n h_{n-1}} \int_{-h_n}^{h_n} (f_n^*(x))^P\,dx} =$$

$$\sqrt[P]{\frac{m_n}{\nu_n} \cdot \frac{1}{2\,h_n} \int_{-h_n}^{h_n} (f_n^*(x))^P\,dx} = \sqrt[P]{\frac{m_n}{\nu_n}} \cdot \sqrt[P]{M\{(f_n(x))^P\}},$$

the relation $A_P(n) \to \infty$ obviously involves the relation $\frac{m_n}{\nu_n} \to \infty$, since, on account of $\sqrt[P_n]{M\{(f_n(x))^{P_n}\}} \to 0$, we have $\sqrt[P]{M\{(f_n(x))^P\}} \to 0$.

As the p-integral of a tower of type n is equal to $\varepsilon_n^{\frac{\alpha-p}{\alpha-1}}$ we have

$$A_p(n)=\sqrt[p]{\frac{1}{\nu_n}d_{n-1}\,\varepsilon_n^{\frac{\alpha-p}{\alpha-1}}}=\sqrt[p]{\frac{1}{\nu_n}d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{p-\alpha}{\alpha-1}}}$$

where d_{n-1} has the same meaning as in main example 3, i. e. indicates the relative density of the empty intervals $\eta \leqq x < \eta + 1$ in the function $F_{n-1}(x)$. Thus, denoting $d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{p-\alpha}{\alpha-1}}$ by $B_p(n)$, we have

$$A_p(n)=\sqrt[p]{\frac{1}{\nu_n}B_p(n)}.$$

Obviously we can choose a sequence of numbers $(\alpha <)p_1 < p_2 < \cdots \to P$ which converges so slowly to P that

$$\frac{B_P(n)}{B_{p_n}(n)}=\frac{d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}}{d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{p_n-\alpha}{\alpha-1}}}=\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-p_n}{\alpha-1}}\to\infty \quad \text{for} \quad n\to\infty.$$

We shall show that as our ν_n we may, from a certain step N (to be indicated below), use

$$\nu_n=\left[\frac{1}{\log\frac{B_P(n)}{B_{p_n}(n)}}B_P(n)\right]$$

(where $[x]$ denotes the greatest integer $\leqq x$). In fact we shall show that, choosing ν_n in this manner, $A_p(n)\to 0$ for $p<P$, $A_P(n)\to\infty$ and (for N sufficiently large) $\nu_n < \frac{m_n}{2}$.

We start by observing that $\nu_n\to\infty$ (and thus especially $\nu_n \geqq 1$ for n sufficiently large). This results from

$$\nu_n=\left[\frac{1}{\log\frac{B_P(n)}{B_{p_n}(n)}}B_P(n)\right]=\left[\frac{1}{\frac{P-p_n}{\alpha-1}\log\frac{1}{\varepsilon_n}}d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}\right]=\left[\frac{1}{\frac{P-p_n}{\alpha-1}}d_{n-1}\frac{\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}}{\log\left(\frac{1}{\varepsilon_n}\right)}\right]$$

where

$$\frac{P-p_n}{\alpha-1}\to 0, \qquad d_{n-1}\to\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots>0$$

and

$$\frac{\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}}{\log\frac{1}{\varepsilon_n}}\to\infty.$$

In particular

$$\nu_n\sim\frac{1}{\log\frac{B_P(n)}{B_{p_n}(n)}}B_P(n) \quad \text{for} \quad n\to\infty.$$

Then we have

$$A_{p_n}(n)=\sqrt[p_n]{\frac{1}{\nu_n}B_{p_n}(n)}\sim\sqrt[p_n]{\log\frac{B_P(n)}{B_{p_n}(n)}\cdot\frac{B_{p_n}(n)}{B_P(n)}}\to 0,$$

since

$$\frac{B_P(n)}{B_{p_n}(n)}\to\infty,$$

so that $A_{p_n}(n)\to 0$. Now, for a fixed $p<P$ and n being chosen so large that $p_n>p$ (and $\nu_n\geqq 1$), we have

$$A_p(n)\leqq A_{p_n}(n);$$

in fact, as

$$A_p(n)=\sqrt[p]{\frac{1}{2\,\nu_n\,h_{n-1}}\int_{-h_n}^{h_n}(f_n^*(x))^p\,dx},$$

the inequality $A_p(n)\leqq A_{p_n}(n)$ follows from Hölder's inequality, since $f_n^*(x)$ is different from 0 at most on intervals with total length $\leqq 2h_{n-1}$ and a fortiori with total length $\leqq 2\nu_n h_{n-1}$. Thus, as $A_{p_n}(n)\to 0$, we have

$$A_p(n)\to 0 \quad \text{for} \quad p<P.$$

Further we have

$$A_P(n)=\sqrt[P]{\frac{1}{\nu_n}B_P(n)}\sim\sqrt[P]{\log\frac{B_P(n)}{B_{p_n}(n)}}\to\infty$$

so that

$$A_P(n)\to\infty.$$

As mentioned above this involves $\frac{\nu_n}{m_n} \to 0$, and therefore we can determine our N so large that the last claim $\nu_n < \frac{m_n}{2}$ for $n \geqq N$ is satisfied. For $n < N$ we choose the ν_n arbitrarily so that merely $\nu_n < \frac{m_n}{2}$.

We shall now show that by this choice of the numbers ν_n the function $F(x)$ gets the desired properties.

First we shall see that $F(x)$ is B^p-a. p. for $p < P$. This results from (1) and (3), since for $h_{n+q} \leqq T < h_{n+q+1}$ and n so large that $P_{n+1} > p$ we have

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T}(F(x)-F_n(x))^p\,dx} \leqq$$

$$\sqrt[p]{2}\left(\sqrt[p]{M\{(f_{n+1}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}}\right) + A_p(n+q+1) \leqq$$

$$\sqrt[p]{2}\left(\sqrt[P_{n+1}]{M\{(f_{n+1}(x))^{P_{n+1}}\}} + \cdots + \sqrt[P_{n+q}]{M\{(f_{n+q}(x))^{P_{n+q}}\}}\right) + A_p(n+q+1) \leqq$$

$$\sqrt[p]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots + \delta_{n+q}) + A_p(n+q+1),$$

and hence, letting $q \to \infty$,

$$D_{B^p}[F(x), F_n(x)] \leqq \sqrt[p]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots).$$

Thus $D_{B^p}[F(x), F_n(x)] \to 0$ for $n \to \infty$, and consequently $F(x)$ is a B^p-a. p. function for $p < P$.

Next we show that the W^α-point around $F(x)$ does not contain any B^P-function. Proceeding indirectly, we suppose that there is a W^α-zero function $J(x)$ so that $D^*_{B^P}[F(x) + J(x)] < \infty$. Denoting by $J^*(x)$ the function which is equal to $J(x)$ where $j(x) \neq 0$ and equal to 0 elsewhere, we have

$$D^*_{B^P}[j(x) + J^*(x)] \leqq D^*_{B^P}[F(x) + J(x)] < \infty,$$

since $F(x) = j(x)$ where $j(x) \neq 0$. Let $D^*_{B^P}[j(x) + J^*(x)] = K$. For $n \geqq$ some sufficiently large N_1 we have

$$\sqrt[P]{\frac{1}{(\nu_n+1)h_{n-1}}\int_0^{(\nu_n+1)h_{n-1}} |j(x) + J^*(x)|^P\,dx} \leqq 2K;$$

in particular we have, denoting by $J_n^*(x)$ the function which is equal to $J^*(x)$ where $f_n^*(x) \neq 0$ and equal to 0 elsewhere,

$$\sqrt[p]{\frac{1}{(\nu_n+1)h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} |f_n^*(x) + J_n^*(x)|^p \, dx} \leqq 2K.$$

For $n \to \infty$ we have also

$$\sqrt[p]{\frac{1}{(\nu_n+1)h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} (f_n^*(x))^p \, dx} =$$

$$\sqrt[p]{\frac{1}{2(\nu_n+1)h_{n-1}} \int_{-h_n}^{h_n} (f_n^*(x))^p \, dx} \sim \sqrt[p]{\frac{1}{2\nu_n h_{n-1}} \int_{-h_n}^{h_n} (f_n^*(x))^p \, dx} = A_p(n) \to \infty$$

so that for $n \geqq$ some sufficiently large N_2

$$\sqrt[p]{\frac{1}{(\nu_n+1)h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} (f_n^*(x))^p \, dx} \geqq 4K.$$

Thus for $n \geqq \max(N_1, N_2)$

$$\sqrt[p]{\frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} |f_n^*(x) + J_n^*(x)|^p \, dx} \leqq \frac{1}{2} \sqrt[p]{\frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} (f_n^*(x))^p \, dx}.$$

Hence, by help of the lemma on page 116, we get

$$\frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} |J_n^*(x)|^\alpha \, dx \geqq \left(\frac{1}{2}\right)^p \frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} (f_n^*(x))^\alpha dx = \left(\frac{1}{2}\right)^p d_{n-1} \geqq \left(\frac{1}{2}\right)^p \left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right) \cdots$$

and therefore further

$$\frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} |J(x)|^\alpha \, dx \geqq \left(\frac{1}{2}\right)^p \left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right) \cdots.$$

This inequality, holding for every sufficiently large n, contradicts the fact that $J(x)$ is a W^α-zero function.

Main Example VII b.

Here we wish to choose the numbers $\nu_1, \nu_2, \ldots$ so that $F(x)$ becomes a B^P-function and so that the W^α-point around $F(x)$ does not contain any B^P-a. p. function.

We begin by proving that $\nu_1, \nu_2, \ldots$ can be chosen so that

$$A_P(n) \to k \quad \text{for} \quad n \to \infty$$

where k is a constant > 0. In a similar way as in main example VII a it is seen that a necessary condition for $A_P(n) \to k$ is that $\frac{\nu_n}{m_n} \to 0$. For all n from a certain step N (which will be indicated below) we put

$$\nu_n = \left[\frac{B_P(n)}{d_{n-1}}\right] = \left[\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}\right].$$

We observe immediately that $\nu_n \geqq 1$. If $P = \alpha$, we have $B_P(n) = d_{n-1}$, $\nu_n = 1$, and hence

$$A_P(n) = \sqrt[P]{\frac{1}{\nu_n} B_P(n)} = \sqrt[P]{d_{n-1}} \to \sqrt[P]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots}.$$

If $P > \alpha$, we have

$$\frac{B_P(n)}{d_{n-1}} = \left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}} \to \infty$$

so that

$$\nu_n \sim \frac{B_P(n)}{d_{n-1}}$$

and therefore

$$A_P(n) = \sqrt[P]{\frac{1}{\nu_n} B_P(n)} \sim \sqrt[P]{d_{n-1}} \to \sqrt[P]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots}.$$

As mentioned we have then $\frac{\nu_n}{m_n} \to 0$ and therefore our above N can be determined so that the claim $\nu_n < \frac{m_n}{2}$ is satisfied for $n \geqq N$. For $n < N$ the numbers ν_n are chosen arbitrarily so that merely $\nu_n < \frac{m_n}{2}$ is satisfied.

We shall show that by this choice of the numbers ν_n the function $F(x)$ gets the desired properties.

Firstly $F(x)$ is a B^P-function. In fact for $h_{n+q} \leqq T < h_{n+q+1}$ and sufficiently large n we have

$$\sqrt[P]{\frac{1}{2T}\int_{-T}^{T}(F(x)-F_n(x))^P\,dx} \leqq$$

$$\sqrt[P]{2}\left(\sqrt[P]{M\{(f_{n+1}(x))^P\}}+\cdots+\sqrt[P]{M\{(f_{n+q}(x))^P\,dx\}}\right)+A_P(n+q+1) \leqq$$

$$\sqrt[P]{2}\left(\sqrt[P_{n+1}]{M\{(f_{n+1}(x))^{P_{n+1}}\}}+\cdots+\sqrt[P_{n+q}]{M\{(f_{n+q}(x))^{P_{n+q}}\}}\right)+A_P(n+q+1) \leqq$$

$$\sqrt[P]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots+\delta_{n+q})+2k<\sqrt[P]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots)+2k.$$

Letting $q \to \infty$, we get

$$D_{B^P}[F(x), F_n(x)] \leqq \sqrt[P]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots)+2k$$

and thus

$$D_{B^P}[F(x)] \leqq D_{B^P}[F_n(x)]+\sqrt[P]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots)+2k.$$

Secondly we shall show that the W^α-point around $F(x)$ does not contain any B^P-a. p. function. Proceeding indirectly we assume that there exists a W^α-zero function $J(x)$ so that $F(x)+J(x)$ is a B^P-a. p. function or, which is equivalent (as $F(x)$ is B-a. p.), that

$$(F(x)+J(x))_N \xrightarrow{B^P} F(x)+J(x).$$

Denoting by $J^*(x)$ the function which is equal to $J(x)$ in the points where $j(x) \neq 0$ and equal to 0 elsewhere, we have

$$(j(x)+J^*(x))_N \xrightarrow{B^P} j(x)+J^*(x),$$

since $F(x)=j(x)$ in the points where $j(x) \neq 0$. $J(x)$ being a W^α-zero function, $J^*(x)$ is also a W^α-zero function. Then $j(x)+J^*(x)$ is a W-zero function, in particular a B-zero function. Consequently $(j(x)+J^*(x))_N$ is a B^p-zero function for all p and especially a B^P-zero function. As the B^P-point around 0, considered as a set of functions, is B^P-closed, the function $j(x)+J^*(x)$ is also a B^P-zero function. Consequently we have

$$\sqrt[P]{\frac{1}{(\nu_n+1)\,h_{n-1}}\int_0^{(\nu_n+1)\,h_{n-1}}|j(x)+J^*(x)|^P\,dx} \leqq \frac{k}{8} \quad \text{for} \quad n \geqq \text{some } N_1,$$

in particular

$$\sqrt[P]{\frac{1}{(\nu_n+1)\,h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} |f_n^*(x)+J_n^*(x)|^P\,dx} \leqq \frac{k}{8} \quad \text{for} \quad n \geqq N_1,$$

where $J_n^*(x)$ denotes the function which is equal to $J(x)$ in the points where $f_n^*(x) \neq 0$ and 0 elsewhere. Further, for sufficiently large n,

$$\sqrt[P]{\frac{1}{(\nu_n+1)\,h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} (f_n^*(x))^P\,dx} = \sqrt[P]{\frac{1}{2\,(\nu_n+1)\,h_{n-1}}\int\limits_{-h_n}^{h_n} (f_n^*(x))^P\,dx} =$$

$$\sqrt[P]{\frac{\nu_n}{\nu_n+1}}\,\sqrt[P]{\frac{1}{2\,\nu_n h_{n-1}}\int\limits_{-h_n}^{h_n} (f_n^*(x))^P\,dx} = \sqrt[P]{\frac{\nu_n}{\nu_n+1}}\,A_P(n) \geqq \frac{k}{4}$$

and thus

$$\sqrt[P]{\frac{1}{(\nu_n+1)\,h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} (f_n^*(x))^P\,dx} \geqq \frac{k}{4} \quad \text{for} \quad n \geqq \text{some } N_2.$$

Hence for $n \geqq \max\,(N_1, N_2)$

$$\sqrt[P]{\frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} |f_n^*(x)+J_n^*(x)|^P\,dx} \leqq \frac{1}{2}\sqrt[P]{\frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} (f_n^*(x))^P\,dx}.$$

By help of the lemma of page 116 we get

$$\frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} |J_n^*(x)|^\alpha\,dx \geqq \left(\frac{1}{2}\right)^P \frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} (f_n^*(x))^\alpha\,dx \geqq \left(\frac{1}{2}\right)^P \left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots.$$

Hence, for sufficiently large n,

$$\frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} |J(x)|^\alpha\,dx \geqq \left(\frac{1}{2}\right)^P \left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots.$$

which contradicts the fact that $J(x)$ is a W^α-zero function.

Main Example VII c.

Finally, in this main example, we wish to choose the numbers $\nu_1, \nu_2, \ldots$ such that $F(x)$ becomes B^P-a. p. and so that the W^α-point around $F(x)$ does not contain B^p-functions for $p > P$.

We shall show that we can determine $\nu_1, \nu_2, \ldots$ such that

$$A_P(n) \to 0 \quad \text{for} \quad n \to \infty$$

and

$$A_p(n) \to \infty \quad \text{for} \quad p > P \quad \text{and} \quad n \to \infty.$$

A necessary condition for this last relation is that $\frac{\nu_n}{m_n} \to 0$. We have

$$A_p(n) = \sqrt[p]{\frac{1}{\nu_n} d_{n-1} \left(\frac{1}{\varepsilon_n}\right)^{\frac{p-\alpha}{\alpha-1}}} = \sqrt[p]{\frac{1}{\nu_n} B_p(n)}.$$

First we choose $p_1 > p_2 > \cdots \to P$ converging so slowly to P that

$$\frac{B_{p_n}(n)}{B_P(n)} = \left(\frac{1}{\varepsilon_n}\right)^{\frac{p_n - P}{\alpha-1}} \to \infty.$$

Then from a certain step N (which will be indicated below) we put

$$\nu_n = \left[\frac{1}{\log \frac{B_{p_n}(n)}{B_P(n)}} B_{p_n}(n)\right].$$

Then on the one hand, as $\nu_n \to \infty$ and therefore

$$\nu_n \sim \frac{1}{\log \frac{B_{p_n}(n)}{B_P(n)}} B_{p_n}(n),$$

we have

$$A_P(n) = \sqrt[P]{\frac{1}{\nu_n} B_P(n)} \sim \sqrt[P]{\log \frac{B_{p_n}(n)}{B_P(n)} \cdot \frac{B_P(n)}{B_{p_n}(n)}} \to 0,$$

while on the other hand

$$A_{p_n}(n) = \sqrt[p_n]{\frac{1}{\nu_n} B_{p_n}(n)} \sim \sqrt[p_n]{\log \frac{B_{p_n}(n)}{B_P(n)}} \to \infty,$$

which involves that $A_p(n) \to \infty$ for $p > P$. From this it follows as mentioned that $\frac{\nu_n}{m_n} \to 0$ for $n \to \infty$, and we can therefore determine our above N so that the claim $\nu_n < \frac{m_n}{2}$ for $n \geqq N$ is satisfied. For $n < N$ we choose ν_n arbitrarily so that merely $\nu_n < \frac{m_n}{2}$.

Now it results (in a way quite analogous to main example VII a) that $F(x)$ is B^P-a. p. and that the W^α-point around $F(x)$ does not contain B^p-functions for $p > P$.