DEFORMATION CLASSES OF MEROMORPHIC FUNCTIONS AND THEIR EXTENSIONS TO INTERIOR TRANSFORMATIONS.

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§ 1. *Introduction.* The first objective of this paper is to use topological methods and concepts to enlarge the store of knowledge of meromorphic functions. Deformation classes of meromorphic functions are defined. The extension to interior transformations results in new homotopy theorems and contrasts between interior and conformal maps. Earlier papers have shown that there is a considerable body of theorems which can be formulated so as to retain meaning and validity after arbitrary homeomorphisms of the z- or w-spheres. Many theorems involving the relations between zeros, poles, and branch point antecedents and the images under f of boundaries are of this character. See Morse and Heins, (1) and Morse (2).

The second objective is to distinguish between the properties of meromorphic functions which are shared by interior transformations and those which are not. For the transformations from $\{|z| < 1\}$ to the w-sphere which are considered we find no difference[1] between meromorphic functions and interior transformations with respect to the invariants necessary to characterize a deformation class of functions with prescribed zeros, poles, and branch point antecedents. However, sequences $[f_k(z)]$ of meromorphic functions properly taken from different deformation classes cover the w-plane in a manner suggestive of the Picard theorem on essential singularities but with no counterpart for sequences of interior transformations. The discovery of such properties points to the problem of finding the non-topological assumptions which must be imposed upon interior transformations in order that they may share the non-topological properties discovered.

[1] For domains for z other than the disc differences may arise.

Topologically we are concerned with the homotopy classes of open simply-connected Riemann surfaces with prescribed properties, conformally with the existence of deformations of prescribed type through meromorphic functions when it is known that such deformations are possible through interior transformations. We seek to construct models of all deformation classes by »composing» homeomorphisms of $\{|z| < 1\}$ with an interior transformation with the prescribed zeros, poles, and branch point antecedents. Conformally this is possible only in trivial cases.

Interior transformations are used essentially in the sense of Stoilow. (4) With Whyburn (3) they are »interior» and »light». Whyburn has studied the underlying point set characteristics of these transformations in a very general setting. The uniformization theorems of Stoilow are found useful. See also, v. Kerékjártó, (5) pp. 173—184.

The following section gives a summary of the principal results without details or proofs. A detailed exposition follows.

§ 2. *The problem and the principal results.* In the sense in which we shall use the term an *interior transformation* $w = f(z)$ will be a sense-preserving continuous map of the open disc $S\{|z| < 1\}$ into the complex w-sphere with the following characteristic property. If z_0 is any point on S there exists a sense-preserving homeomorphism $\varphi(z)$ from a neighborhood N of z_0 to another neighborhood N_1 of z_0 with z_0 fixed, such that the function $f[\varphi(z)] = F(z)$ is analytic on N except at most for a pole at z_0, and is not identically constant. The trans, formation f is said to have a zero or pole at z_0 if F has a zero or pole at z_0- more precisely f is said to have a *zero* or *pole* of the *order* of the zero or pole of F at z_0.

If z_0 is the antecedent of a branch point of the r-th order of the inverse of F, z_0 is said to be an antecedent of a *branch point* of the r-th order of the inverse of f. In any case it is seen that $r + 1$ is the number of times a neighborhood of $w_0 = f(z_0)$ is covered by $f(z)$ for $|z - z_0|$ sufficiently small.

We shall restrict ourselves to the case in which f has a finite set of zeros

$$a_0, a_1, \ldots, a_r \qquad (r > 0), \tag{2.1}$$

and poles

$$a_{r+1}, \ldots, a_n \qquad (n > 1)^1, \tag{2.2}$$

[1] In all cases $m = n + 1$ shall denote the total number of zeros and poles. The cases $m = 1$ and $m = 0$ are treated in § 14.

with branch point antecedents

$$b_1, \ldots, b_\mu \qquad (\mu \geqq 0), \tag{2.3}$$

with $n > 1$ until § 14 is reached. In the principal study we shall assume that the zeros, poles, and branch points are *simple*, that is, have the order 1. The zeros, poles, and branch point antecedents will form a set of points

$$(\alpha) = (a_0, \ldots, a_n, b_1, \ldots, b_\mu)$$

termed *characteristic*. Two points in the set (α) both of which are zeros, or poles, or branch point antecedents are said to be of *like character*. Any reordering of the points of (α) in which points of like character are reordered among themselves will be termed *admissible*. Except in the case $\mu = 0$, $m = 2$ we shall use no reordering in which a_0 is changed in relative order.

Admissible f-deformations. We shall admit deformations D of f of the form

$$w = F(z, t) \qquad (|z| < 1) \qquad (0 \leqq t \leqq 1)$$

in which t is the deformation parameter and

$$F(z, 0) \equiv f(z). \qquad (|z| < 1)$$

We require that F map (z, t) continuously into the w-sphere and reduce to an interior transformation for each fixed t. Let (α^t) be the characteristic set of F at the time t. We require that the points of (α^t) vary continuously with t, remain simple, distinct and constant in number and character as t varies from 0 to 1, return respectively, when $t = 1$, to some one but not necessarily the same one of the characteristic points of $f(z)$ of like character. The terminal transformation $F(z, 1)$ has the same characteristic set as f with a possible reordering.

If (α^t) is independent of t the deformation is termed *restricted*. If $(\alpha^0) = (\alpha^1)$, the deformation is termed *terminally restricted*. If (α^1) is an admissible reordering of (α^0), D is termed *semi-restricted*. We let X denote any one of these three types of admissible deformations. Interior transformations which admit a deformation into each other of type X will be said to belong to the *same X-deformation class* and to be *X-equivalent*.

The invariants J_i. We shall define a set (J) of n numbers $J_i(f, \alpha)$ $(i = 1, \ldots, n)$ associated respectively with the respective pairs (a_0, a_i). The J_i's are invariant under any restricted deformation of f. The set (J) is thus associated with an ordered set (α) in which the first zero a_0 plays a special role. If F is a transformation with the ordered set (α), then all sets (J) belonging to transformations f with the same ordered set (α) have the form

$$J_i(f, \alpha) = J_i(F, \alpha) + r_i \qquad (i = 1, \ldots, n) \tag{2.4}$$

where r_i is an arbitrary integer. All such sets are realizable.

Topological models with prescribed sets (α) *and* (J). One begins by exhibiting an interior transformation F with the given ordered set (α). Then (2.4) defines the ensemble of sets (J) which are »associated» with (α). Any one of these sets (J) associated with (α) belongs to a transformation f obtainable from F in a simple way.

To obtain these models use is made of sense-preserving homeomorphisms $\eta(z)$ of the disc $S = \{|z| < 1\}$ onto itself. We term η *restricted* if (α) is point-wise invariant under η, *semi-restricted* if points of (α) are transformed into points of (α) of like character. Set $n + 1 = m$. Except[1] when $m = 2$ and $\mu = 0$, one can prescribe the ordered set (α) and any one of the associated sets (J) and affirm the existence of a semi-restricted homeomorphism η, dependent on (α) and (J), such that the function[2] $F\eta$ has (α) as an ordered characteristic set and (J) as its set of invariants. The case $m = 2$, $\mu = 0$ is exceptional.

Meromorphic models. The preceding models f are not in general meromorphic. Nevertheless there exists an explicit formula for a meromorphic function f with a prescribed ordered characteristic set (α) and associated invariants (J).

Restricted deformation classes C. A necessary and sufficient condition that two transformations f_1 and f_2 with the same ordered set (α) be in the same class C is that f_1 and f_2 possess the same invariants (J) with respect to (α). If f_1 and f_2 are meromorphic, the restricted deformation of f_1 into f_2 which is affirmed to exist can be made through meromorphic functions. This is equally true of the deformation classes C' and C'' to which we now refer.

Terminally restricted deformation classes C'. A new invariant is needed here. Two transformations f with the same ordered set (α) will be said to belong to the same *category* if the sums

$$J_1 + \cdots + J_n$$

for the two functions are equal mod. 2. There are three principal cases (assuming $m > 1$ until § 14).

[1] We are excluding the case $m < 2$ until § 14 in order to avoid complexity of statement. When $m < 2$ there are no invariants (J).

[2] We write $F(\eta(z)) = F\eta$.

I $\mu > 0$ or m odd.

II $\mu = 0$, $m = 4, 6, 8, \ldots$

III $\mu = 0$, $m = 2$.

In Case I any two functions f_1 and f_2 with the same set (α) are in the same class C'. In Case II, f_1 and f_2 are in the same class C' if and only if they are of the same category. In Case III f_1 and f_2 are in the same class C' if and only if they have the same invariant J_1 with respect to (α).

Semi-restricted deformation classes C''. If $m > 2$ or $\mu > 0$ there is but one class C'' with the given set (α). The case $\mu = 0$ and $m = 2$ is again exceptional.

Other exceptional cases. The cases in which $m < 2$, are excluded until § 14. We have required that $r > 0$ in (2. 1), thus excluding the possibility that f have poles but no zeros. This exceptional case is readily brought under the preceding by replacing f by its reciprocal. The case where there are no poles is admitted.

Cases in which the characteristic points are not simple can be treated essentially as in the simple case provided the deformations which are admitted are required to preserve the multiplicity of the respective characteristic points. If one permits the multiplicities of the characteristic points to change by virtue of various types of coalescence, an interesting theory of degeneracy arises analogous to the theory which describes the degeneracy of elliptic functions into trigonometric functions. It is planned to return to this problem in a later paper.

Topological and conformal differences. A first difference has already been indicated. Except when $m = 2$ and $\mu = 0$ models of all restricted deformation classes with a given set (α) can be obtained as functions F by composing a particular model F_0 with characteristic set (α) with suitably chosen semi-restricted homeomorphisms η of S. This is impossible if one operates only with meromorphic functions.

A second difference appears in the extent to which the w-sphere is covered by an infinite sequence $[f_k]$ of transformations with a prescribed set (α) and at most one representative from each restricted deformation class. We term $[f_k]$ a *model sequence.* Let R be any connected region of the w-sphere which contains $w = 0$ and $w = \infty$ and whose closure does not cover the w-sphere. If the models $[f_k]$ are not required to be meromorphic a model sequence $[f_k]$ can be defined so as to cover no points on the complement of R and with no subsequence converging continuously on S to 0 or ∞. This is impossible if the functions f_k are meromorphic.

To give a more revealing statement in the meromorphic case let a point z_0 of S be termed a *covering point* of $[f_k]$ if corresponding to any arbitrary neighborhood N of z_0 the set of images of N under $[f_k]$ covers the finite w-plane ($w = 0$ excepted) infinitely many times. Let H be any connected neighborhood on S of the points

$$(a_0, a_1, \ldots, a_n) = (a)$$

with the points (a) excluded. Any model sequence $[f_k]$ of meromorphic functions no subsequence of which »converges continuously» to 0 or to ∞ on H possesses at least one covering point on H.

This theorem follows readily from a theorem of Julia on normal families (see Montel (6), p. 37) once the appropriate properties of meromorphic models have been derived. Other theorems concerning the covering points z_0 of $[f_k]$ are obtained.

We begin the detailed study with the development of homotopy properties of locally simple arcs k and introduce invariants $d(k)$ designed to make possible a topological definition of the invariants J_i.

Part I. The Topological Theory.

§ 3. *The difference order $d(k)$ of a locally simple arc k.* The object of this section is to attach a number $d(k)$ to a locally simple, sensed arc k with prescribed end points in such a manner that $d(k)$ remains invariant under a class of deformations of k (to be defined) and characterizes k as representative of its deformation class.

Local simplicity. See Morse and Heins (1) I. The arcs k which are admitted are »represented» by continuous and locally 1—1 images

$$w(t) = u(t) + iv(t) \qquad (0 \leqq t \leqq t_0)$$

of a line interval $(0, t_0)$ and shall intersect their end points $w(0)$ and $w(t_0)$ only when $t = 0$ and t_0 respectively. The condition of local simplicity implies that there exists a constant $e > 0$ such that any subarc of k whose diameter is less than e is simple; such a constant e is called a *norm of local simplicity of k*. Any set of locally simple curves which admit the same norm of local simplicity will be termed *uniformly* locally simple.

Strictly speaking, $w(t)$ is a *representation* of k and not identical with k. We admit any other representation of k obtained by mapping the interval $(0 \leqq t \leqq t_0)$ homeomorphically onto another interval $(0, t_1)$. The arc k can be identified with a class of such representations. The property of local simplicity of an arc k is independent of the representation of k which is used, because the existence of a norm of local simplicity is independent of the representation used.

Let the end points of k be $w = a$ and $w = b$ respectively.

Admissible deformations of k. We shall admit deformations D of k with the following properties. The arcs of D shall be represented in a locally $1-1$ way in the form

$$w = H(t, \lambda) \qquad (0 \leqq t \leqq t_0) \qquad (0 \leqq \lambda \leqq \lambda_0) \tag{3.1}$$

where H maps the (t, λ) rectangle continuously into the finite w-plane with

$$H(0, \lambda) \equiv a \qquad H(t_0, \lambda) \equiv b$$

The arc k shall be represented by $H(t, 0)$. The arc k is »deformed» at the »time» λ into the arc k^λ represented by $H(t, \lambda)$. *The arcs k^λ shall be uniformly locally simple and shall intersect a and b only as end points.*

If the requirement of uniform local simplicity were replaced by the requirement that the arcs k^λ be separately locally simple, there would be but one deformation class of arcs with the end points a and b, instead of the countably infinite set of deformation classes which actually exists.[1]

Any two arcs k^λ appearing in the above deformation are said to be in the *same deformation class.* Actually the deformation is one of »representations», but it is immediately obvious that any two representations of the same arc can be admissibly deformed into each other. See (1) I p. 603. Accordingly the property of two arcs of being in the same deformation class is independent of their representations.

To avoid misunderstanding, the following should be pointed out. The possibility that $a = b$ is admitted. Let k_a and k_b be proper simple subarcs of k with a the initial point of k_a and b the terminal point of k_b. The arcs k_a and k_b can intersect in infinitely many points, even coincide. Fig. 1 shows four examples of arcs k in the same deformation class in the case $a = b$. The figures must of course be superimposed so that the point $w = a$ coincides with $w = b$ in all four cases. The lower right curve is designed to indicate the possibility of a curve

[1] Cf. Lemma 3. 3.

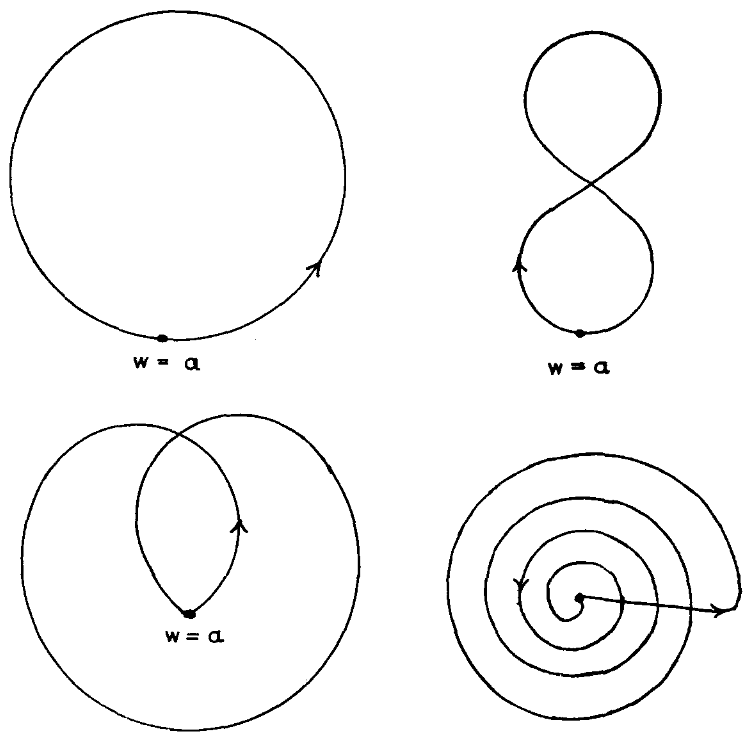

Figure 1.

in the given deformation class with a simple spiral terminal arc. Another figure could be drawn indicating an arc in the same deformation class with spiral like simple subarcs at both ends. Reversed in sense the four arcs in Fig. 1 belong to a second and different deformation class.

The regular case $a \neq b$. The arc k is termed *regular* if it admits a representation $w(t)$ in which $w'(t)$ exists, is continuous and is never zero. For many purposes it will be convenient to measure angles in *rotational units*, that is, in units which equal 2π radians. The algebraic increment in

$$\frac{1}{2\pi} \arg w'(t) \qquad (0 \leqq t \leqq t_0) \tag{3.2}$$

as t increases from 0 to t_0 and $\arg w'(t)$ varies continuously will be denoted by $P(k)$. The angle P is in rotational units and represents the total angular variation of the tangent to k as the point of tangency $w(t)$ traverses k. Let c be either end point of k. The limiting algebraic increment of

$$\frac{1}{2\pi} \arg [w(t) - c] \qquad [0 < t < t_0]$$

as t increases from 0 to t_0 and the argument varies continuously will be denoted by $Q_c(k)$.

In the regular case with $a \neq b$ *we tentatively define* $d(k)$ *by the equation*

$$d(k) = P(k) - Q_a(k) - Q_b(k) \qquad (a \neq b) \tag{3.3}$$

and show that $d(k)$ *is an integer.*

The number $d(k)$ equals the rotational units in the total angular variation of a unit vector X which varies continuously as follows from

$$\frac{b-a}{|b-a|} \tag{3.4}$$

back to the same vector. Let X start with the vector (3.4) and coincide with

$$\frac{w(t)-a}{|w(t)-a|} \qquad (t_0 \geqq t > 0) \tag{3.5}$$

as t decreases from t_0 to 0, excluding 0. Let X then coincide with

$$\frac{w'(t)}{|w'(t)|} \qquad (0 \leqq t \leqq t_0) \tag{3.6}$$

as t increases from 0 to t_0. Finally let X coincide with

$$\frac{b-w(t)}{|b-w(t)|} \qquad (t_0 > t \geqq 0) \tag{3.7}$$

as t decreases from t_0 to 0. The initial and final vectors coincide with (3.4). The algebraic increments of $\arg\left(\frac{X}{2\pi}\right)$ corresponding to the variations associated with (3.5), (3.6) and (3.7) respectively are

$$-Q_a, \quad P, \quad -Q_b,$$

so that the resultant algebraic increment in angle is given by (3.3). The variation in X is continuous so that the statement in italics follows.

The general case $a \neq b$. The arc k is no longer assumed regular. A subarc of k will be defined by an interval (σ, τ) for t. If this subarc is simple the vector

$$w(\tau) - w(\sigma) \tag{3.8}$$

has a well defined direction. This direction will vary continuously with σ and τ so long as the arc (σ, τ) remains simple and $\sigma < \tau$. We term such a variation an *admissible chord variation*. In such a variation we suppose that the angle

$$\frac{1}{2\pi} \arg [w(\tau) - w(\sigma)] \tag{3.9}$$

has been chosen so as to vary continuously with σ and τ. For an admissible chord variation the algebraic increment of (3.9) depends only on the initial and final simple subarcs.

Let k_a and k_b be respectively proper simple subarcs of k of which the initial point of k_a is a and the terminal point of k_b is b. Let the chord subtending k_a vary admissibly into the chord subtending k_b. Let

$$P(k, k_a, k_b)$$

represent the resulting algebraic increment of (3.9). Let the algebraic increment of

$$\frac{1}{2\pi} \arg [w(t) - a] \tag{3.10}$$

as t increases from its terminal value t_a on k_a to t_0 be denoted by $Q_a(k, k_a)$ and let the algebraic increment of

$$\frac{1}{2\pi} \arg [w(t) - b] \tag{3.11}$$

as t increases from 0 to its initial value t_b on k_b be denoted by $Q_b(k, k_b)$.

The difference order $d(k)$ when $a \neq b$ is defined by the equation

$$d(k) = P(k, k_a, k_b) - Q_a(k, k_a) - Q_b(k, k_b) \qquad (a \neq b) \tag{3.12}$$

We state the lemma.

Lemma 3.1. *The difference order $d(k)$ as defined by (3.12), where $a \neq b$, is an integer which is independent of the choice of k_a and k_b among proper simple subarcs of k with end points as prescribed.*

The proof that $d(k)$ is an integer is essentially the same as in the regular case. As before, $d(k)$ measures the angular variation of a unit vector which initially and terminally coincides with (3.4) and may be broken up into the variations which define $-Q_a$, P, and $-Q_b$ respectively in (3.12).

It should be noted that any one of these three component angular variations may be arbitrarily large in numerical value and increase without limit as k_a or k_b tend to a or b respectively. This would certainly happen if k had spiral like terminal simple subarcs. We have the following lemma.

Lemma 3.2. *If k is regular*

$$d(k) = P - Q_a - Q_b \qquad (a \neq b) \tag{3.13}$$

consistent with the earlier definition (3.3).

The following lemma is also immediately obvious.

Lemma 3.3. *The difference order $d(k)$ $(a \neq b)$ is independent of any admissible deformation of k. There are accordingly at least as many deformation classes as there are different values of $d(k)$ for admissible arcs joining a to b.*

For any integer m a »model» arc K_m joining a to b and with $d(k) = m$ can be defined as follows. For $n = 0$ take K_0 as the straight line segment from a to b. Let c be the mid point of K_0 and C be a unit circle tangent to K_0 at c and to the left of K_0. To define K_n for $n > 0$ trace K_0 from a to c, then trace C n times in the positive sense, and finally trace K_0 from c to b. Take K_{-n} as the reflection of K_n in K_0. That $d(K_m) = m$ follows from (3.13). Since $d(k)$ is invariant under admissible deformations no two of the models K_m are in the same deformation class. It can be shown[1] that each admissible curve k which joins a to b and for which $d(k) = m$ is admissibly deformable into K_m.

The case $b = \infty$. We suppose that a is finite as previously, and make the obvious extensions of previous definitions as follows.

The arc k may be given in the w-plane. By convention it joins $w = a$ to $w = \infty$ if its closure $\bar{k}$ on the w-sphere joins $w = a$ to $w = \infty$. It is termed locally simple if $\bar{k}$ is locally simple on the w-sphere. Admissible deformations are defined as previously but on the w-sphere. If k is locally simple there exists a value τ with $0 < \tau < t_0$ such that the subarc $\tau \leqq t < t_0$ is simple in the finite w-plane. On this arc $|w(t)|$ becomes infinite as t tends to t_0.

To define $d(k)$ let k^s represent the subarc $(0, s)$ of k, with $0 < s < t_0$. Then $d(k^s)$ is well defined. It is clearly independent of s provided $s > \tau$. We accordingly set

$$d(k) = d(k^s) \qquad (\tau < s < t_0).$$

That $d(k)$ is invariant under admissible deformations follows as previously.

The arc k is termed *regular* if its closure $\bar{k}$ on the w-sphere is regular; one then has the lemma.

[1] We make no use of this theorem and accordingly omit the proof.

Lemma 3.4. *In case $\bar{k}$ is regular on the w-sphere, a is finite and $b=\infty$, then*

$$d(k) = P(k) - Q_a(k). \tag{3.14}$$

In case $\bar{k}$ is regular on the w-sphere k has a definite *asymptote* in the finite w-plane as t tends to t_0. The subarc k^s of k is regular, and if $w(t)$ represents k,

$$d(k^s) = P(k^s) - Q_a(k^s) - Q_{w(s)}(k^s).$$

On letting s tend to t_0 the last term tends to zero and the first two terms tend to the corresponding terms in (3.14). Thus Lemma 3.4 holds as stated.

The case $a=b$. We here suppose that a is finite. The values

$$Q_a(k, k_a) \qquad Q_b(k, k_b)$$

are undefined since they involve the null vector $b-a$. Moreover, the value of $d(k)$ which would be obtained by taking an appropriate limit as a tends to b is not the invariant which is useful. Instead we proceed as follows.

In the regular case one sets

$$d(k) = P(k) - Q_a(k). \qquad (a=b) \tag{3.15}$$

In the general case let

$$Q_a(k, k_a, k_b)$$

equal the algebraic increment of

$$\frac{1}{2\pi} \arg [w(t) - a] \tag{3.16}$$

as t varies monotonically from its terminal value t_a on k_a to its initial value t_b on k_b. In the general case one sets

$$d(k) = P(k, k_a, k_b) - Q_a(k, k_a, k_b). \qquad (a=b) \tag{3.17}$$

The first relevant facts are as follows:

Lemma 3.5. *The value of $d(k)$ when $a=b$ equals $\frac{1}{2}$ mod 1. It is independent of the choice of k_a and k_b among proper simple subarcs of k with end points as prescribed, and is invariant under admissible deformations of k.*

The angular variation which defines $d(k)$ is that of a vector Y which starts with $w(t_a)-a$ and ends with this vector reversed in sense. In fact it is sufficient if Y first varies admissibly as a chord from its initial position to the vector $b-w(t_b)$ subtending k_b; this variation gives P in (3.17). One continues with a variation of

$$b - w(t)$$

in which t decreases from t_b to t_a and in which the angular variation is measured by Q_a in (3.17). But the terminal vector Y coincides with the initial vector Y reversed in sense so that $d(k)$, measured in rotational units, equals $\frac{1}{2}$ mod 1.

The independence of $d(k)$ of the choice of k_a and k_b is clear as is its invariance under admissible deformations.

We note the following.

Lemma 3.6. *If k is regular and if $a = b$ (finite), or if a is finite and $b = \infty$, then*

$$d(k) = P(k) - Q_a(k). \tag{3.18}$$

Models for the different deformation classes when $b = \infty$ are obtainable from those for which $b = b_1$ (b_1 finite) by making a directly conformal transformation of the w-sphere which leaves a fixed and carries b_1 to ∞.

In case $a = b$ (finite) the values of $d(k)$ are found to be

$$\ldots, \frac{-3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$$

A positively sensed circle C through a has a difference order $\frac{1}{2}$. Reversing sense always changes the sign of the difference order. To obtain a curve K_r with a difference order $n = \frac{2r+1}{2}$, and with $r > 0$, one attaches a small positively sensed circle C_1 to C within C, and tangent to C at some point c other than $a = b$; one then traces C until c is reached, then traces C_1 r times in the positive sense, and continues to b on C. To obtain a curve with difference order $-n$ one can reverse the sense of K_n, or more symmetrically reflect K_n in the tangent to C at a.

It can be shown that these models represent all possible deformation classes of admissible curves when $a = b$. We shall not use this fact.

§ 4. *Three deformation lemmas.* Let h be a simple arc joining two different points z_1 and z_2 in the finite z-plane. »Admissible» deformations of such an arc keep z_1 and z_2 fixed. If in addition only simple arcs are employed, the deformation will be termed an *isotopic* deformation of h. A first lemma is as follows.

Lemma 4.1. *Any simple arc h, joining z_1 and z_2 ($z_1 \neq z_2$) in the finite z-plane, can be isotopically deformed on an arbitrary neighborhood N of h into a simple, regular, analytic arc joining z_1 to z_2.*

We begin by proving the following:

(a) *The arc h can be isotopically deformed on N into a simple arc h^* on which sufficiently restricted terminal subarcs are straight.*

For simplicity take $z_1 = 0$. Choose e so that $0 < 2e < |z_2|$. Let C be the circle $|z| = e$ and E the open disc $\{|z| < e\}$. Let h_1 be a maximal connected subarc of h on E with $z = 0$ as an initial point. Let h_2 be a simple open arc of which h_1 is a subarc, which lies on E and whose closure is an arc joining diametrically opposite points of C. There exists a sense-preserving homeomorphism T of $\bar{E}$ which leaves $z = 0$ and C pointwise fixed, and maps $\bar{h}_2$ onto a diameter of $\bar{E}$. It follows from a theorem of Tietze (7) that T may be generated by an isotopic deformation Δ of $\bar{E}$ from the identity leaving C and $z = 0$ pointwise fixed.

If e is sufficiently small, Δ will deform $h \vee \bar{E}$ only on N and yield an image of h with a straight initial subarc. Such a deformation of h will be isotopic. Neighboring z_2, h can be similarly deformed, so that (a) follows.

To complete the proof of the lemma, let g be a closed Jordan curve of which the arc h^* in (a) is a subarc, and which is analytic and regular neighboring z_1 and z_2. Let R be the Jordan region bounded by g. Let $\bar{R}$ be mapped homeomorphically onto a circular disc, conformally at points of R. On the disc the pencil of circles through the images of z_1 and z_2 will have antecedents on $\bar{R}$ which will suffice to deform h^* isotopically on N into a simple, regular, analytic arc joining z_1 to z_2.

This completes the proof of the lemma.

In deriving deformation theorems for h when h joins z_1 to z_2 on S and $z_1 \neq z_2$, no generality is lost if it is supposed that a is real and positive and that, with $a < 1$,

$$z_1 = -a \qquad z_2 = a \qquad\qquad (a \neq 0).$$

For S can be mapped conformally onto itself so that z_1 and z_2 go into $-a$ and a respectively for a suitable value of a.

A non-singular transformation of coordinates. Let the z-plane be referred to polar coordinates (r, θ). For the above a and for each constant b on the interval $0 \leqq b < 1$ a transformation from polar coordinates (ϱ, φ) to (r, θ) will be defined by the equations

$$(4.1) \qquad r = \varrho\left(\frac{1 - a^2 b^2}{1 - \varrho^2 b^2}\right), \quad \theta = \varphi \qquad (0 \leqq \varrho^2 b^2 < 1).$$

For $b = 0$, (4. 1) reduces to the identity. For $b \neq 0$ the interval $0 \leqq r < \infty$ and the interval

$$0 \leqq \varrho < \frac{1}{b} \tag{4.2}$$

correspond in a $1-1$ manner. The circle $\varrho = a$ corresponds to the circle $r = a$. Subject to (4. 2) the rectangular transformation of coordinates defined by (4. 1) is $1-1$, analytic and non-singular. The circle $\varrho = 1$ corresponds to the circle for which

$$r = \frac{1 - a^2 b^2}{1 - b^2}. \tag{4.3}$$

We shall make use of the transformation (4. 1) and prove the following lemma.

Lemma 4. 2. *A simple, regular, analytic arc h joining $-a$ to a on S can be isotopically deformed on S through regular analytic arcs h^t into a circular arc joining $-a$ to a on S under a deformation in which the point with length parameter s on h corresponds to a point*

$$z = z(s, t) \qquad \begin{Bmatrix} 0 \leqq t \leqq 1 \\ 0 \leqq s \leqq s_0 \end{Bmatrix} \tag{4.4}$$

at the time t, where $z(s, t)$ is analytic in the real variables (s, t), and $z_s \neq 0$.

To define the deformation (4. 4) let g be a Jordan curve on S which includes h as a subarc and which is analytic and regular neighboring a and $-a$. Let R be the Jordan region on S bounded by g. Let C be a circle on R and let R_1 be the subregion of R exterior to C. The region R_1 can be mapped $1-1$ and conformally on an annulus A in such a manner that the mapping can be extended $1-1$ and conformally over C and h. Suppose that the image k of h is on the outer circular boundary of A. We shall deform k through concentric circular arcs k_t, $0 \leqq t \leqq 1$, on A into a prescribed arc k_1 on the inner circular boundary of A. Such a deformation is readily defined in terms of the polar coordinates of the annulus so as to have a regular analytic representation in terms of t and the arc length on k. The antecedent h_t on R_1 of k_t on A will deform $h = h_0$ into a subarc h_1 of C.

During the deformation h_t of h the end points of h_t move. To remedy this defect let h_t be carried by a linear transformation $az + d$ (unique) into an arc H_t joining $-a$ to a. Note that $H_0 = h_0 = h$, and that H_1 will be *on* S provided the above arc k_1 has been chosen sufficiently short, and this we suppose done.

The deformation H_t satisfies the lemma except that H may not remain on S, although it begins and ends on S. We shall accordingly modify H_t as follows. Let $r(t)$ be the maximum value of r on H_t. Note that $r(0)$ and $r(1)$ are < 1. Let $[t]$ be the set of values of t on which $r(t) \geqq 1$. Let $b(t)$ be a real analytic function of t for $0 \leqq t \leqq 1$ such that

$$0 \leqq b(t) < 1 \qquad b(0) = b(1) = 0, \tag{4.5}$$

and so near 1 on $[t]$ that

$$r(t) < \frac{1 - a^2 b^2(t)}{1 - b^2(t)} \qquad (0 \leqq t \leqq 1). \tag{4.6}$$

With $b(t)$ so chosen let T_t be the transformation from the z-plane of (r, θ) defined by the inverse of (4.1) when $b = b(t)$, and let h^t be the image $T_t H_t$ of H_t at the time t. It follows from the choice of $b(t)$ and from (4.6) that $\varrho < 1$ on h^t, so that h^t is on S. Moreover, $h^0 = h$ and for every t the end points of h^t are a and $-a$ respectively. The arc h^1 is the circular arc H_1.

The deformation h^t, suitably represented, satisfies the lemma. The representation of h^t is completely determined by the requirement that $z(s, t)$ represent the point on h^t at the time t into which the point s on h has been deformed. The condition $z_s \neq 0$ is obviously satisfied and the proof of the lemma is complete.

(b) *In the preceding lemma at most a finite number of arcs h^t pass through any point of S not on h.*

To verify (b) let z_0 be a point on S not on h. The set of pairs (s, t) on the (s, t) rectangle which satisfy the condition $z(s, t) = z_0$ is empty, or consists of a finite number of pairs, or includes at least one analytic arc $s = s(t)$. In the last case the fact that $z_s(s, t) \neq 0$ insures that the arc $s(t)$ can be continued analytically, in either sense, and in particular in the sense of decreasing t until a boundary point (s^*, t_0) of the (s, t) rectangle is reached. But $z(s^*, t_0)$ is then a point of h, so that z_0 is a point of h, contrary to hypothesis. Hence (b) holds as stated.

The following lemma is a consequence of Lemma 4.2 and (b).

Lemma 4.3. *Any two simple, regular, analytic arcs h_1 and h_2 which join $-a$ and a on S $(a \neq 0)$ can be isotopically deformed into each other on S through simple, regular, analytic arcs no more than a finite number of which pass through any point of S not on h_1 or h_2.*

The arc h_i, $i = 1, 2$, can be deformed on S in the manner stated in Lemma 4.1 and in (b), into a circular arc k_i joining $-a$ to a. But k_1 can be deformed into k_2 through a pencil of circles joining $-a$ to a on S. Lemma 4.3 follows.

§ 5. *The invariants* J_i. Before coming to the definition of the invariants J_i a theorem on interior transformations will be recalled. See ref. (1) II p. 653. Let g be a locally simple, sensed, closed plane curve: $w = w(t)$, with $w(t + 2\pi) \equiv w(t)$. If e_1 is a sufficiently small positive constant, and $0 < e < e_1$

$$\frac{1}{2\pi} \arg [w(t + e) - w(t)] \qquad (0 \leqq t \leqq 2\pi)$$

will be well defined,[1] and as t increases from 0 to 2π, will change by an integer $p(g)$ independent of $e < e_1$. We term p the *angular order* of g. If g does not pass through $w = 0$, its ordinary order with respect to $w = 0$ will be denoted by $q(g)$. The theorem which we shall use is as follows:

Theorem 5.1. *Let B be a closed Jordan curve in the z-plane with interior G and let $F(z)$ be an interior transformation of some region R which includes $\bar{G}$ in its interior, and which maps R into the w-sphere. If B is free from branch point antecedents and has an image g which does not intersect $w = 0$ or ∞, then*

$$n(0) + n(\infty) - \mu = 1 + q(g) - p(g) \tag{5.1}$$

where $n(0)$, $n(\infty)$ and μ are respectively the numbers of zeros, poles, and branch point antecedents of F on G counting these points with their multiplicities.

To come to the definition of the invariants J_i, let f be an interior transformation from $S = \{|z| < 1\}$ to the w-sphere with the characteristic set

$$(\alpha) = (a_0, a_1, \ldots, a_n, b_1, \ldots, b_\mu) \qquad (m = n + 1).$$

Let h_i be a simple curve joining a_0 to a_i on S with $i > 0$. Two curves h_i will be said to be of the same *topological type* if they can be isotopically deformed into each other on S without intersecting the set (α) other than in h_i's end points a_0 and a_i. Let h_i^f denote the image of h_i under f. The curve h_i^f will join $f(a_0)$ to $f(a_i)$ in the w-plane and be locally simple. If h_i is isotopically deformed for $0 \leqq t \leqq 1$ through curves of the same topological type, h_i^f will be admissibly deformed in the sense of § 3 through a uniformly locally simple family of arcs joining $f(a_0)$ to $f(a_i)$ and intersecting $f(a_0)$ and $f(a_i)$ only as end points. The difference order $d(h_i^f)$ is accordingly invariant under such deformations.

However $d(h_i^f)$ will change in general with the topological type of h_i. It is possible to define another function $V(h_i)$ of h_i which changes with the type of h_i exactly as does $d(h_i^f)$. To that end set

[1] Here and elsewhere the argument of any continuously varying non-null function F will be taken as a branch which varies continuously with F.

$$A(z, \alpha) = (z - a_0)(z - a_1) \dots (z - a_n)$$

$$(5.2) \qquad B(z, \alpha) = (z - b_1)(z - b_2) \dots (z - b_\mu) \qquad (\mu > 0)$$

$$B(z, \alpha) \equiv 1 \qquad (\mu = 0)$$

$$(5.3) \qquad C_i(z, \alpha) = \frac{(z - a_0)(z - a_i) B(z, \alpha)}{A(z, \alpha)} \qquad (i > 0).$$

The right member of (5.3) has a removable singularity at $z = a_0$ and at $z = a_i$. We suppose that $C_i(z, \alpha)$ takes on its limiting values at a_0 and a_i so that

$$(5.4) \qquad C_i(a_0, \alpha) = \frac{(a_0 - a_i) B(a_0, \alpha)}{A'(a_0, \alpha)}$$

$$(i > 0)$$

$$(5.5) \qquad C_i(a_i, \alpha) = \frac{(a_i - a_0) B(a_i, \alpha)}{A'(a_i, \alpha)}.$$

Corresponding to a variation of z along h_i set

$$(5.6) \qquad V(h_i) = \frac{1}{2\pi} [\arg C_i(z, \alpha)]_{z = a_0}^{z = a_i}.$$

Regardless of the arguments used

$$(5.7)' \qquad V(h_i) \equiv \frac{1}{2\pi} [\arg C_i(a_i, \alpha) - \arg C_i(a_0, \alpha)] \qquad (\mathrm{mod}\ 1).$$

We shall prove the following theorem:

Theorem 5.2. *The value of the difference*

$$(5.7) \qquad d(h_i^f) - V(h_i) \qquad (i = 1, \dots, n)$$

is independent of h_i among simple curves which join a_0 to a_i without intersecting the other points of the characteristic set (α).

The value of the difference (5.7) is clearly independent of isotopic deformations of h_i through curves of the same topological type, since this is true of both terms in (5.7). By virtue of Lemma 4.1 we can accordingly restrict attention to arcs h_i which are admissible in the lemma and in addition are regular and analytic. If h_i^0 and h_i^1 are two such curves we seek to prove that (5.7) has the same value for h_i^0 as for h_i^1.

In accordance with Lemma 4.3 there exists an isotopic deformation of h_i^0 into h_i^1 through simple, regular, analytic arcs h_i^t $(0 \leqq t \leqq 1)$ such that h_i^t intersects the set $(\alpha) - (a_0, a_i)$ for at most a finite set of values t_0 of t. It is only as t passes through such a value t_0 that

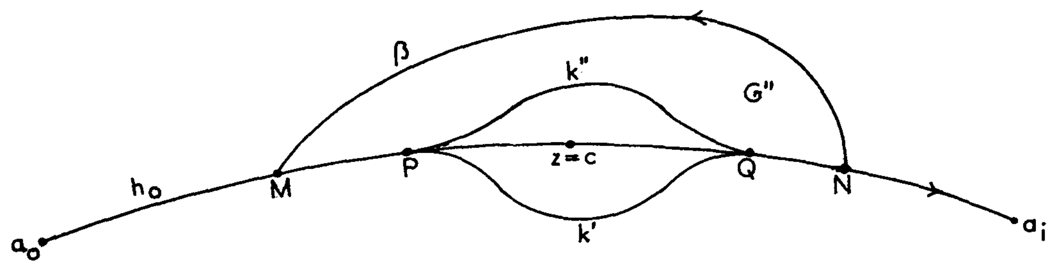

Figure 2.

$$d(h_i^{t,f}) - V(h_i^t) \tag{5.8}$$

could possibly change. Suppose that $h_i^{t_0}$ passes through just one point $z = c$ of the set $(\alpha) - (a_0, a_i)$. The proof in the case in which there are several points $z = c$ on $h_i^{t_0}$ will be seen to be similar. We shall show that there is no net change in (5.8) as $t - t_0$ changes sign.

Let h', h'', and h^0 be respectively the arcs h_i^t for which $t = t_0 - e$, $t_0 + e$ and t_0. For e_1 sufficiently small and $0 < e < e_1$, the curves h' and h'' will intersect (α) in a_0 and a_i only. We suppose that $e < e_1$. Without loss of generality we can suppose that h' passes[1] $z = c$ to the right of h^0. If h'' likewise passes $z = c$ to the right of h^0, it is possible to deform h' into h'' without intersecting $z = c$. (One moves each point of h' along a normal[2] to h^0 until h'' is met.) In this case (5.7) has the same value on h' as on h''.

Suppose then that h'' passes $z = c$ to the left of h^0. Without changing the topological type of h' one can deform the points of h' along normals to h^0 so that h' comes to coincide with h_0 except on a short open arc k' which lies to the right of h^0 neighboring $z = c$. Similarly one can deform h'' so that it comes to coincide with h_0 except on a short open arc k'' which lies to the left of h^0 neighboring $z = c$. We can also suppose that the end points of k' and k'' on h_0 coincide in points P and Q. See Fig. 2.

Let M and N be points of h^0 such that the arc (MN) of h^0 contains the arc (PQ) of h^0 on its interior. Let β be a simple open arc which joins M to N to the left of h'' so near h'' that

$$B'' = \beta(MP)k''(QN)$$

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[1] More definitely, we suppose that h' intersects the normal to h^0 at $z = c$ to the right of h_0.
[2] Provided e_1 is sufficiently small.

is a Jordan curve with no points of (α) on the closure of its interior. We refer to the subarcs (MP) and (QN) of h^0. The closed curve

$$B' = \beta(MP)k'(QN)$$

is then simple and contains no point of (α) other than $z = c$. Let B' and B'' be taken in their positive senses relative to their interiors. Let G' be the region bounded by B', and G'' the region bounded by B''. We shall apply Theorem 5. 1 to f as defined on $\bar{G}'$, and to f as defined on $\bar{G}''$.

Case I. f on G'. Let g' be the image of B' under f. The point $z = c$ is the only zero, pole, or branch point antecedent on G'. The numbers $n(0)$, $n(\infty)$ and μ refer to $z = c$. One only of these numbers differs from zero. In accordance with Theorem 5. 1,

$$n(0) + n(\infty) - \mu = 1 + q(g') - p(g'). \tag{5.9}$$

Case II. f on G''. Let g'' be the image of B' under f. There are no zeros, poles or branch point antecedents on G'' so that

$$0 = 1 + q(g'') - p(g''). \tag{5.10}$$

From (5. 9) and (5. 10) one obtains the relation

$$n(0) + n(\infty) - \mu = [p(g'') - q(g'')] - [p(g') - q(g')]. \tag{5.11}$$

Equation (5. 11) will give us our basic equality.

On taking account of the fact that B' coincides with B'' except along the arcs k' and k'' respectively, and that h' (as altered) similarly coincides with h'' (as altered) except along k' and k'' respectively, it appears that the right member of (5. 11) has the value

$$d(h''^f) - d(h'^f) \tag{5.12}$$

in accordance with the definition of the difference order d.

On the other hand, $V(h') - V(h'')$ reduces to the variation of arg C_i along the closed curve $k'k''$, and so equals the number of zeros minus the number of poles of C_i within this curve. Thus

$$V(h') - V(h'') = \mu - n(0) - n(\infty) \tag{5.13}$$

in accordance with the definition of C_i. From (5. 11) then

$$d(h''^f) - d(h'^f) = V(h'') - V(h'). \tag{5.14}$$

There is thus no change in (5.7) as t passes through t_0, and the theorem follows.

The invariants J_i. As suggested by Theorem 5.2 we set

$$J_i(f, \alpha) = d(h_i^f) - V(h_i) \qquad (i = 1, \ldots, n) \tag{5.15}$$

for any simple arc h_i which joins a_0 to a_i without intersecting $(\alpha) - (a_0, a_i)$. The numbers J_i are independent of the choice of h_i among admissible arcs h_i and of restricted deformations of f.

A necessary condition that two interior transformations with the same characteristic set (α) be in the same restricted deformation class is accordingly that their invariants J_i be respectively equal. It will be shown in § 12 that this condition is sufficient.

If (β) is an admissible reordering of (α), then

$$J_i(f, \alpha) \neq J_i(f, \beta)$$

in general. In case of ambiguity we shall refer to $J_i(f, \alpha)$ as the i^{th} invariant *J_i with respect to (α).*

The value of $V(h_i)$ is independent of f and depends on (α). The values of $d(h_i^f)$ obtainable by changing f, differ by integers. One can accordingly include all values of J_i with respect to (α) in the set

$$J_i = J_i(f^0, \alpha) + m_i \tag{5.16}$$

where m_i is an integer and f^0 an interior transformation with the characteristic set (α). We thus have the theorem,

Theorem 5.3. *The invariants $J_i(f, \alpha)$ for a given i belonging to two different transformations f with the same characteristic set (α) differ by an integer m_i.*

It will be seen that there are functions with the prescribed set (α) for which the integers m_i are prescribed.

We term the sets (J) given by (5.16) with m_i an arbitrary integer the sets (J) associated with (α).

§ 6. *The existence of at least one interior transformation f with a prescribed characteristic set (α).* In this section we shall establish the existence of at least one f with the characteristic set (α) by exhibiting the Riemann image of S with respect to f. In the next section we shall show that except when $\mu = 0$ and

$m = 2$, f can be composed with suitably chosen semi-restricted homeomorphisms of S to obtain composite functions $f\eta$ with invariants $J_i(f, \alpha) + r_i$ where r_i is an arbitrary integer. These models are meromorphic only in special cases. Meromorphic models will be described by formula in § 10.

The following lemma will be used.

Lemma 6.1. *There exists a sense-preserving homeomorphism T of S onto itself in which the image of an arbitrary point set*

$$z_1, \ldots, z_s \tag{6.0}$$

of s distinct points on S is a prescribed set

$$w_1, \ldots, w_s \tag{6.1}$$

of s distinct points on S.

Let g be a simple arc which joins two points on the boundary of S but which otherwise lies on S and passes through the points (6.0) in the order written. Let k be a similar arc passing through the points (6.1). The closed domains into which g divides $\bar{S}$ can be carried respectively by sense-preserving homeomorphisms T_1 and T_2 into the closed domains into which k divides $\bar{S}$ and these homeomorphisms can then be modified so as to transform g into k and in particular to carry the respective points z_i into the corresponding points w_i. The lemma follows.

Theorem 6.1. *There exists at least one interior transformation f of S with a prescribed characteristic set (α).*

The set (α) is proposed as an ordered set of r zeros, s poles and μ branch point antecedents.

The function f affirmed to exist will be defined by describing the Riemann image H of S with respect to f over the w-sphere Σ. As defined H will be the homeomorph of S, will cover the point $w = 0$ of Σ r times, the point $w = \infty$ s times, and possess μ simple branch points covering points of Σ distinct from $w = 0$ and $w = \infty$. One starts with an arbitrary open disc-like piece k of Σ which does not cover $w = 0$ or $w = \infty$. One then extends $r+s$ narrow open tongues from k over Σ with tips covering $w = 0$ r times and $w = \infty$ s times, keeping the extended surface free from branch points and simply connected. To the boundary of k so extended one joins μ two-sheeted branch elements making each junction along a short simple boundary arc of the branch element so that these elements do not cover

$w = 0$ or $w = \infty$. We can suppose that the boundary of the resulting open Riemann surface does not cover $w = 0$ or $w = \infty$.

The Riemann surface H so obtained can be mapped homeomorphically in a sense-preserving fashion onto S and, by virtue of the preceding lemma, in such a manner that the r points of H covering $w = 0$, the s points covering $w = \infty$, and the μ branch points of H go respectively into the proposed zeros, poles, and branch point antecedents of the given set (α).

The proof of the theorem is complete.

The case $\mu = 0$ $m = 2$. In all cases except this one the composition of a particular transformation f possessing a prescribed characteristic set, with a suitably chosen homeomorphism η of S will yield (cf. § 8) a transformation $f\eta$ with the given set (α) and invariants J_i differing from those of f by arbitrary integers.

This method of composition fails when $\mu = 0$ and $m = 2$. In this case there is but one function $C_i(z, \alpha)$, namely $C_1 \equiv 1$, so that $V(h_i)$ as given by (5.6) reduces to 0. There is but one invariant J_i, namely J_1, and

$$J_1 = d(h_1^f). \tag{6.2}$$

The set (α) reduces to (a_0, a_1) and there are two cases according as a_1 is a zero or pole. In the case of two zeros a_0 and a_1, the only possible values of J_1 are

$$\cdots -\frac{3}{2},\ -\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2},\ \cdots \tag{6.3}$$

and in the case of a zero a_0 and a pole a_1

$$\cdots -2,\ -1,\ 0,\ 1,\ 2, \cdots. \tag{6.4}$$

We state the following theorem:

Theorem 6.2. *In case* $\mu = 0$ *and* $m = 2$ *there exists an interior transformation* f *of* S *with a prescribed characteristic set* (a_0, a_1) *with* J_1 *arbitrarily chosen from among the values* (6.3) *when* a_1 *is a zero, and from among the values* (6.4) *when* a_1 *is a pole.*

A proof of this theorem may be given by constructing a Riemann surface for the inverse of a function of the required type. However, the theorem is also established by the formulas of § 10 and the more topological proof will be omitted.

§ 7. *The variation in* $J_i(f, \alpha)$ *caused by variation in* (α). Let f be an interior transformation of S with the characteristic set (α). Let f^t, $0 \leqq t \leqq 1$, be an admissible deformation of f in which (α^t) is the characteristic set of f^t at the time t. Recall that

$$J_i(f, \alpha) = d(h_i^f) - V(h_i), \tag{7.0}$$

by definition. As f is deformed, $d(h_i^f)$ remains invariant while $V(h_i)$ depends on (α) but not on f. We have the important result:

The algebraic increment

$$\Delta J_i = J_i(f^1, \alpha^1) - J_i(f^0, \alpha^0) \tag{7.1}$$

in the invariants J_i *in an admissible deformation* f^t *depends only on the path* (α^t) *and not on* f.

More explicit formulas for J_i and ΔJ_i are needed. To that end set

$$d(h_i^f) \equiv u_i \qquad (\text{mod } 1), \tag{7.2}$$

and recall that u_i can be taken as $\frac{1}{2}$ when a_i is a zero, and 0 when a_i is a pole. Let e_i be 1 or -1 according as a_i is a zero or a pole. Then

$$u_i - \frac{1}{2\pi} \arg e_i \equiv \frac{1}{2} \qquad (\text{mod } 1). \tag{7.3}$$

To make the formula for J_i more definite we shall use a branch $\overline{\arg}\, X$ of the argument for which

$$0 \leqq \overline{\arg}\, X < 2\pi$$

signalling this branch by the addition of the bar. In accordance with the definition of C_i in (5.3)

$$C_i(a_i, \alpha) = \frac{B(a_i, \alpha)}{A'(a_i, \alpha)}(a_i - a_0) \qquad (i \neq 0)$$

$$C_i(a_0, \alpha) = \frac{B(a_0, \alpha)}{A'(a_0, \alpha)}(a_0 - a_i).$$

On referring to the formula $(5.7)'$ for $V(h_i)$ and making use of (7.0), (7.2) and (7.3), one finds that

$$J_i(f, \alpha) = \frac{\overline{\arg}}{2\pi}\left[\frac{e_i A'(a_i, \alpha)}{B(a_i, \alpha)}\right] - \frac{\overline{\arg}}{2\pi}\left[\frac{A'(a_0, \alpha)}{B(a_0, \alpha)}\right] + I_i \qquad (i = 1, \ldots, n) \tag{7.4}$$

where I_i is an integer, and $e_i = 1$ when a_i is a zero, and -1 when a_i is a pole.

The integers $I_i(f, \alpha)$ are invariant under restricted deformations of f and are uniquely determined by f and (α). They are fundamental in the meromorphic theory.

From (7.4) and the continuity of $J(f^t, \alpha^t)$ one obtains the following result:

Lemma 7.0. *If f^t, $0 \leqq t \leqq 1$, is an admissible deformation of a transformation f with characteristic set (α) and with (α^t) the characteristic set of f^t, then the difference*

$$\Delta J_i = J_i(f^1, \alpha^1) - J_i(f^0, \alpha^0) \qquad (i = 1, \ldots, n) \tag{7.5}$$

is given by

$$\theta_i(\lambda) = \left[\sum_k \frac{\arg}{2\pi}\left(\frac{a_i^t - a_k^t}{a_0^t - a_k^t}\right) - \sum_j \frac{\arg}{2\pi}\left(\frac{a_i^t - b_j^t}{a_0^t - b_j^t}\right)\right]_{t=0}^{t=1} \tag{7.6}$$

where

$$k = 1, \ldots, i-1, i+1, \ldots, n; \; j = 1, \ldots, \mu;$$

where λ represents the path (α^t); and where the argument in (7.6) is immaterial as long as a branch which varies continuously with t is used.

A path (α^t), $0 \leqq t \leqq 1$, which leads from a set (β) back to (β) will be called an *α-circuit.* We shall also use paths λ in which (α^1) is an admissible reordering of $(\alpha^0) = (\beta)$. We term such a path *an admissible α-circuit* mod β. As previously, admissibility of a path λ requires that $a_0^1 = a_0^0$.

The difference ΔJ_i in (7.5) will concern us not so much as the difference

$$\delta J_i = J_i(f^1, \beta) - J_i(f^0, \beta) \qquad (i = 1, \ldots, n)$$

because it is necessary to compare the J_i's with reference to the same set (β). When λ is an α-circuit,

$$\delta J_i = \Delta J_i,$$

and in this case δJ_i depends only on λ and not on the initial and final sets (J). If λ is an admissible α-circuit mod β which is not an α-circuit, this is never the case, as we shall see.

The group Ω of J-displacement vectors. Let $\{J, \beta\}$ denote the set of invariants (J) realizable as invariants of an interior transformation with the characteristic set (β). It will presently be seen that the set $\{J, \beta\}$ is identical with the complete set of (J)'s »associated» with (β) in (5.16).

When (α^t), $0 \leqq t \leqq 1$, represents an α-circuit λ leading from (β) to (β), the differences ΔJ_i given by (7.6) are integers r_i. If one sets

$$J_i = J_i(f^0, \beta) \qquad T_\lambda(J_i) = J_i(f^1, \beta)$$

it is seen that the α-circuit λ *induces* a transformation

$$T_\lambda(J) = (J) + (r)$$

of $\{J, \beta\}$ onto itself, in fact a translation. We term (r) the J-displacement vector *induced* by λ. The J-displacement vectors induced by α-circuits from (β) to (β) form an *additive abelian group* Ω. The group Ω is a subgroup of the additive group G of all integral vectors (r); Ω may coincide with G, be a proper subgroup of G and even reduce to the null element.

We shall seek a set of generators of Ω.

To that end let z_p and z_q be any two distinct points of (β). An α-circuit $G(z_p, z_q)$ leading from (β) to (β) will be defined in which all points of (β) except the pair (z_p, z_q) remain fixed while the paths z_p^t, z_q^t $(0 \leqq t \leqq 1)$ are such that $z_p^t - z_q^t$ rotates through an angle -2π. We suppose, moreover, that these paths lie on a topological disc on S which does not intersect the set $(\beta) - (z_p, z_q)$. Such paths clearly exist. The disc can then be isotopically deformed on itself, into a disc arbitrarily close to a point. If this α-circuit is used in (7.6), the only terms which will make a non-null contribution are those which involve $z_p^t - z_q^t$ or $z_q^t - z_p^t$.

The case $\mu > 0$. In this case we introduce the α-circuit

$$\lambda_k = G(a_k, b_1) \qquad (k = 1, \ldots, n)$$

and obtain the following lemma.

Lemma 7.1. *When* $\mu > 0$, *there exist α-circuits* λ_k, $k = 1, \ldots, n$, *for which the components of the corresponding J-displacement vectors are* δ_i^k,[1] $i = 1, \ldots, n$. *These vectors generate Ω as the complete group G of integral vectors* (r).

The case $\mu = 0$ *and* $m > 2$. In this case the α-circuits

$$\lambda_{rs} = G(a_r, a_s) \qquad (r < s)$$

are introduced. The following lemma results.

Lemma 7.2. *When* $\mu = 0$ *and* $m > 2$ *there exist α-circuits* $\lambda_{rs}(r, s = 0, 1, \ldots, n;$ $r < s)$, *for which the corresponding J-displacement vector* D_{rs} *has the components*

$$-\delta_i^r - \delta_i^s \qquad (i = 1, 2, \ldots, n) \tag{7.7}$$

when $rs \neq 0$, *and when* $r = 0$ *has all components 1 except the s-th, which is zero.*

The vectors D_{rs} *generate the group* Ω. *When m is odd, Ω is the group G of all integral vectors* (r). *When* $m = 4, 6, 8, \ldots$, *Ω is the subgroup of G of vectors (r) for which* Σr_i *is even.*

[1] The Kronecker delta gives the i-th component.

The components of D_{rs} are obvious from (7.6).

When m is odd, the matrix whose columns are the components of the vectors

$$D_{12}, D_{23}, \ldots \quad D_{n-1\,n}\, D_{01} \qquad (m = n + 1) \tag{7.8}$$

has the determinant $\omega = -1$. For example, when $m = 5$:

$$\omega = \begin{vmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix} = -1. \tag{7.9}$$

When m is odd, the vectors accordingly generate G and hence Ω.

To treat the case $m = 4, 6, 8, \ldots$, let us term a vector (r) for which Σr_i is even, of *even* category, otherwise of *odd* category. When m is even, each vector D_{rs} is of even category, and hence the vectors generated by the vectors D_{rs} are of even category.

The vectors D_{rs} generate the group Ω.

To see this let λ be an arbitrary admissible α-circuit. The vector $a_r^t - a_s^t$ $(r < s)$ rotates through 2π an integral number m_{rs} of times (possibly zero) as t increases from 0 to 1. It follows from (7.6) that the J-displacement vector induced by λ has the form

$$-\Sigma m_{rs} D_{rs}$$

where the summation extends over the pairs (r, s) with $r < s$. Thus the vectors D_{rs} generate Ω.

It remains to show that every vector (r) of even category is in Ω. To that end we introduce a vector E whose components are δ_i^1. The matrix whose columns are the components of the vectors

$$D_{12}, D_{23}, \ldots, D_{n-1\,n}, E \tag{7.10}$$

is of odd order n and has a determinant 1 so that the vectors (7.10) generate G. Thus (r) is of the form

$$(r) = sE + D \tag{7.11}$$

where s is an integer and D is in Ω. Observe that

$$D_{12} - D_{23} + D_{34} - + \cdots - D_{n-1\,n} + D_{1\,n} = -2E$$

so that s in (7.11) can be taken as 1 or 0. The vector D is of even category, and if (r) is of even category, s in (7.11) cannot be 1. Thus (r) is in Ω, if of even category. This completes the proof when $m = 2, 4, 6, 8, \ldots$.

The case $\mu = 0$, $m = 2$. In this case there is but one value of i in (7.6), and $\varDelta J_1 = 0$. Hence Ω reduces to the vector $(r) = 0$.

§ 8. *The generation of interior transformations by composition $f\eta$ with restricted homeomorphisms η.* We have seen in § 6 that there is at least one interior transformation f with a prescribed characteristic set (β). We shall see to what extent one can choose restricted homeomorphisms η of S onto itself so that $f\eta$ has invariants (J) arbitrarily prescribed from those associated with (β).

To that end we first connect restricted homeomorphisms η leaving (β) fixed with α-circuits from (β) to (β).

Any sense-preserving homeomorphism η of S may be generated as the terminal homeomorphism of an isotopic deformation η^t of S from the identity. More explicitly there exists a 1-parameter family η^t of homeomorphisms S onto S of the form

$$\eta^t \equiv \varphi(z, t) \qquad (0 \leqq t \leqq 1), \tag{8.1}$$

where φ is continuous in z and t,

$$z \equiv \varphi(z, 0)$$

and

$$\eta(z) \equiv \varphi(z, 1).$$

Let (α^t) be the antecedent of (β) under η^t. If η is a restricted homeomorphism leaving (β) fixed, (α^t) determines an α-circuit λ from (β) to (β). We shall say that η *induces* this α-circuit. If f is an interior transformation with the characteristic set (β), the composite function of z, $f\eta^t$, affords a terminally restricted deformation f^t of f in which the characteristic set of f^t at the time t is (α^t).

Formula (7.6) is applicable to the f-deformation $f^t = f\eta^t$ with its associated α-circuit (α^t), $0 \leqq t \leqq 1$, and yields the result

$$J_i(f\eta, \beta) - J_i(f, \beta) = r_i \qquad (i = 1, 2, \ldots, n) \tag{8.2}$$

where (r) is the J-displacement vector determined by (α^t). This displacement vector is independent of the choice of α-circuits λ induced by η since for the same f, (β), and η in (8.2), a second choice of a λ induced by η cannot change (r). The vector (r) is a J-displacement vector $D_1(\eta)$ *determined* by η in the group Ω. If $D_2(\lambda)$ is the vector in Ω *determined* by the α-circuit λ, then

$$D_1(\eta) = D_2(\lambda)$$

whenever λ is induced by η.

It can be shown that any α-circuit (α^t), $0 \leqq t \leqq 1$, from (β) to (β) is induced by *some* restricted homeomorphism of S leaving (β) fixed; to establish this one must show that there exists an isotopic deformation η^t of S from the identity in which (α^t) is the antecedent of (β) at the time t and in which η^1 is a restricted homeomorphism leaving (β) fixed. The details of a proof of this need not be given. It is sufficient to suggest to the reader that η^t can be defined by a sequence of deformations in each of which just one point of (α^t) is moved from an initial point z_0 to a nearby point z_1. One can make use of a deformation Δ from the identity of a small circular neighborhood N of z_0, defining the deformation Δ as the identity outside of N.

We summarize as follows:

Lemma 8.1. *Each restricted homeomorphism η of S leaving (β) fixed induces a class of α-circuits λ leading from (β) to (β), and every α-circuit leading from (β) to (β) is induced by a class of restricted homeomorphisms η leaving (β) fixed. If η induces λ and (r) is the J-displacement vector determined in* (7.6) *by λ, then* (8.2) *holds for every interior transformation with (β) as a characteristic set. The vector (r) depends only on η and not on the choice of an α-circuit λ induced by η.*

The reciprocal relations between restricted homeomorphisms η and their induced α-circuits and the theorems on the nature of the group Ω of J-displacement vectors induced by α-circuits yield the following theorem:

Theorem 8.1. *If f^0 is an interior transformation of S with the characteristic set (β) and invariants (J^0), suitably chosen restricted homeomorphisms η of S leaving (β) fixed will yield interior transformations $f^0\eta$ with invariants $(J^0) + (r)$ where*

(1) *(r) is an arbitrary integral vector when $\mu > 0$, or when $\mu = 0$ and m is odd,*

(2) *(r) is an arbitrary integral vector of even category when $\mu = 0$ and $m = 4, 6, 8, \ldots$,*

(3) *$(r) = (0)$ only, when $\mu = 0$ and $m = 2$.*

No other values of (J) can be obtained by composition $f^0\eta$ of f^0 with restricted homeomorphisms η.

§ 9. *α-circuits mod (β) and semi-restricted homeomorphisms*[1] *η of S.* We resume the theory of admissible α-circuits mod (β) initiated in § 7. As in § 7 we are concerned with an admissible deformation f^t, $0 \leqq t \leqq 1$, for which (α^t) is the characteristic set of f^t at the time t. We suppose that (α^1) is an admissible reordering of (α^0). In particular $a_0^1 = a_0^0$.

[1] The present section could be omitted by a reader who wishes to comprehend first the main theory.

Such a reordering defines a permutation π of $(1, \ldots, n)$ in which i is replaced by $\pi(i)$ and

$$a_i^1 = a_{\pi(i)}^0 \qquad (i = 1, \ldots, n). \tag{9.1}$$

For any interior transformation f with characteristic set (a^0),

$$J_i(f, a^1) = J_{\pi(i)}(f, a^0) \qquad (i = 1, \ldots, n) \tag{9.2}$$

in accordance with the definition of (J). If $(x_1, \ldots, x_n)$ is an arbitrary set of n symbols, we shall write

$$(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \pi(x).$$

Thus (9.2) takes the form

$$[J(f, a^1)] = \pi[J(f, a^0)].$$

It follows from (7.6) that

$$J_i(f^1, a^1) = J_i(f^0, a^0) + \theta_i(\lambda) \qquad [\lambda = (a^t)]. \tag{9.3}$$

From (9.2) and (9.3) one sees that

$$J_{\pi(i)}(f^1, \beta) = J_i(f^0, \beta) + \theta_i(\lambda) \qquad (\beta) = (a^0). \tag{9.4}$$

Equations (9.4) may be written in the vector forms

$$\pi[J(f^1, \beta)] = [J(f^0, \beta)] + [\theta]. \tag{9.5}$$

$$[J(f^1, \beta)] = \pi^{-1}\{J(f^0, \beta) + [\theta]\}. \tag{9.6}$$

We thus have the following lemma:

Lemma 9.1. *Any admissible a-circuit mod (β) of the form $\lambda = \{(a^t),\ 0 \leqq t \leqq 1\}$ in which*

$$(a_1^1, \ldots, a_n^1) = \pi(a_1^0, \ldots, a_n^0),$$

induces a transformation

$$T_\lambda(J) = \pi^{-1}\{(J) + (\theta(\lambda))\} \tag{9.7}$$

of $\{J, \beta\}$ such that for any admissible deformation f^t of an interior transformation in which (a^t) is the characteristic set of f^t the invariants (J^1) of f^1 with respect to (β) are the transforms $T_\lambda(J^0)$ of the invariants (J^0) of f^0 with respect to (β).

The transformation T_λ of $\{J, \beta\}$ is compounded of a translation $(J) + (\theta)$ and a permutation π^{-1} of the components of the translated vector. It is a translation if and only if π is the identity. If π is not the identity, the numbers $\theta_i(\lambda)$ are not integers in general, so that $(J) + (\theta)$ is not in $\{J, \beta\}$ in general.

It follows from (9.4) that

$$\sum_i [J_i(f^1, \beta) - J_i(f^0, \beta)] = \Sigma\theta_i = q(\lambda), \tag{9.8}$$

introducing q. Here q is an odd or even integer depending only on λ. The transformation T_λ is termed of *odd* or *even category* according as q is odd or even. When $\mu = 0$ and $m = 4, 6, 8, \ldots$, all translations of $\{J, \beta\}$ induced by α-circuits have been seen to be of even category. Transformations of odd category are sought in this case. We shall prove the following lemma:

Lemma 9.2. *When $\mu = 0$ and m is even, a necessary and sufficient condition that an admissible α-circuit λ mod (β) induce a transformation U_λ of $\{J, \beta\}$ of odd category is that (α^1) be an admissible odd permutation of (α^0).*

To prove the lemma we evaluate $q(\lambda)$ in (9.8). When $\mu = 0$, a summation of the right members of (7.6) yields the result

$$q(\lambda) = \left[\frac{\arg}{\pi}\prod_{i,j}(a_i^t - a_j^t) - \frac{(n-1)}{2\pi}\arg\prod_k (a_0^t - a_k^t)\right]_{t=0}^{t=1} \tag{9.9}$$

where $i, j, k = 1, 2, \ldots, n$ with $i < j$. Since $a_0^1 = a_0^0$ and $n - 1$ is even, the contribution of the last term in (9.9) is an even integer. Observe that

$$\prod_{i,j}(a_i^1 - a_j^1) = \pm \prod_{i,j}(a_i^0 - a_j^0) \tag{9.10}$$

according as (α^1) is an even or odd permutation of (α^0) so that $q(\lambda)$ is correspondingly odd or even. This completes the proof of the lemma.

It is a consequence of this lemma that, when $\mu = 0$ and $m = 4, 6, 8, \ldots$, there exists a transformation U_λ of $\{J, \beta\}$ of odd category. For two at least of the points a_1, a_2, a_3, are of like type (zeros or poles) and there accordingly exists an admissible α-circuit mod (β) which interchanges these two points but which otherwise leaves (β) fixed.

The following lemma can now be proved:

Lemma 9.3. *Let J^0 be an arbitrary set in $\{J, \beta\}$ and let (r) be an arbitrary set of n integers. When $\mu = 0$ and $m = 4, 6, 8, \ldots$, there exists a transformation T_λ of $\{J, \beta\}$ induced by an admissible α-circuit λ mod (β) such that the relation*

$$T_\lambda(J) = (J) + (r) \tag{9.11}$$

holds for $(J) = (J^0)$ and the given set (r).

The lemma is a consequence of Lemma 7.2 when (r) is of even category. There then exists a T such that (9.11) holds for *every* (J) in $\{J, \beta\}$.

Suppose then that (r) is of odd category. It follows from the preceding lemma that, when $\mu = 0$ and $m = 4, 6, 8, \ldots$, there exists a transformation U of $\{J, \beta\}$ of odd category induced by an admissible α-circuit mod (β) which replaces (β) by an admissible reordering (β'). Set

$$U(J^0) - (J^0) = (s). \tag{9.12}$$

The set (s) is of odd category and hence $(r) - (s)$ is of even category. In accordance with Lemma 7.2 there exists a translation V of $\{J, \beta'\}$ of even category induced by an α-circuit from (β') to (β') such that

$$V[J^0 + (s)] = [J^0 + (s)] + (r) - (s).$$

On making use of (9.12) it is seen that

$$VU(J^0) = (J^0) + (r).$$

Hence the transformation $T = VU$ satisfies the lemma.

The case $\mu = 0$, $m = 2$. In this case there is but one invariant J, and this invariant has the form

$$J(f, a) = J_1(f, a_0, a_1) = d(h_1^f)$$

where h_1 is a simple curve joining a_0 to a_1 on S. The term $V(h_1) = 0$ since $C_1(z) \equiv 1$ in this case. When a_0 is a zero and a_1 is a pole, we admit no relative α-circuits which interchange a_0 and a_1.

When both a_0 and a_1 are zeros, we shall admit relative α-circuits which interchange a_0 and a_1. When $m = 2$ and $\mu = 0$, the right member of (7.6) is devoid of terms so that

$$J_1(f^1, a_1, a_0) = J_1(f^0, a_0, a_1) \tag{9.13}$$

if (a^t) interchanges a_1 and a_0 during the deformation f^t.

A simple but striking illustration may be given. Let f be an interior transformation of S with $\mu = 0$, $m = 2$, and with zeros at real points a and $-a$. $(0 < a < 1)$. Let $F(z) = f(-z)$. There exists no admissible deformation of f into F which returns each zero into its initial position. This is a consequence of the fact that

$$J_1(F, a_0, a_1) = -J_1(f, a_0, a_1) \neq 0 \tag{9,13$'$}$$

while the existence of the deformation would require that

$$J_1(F, a_0, a_1) = J_1(f, a_0, a_1)$$

contrary to (9.13)'.

Let k_1 be h_1 reversed in sense. Then in accordance with the definition of J_1

$$J_1(f, a_0, a_1) = d(h_1^f) = - d(k_1^f) = - J_1(f, a_1, a_0). \tag{9.14}$$

It follows from (9. 13) and (9. 14) that when (a^t) interchanges the zeros in a deformation f^t

$$J_1(f^1, a_0, a_1) = - J_1(f^0, a_0, a_1). \tag{9.15}$$

We summarize in the lemma:

Lemma 9. 4. *When $\mu = 0$ and $m = 2$, admissible deformations f^t with characteristic sets (α^t), $0 \leqq t \leqq 1$, may interchange two zeros a_0 and a_1 but must return a zero a_0^t and pole a_1^t to their initial positions. In any case*

$$J_1(f^1, a_0, a_1) = \pm J_1(f^0, a_0, a_1) \tag{9.16}$$

where the minus sign prevails if and only if (α^t) interchanges two zeros.

Contrast this result with the fact that, when f is an even admissible function with just two zeros on S and no poles, f can be restrictedly deformed into $f(-z)$; in fact by the identity. The reason is found in the fact that an even function has a branch point antecedent at the origin so that $\mu > 0$.

Semi-restricted homeomorphisms η. The developments of § 8 showing the reciprocal relationship between α-circuits, (α^t) and restricted homeomorphisms η which induce them is paralleled here by the relationship between admissible α-circuits λ mod β and semi-restricted homeomorphisms η which admissibly reorder (β). Let η^t be an isotopic deformation of S generating η and let (α^t) be the antecedent of (β) under η^t. We say that η *induces* the relative α-circuit $\lambda = \{(\alpha^t)\ 0 \leqq t \leqq 1\}$. The f-deformation $f^t = f\eta^t$ has the characteristic set (α^t) and with λ comes under Lemma 9. 1, so that for the transformation T_λ of $\{J, \beta\}$ given by (9. 7) and induced by λ

$$[J(f\eta, (\beta)] = T_\lambda [J(f, \beta)]. \tag{9.17}$$

As in § 8 we infer that each such relative α-circuit λ is »induced» by some semi-restricted homeomorphism η. Hence if λ is an arbitrary admissible α-circuit mod β, there exists a semi-restricted homeomorphism η admissibly reordering (β) such that (9. 17) holds for every f with the characteristic set (β).

The results of the present section on transformations T_λ of $\{J, \beta\}$ induced by relative α-circuits λ together with the results of the preceding section on translations in the group Ω lead to the following theorem:

Theorem 9.1. *Let f be an interior transformation of S with the characteristic set (β) and let (r) be an arbitrary set of n integers. Except in the case in which $\mu = 0$ and $m = 2$ there exists a semi-restricted homeomorphism η of S, admissibly reordering (β) but leaving a_0 fixed, such that*

$$J_i(f\eta, \beta) = J_i(f, \beta) + r_i \qquad (i = 1, \ldots, n). \tag{9.18}$$

When $\mu = 0$, $m = 2$, and η is an arbitrary homeomorphism leaving a_0 and a_1 fixed, or interchanging a_0 and a_1 in case a_0 and a_1 are zeros, then

$$J_1(f\eta, a_0, a_1) = \pm J_1(f, a_0, a_1)$$

where the minus sign holds if and only if η interchanges a_0 and a_1.

It is of interest to add that when η in (9.18) is restricted, and (r) is fixed, (9.18) holds for all transformations f with the characteristic set (β) if it holds for one such f. This is not the case in general if η is not restricted.

Part II. Meromorphic Functions.

§ 10.1. *The residual function $\varphi(z)$ and the canonical functions $F(z, \alpha, r)$.* Suppose that f is meromorphic on S and possesses the characteristic[1] set (α). The function $\varphi(z)$ defined by the equation

$$\frac{f'(z)}{f(z)} = \varphi(z) \frac{B(z, \alpha)}{A(z, \alpha)} \tag{10.1}$$

is analytic on S except for removable singularities, and never zero. We term $\varphi(z)$ the *residual function* of f.

The algebraic increment of arg φ along any simple regular arc h_i joining a_0 to a_i on S equals $2\pi J_i(f, \alpha)$, as we shall see. To establish this fact a lemma is needed.

Let h_i be referred to its arc length s measured from $z = a_0$. Suppose that the total length of h_i is σ. Let e be a constant with $0 < e < \sigma$. Suppose that $z(s)$ and $z_1(s)$ are functions of s of which $z(s)$ represents the point s on h_i and $z_1(s)$ the point $s + \Delta s$ on h_i, where $\Delta s = e$. The parameter s shall vary on the interval

$$0 \leqq s \leqq \sigma - e. \tag{10.2}$$

[1] In the preceding we have supposed that $m > 1$. The results of the present section hold for $m = 1$ and with obvious interpretations for $m = 0$.

With $z(s)$ and $z_1(s)$ so determined, set

$$\Delta z = z_1(s) - z(s). \tag{10.3}$$

We shall consider the increment of angle given by

$$U(h_i, e) = \left[\arg \frac{\Delta z}{(z_1(s) - a_0)(z(s) - a_i)}\right]_{s=0}^{s=\sigma-e}. \tag{10.4}$$

Lemma 10.1. *The value of $U(h_i, e)$ is zero.*

Reference to the definition (3.12) of the difference order shows that

$$2\pi d(h_i) = U(h_i, e)$$

regardless of the choice of $e < \sigma$. On letting e tend to σ it appears that $U(h_i, e) = 0$.

The following theorem is one of the bridges between the theory of interior transformations and meromorphic functions.

Theorem 10.1. *The algebraic increment $E(h_i)$ of the argument of the residual function φ of f as z traverses a simple, regular arc h_i leading from a_0 to a_i on S equals $2\pi J_i(f, a)$.*

The value of $E(h_i)$ is independent of the choice of h_i among regular arcs leading from a_0 to a_i since $\varphi \neq 0$ on S. Hence no generality will be lost if h_i does not intersect the set $(a) - (a_0, a_i)$. We suppose h_i so chosen.

We make use of the terminology preceding Lemma 10.1. Set

$$\Delta f = f[z_1(s)] - f[z(s)] \qquad (0 \leq s \leq \sigma - e).$$

Recall that

$$\varphi = \frac{f'}{f}\frac{A}{B}.$$

It is clear that $E(h_i)$ is the limit as e tends to 0 of

$$\left[\arg \frac{\Delta f}{\Delta z} - \arg f - \arg \frac{B}{A}\right]_{s=0}^{s=\sigma-e}$$

where $z = z(s)$ in f, B and A.[1] Recall that

$$\frac{B}{A} = \frac{C_i(z, a)}{(z - a_0)(z - a_i)}.$$

It thus appears that $E(h_i)$ is equally the limit of

$$[\arg \Delta f - \arg f - \arg C_i - U(h_i, e)]_{s=0}^{s=\sigma-e}. \tag{10.5}$$

[1] Strictly $\arg f$ and $\arg \frac{B}{A}$ are not defined for $s = 0$, but limiting values exist. The appropriate conventions are understood.

But $U(h_i, e) = 0$ according to Lemma 10. 1, and

$$V(h_i) = \frac{1}{2\pi} \arg C_i \Big|_{s=0}^{s=\sigma}.$$

In the terminology of (3. 3), it follows that

$$E(h_i) = 2\pi \{P(h_i^f) - Q_0(h_i^f) - V(h_i)\}.$$

By virtue of Lemma 3. 6,

$$d(h_i^f) = P(h_i^f) - Q_0(h_i^f)$$

so that

$$E(h_i) = 2\pi [d(h_i^f) - V(h_i)] = 2\pi J_i(z, \alpha).$$

This completes the proof of the theorem.

On multiplying the members of (10. 1) by $z - a_j$ and letting z tend to a_j as a limit, one finds that

$$\varphi(a_j) = e_j \frac{A'(a_j, \alpha)}{B(a_j, \alpha)} \qquad (j = 0, 1, \ldots, n) \tag{10. 6}$$

where $e_j = 1$ if a_j is a zero and -1 if a_j is a pole.

The following lemma is basic.

Lemma 10. 2. *Corresponding to an arbitrary admissible characteristic set* (α) *and to a function* $\psi(z)$ *which is non-null and analytic on* S *and satisfies* (10. 6) *in terms of* (α), *there exists a function* $F(z)$ *which is meromorphic on* S, *possesses the characteristic set* (α), *and for which the residual function is* $\psi(z)$.

Such a function F is obtained as an integral of (10. 1) in the form

$$F = C e^{\int \frac{\psi B}{A} dz} \qquad (C = \text{const} \neq 0) \tag{10. 7}$$

upon removing singularities at the proposed zeros and poles of (α), and any function F with the residual function ψ and characteristic set (α) is of this form. Because $\psi(z)$ satisfies (10. 6) the residue of $\dfrac{\psi B}{A}$ at a_j is e_j, so that a_j is a zero or pole of F as required. Moreover

$$\frac{F'}{F} = \psi \frac{B}{A},$$

and since $\psi \neq 0$, $F' = 0$ only at the zeros of B. Thus F possesses the characteristic set (α) and has the residual function $\psi(z)$.

It will simplify the notation if one sets

$$e_j \frac{A'(a_j, \alpha)}{B(a_j, \alpha)} = g_j(\alpha) \qquad (j = 0, \ldots, n) \tag{10.8}$$

where $e_j = 1$ if a_j is a zero and -1 if a_j is a pole. The g_j's are functions of (α) which one can suppose given before one has knowledge of the existence of a function f from which they are derived. In taking arguments of these g_j's we have referred to a choice of the argument for which

$$0 \leqq \overline{\arg} X < 2\pi \tag{10.9}$$

signalling this choice by adding the bar. Similarly we shall write

$$\overline{\log} X = \log |X| + \sqrt{-1}\, \overline{\arg} X. \tag{10.10}$$

With this terminology we write (7.4) *in the basic form*

$$J_i(f, \alpha) = \frac{\overline{\arg} g_i(\alpha)}{2\pi} - \frac{\overline{\arg} g_0(\alpha)}{2\pi} + I_i(f, \alpha) \tag{10.11}$$

recalling that I_i is an integer. Given (α), (I) uniquely determines (J) and conversely.

The fundamental existence theorem for meromorphic functions follows.

Theorem 10.2. *Corresponding to any admissible characteristic set (α) and arbitrary set (r) of n integers there exists a function $F(z, \alpha, r)$ which is meromorphic in z on S, whose characteristic set is (α) and whose invariants $[I(F, \alpha)] = (r)$.*

In terms of the given integers (r) with $r_0 = 0$ adjoined, set

$$c_j(\alpha, r) = \overline{\log}\, g_i(\alpha) + 2\pi r_j \sqrt{-1} \qquad (j = 0, \ldots, n). \tag{10.12}$$

The Lagrange interpolation formula suffices to yield a polynomial[1] $P(z)$ such that

$$P(a_j) = c_j \qquad (j = 0, \ldots, n).$$

For fixed (α) and (r) we shall show that the function[1]

$$\psi(z) = e^{P(z)} \tag{10.13}$$

is admissible as a residual function and yields a solution F of our problem. In fact,

$$\psi(a_j) = e^{P(a_j)} = e^{c_j} = g_j = e_j \frac{A'(a_j, \alpha)}{B(a_j, \alpha)} \qquad (j = 0, 1, \ldots, n)$$

[1] More explicitly we could write

$$P = P(z, \alpha, r) \qquad \psi = \psi(z, \alpha, r).$$

so that conditions (10.6) on a residual function are satisfied. To apply Lemma 10.2 we note also that $\psi(z)$ is non-vanishing and analytic on S. Hence ψ is the residual function of a function F with the given characteristic set (α).

It remains to prove that $I_i(F, \alpha) = r_i$. Since $\psi \neq 0$ on S there exists a single-valued continuous branch of $\arg \psi$ over S, and for any such continuous branch

$$2\pi J_i(F, \alpha) = \arg \psi(a_i) - \arg \psi(a_0) \tag{10.14}$$

in accordance with Theorem 10.1. For the moment let $\hat{X}$ denote the coefficient of $\sqrt{-1}$ in X. Then $\hat{P}(z)$ is a continuous branch of $\arg \psi$ by virtue of (10.13).

On setting $\arg \psi = \hat{P}(z)$ in (10.14) we find that

$$2\pi J_i(F, \alpha) = \hat{P}(a_i) - \hat{P}(a_0) = \hat{c}_i - \hat{c}_0.$$

From the definition of c_i it then follows that

$$J_i(F, \alpha) = \left(\frac{\overline{\arg g_i}}{2\pi} + r_i\right) - \frac{\overline{\arg g_0}}{2\pi}.$$

Reference to (10.11) shows that

$$I_i(F, \alpha) = r_i \qquad (i = 1, \ldots, n).$$

Thus the characteristic set of F is (α), and its invariants (I) with respect to (α) equal (r). This completes the proof of the theorem.

The preceding polynomials P will be taken in the explicit Lagrangian form

$$P(z, c) = A\left[\frac{c_0}{(z - a_0) A_0'} + \cdots + \frac{c_n}{(z - a_n) A_n'}\right] \tag{10.15}$$

where

$$A = (z - a_0)(a - a_1) \ldots (z - a_n)$$

and A_j' is the value of A' when $z = a_j$. With ψ given by (10.13) we shall refer to the *canonical functions*

$$F(z, \alpha, r) = C e^{\int \frac{\psi B}{A} dz} \tag{10.16}$$

and will determine C by the condition that

$$F_z(a_0, \alpha, r) = 1.$$

In (10.16)

$$A = A(z, \alpha) \quad B = B(z, \alpha) \quad \psi = \psi(z, \alpha, r).$$

§ 11. *The monogenic continuation with respect to* (α) *of the functions* $F(z,\alpha,r)$. Starting with a particular value (β) of (α), a point z and set (r), one can continue $F(z,\alpha,r)$ as a function of (α), into a monogenic function $M(z,\alpha)$ of (α). As we shall see, $M(z,\alpha)$ may be single-valued in (α), as happens in the case $\mu = 0$, $m = 2$, or it may be infinitely multiple-valued. The branches of $M(z,\alpha)$ form a subset of the functions $F(z,\alpha,r)$ including the extreme possibilities that this subset consists of all of the functions $F(z,\alpha,r)$ or just one. For different sets (r) no two functions $F(z,\alpha,r)$ are identical in z; identity of the functions would imply identity of their invariants (J) and through (10.11), identity of their invariants $(I) = (r)$.

The mechanism of the continuation is best understood by noting that $M(z,\alpha)$ is a function of (α) defined by (10.16) through the mediation of the c_j's. The c_j's enter through the polynomial $P(z,c)$ of (10.15). Finally, the c_j's are given as functions

$$c_j(\alpha, r) = \overline{\log}\, g_j + 2\pi i r_j \qquad (r_0 = 0). \tag{11.1}$$

For each set of integers $(r) = (r_1, \ldots, r_n)$, (11.1) defines $[c(\alpha,r)]$ as an analytic vector function, more precisely as a branch of an infinitely multiple-valued function of (α). It is not implied that all elements $[c(\alpha,r)]$ are branches of the same monogenic function of (α). As (α) varies continuously through a point (β) the functions $c_j(\alpha,r)$ may suffer jumps of the form

$$\varDelta c_j = 2\pi i \sigma_j \qquad (j = 0, \ldots, n) \tag{11.2}$$

where $\sigma_j = \pm 1$. We need the following lemma.

Lemma 11.1. *A jump of the* $c_j(\alpha,r)$*'s in which all* c_j*'s change by* $2\pi i$ *or by* $-2\pi i$ *corresponds to no singularity in the right member of* (10.16).

Corresponding to any change in the c_j's in which $\varDelta c_j$ is a constant k independent of j set $c_j' = c_j + k$. Then

$$P(z,c) + k = P(z,c').$$

When in particular $k = \pm 2\pi i$ the new residual function is

$$\psi(z,c') = e^{P(z,c) \pm 2\pi i} = \psi(z,c)$$

so that there is no change in the right member of (10.16).

Any continuation of an element $[c(\alpha,r)]$ by itself alone or through other elements $[c(\alpha,s)]$ in which the only singularities are a finite number of jumps of the type in Lemma 11.1 will be termed *effectively analytic;* such a continuation causes no singularity in the right member of (10.16). The set of elements $[c(\alpha, r)]$ defined by (11.1) will not in general include all of the analytic continuations with respect to (α) of a given element $[c(\alpha, s)]$, because we have taken $r_0 \equiv 0$. However, the set of elements $[c(\alpha, r)]$ does permit »effective» analytic continuation of any given element $[c(\alpha, s)]$. A continuous variation of (α) in which the functions $c_j(\alpha, s)$ suffer jumps (11.2) at a point (β) will correspond to no singularity in a continuation of $M(z, \alpha)$ in which one changes at (β) from $[c(\alpha, s)]$ to $[c(\alpha, r)]$ with

$$r_i = s_i - \sigma_i + \sigma_0 \qquad (i = 1, \ldots, n)$$

in (10.12).

The following theorem is a consequence of this result.

Theorem 11.1. *Any one of the monogenic functions of (α) obtained by analytic α-continuation of a particular element $F(z, \alpha, s)$ is composed of single-valued branches forming a subset of the canonical functions $F(z, \alpha, r)$.*

There remains the problem of determining how the functions $F(z, \alpha, r)$ combine to make up the monogenic functions $M(z, \alpha)$ of (α). A necessary and sufficient condition that $F(z, \alpha, r)$ be continuable with respect to (α) into $F(z, \alpha, s)$ is that there exist an α circuit $\lambda = \{(\alpha^t),\ 0 \leqq t \leqq 1,\}$ from (β) to (β) and a function $M(z, \alpha)$ monogenic in (α) such that the family of functions

$$f^t = M(z, \alpha^t) \tag{11.3}$$

obtained by analytic α-continuation from the branch

$$M(z, \beta) = F(z, \beta, r)$$

terminate with the branch

$$M(z, \beta) = F(z, \beta, s).$$

This is possible if and only if there exists an α-circuit λ from (β) to (β) for which the induced J-displacement vector in Ω is

$$\varDelta J = \varDelta I = (s) - (r). \tag{11.4}$$

The necessity of this condition is obvious since f^t is merely a special case of the more general restricted f-deformations considered in § 7. The existence of such an α-circuit $\lambda = (\alpha^t)$ is also sufficient. For one can continue $F(z, \alpha, r)$ along the path (α^t) through the family (11.3). Relation (11.4) will then hold so that the invariants (I), with respect to (β), of the terminal canonical function must be (s).

The lemmas of § 7 on the group Ω accordingly yield the following theorem.

Theorem 11. 2. *The canonical functions* $F(z, \alpha, r)$ *combine as branches of functions* $M(z, \alpha)$ *monogenic in* (α) *as follows:*

(1) *When* $\mu > 0$, *or* $\mu = 0$ *and* m *is odd, all canonical functions* F *are branches of a single monogenic function* $M(z, \alpha)$;

(2) *When* $\mu = 0$ *and* $m = 4, 6, 8, \ldots$, *all canonical functions* $F(z, \alpha, r)$ *for which* (r) *is of even category are branches of one monogenic function of* (α), *while all for which* (r) *is of odd category are branches of another;*

(3) *When* $\mu = 0$ *and* $m = 2$, *each canonical function* $F(z, \alpha, r)$ *is of itself monogenic in* (α).

§ 12. *Equivalence of meromorphic functions.* Restricted, terminally restricted, and semi-restricted equivalence of two interior transformations f_1 and f_2 of S have been defined in § 2, in terms of admissible f-deformations. If the deformations employ meromorphic functions only, the equivalence is said to be of *meromorphic type*. The first theorem on equivalence follows.

Theorem 12. 1. *Necessary and sufficient conditions that meromorphic functions* f_1 *and* f_2 *with the same characteristic set* (a) *be restrictedly and meromorphically equivalent are that*

$$J_i(f_1, \alpha) = J_i(f_2, \alpha) \qquad (i = 1, \ldots, n). \tag{12.1}$$

That the conditions are necessary has already been established. We shall accordingly prove the conditions sufficient assuming that (12. 1) holds.

Let φ_1 and φ_2 be residual functions of f_1 and f_2 respectively. Set

$$\psi(z, t) = e^{(1-t)\log\varphi_1 + t\log\varphi_2} \qquad (0 \leqq t \leqq 1) \tag{12.2}$$

using continuous branches of $\log\varphi_1$ and $\log\varphi_2$ with

$$\log\varphi_1(a_0) = \log\varphi_2(a_0). \tag{12.3}$$

The condition (12. 3) can be fulfilled since

$$\varphi_1(a_0) = \varphi_2(a_0)$$

in accordance with (10. 6). The resulting function $\psi(z, t)$ satisfies the conditions on a residual function in Lemma 10. 2 regardless of the value of t, as will now be shown.

First, for $0 \leqq t \leqq 1$ and for z on S, $\psi(z, t)$ is non-null and analytic, since φ_1 and φ_2 are non-null and analytic on S. The proof that conditions (10.6) are satisfied by $\psi(z, t)$ is as follows.

The relation

$$\log \varphi_1 \Big|_{a_0}^{a_i} = \log \varphi_2 \Big|_{a_0}^{a_i} \qquad (i = 1, \ldots, n) \tag{12.4}$$

will first be verified. The real parts of the logarithms are equal at a_0 as well as at a_i, since φ_1 and φ_2 satisfy (10.6).

The hypothesis (12.1) in the form

$$\arg \varphi_1 \Big|_{a_0}^{a_i} = \arg \varphi_2 \Big|_{a_0}^{a_i}$$

then insures the truth of (12.4). It follows from (12.4) that for branches of the logarithm for which (12.3) holds

$$\log \varphi_1(a_i) = \log \varphi_2(a_i) \qquad (i = 1, \ldots, n). \tag{12.5}$$

Using (12.5) in (12.2) we find that

$$\psi(a_i, t) = e^{\log \varphi_1(a_i)} = \varphi_1(a_i) \qquad (i = 1, \ldots, n).$$

Thus $\psi(z, t)$ satisfies (10.6).

In accordance with Lemma 10.2 $\psi(z, t)$ is the residual function of a meromorphic function of z

$$f(z, t) = e^{\int \frac{\psi B}{A} dz} \qquad (0 \leqq t \leqq 1) \tag{12.6}$$

with characteristic set (α).

It follows from (12.6) that

$$f(z, 0) = C_1 f_1(z) \qquad C_1 \neq 0$$

$$f(z, 1) = C_2 f_2(z) \qquad C_2 \neq 0$$

where C_1 and C_2 are constants. The function

$$\frac{f(z, t)}{C_1^{1-t} C_2^t} \qquad (0 \leqq t \leqq 1) \tag{12.7}$$

gives the required meromorphic deformation of f_1 into f_2.

The canonical (α)-projection of a transformation f. If an interior transformation f has a characteristic set (α) and integral invariants (I) with respect to (α), $F(z, \alpha, I)$ will be called the *(α)-projection* of f. If f is meromorphic it has just been seen that f admits a restricted deformation of meromorphic type into its (α)-projection.

If (α^1) is any admissible reordering of (α) it will be shown that the (α)- and (α^1)-projections of f are identical as functions of z. If (I^1) is the set of integral invariants of f with respect to (α^1) this statement takes the form of an identity in z.

$$F(z, \alpha, I) \equiv F(z, \alpha^1, I^1). \tag{12.8}$$

To establish (12.8) recall that the definition of a canonical function involves the functions g_i and c_i of (α) as well as I_i. If

$$(a_1^1, \ldots, a_n^1) = \pi(a_1, \ldots, a_n),$$

one verifies that in passing from the (α)- to the (α^1)-projection of f a similar permutation π of g_i, c_i and I_i, $(i = 1, \ldots, n)$ is made with g_0 and c_0 unchanged. The Lagrange polynomial $P(z, c)$ is invariant under this change and (12.8) follows.

If f is admissibly deformed through a family f^t, $0 \leqq t \leqq 1$, of interior transformations with (α^t) the characteristic set of f^t, the (α^t)-projection F^t of f^t will admissibly deform the (α)-projection F^0 of f through a family of canonical functions. If $M(z, \alpha)$ is the monogenic function of (α) of which F^0 is a branch, then

$$F^t = M(z, \alpha^t) \qquad 0 \leqq t \leqq 1$$

provided the appropriate branches of $M(z, \alpha^t)$ are selected for each t. This deformation F^t is termed *canonical*. In these terms one can state that a necessary and sufficient condition that two meromorphic functions f^1 and f^2 with the same characteristic set (β) be terminally restrictedly and meromorphically equivalent is that their (β)-projections admit a canonical deformation into each other. The latter condition is merely that the two (β)-projections be branches of the same function $M(z, \alpha)$ monogenic in (α). Hence Theorem 11.2 yields the following.

Theorem 12. 2. *Necessary and sufficient conditions that two admissible meromorphic functions f^1 and f^2 with the same characteristic set (β) be terminally restrictedly and meromorphically equivalent are that either*

(i) *$\mu > 0$ or $\mu = 0$ and m be odd, or*

(ii) *$\mu = 0$, $m = 4, 6, 8, \ldots$, and the invariants (I^1) and (I^2) of f^1 and f^2 with respect to (α) be in the same category, or*

(iii) *$\mu = 0$, $m = 2$ and $(I^1) = (I^2)$.*

We complete this section with the following theorem.

Theorem 12. 3. (a) *Any two admissible meromorphic functions f^1 and f^2 with the same characteristic set (β) are semi-restrictedly and meromorphically equivalent except in the case $\mu = 0$ and $m = 2$.*

(b) *In case* $\mu = 0$ *and* $m = 2$, *and* a_1 *is a pole*, f^1 *and* f^2 *are equivalent in the sense of* (a) *if and only if their invariants* J_1 *with respect to* (a_0, a_1) *are equal. If* a_1 *is a zero*, f^1 *and* f^2 *are equivalent if and only if their invariants* J_1 *with respect to* (a_0, a_1) *are equal in absolute value.*

Statement (a) follows from Theorem 12.2 except when $\mu = 0$, $m = 4, 6, 8, \ldots$, and (I^1) and (I^2) are of opposite category. In this case one first deforms f^1 into its (β)-projection F^1. One then continues F^1 with respect to (α) along any admissible path (α^t) that leads to a set (α^1) which is an admissible odd permutation of (β). In accordance with Lemma 9.2, the canonical function F^* into which F^1 is thereby continued will have an invariant set (I^*) with respect to (β) of a category opposite to that of (I^1), and hence of a category which is the same as that of (I^2). It follows from Theorem 12.2 (ii) that F^* can be terminally restrictedly and meromorphically deformed into f^2. This completes the proof of (a).

The necessary conditions for equivalence of f^1 and f^2 in (b) have already been affirmed in Lemma 9.4.

We come to sufficient conditions. When the invariants J_1 with respect to (β) are equal, a restricted deformation carries f^1 into f^2. There remains the case of two zeros in which the invariants J_1 with respect to (β) are equal but opposite in sign. In this case we regard S as a hyperbolic plane and make a hyperbolic rotation about the hyperbolic mid-point of a_0 and a_1, carrying a_0 into a_1. Let η^t be the analytic, isotopic deformation of S thereby defined, $0 \leqq t \leqq 1$. The composite function $f^1 \eta^t$ deforms f^1 into a function f^* which in accordance with (9.15) has the same invariant J_1 with respect to (β) as does f^2. Hence f^* can be restrictedly deformed into f^2.

Statement (b) follows.

§ 13. *Equivalence of interior transformations.* Let f be an interior transformation of S with a characteristic set (β). By virtue of a theorem of Stoilow (4) there exists a homeomorphism ζ from a simply-connected region R of the z-plane to S such that the composite function $f\zeta$ is meromorphic on R. When the boundary of R consists of more than a point, R can be mapped directly conformally onto S. In such a case we can suppose that ζ is a homeomorphism from S to S, and we say that f then comes under *Case A*.

When the boundary of R consists of a point, one can first restrictedly deform f on S into a function f_1 that comes under Case A. To this end let e be so small a positive constant that all points of (β) lie on the disc $\{|z| < 1 - 2e\}$.

Let ζ^t be an isotopic deformation of S onto $S' = \{|z| < 1 - e\}$ which leaves points of S at which $|z| \leqq 1 - 2e$ fixed. Thus ζ^1 is a homeomorphism from S to S'. The composite function $f\zeta^t$ deforms f into a function f_1 again defined over S. The function f_1 comes under Case A. For the Riemann image of S under $w = f_1(z)$ simply covers the Riemann image under $w = f(z)$ of S'. This Riemann image is then conformally equivalent to a subregion of R whose boundary consists of more than one point.

We have the following lemma concerning admissible interior transformations of S.

Lemma 13.1. *Any such transformation f^* of S admits a restricted deformation into a function f for which a homeomorphism ζ of S exists such that $f\zeta$ is meromorphic on S.*

Use will be made of the canonical functions $M(z, \alpha)$ monogenic in (α) of the preceding sections. Let η^t be an isotopic deformation of S from the identity and let (α^t) be the image of (β) under η^t. The characteristic set of

$$M(\eta^t, \alpha^t) \qquad (0 \leqq t \leqq 1) \tag{13.1}$$

remains constantly (β). For example, if a_i represents a zero of (β), then $\eta^t(a_i) = a_i^t$ and

$$M(a_i^t, \alpha^t) = 0$$

since (α^t) is the characteristic set of $M(z, \alpha^t)$.

The basic theorem here is as follows.

Theorem 13.1. *Any interior transformation f with a characteristic set (β) admits a restricted deformation into its canonical β-projection.*

In accordance with Lemma 13.1 one can suppose that a homeomorphism ζ of S exists such that $f\zeta$ is meromorphic on S.

The proof of the theorem is relatively simple once the appropriate auxiliary functions are defined. These functions are as follows:

$\zeta(z) =$ A homeomorphism of S such that $f\zeta$ is meromorphic on S.

$\eta(z) =$ The inverse of ζ on S.

$\lambda(z) =$ The meromorphic function $f\zeta$. The characteristic set of λ is the image of (β) under η.

$\eta^t =$ An isotopic deformation of S from the identity, generating η. $(0 \leqq t \leqq 1)$.

$(\alpha^t) =$ The image of (β) under η^t at the time t. The characteristic set of λ is (α^1).

$F_1 =$ The (α^1)-projection $F(z, \alpha^1, r)$ of λ.

$\lambda^t =$ A restricted deformation of meromorphic type of λ into F_1. $(0 \leqq t \leqq 1)$.

$M(z, \alpha) =$ The function $M(z, \alpha)$ monogenic in (α) of which $F(z, \alpha, r)$ is a branch.

The required deformation of f will be given as a sequence of two deformations of t in both of which (β) remains the characteristic set.

The first deformation is defined by the composite function,

$$\lambda^t \eta \qquad (0 \leqq t \leqq 1). \tag{13.2}$$

When $t = 0$, this function reduces to f and when $t = 1$ it becomes $F_1 \eta$. The characteristic set of $\lambda^t \eta$ remains constantly that of $\lambda \eta$, namely (β).

The second deformation is defined by the functions

$$M(\eta^t(z), \alpha^t) \qquad (0 \leqq t \leqq 1) \tag{13.3}$$

with t however decreasing from 1 to 0. As noted in connection with (13.1), the characteristic set remains constantly (β). When $t = 1$ one starts with the branch

$$M(z, \alpha) = F(z, \alpha, r)$$

at the set (α^1). When $t = 1$, the function (13.3) reduces to

$$M(\eta, \alpha^1) = F_1 \eta.$$

When $t = 0$, the function (13.3) becomes $M(z, \beta)$. This is the (β)-projection of f. It has the characteristic set (β); it also has the invariants (I) of f since it is obtained from f by a restricted deformation.

The deformation (13.2) followed by the deformation (13.3), with decreasing t, thus deforms f into its (β)-projection, as required.

The deformation classes into which meromorphic functions may be divided are, in the sense of the following theorem, not affected by the inclusion or exclusion of interior transformations which are not meromorphic.

Theorem 13.2. *Two meromorphic functions f_1 and f_2 with the same characteristic set (β) are restrictedly, terminally restrictedly, or semi-restrictedly equivalent only if they are meromorphically equivalent in the same sense.*

Let F_1 and F_2 be the (β)-projections of f_1 and f_2 respectively. The functions f_1 and f_2 are restrictedly and meromorphically deformable into F_1 and F_2 respectively. Any admissible deformation f^t of f_1 into f_2 in which (α^t) is the characteristic set of f^t implies a deformation F^t of F_1 into F_2 through the (α^t)-projection of f^t. The deformation F^t will be restricted, terminally restricted, or semi-restricted in the same sense as f^t. In addition, F^t is of meromorphic type.

The theorem follows.

Corollary 13.1. *Conditions on the invariants (J) of two interior transformations f_1 and f_2 with the same characteristic set (β) together with conditions on μ and m which are necessary and sufficient for the equivalence of f_1 and f_2 in any one of the senses of the theorem are precisely those stated in § 12 for equivalence in the same sense of meromorphic functions.*

The functions f_1 and f_2 have the same characteristic set (β), and characteristic integers μ and m, and invariants (J^1) and (J^2) with respect to (β) as their respective (β)-projections F_1 and F_2. In accordance with Theorem 13.2 the equivalence of f_1 and f_2 in a given sense implies and is implied by the meromorphic equivalence in the same sense of F_1 and F_2. The conditions for the latter equivalence in terms of μ, m, (J^1) and (J^2) are thus the conditions for the equivalence of f_1 and f_2 in the given sense.

The theorem follows.

§ 14. *The special cases $m = 0$ and $m = 1$.* We admit μ branch point antecedents as previously. In case $m = 0$ or 1 there are no invariants J_i. The construction of a function with the given characteristic set as given in § 6 is still valid here. When $m = 1$ we assume that a_0 is a zero. The case of a pole is treated by considering f^{-1}.

In the meromorphic case the residual function φ is defined by the equation

$$\frac{f'}{f} = \frac{\varphi B}{(z - a_0)^m}$$

both when $m = 0$ and $m = 1$. A canonical function with a given characteristic set may be given in the form

$$F_m(z, a) = e^{\int \frac{B(z)}{B(c_0)} \frac{dz}{(z - a_0)^m}} \qquad (m = 0 \text{ or } 1).$$

The residual function must satisfy the condition

$$\varphi(a_0) = \frac{1}{B(a_0)} \qquad (m = 1);$$

it must be analytic and non-null on S.

If f_1 and f_2 are two meromorphic functions with the same characteristic set (a) and residual functions φ_1 and φ_2 the function

$$e^{\int \frac{\varphi_1^{1-t} \varphi_2^t B \, dz}{(z - a_0)^m}} \qquad (0 \leqq t \leqq 1)$$

suffices to deform restrictedly and meromorphically a constant multiple of f_1 into a constant multiple of f_2. Hence f_1 and f_2 are restrictedly and meromorphically equivalent. Any interior transformation f with the characteristic set (α) can be restrictedly deformed into its (α)-projection $F_m(z, \alpha)$ following the proof of Theorem 13. 1.

We summarize what is essential as follows:

Theorem 14. 1. *When* $m = 0$ *or* 1, *any interior transformation* f *with the characteristic set* (α) *can be restrictedly deformed into its* (α)*-projection* $F_m(z, \alpha)$, *meromorphically if* f *is meromorphic.*

In the very special case in which $m = 0$ and $\mu = 0$ the characteristic set (α) is empty, $B(z)$ may be taken as 1 and e^z as the canonical function into which each interior transformation with an empty characteristic set can be admissibly deformed.

§ 15. *Covering properties of sequences of meromorphic transformations of* S. We shall be concerned with infinite sequences $[f_k]$ of meromorphic transformations of S with the same characteristic set (α). The number m of zeros and poles in the set

$$(15. 1) \qquad (a_0, a_1, \ldots, a_n) = (a)$$

shall be at least two. A sequence $[f_k]$ in which each of the functions f_k is meromorphic and no two functions f_k are in the same restricted deformation class will be termed a *model sequence*. We shall be concerned with the set W of points $w = f_k(z)$ on the w-sphere given by a model sequence for z on S. When the set (15. 1) includes both zeros and poles, we shall see that the set W covers each point of the w-sphere infinitely many times. When the set (15. 1) consists of zeros alone, the set W will cover each point of the w-sphere infinitely many times ($w = \infty$ excepted) provided f_k does not converge uniformly to zero on every compact subset of S.

A term due to Carathéodory is convenient. A sequence $[F_n]$ of functions of z meromorphic on a region R is said to *converge continuously* to F on R if, with respect to the metric on the w-sphere, $[F_n]$ converges uniformly to F on every closed subset of R. The function F is necessarily meromorphic on R, or reduces to the function $F = \infty$.

Suppose that each function F_n is meromorphic on S and has the characteristic set (α). If $[F_n]$ converges continuously on S to F, then F is identically o or ∞ or else has the characteristic set (a). If $[F_n]$ converges continuously on $S-(a)$ to a function F not identically o or ∞, then $[F_n]$ converges continuously on S without exception. If F_n has no poles, the preceding statement remains valid with the condition $F \not\equiv 0$ omitted. These results follow from classical theorems.

The principal results of this section are an immediate consequence of theorems on normal families of functions. A family M of functions meromorphic on a region R of the z-plane is termed *normal* if corresponding to an arbitrary sequence $[F_n]$ of functions of the family there exists a subsequence which converges continuously on R to a meromorphic function F or to ∞. According to a theorem of Julia a necessary and sufficient condition that M be normal on R is that M be normal on some neighborhood of each point of R. Cf. (6), p. 37. The property which connects the theory of normal families with Picard's theorem is the following. If the family M is not normal on R, then every point w on the w-sphere, with the possible exception of two points, is covered infinitely many times by points $f(z)$ defined by members of the family for z on R. In particular if M is not normal on R, there exists at least one point z_0 on R in no neighborhood of which M is normal. Such a point z_0 is called *a point* J, and the images under the functions of M of an arbitrary neighborhood of a point J cover the w-sphere infinitely many times, two points at most excepted.

Let R be a subregion of S. A model sequence $[f_k]$ no subsequence of which converges continuously to o or ∞ on R will be termed *proper on* R. We shall prove the following lemma.

Lemma 15.1. *A model sequence which is proper on $S-(a)$ is not a normal family on $S-(a)$.*

Suppose the lemma false and that a subsequence $[F_n]$ of the given model sequence converges continuously on $S-(a)$ to a function F. The function F is analytic on $S-(a)$ and never zero. It follows from the definition of a residual function that the sequence $[\varphi_n]$ of residual functions of the functions F_n converges continuously on $S-(a)$ to a function φ which satisfies the relation

$$\frac{F'}{F} = \varphi \frac{B}{A} \tag{15.2}$$

on $S-(a)$, and is accordingly analytic and never zero on $S-(a)$. Hence the functions φ_n converge continuously on S without exception to a function φ that is analytic and never zero on S. The invariants J_i^n of F_n must then converge in accordance with Theorem 10.1. Since the invariants J_i^n differ for different values of n by integers, it follows that for n exceeding some integer n_0, J_i^n is independent of n.

This is contrary to the hypothesis that the members of the sequence $[f_k]$ belong to different restricted deformation classes. We infer the truth of the lemma.

We state a lemma concerning quasi-normal sequences. Cf. (6) p. 66.

Lemma 15.2. *Let $[F_n]$ be a sequence of functions which are meromorphic on a neighborhood N of a point z_0 and analytic and not zero on $N - z_0$. If $[F_n]$ converges continuously to ∞ (0) on $N - z_0$ but fails to so converge on N, then for n sufficiently large*[1] *and for some z on N, F_n assumes any given value w except ∞ (0).*

The two cases involved in this lemma are reducible to each other on replacing F_n by its reciprocal. The case in which $F \equiv \infty$ is treated in the above reference.

A point z_0 will be called a *covering point* relative to a model sequence $[f_k]$ if the totality of w-images of the functions f_k, $k = 1, 2, \ldots$, for z on an arbitrary neighborhood of z_0 cover the points of the w-sphere infinitely many times, $w = 0$ and $w = \infty$ at most excepted. The first covering theorem follows.

Theorem 15.1. *Let $[f_k]$ be a model sequence of meromorphic functions.*

I. *If $[f_k]$ is proper on $S - (a)$, there exists at least one covering point on $S - (a)$ relative to $[f_k]$.*

II. *If a subsequence of $[f_k]$ converges continuously to ∞ [0] on $S - (a)$, then any zero [pole*[2]*] a_i is a covering point relative to $[f_k]$.*

Case I *or* II *always occurs.*

In Case I $[f_k]$ is not normal on $S - (a)$ in accordance with Lemma 15.1. There accordingly exists a point z_0 of type J on $S - (a)$. For any neighborhood of z_0 relative to $S - (a)$, $w = 0$ and $w = \infty$ are values not taken on by $[f_k]$. Hence every other value is taken on infinitely often. Thus z_0 is a covering point relative to $[f_k]$. In Case II the theorem follows from Lemma 15.2.

Corollary. *If $[f_k]$ is a model sequence there exists at least one covering point on S, excepting only the case in which there are no poles in (a) and $[f_k]$ converges continuously to zero on S.*

Recall that the case in which there are no zeros in (a) has been excluded. If admitted, this case would parallel the case in which there are no poles.

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[1] That is for n exceeding an integer n_0 depending on N and w.

[2] If there are any poles.

Given a model sequence $[f_k]$ which is not proper, a proper model sequence can be readily obtained by replacing each f_k by a constant multiple $c_k f_k$ with c_k suitably chosen. In particular it is sufficient to choose the constants c_k so that $|c_k f_k|$ is bounded from 0 and ∞ at some point not in (a).

The following theorem is a consequence of Bloch's theorem. Cf. (9) p. 230.

Theorem 15.2. *If $[f_k]$ is a model sequence no subsequence of which converges to 0 on S, and if the characteristic set (a) includes no poles, there then corresponds to any positive constant r, no matter how large, a member $f_{k(r)}$ of the sequence and a circular disc D_r of radius r in the w-plane such that D_r is the one-to-one image under $f_{k(r)}$ of some subdomain of S.*

This theorem follows from Bloch's theorem provided the values

$$|f_k'| \qquad (k = 1, \ldots,) \tag{15.3}$$

are unbounded on some compact subset of S.

Let S_c be the subdisc of S concentric with S and of radius c. If the values (15.3) admitted a bound M_c on S_c for each c, $0 < c < 1$, $|f_k|$ would be bounded independently of k on each S_c and $[f_k]$ would be normal on S. Since no subsequence of $[f_k]$ converges continuously to zero by hypothesis, the sequence $[f_k]$ would then be proper. This is contrary to Lemma 15.1. Thus for some $c < 1$ the values (15.3) are unbounded on S_c. Hence there are points z on this S_c and values of k for which $|f_k'|$ is arbitrarily large.

The theorem follows.

The preceding theorems can be extended as follows. Let J_0 be defined as 0. Let a sequence $[f_k]$ of meromorphic functions with the given characteristic set be termed *model* with respect to two points (a_r, a_s) in (a) if no two pairs (J_r, J_s) and (J_r^*, J_s^*) belonging to different functions in the sequence $[f_k]$ have the property that

$$J_s^* - J_r^* = J_s - J_r. \tag{15.4}$$

If a_r and a_s are not both poles, we could suppose that a_r is a zero and change the notation so that $a_r = a_0$. The condition (15.5) would then take the form

$$J_s^* = J_s.$$

If a_r and a_s are both poles one could replace each function f_k by its reciprocal, noting that the residual function of f_k and its reciprocal are negatives of each other.

Lemma 15.1 can now be replaced by the following.

Lemma 15.3. *A sequence $[f_k]$ which is model with respect to a_r and a_s and which is proper on $N-(a_r, a_s)$, where N is any connected neighborhood of a_r and a_s, is not a normal family on $N-(a_r, a_s)$.*

With a_r taken as a_0 the proof is essentially the same as that of Lemma 15.1. Lemma 15.2 is unchanged for present purposes. Theorem 15.1 takes the following form.

Theorem 15.3. *Let $[f_k]$ be a sequence which is model with respect to a_r and a_s and let N be any connected neighborhood of a_r and a_s.*

I. *If $[f_k]$ is proper on $N-(a_r, a_s)$, there exists at least one covering point on $N-(a_r, a_s)$ relative to $[f_k]$.*

II. *If a subsequence of $[f_k]$ converges continuously to ∞ [0] on $N-(a_r, a_s)$, then any zero [pole] in the pair (a_r, a_s) is a covering point relative to $[f_k]$.*

Case I *or* II *always occurs.*

The set of points J taken relative to a model sequence $[f_k]$ is *closed* on S by virtue of the definition of a point J. One can then establish the following.

Theorem 15.4. *If a model sequence $[f_k]$ is proper on every subregion of a region R then the set E of points J on R is perfect (possibly empty) relative to R. Each non-empty component E_1 of E contains at least one zero and one pole of (a), or else has a limit point on the boundary of R.*

No point z_0 of E can be isolated in E. To see this let N be a neighborhood of z_0 such that $N-z_0$ contains no point of (a). Continuous convergence of a subsequence $[F_n]$ of $[f_k]$ on $N-z_0$ to a function F implies that F is analytic and not null on $N-z_0$ since $[f_k]$ is proper on $N-z_0$ and no f_k has a zero or pole on $N-z_0$. It follows that $[F_n]$ converges continuously on N so that z_0 cannot be a point J. We infer that z_0 is not isolated in E and that E is accordingly perfect relative to R.

Suppose that $\bar{E}_1$ is on R. We shall show that E_1 must contain at least one zero and one pole in (a). Suppose in particular that E_1 contained no pole in (a). Then E_1 could be separated on R from the poles in (a) and from the boundary of R by a finite set (g) of regular Jordan curves on R which do not intersect E or (a) and bound a subregion R_1 of R on which E_1 lies. The sequence $[f_k]$ would be normal on a neighborhood N of (g). On N a subsequence $[F_n]$ of $[f_k]$ would converge continuously to a function F which would be analytic and never zero on $N-(a)$. It follows that $[F_n]$ would converge continuously on R_1 so that E_1 would be empty.

The supposition that $\bar{E}_1$ is on R and contains no zero in (a) would similarly lead to the conclusion that E_1 is empty. The theorem follows.

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