DETERMINATION OF THE DENOMINATOR OF FREDHOLM IN SOME TYPES OF INTEGRAL EQUATIONS.

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Contents.

Introduction.

1. Consider a homogeneous integral equation of the Fredholm type with kernel $K(x, y)$ and interval of integration $(0, 1)$:

$$\varphi(x) = \lambda \int_0^1 K(x, y)\, \varphi(y)\, dy. \tag{1.1}$$

x is restricted to the interval $0 \le x \le 1$, and thus the kernel varies in the square $0 \le x \le 1$, $0 \le y \le 1$, its »existence-square». Suppose that $K(x, y)$ is 0 on the one side of a certain curve in this square and of the form

$$K(x, y) = P_1(x)\, Q_1(y) + P_2(x)\, Q_2(y) + \cdots + P_N(x)\, Q_N(y) \tag{1.2}$$

on the other side. The purpose of this paper is to give a method of determining the characteristic values of the corresponding integral equations. They will be obtained as the zeros of certain expressions which will be shown to coincide with the denominators of Fredholm.

An equation of the Volterra type belongs to this class if it has a kernel of the form (1.2), because $K(x, y)$ is 0 above the line $y = x$, a diagonal in the square. It is well-known that this integral equation has no characteristic values.

Our class of integral equations might be considered as a generalisation in a certain direction of the equation of Volterra. There will appear simple relations between the number of characteristic values, the form of the curve, and the number N characterizing $K(x, y)$ in (1.2).

Integral equations of a type including this have been studied by G. Andreoli: »Sulle equazioni integrali», Rendiconti del Circolo Matematico di Palermo 37 (1914), p. 76—112. He investigates the non-homogeneous equations and does not obtain explicit expressions for the denominators of Fredholm.

In Chapter I we shall study a simple case, supposing that we have a line parallel to the diagonal $y = x$ and situated above it and that the kernel is 0 above the line. We first treat the case where the kernel $K(x, y) = 1$ below the line and then generalize to $K(x, y) = P(x)\, Q(y)$ and finally to $K(x, y)$ of the form (1.2).

Chapter II is devoted to kernels, the boundary line in which is still straight but no longer parallel to the diagonal $y = x$. We assume that the line is situated completely above the diagonal and that the kernel is 0 above the line. These kernels naturally include those of Chapter I. Special attention has to be paid to the case where the line goes through the point (0. 0) or (1. 1).

In Chapter III we shall suppose that the line is still straight but crosses the diagonal $y = x$.

In Chapter IV we shall determine the characteristic values of an integral equation with a symmetric kernel that is 1 in the strip between two lines symmetric with respect to the diagonal $y = x$ and 0 in the rest of the square.

Chapter V generalizes the results of Chapters I—III to lines no longer straight. Further we shall show that the expressions obtained are the denominators of Fredholm of the integral equations.

CHAPTER I.

The Line $y = x + a$. $(0 < a \leq 1)$.

2. $K(x,y) = 1$.

We start with a simple case: suppose that the kernel is 0 above a straight line parallel to the diagonal $y = x$ and 1 below it. Choose the line (fig. 1):

$$y = x + a; \ (0 \leq a \leq 1).$$

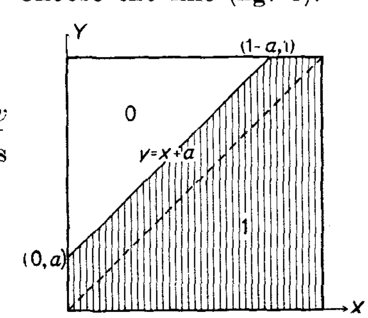

Fig. 1.

The following notation will be useful: let $\underline{u, v}$ be the lesser and $\overline{u, v}$ the greater of the real numbers u and v.

The integral equation may be written:

$$\varphi(x) = \lambda \int_0^{\underline{x+a,\,1}} \varphi(y)\, dy. \tag{2. 1}$$

Derivation gives:

$$\varphi'(x) = \begin{cases} \lambda\, \varphi(x + a), & \text{if } 0 < x < 1 - a; \\ 0, & \text{if } 1 - a < x < 1. \end{cases} \tag{2. 2}$$

From (2. 1) follows that $\varphi(x)$ is a continuous function because $\underline{x + a,\ 1}$ is continuous. Because $\varphi'(x)$ is 0 in the interval $1 - a < x < 1$, $\varphi(x)$ is constant there: $\varphi(x) = \varphi(1 - a) = \varphi(1)$. In the interval $1 - 2a \leq x \leq 1 - a$ (2. 2) gives:

$$\varphi'(x) = \lambda\, \varphi(x + a) = \lambda\, \varphi(1).$$

By integration we obtain $\varphi(x)$ as a polynomial in x and λ of degree 1. The constant of integration is determined so as to make $\varphi(x)$ continuous at the point $x = 1 - a$.

In the same way, in the interval $1 - 3a \leq x \leq 1 - 2a$ $\varphi(x)$ is obtained by integration of the expression for $\varphi(x)$ in $1 - 2a \leq x \leq 1 - a$ and adapting the constant of integration to make $\varphi(x)$ continuous at $x = 1 - 2a$. It is a polynomial in x and λ of degree 2.

So we can go on determining $\varphi(x)$ in the intervals $1 - 4a \leq x \leq 1 - 3a$, $1 - 5a \leq x \leq 1 - 4a, \ldots$. Finally $\varphi(x)$ is obtained in the whole interval $0 \leq \leq x \leq 1$ and expressed as $\varphi(1)$. Putting this into the equation:

$$\varphi(1) = \lambda \int_0^1 \varphi(y)\, dy,$$

we obtain the condition that λ has to satisfy in order to be a characteristic value.

The calculations are simplified by using integration by parts.

Integrate (2. 1) by parts:

$$\varphi(x) = \lambda \Big[y\,\varphi(y)\Big]_0^{\overline{x+a,\,1}} - \lambda \int_0^{\overline{x+a,\,1}} y\,\varphi'(y)\, dy.$$

The expression in square brackets vanishes at the lower limit, therefore the lower limit will be omitted here and in similiar expressions in the sequel. In the integral substitute (2. 2) for $\varphi'(x)$:

$$\varphi(x) = \lambda \Big[y\,\varphi(y)\Big]^{\overline{x+a,\,1}} - \lambda^2 \int_0^{\overline{x+a,\,1-a}} \varphi(y + a)\, d\frac{y^2}{2!}.$$

Note that the upper limit in the integral is reduced to $\underline{x + a,\ 1 - a}$. Integrating by parts again, the result is:

$$\varphi(x) = \lambda \Big[y\,\varphi(y)\Big]^{\overline{x+a,\,1}} - \frac{\lambda^2}{2!}\Big[y^2\,\varphi(y + a)\Big]^{\overline{x+a,\,1-a}} + \lambda^3 \int_0^{\overline{x+a,\,1-2a}} \varphi(y + 2a)\, d\frac{y^3}{3!}.$$

Hence each integration by parts reduces the upper limit in the integral. Repeating the process n times, n being determined by the inequality

$$\frac{1}{n} \leq a < \frac{1}{n - 1}, \tag{2. 3}$$

the integral vanishes and the following formula is obtained:

$$(2.4)\qquad \varphi(x) = \lambda\Big[y\,\varphi(y)\Big]^{\overline{x+a,\,1}} - \frac{\lambda^2}{2!}\Big[y^2\varphi(y+a)\Big]^{\overline{x+a,\,1-a}} + $$
$$+ \frac{\lambda^3}{3!}\Big[y^3\varphi(y+2a)\Big]^{\overline{x+a,\,1-2a}} - \cdots + (-1)^{n-1}\frac{\lambda^n}{n!}\Big[y^n\varphi[y+(n-1)a]\Big]^{\overline{x+a,\,1-(n-1)a}}.$$

Writing this for the various intervals, we get:

$$(2.4')\qquad 1-a \le x \le 1 : \varphi(x) = \lambda\varphi(1) - \frac{\lambda^2}{2!}(1-a)^2\varphi(1) +$$
$$+ \frac{\lambda^3}{3!}(1-2a)^3\varphi(1) - \cdots + (-1)^{n-1}\frac{\lambda^n}{n!}[1-(n-1)a]^n\varphi(1);$$

$$1-2a \le x \le 1-a : \varphi(x) = \lambda(x+a)\varphi(x+a) - \frac{\lambda^2}{2!}(1-a)^2\varphi(1) +$$
$$+ \frac{\lambda^3}{3!}(1-2a)^3\varphi(1) - \cdots + (-1)^{n-1}\frac{\lambda^n}{n!}[1-(n-1)a]^n\varphi(1);$$

$$1-3a \le x \le 1-2a : \varphi(x) = \lambda(x+a)\varphi(x+a) - \frac{\lambda^2}{2!}(x+a)^2\varphi(x+2a) +$$
$$+ \frac{\lambda^3}{3!}(1-2a)^3\varphi(1) - \cdots + (-1)^{n-1}\frac{\lambda^n}{n!}[1-(n-1)a]^n\varphi(1);$$

$$\cdots\qquad\cdots\qquad\cdots\qquad\cdots\qquad\cdots\qquad\cdots$$

$$0 \le x \le 1-(n-1)a : \varphi(x) = \lambda(x+a)\varphi(x+a) - \frac{\lambda^2}{2!}(x+a)^2\varphi(x+2a) +$$
$$+ \frac{\lambda^3}{3!}(x+a)^3\varphi(x+3a) - \cdots + (-1)^{n-2}\frac{\lambda^{n-1}}{(n-1)!}(x+a)^{n-1}\varphi[x+(n-1)a] +$$
$$+ (-1)^{n-1}\frac{\lambda^n}{n!}[1-(n-1)a]^n\varphi(1).$$

This gives different analytic expressions for $\varphi(x)$ in the intervals

$$1-a \le x \le 1,\quad 1-2a \le x \le 1-a,\quad \ldots,\quad 0 \le x \le 1-(n-1)a,$$

and we are able to calculate them successively. Putting $x = 1$, the first equation of $(2.4')$ becomes:

$$\varphi(1) = \lambda\varphi(1) - \frac{\lambda^2}{2!}(1-a)^2\varphi(1) +$$
$$+ \frac{\lambda^3}{3!}(1-2a)^3\varphi(1) - \cdots + (-1)^{n-1}\frac{\lambda^n}{n!}[1-(n-1)a]^n\varphi(1).$$

But $\varphi(1) = 0$ implies $\varphi(x) \equiv 0$. Hence the characteristic values necessarily satisfy:

$$(2.5) \qquad 1 - \lambda + \frac{\lambda^2}{2!}(1-a)^2 - \\ - \frac{\lambda^3}{3!}(1-2a)^3 + \cdots + \frac{(-\lambda)^n}{n!}[1-(n-1)a]^n = P_1(\lambda) = 0.$$

Every root of (2.5) is a characteristic value. This follows from the fact that, λ being a root of (2.5), the corresponding characteristic function is determined by (2.4'). We also see that this function is unique.

$P_1(\lambda)$ is a polynomial of degree n, and it will be shown in **10** that it is just the denominator of Fredholm. When a varies and is not the inverted value of an integer the characteristic values vary continuously with a. When a decreasing passes through the inverted value of an integer a new characteristic value enters from $+\infty$. Thus, when a tends to 0, the number of characteristic values grows infinitely and their magnitude, too. In the limiting case, the Volterra integral equation, there are no characteristic values.

3. $K(x,y) = P(x)\, Q(y)$.

We now generalize the integral equation (2.1) supposing that the kernel is a function of the form $K(x,y) = P(x)\, Q(y)$ under the line $y = x + a$. Assume that $P(x)\, Q(x)$ is integrable. The integral equation is:

$$(3.1) \qquad \varphi(x) = \lambda P(x) \int_0^{\overline{x+a,1}} Q(y)\, \varphi(y)\, dy.$$

The transformation $\varphi_1(x) = \int_0^{\overline{x+a,1}} Q(y)\, \varphi(y)\, dy$ implies $\varphi(x) = \lambda P(x)\, \varphi_1(x)$ and thus changes (3.1) into:

$$(3.2) \qquad \varphi_1(x) = \lambda \int_0^{\overline{x+a,1}} P(y)\, Q(y)\, \varphi_1(y)\, dy.$$

It is evident that (3.1) and (3.2) have the same characteristic values.

The derivative of $\varphi_1(x)$ is:

$$\varphi_1'(x) = \begin{cases} \lambda P(x+a)\, Q(x+a)\, \varphi_1(x+a), & \text{if } 0 < x < 1-a; \\ 0 & \text{, if } 1-a < x < 1. \end{cases}$$

Introduce the following notations:

$$
\begin{cases}
f_1(x) = \int\limits_0^x P(y)\; Q(y)\,dy; \\
f_2(x) = \int\limits_0^x f_1(y)\; P(y+a)\; Q(y+a)\,dy; \\
\dots \qquad \dots \qquad \dots \\
f_\nu(x) = \int\limits_0^x f_{\nu-1}(y)\; P(y+(\nu-1)a)\; Q(y+(\nu-1)a)\,dy; \\
\dots \qquad \dots \qquad \dots
\end{cases}
$$

We integrate (3. 2) by parts (observe that all expressions in brackets vanish at the lower limit):

$$
\varphi_1(x) = \lambda \int\limits_0^{\overline{x+a,\,1}} \varphi_1(y)\,df_1(y) = \lambda \Big[f_1(y)\,\varphi_1(y)\Big]^{\overline{x+a,\,1}} -
$$

$$
- \lambda \int\limits_0^{\overline{x+a,\,1}} f_1(y)\,\varphi_1'(y)\,dy = \lambda\Big[f_1(y)\,\varphi_1(y)\Big]^{\overline{x+a,\,1}} - \lambda^2 \int\limits_0^{\overline{x+a,\,1-a}} f_1(y)\;\; P(y+a)\;\; Q(y+a)\;\varphi_1(y+a)\,dy =
$$

$$
= \lambda\Big[f_1(y)\,\varphi_1(y)\Big]^{\overline{x+a,\,1}} - \lambda^2 \int\limits_0^{\overline{x+a,\,1-a}} \varphi_1(y+a)\,df_2(y).
$$

Repeat the process n times, n being the number determined by (2. 3). The result is:

$$
\begin{aligned}
(3.\,3) \qquad \varphi_1(x) = {} & \lambda\Big[f_1(y)\,\varphi_1(y)\Big]^{\overline{x+a,\,1}} - \lambda^2\Big[f_2(y)\,\varphi_1(y+a)\Big]^{\overline{x+a,\,1-a}} + \\
& + \lambda^3\Big[f_3(y)\,\varphi_1(y+2a)\Big]^{\overline{x+a,\,1-2a}} - \cdots + (-1)^{n-1}\lambda^n\Big[f_n(y)\,\varphi_1[y+(n-1)a]\Big]^{\overline{x+a,\,1-(n-1)a}}.
\end{aligned}
$$

This is a relation of the same kind as (2. 4) and may be written as (2. 4′). We see immediately that to every characteristic value there corresponds a single characteristic function. The characteristic values are obtained by putting $x = 1$ in (3. 3):

$$
\begin{aligned}
1 - \lambda f_1(1) + \lambda^2 f_2(1-a) - \lambda^3 f_3(1-2a) + \cdots + (-\lambda)^n f_n[1-(n-1)a] = {} & \\
= P_2(\lambda) = 0. &
\end{aligned}
$$

$P_2(\lambda)$ is a polynomial in λ of degree at most n whose zeros are the characteristic values of (3. 1) and (3. 2). In **12** the identity of $P_2(\lambda)$ and the Fredholm denominators of (3. 1) and (3. 2) will be shown.

4. $K(x,y) = \sum_{\nu=1}^{N} P_\nu(x)\, Q_\nu(y)$.

Next consider a kernel of the form (1. 2) below the line $y = x + a$. The corresponding integral equation is:

$$\varphi(x) = \lambda \int_0^{\overline{x+a,\,1}} \left(\sum_{\nu=1}^{N} \{ P_\nu(x)\, Q_\nu(y) \} \right) \varphi(y)\, dy. \tag{4. 1}$$

We introduce:

$$\varphi_\nu(x) = \int_0^{\overline{x+a,\,1}} Q_\nu(y)\, \varphi(y)\, dy.$$

This gives:

$$\varphi(x) = \lambda \sum_{\nu=1}^{N} P_\nu(x)\, \varphi_\nu(x).$$

(4. 1) is transformed into the following system of integral equations:

$$\varphi_\mu(x) = \lambda \int_0^{\overline{x+a,\,1}} \left(\sum_{\nu=1}^{N} P_\nu(y)\, Q_\mu(y)\, \varphi_\nu(y) \right) dy. \tag{4. 2}$$

$$(\mu = 1, 2, \ldots, N).$$

Derivation gives:

$$\varphi_\mu'(x) = \begin{cases} \lambda \sum_{\nu=1}^{N} P_\nu(x+a)\, Q_\mu(x+a)\, \varphi_\nu(x+a), & \text{if } 0 < x < 1 - a; \\ 0, & \text{if } 1 - a < x < 1. \end{cases}$$

The following notations will be used:

$$f_{\nu\mu}^{(1)}(x) = \int_0^x P_\nu(y)\, Q_\mu(y)\, dy;$$

$$f_{\nu\mu}^{(2)}(x) = \int_0^x \left(\sum_{\tau=1}^{N} f_{\tau\mu}^{(1)}(y)\, P_\nu(y+a)\, Q_\tau(y+a) \right) dy;$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$f_{\nu\mu}^{(\alpha)}(x) = \int_0^x \left(\sum_{\tau=1}^{N} f_{\tau\mu}^{(\alpha-1)}(y)\, P_\nu[y + (\alpha-1)a]\, Q_\tau[y + (\alpha-1)a] \right) dy;$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Integrate (4. 2) by parts:

$$\varphi_\mu(x)=\lambda\int\limits_0^{\underline{x+a,\,1}}\left(\sum_{\nu=1}^N \varphi_\nu(y)\,d f^{(1)}_{\nu\mu}(y)\right)=\lambda\left[\sum_{\nu=1}^N \varphi_\nu(y) f^{(1)}_{\nu\mu}(y)\right]^{\underline{x+a,\,1}}-$$

$$-\lambda\int\limits_0^{\underline{x+a,\,1}}\left(\sum_{\nu=1}^N f^{(1)}_{\nu\mu}(y)\,\varphi'_\nu(y)\right)dy=\lambda\left[\sum_{\nu=1}^N \varphi_\nu(y) f^{(1)}_{\nu\mu}(y)\right]^{\underline{x+a,\,1}}-$$

$$-\lambda^2\int\limits_0^{\underline{x+a,\,1-a}}\left\{\sum_{\nu=1}^N f^{(1)}_{\nu\mu}(y)\sum_{\tau=1}^N P_\tau(y+a)\,Q_\nu(y+a)\,\varphi_\tau(y+a)\right\}dy=$$

$$=\lambda\left[\sum_{\nu=1}^N \varphi_\nu(y) f^{(1)}_{\nu\mu}(y)\right]^{\underline{x+a,\,1}}-\lambda^2\int\limits_0^{\underline{x+a,\,1-a}}\left(\sum_{\nu=1}^N \varphi_\nu(y+a)\,d f^{(2)}_{\nu\mu}(y)\right).$$

The upper limit in the integral is reduced to $\underline{x+a,\,1-a}$. After n integrations by parts (n being determined by (2. 3)), the integral vanishes:

$$\varphi_\mu(x)=\lambda\left[\sum_{\nu=1}^N \varphi_\nu(y) f^{(1)}_{\nu\mu}(y)\right]^{\underline{x+a,\,1}}-\lambda^2\left[\sum_{\nu=1}^N \varphi_\nu(y+a) f^{(2)}_{\nu\mu}(y)\right]^{\underline{x+a,\,1-a}}+ \tag{4. 3}$$

$$+\lambda^3\left[\sum_{\nu=1}^N \varphi_\nu(y+2a) f^{(3)}_{\nu\mu}(y)\right]^{\underline{x+a,\,1-2a}}-\cdots+(-1)^{n-1}\lambda^n\left[\sum_{\nu=1}^N \varphi_\nu[y+(n-1)a] f^{(n)}_{\nu\mu}(y)\right]^{\underline{x+a,\,1-(n-1)a}}.$$

This formula corresponds to (2. 4) and might be written as (2. 4′) giving the different analytic expressions for $\varphi_\mu(x)$, $\mu=1,2,\ldots N$, in each of the intervals $1-a\le x\le 1$, $1-2a\le x\le 1-a$, $\ldots$, $0\le x\le 1-(n-1)a$. Reasoning as in **2** we see that the characteristic values are obtained by putting $x=1$:

$$\varphi_\mu(1)=\lambda\sum_{\nu=1}^N \varphi_\nu(1) f^{(1)}_{\nu\mu}(1)-\lambda^2\sum_{\nu=1}^N \varphi_\nu(1) f^{(2)}_{\nu\mu}(1-a)+ \tag{4. 4}$$

$$+\lambda^3\sum_{\nu=1}^N \varphi_\nu(1) f^{(3)}_{\nu\mu}(1-2a)-\cdots+(-1)^{n-1}\lambda^n\sum_{\nu=1}^N \varphi_\nu(1) f^{(n)}_{\nu\mu}[1-(n-1)a].$$

$$(\mu=1,\,2,\,\ldots,\,N).$$

Abbreviate:

$$P_{\nu\mu}(\lambda) = \lambda f_{\nu\mu}^{(1)}(1) - \lambda^2 f_{\nu\mu}^{(2)}(1 - a) + \cdots + (-1)^{n-1} \lambda^n f_{\nu\mu}^{(n)}[1 - (n-1)a].$$

Then (4. 4) becomes:

$$\begin{cases} \varphi_1(1) = \varphi_1(1) P_{11}(\lambda) + \varphi_2(1) P_{21}(\lambda) + \cdots + \varphi_N(1) P_{N1}(\lambda); \\ \varphi_2(1) = \varphi_1(1) P_{12}(\lambda) + \varphi_2(1) P_{22}(\lambda) + \cdots + \varphi_N(1) P_{N2}(\lambda); \\ \quad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ \varphi_N(1) = \varphi_1(1) P_{1N}(\lambda) + \varphi_2(1) P_{2N}(\lambda) + \cdots + \varphi_N(1) P_{NN}(\lambda). \end{cases}$$

This system of equations, linear and homogeneous in $\varphi_1(1), \varphi_2(1), \ldots, \varphi_N(1)$, has a solution different from $\varphi_1(1) = \varphi_2(1) = \cdots = \varphi_N(1) = 0$ when and only when its determinant vanishes. $\varphi_1(1) = \varphi_2(1) = \cdots = \varphi_N(1) = 0$ implies $\varphi_1(x) = \varphi_2(x) = = \cdots = \varphi_N(x) = 0$ everywhere in $0 \leq x \leq 1$. Hence the characteristic values necessarily satisfy:

$$\begin{vmatrix} P_{11}(\lambda) - 1 & P_{21}(\lambda) & \cdots & P_{N1}(\lambda) \\ P_{12}(\lambda) & P_{22}(\lambda) - 1 & \cdots & P_{N2}(\lambda) \\ \cdots & \cdots & & \cdots \\ P_{1N}(\lambda) & P_{2N}(\lambda) & \cdots & P_{NN}(\lambda) - 1 \end{vmatrix} = 0.$$

Every root of this equation is a characteristic value because the corresponding characteristic functions are determined by (4. 3). The left-hand member is a polynomial in λ of degree $n \cdot N$ at most.

CHAPTER II.

The Line $y = bx + a$. $\left(0 \leq a \leq 1,\ 0 \leq \frac{1-a}{b} \leq 1\right)$.

5. The line does not pass through (0, 0) or (1, 1).

Suppose that the line in the square is $y = bx + a$, $\left(0 < a \leq 1,\ 0 \leq \frac{1-a}{b} < 1\right)$. This line goes from the side (0, 0), (0, 1) to the side (0, 1), (1, 1), (fig. 2 a), it is thus situated above the diagonal $y = x$ and avoids the points (0, 0) and (1, 1). We assume the kernel to be 1 below the line and 0 above it. The kernel of 2 is the special case $b = 1$.

The integral equation becomes:

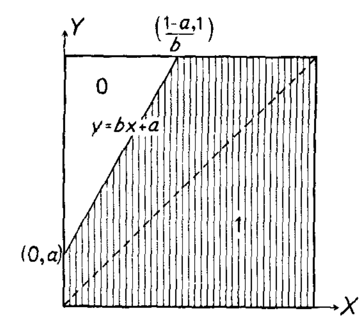

Fig. 2 *a*.

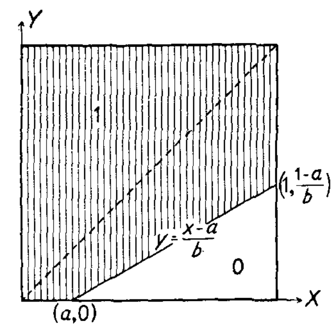

Fig. 2 *b*.

(5. 1)
$$\varphi(x) = \lambda \int_0^{\overline{bx+a,\ 1}} \varphi(y)\, dy.$$

The derivative is:

$$\varphi'(x) = \begin{cases} \lambda\, b\, \varphi(bx+a), & \text{if } 0 < x < \dfrac{1-a}{b}; \\ 0 & , \text{ if } \dfrac{1-a}{b} < x < 1. \end{cases}$$

Integrate (5. 1) by parts:

$$\varphi(x) = \lambda \Big[y\, \varphi(y) \Big]^{\overline{bx+a,\ 1}} - \lambda \int_0^{\overline{bx+a,\ 1}} y\, \varphi'(y)\, dy = \lambda \Big[y\, \varphi(y) \Big]^{\overline{bx+a,\ 1}} - \lambda^2 b \int_0^{\overline{bx+a,\ \frac{1-a}{b}}} \varphi(by+a)\, d\frac{y^2}{2!}.$$

The upper limit in the integral is reduced to $\underline{bx+a,\ \frac{1-a}{b}}$. Repeat the process with the new integral:

$$\lambda^2 b \int_0^{\overline{bx+a,\ \frac{1-a}{b}}} \varphi(by+a)\, d\frac{y^2}{2!} = \frac{\lambda^2}{2!} b \Big[y^2 \varphi(by+a) \Big]^{\overline{bx+a,\ \frac{1-a}{b}}} - \lambda^2 b^{1+1} \int_0^{\overline{bx+a,\ \frac{1-a}{b}}} \frac{y^2}{2!} \varphi'(by+a)\, dy =$$

$$= \frac{\lambda^2}{2!} b \Big[y^2 \varphi(by+a) \Big]^{\overline{bx+a,\ \frac{1-a}{b}}} - \lambda^3 b^{1+2} \int_0^{\overline{bx+a,\ \frac{1-a-ab}{b^2}}} \varphi(b^2 y + ab + a)\, d\frac{y^3}{3!}.$$

Repeat the integration again:

$$\lambda^3 b^{1+2} \int_0^{\overline{bx+a,\ \frac{1-a-ab}{b^2}}} \varphi(b^2 y + ab + a)\, d\frac{y^3}{3!} = \frac{\lambda^3}{3!} b^{1+2} \Big[y^3 \varphi(b^2 y + ab + a) \Big]^{\overline{bx+a,\ \frac{1-a-ab}{b^2}}} -$$

$$-\lambda^3 b^{1+2+2}\int_0^{bx+a,\,\frac{1-a-ab}{b^2}} \frac{y^3}{3!}\varphi'(b^2 y+ab+a)\,dy = \frac{\lambda^3}{3!}b^{1+2}\Big[y^3\varphi(b^2y+ab+a)\Big]^{bx+a,\,\frac{1-a-ab}{b^2}} -$$

$$-\lambda^4 b^{1+2+3}\int_0^{bx+a,\,\frac{1-a-ab-ab^2}{b^3}} \varphi(b^3y+ab^2+ab+a)\,d\frac{y^4}{4!}.$$

Generally:

$$\lambda^\nu b^{1+2+\cdots+(\nu-1)}\int_0^{bx+a,\,\frac{1-a-ab-\cdots-ab^{\nu-2}}{b^{\nu-1}}} \varphi(b^{\nu-1}y+ab^{\nu-2}+ab^{\nu-3}+\cdots+ab+a)\,d\frac{y^\nu}{\nu!} =$$

$$=\frac{\lambda^\nu}{\nu!}b^{1+2+\cdots+(\nu-1)}\Big[y^\nu\varphi(b^{\nu-1}y+ab^{\nu-2}+ab^{\nu-3}+\cdots+ab+a)\Big]^{bx+a,\,\frac{1-a-ab-\cdots-ab^{\nu-2}}{b^{\nu-1}}} -$$

$$-\lambda^{\nu+1}b^{1+2+\cdots+\nu}\int_0^{bx+a,\,\frac{1-a-ab-\cdots-ab^{\nu-1}}{b^\nu}} \varphi(b^\nu y+ab^{\nu-1}+ab^{\nu-2}+\cdots+ab+a)\,d\frac{y^{\nu+1}}{(\nu+1)!}.$$

After n steps the result is, summing up the arithmetic and geometric series:

$$\varphi(x)=\lambda\Big[y\,\varphi(y)\Big]^{bx+a,\,1}-\frac{\lambda^2}{2!}b\Big[y^2\varphi(by+a)\Big]^{bx+a,\,\frac{1-a}{b}}+ \tag{5.2}$$

$$+\frac{\lambda^3}{3!}b^{1+2}\left[y^3\varphi\left(b^2y+a\frac{b^2-1}{b-1}\right)\right]^{bx+a,\,\frac{1-a\frac{b^2-1}{b-1}}{b^2}}-\cdots+$$

$$+(-1)^{n-1}\frac{\lambda^n}{n!}b^{\frac{n(n-1)}{2}}\left[y^n\varphi\left(b^{n-1}y+a\frac{b^{n-1}-1}{b-1}\right)\right]^{bx+a,\,\frac{1-a\frac{b^{n-1}-1}{b-1}}{b^{n-1}}}+$$

$$+(-1)^n\lambda^{n+1}b^{\frac{n(n+1)}{2}}\int_0^{bx+a,\,\frac{1-a\frac{b^n-1}{b-1}}{b^n}}\varphi\left(b^ny+a\frac{b^n-1}{b-1}\right)d\frac{y^{n+1}}{(n+1)!}.$$

If n is determined by the inequality:

$$(5.3) \qquad a\frac{b^{n-1}-1}{b-1} < 1 \leq a\frac{b^n-1}{b-1},$$

the integral in (5.2) vanishes. Then (5.2) is analogous to (2.4) and may be written as (2.4'), giving the n different analytic expressions for $\varphi(x)$ in the n intervals

$$\frac{1-a}{b} \leq x \leq 1, \quad \frac{1-a\dfrac{b^2-1}{b-1}}{b^2} \leq x \leq \frac{1-a}{b}, \quad \cdots, \quad 0 \leq x \leq \frac{1-a\dfrac{b^{n-1}-1}{b-1}}{b^{n-1}}.$$

An argument identical with that of **2** shows that the characteristic values satisfy the equation obtained by putting $x = 1$ in (5.2):

$$(5.4) \qquad 1-\lambda+\frac{\lambda^2}{2!}\frac{1}{b}(1-a)^2-\frac{\lambda^3}{3!}\frac{1}{b^{1+2}}\left(1-a\frac{b^2-1}{b-1}\right)^3+\cdots+$$
$$+\frac{(-\lambda)^n}{n!}\frac{1}{b^{\frac{n(n-1)}{2}}}\left(1-a\frac{b^{n-1}-1}{b-1}\right)^n = P_3(\lambda) = 0.$$

$P_3(\lambda)$ is a polynomial in λ of degree n. To every characteristic value corresponds a single characteristic function. In **10** it will be shown that $P_3(\lambda)$ is the Fredholm denominator of (5.1). Kernels of the form (1.2) instead of 1 under the line are treated as in **3** and **4**. But these results are included in **12** and **13**.

6. The integral equation associated with that of 5.

Naturally it has the same characteristic values, but we shall show how the preceding method may be so modified as to be directly applicable.

The development that we shall obtain will be of interest in **7**.

The kernel is 0 in the square below the line $y = \dfrac{x-a}{b}$ and 1 above it (fig. 2 b). The integral equation is:

$$(6.1) \qquad \varphi(x) = \lambda \int_{0,\frac{x-a}{b}}^{1} \varphi(y)\,dy.$$

The derivative is:

$$\varphi'(x) = \begin{cases} 0, & \text{if } 0 < x < a; \\ \dfrac{-\lambda}{b}\varphi\left(\dfrac{x-a}{b}\right), & \text{if } a < x < 1. \end{cases}$$

Integrate (6. 1) by parts, arranging that the expressions in square brackets vanish at the upper limit:

$$\varphi(x)=\lambda\int_{\overline{0,\frac{x-a}{b}}}^{1}\varphi(y)\,dy=\lambda\int_{\overline{0,\frac{x-a}{b}}}^{1}\varphi(y)\,d\,(y-1)=\lambda\Big[(y-1)\,\varphi(y)\Big]_{\overline{0,\frac{x-a}{b}}}^{1}-\lambda\int_{\overline{0,\frac{x-a}{b}}}^{1}(y-1)\,\varphi'(y)\,dy=$$

$$=\lambda\Big[(y-1)\,\varphi(y)\Big]_{\overline{0,\frac{x-a}{b}}}^{1}+\frac{\lambda^2}{b}\int_{\overline{a,\frac{x-a}{b}}}^{1}\varphi\left(\frac{y-a}{b}\right)d\,\frac{(y-1)^2}{2!}.$$

Repeat the process with the new integral:

$$\frac{\lambda^2}{b}\int_{\overline{a,\frac{x-a}{b}}}^{1}\varphi\left(\frac{y-a}{b}\right)d\,\frac{(y-1)^2}{2!}=\frac{\lambda^2}{2!}\,\frac{1}{b}\left[(y-1)^2\,\varphi\left(\frac{y-a}{b}\right)\right]_{\overline{a,\frac{x-a}{b}}}^{1}-\frac{\lambda^2}{b^{1+1}}\int_{\overline{a,\frac{x-a}{b}}}^{1}\frac{(y-1)^2}{2!}\,\varphi'\left(\frac{y-a}{b}\right)dy=$$

$$=\frac{\lambda^2}{2!}\,\frac{1}{b}\left[(y-1)^2\,\varphi\left(\frac{y-a}{b}\right)\right]_{\overline{a,\frac{x-a}{b}}}^{1}+\frac{\lambda^3}{b^{1+2}}\int_{\overline{a+ab,\frac{x-a}{b}}}^{1}\varphi\left(\frac{y-a-ab}{b^2}\right)d\,\frac{(y-1)^3}{3!}.$$

Generally:

$$\frac{\lambda^\nu}{b^{\frac{\nu(\nu-1)}{2}}}\int_{\overline{a\frac{b^{\nu-1}-1}{b-1},\,\frac{x-a}{b}}}^{1}\varphi\left(\frac{y-a\,\frac{b^{\nu-1}-1}{b-1}}{b^{\nu-1}}\right)d\,\frac{(y-1)^\nu}{\nu!}=\frac{\lambda^\nu}{\nu!}\,\frac{1}{b^{\frac{\nu(\nu-1)}{2}}}\left[(y-1)^\nu\,\varphi\left(\frac{y-a\,\frac{b^{\nu-1}-1}{b-1}}{b^{\nu-1}}\right)\right]_{\overline{a\frac{b^{\nu-1}-1}{b-1},\,\frac{x-a}{b}}}^{1}+$$

$$+\frac{\lambda^{\nu+1}}{(\nu+1)!}\int_{\overline{a\frac{b^{\nu}-1}{b-1},\,\frac{x-a}{b}}}^{1}\varphi\left(\frac{y-a\,\frac{b^{\nu}-1}{b-1}}{b^{\nu}}\right)d\,\frac{(y-1)^{\nu+1}}{(\nu+1)!}.$$

The result after n integrations by parts is:

$$\text{(6. 2)}\qquad \varphi(x)=-\lambda\Big[(y-1)\,\varphi(y)\Big]^{\overline{0,\frac{x-a}{b}}}-\frac{\lambda^2}{2!}\,\frac{1}{b}\left[(y-1)^2\,\varphi\left(\frac{y-a}{b}\right)\right]^{\overline{a,\frac{x-a}{b}}}-$$

$$-\frac{\lambda^3}{3!}\,\frac{1}{b^{1+2}}\left[(y-1)^3\,\varphi\left(\frac{y-a\,\frac{b^2-1}{b-1}}{b^2}\right)\right]^{\overline{a\frac{b^2-1}{b-1},\,\frac{x-a}{b}}}-\cdots-$$

$$-\frac{\lambda^n}{n!}\frac{1}{b^{\frac{n(n-1)}{2}}}\left[(y-1)^n\varphi\left(\frac{y-a\dfrac{b^{n-1}-1}{b-1}}{b^{n-1}}\right)\right]^{\overline{a\frac{b^{n-1}-1}{b-1},\,\frac{x-a}{b}}}+\frac{\lambda^{n+1}}{b^{\frac{n(n+1)}{2}}}\int\limits_{\overline{a\frac{b^n-1}{b-1},\,\frac{x-a}{b}}}^{1}\varphi\left(\frac{y-a\dfrac{b^n-1}{b-1}}{b^n}\right)d\frac{(y-1)^{n+1}}{(n+1)!}.$$

If n is determined by the inequality (5. 3), the integral in (6. 2) vanishes. Then (6. 2) corresponds to (2. 4) and gives the expressions for $\varphi(x)$ in the intervals $0 \leq x \leq a$, $a \leq x \leq a\frac{b^2-1}{b-1}$, $a\frac{b^2-1}{b-1} \leq x \leq a\frac{b^3-1}{b-1}$, . . ., $a\frac{b^{n-1}-1}{b-1} \leq x \leq 1$. $\varphi(x)$ is constant in $0 \leq x \leq a$. Putting $x = 0$ and remarking that $\varphi(0) = 0$ implies $\varphi(x) = 0$ in $0 \leq x \leq 1$, we conclude that the characteristic values satisfy the equation obtained by dividing (6. 2) for $x = 0$ by $\varphi(0)$. We obtain (5. 4) as expected.

7. The line passes through (0, 0) or (1, 1).

In **5** and **6** two cases have been excluded: $a = 0$ and $\frac{1-a}{b} = 1$. Then the inequality (5. 3) determines no finite n and the expansion (5. 2) does not terminate. We shall deduce the equation for the characteristic values first by the preceding method and then by considering integral equations with analytic solutions. Another method will be given in **8** showing further that the functions obtained are the denominators of Fredholm.

(α) The case $a = 0$. ($b > 1$). The integral equation is:

$$\varphi(x) = \lambda\int_0^{\overline{bx,\,1}}\varphi(y)\,dy. \tag{7. 1}$$

The derivative becomes:

$$\varphi'(x) = \begin{cases}\lambda\, b\,\varphi(bx), & \text{if } 0 < x < \frac{1}{b};\\ 0, & \text{if } \frac{1}{b} < x < 1.\end{cases} \tag{7. 2}$$

Expansion (5.2) may be written:

$$\varphi(x) = \lambda\left[y\,\varphi(y)\right]^{\overline{bx,\,1}} - \frac{\lambda^2}{2!}b\left[y^2\varphi(by)\right]^{\overline{bx,\,\frac{1}{b}}} + \frac{\lambda^3}{3!}b^{1+2}\left[y^3\varphi(b^2y)\right]^{\overline{bx,\,\frac{1}{b^2}}} - \cdots + \tag{7. 3}$$

$$+ (-1)^{n-1}\frac{\lambda^n}{n!} b^{\frac{n(n-1)}{2}} \Big[y^n \varphi(b^{n-1} y)\Big]^{bx,\frac{1}{b^{n-1}}} + (-1)^n \lambda^{n+1} b^{\frac{n(n+1)}{2}} \int\limits_0^{bx,\frac{1}{b^n}} \varphi(b^n y)\frac{y^n}{n!}\,dy.$$

We first assume the existence of a characteristic function $\varphi(x)$. Putting $x = 1$ in (7. 3) and substituting t for $b^n y$ in the integral, we obtain:

$$\varphi(1) = \varphi(1)\left[\lambda - \frac{\lambda^2}{2!}\frac{1}{b} + \frac{\lambda^3}{3!}\frac{1}{b^{1+2}} - \cdots + (-1)^{n-1}\frac{\lambda^n}{n!}\frac{1}{b^{\frac{n(n-1)}{2}}}\right] +$$
$$+ (-1)^n \frac{\lambda^{n+1}}{n!}\frac{1}{b^{\frac{n(n+1)}{2}}}\int\limits_0^1 \varphi(t)\, t^n\, dt.$$

The integral $\int\limits_0^1 \varphi(t)\, t^n\, dt$ is bounded for every n because $t^n \leq 1$ in the interval of integration. Hence the last term tends to 0 when n tends to infinity ($b > 1$ in this case). It will be shown below that $\varphi(1) = 0$ implies $\varphi(x) = 0$ in $0 \leq x \leq 1$. Thus the characteristic values necessarily satisfy:

$$(7.4) \qquad 1 - \lambda + \frac{\lambda^2}{2!}\frac{1}{b} - \frac{\lambda^3}{3!}\frac{1}{b^{1+2}} + \cdots + \frac{(-\lambda)^n}{n!}\frac{1}{b^{\frac{n(n-1)}{2}}} + \cdots = H_1(\lambda) = 0.$$

Next it will be proved that every root of (7. 4) is a characteristic value. If λ is a root of (7. 4) we can derive the corresponding characteristic function from (7. 2). Proceeding from the constant value $\varphi(1)$ of $\varphi(x)$ in $\frac{1}{b} \leq x \leq 1$ we determine $\varphi(x)$ successively in the intervals $\frac{1}{b^2} \leq x \leq \frac{1}{b}$, $\frac{1}{b^3} \leq x \leq \frac{1}{b^2}$, $\cdots$, by integrating (7. 2). The constants of integration are determined so as to make $\varphi(x)$ continuous at the points $\frac{1}{b}, \frac{1}{b^2}, \frac{1}{b^3}, \cdots$. So $\varphi(x)$ is uniquely determined for every $x > 0$, and, knowing that $\varphi(0) = 0$, we have only to show that the function just determined is continuous at the point $x = 0$. That means:

$$(7.5) \qquad \lim_{\varepsilon \to 0} \varphi\left(\frac{\varepsilon}{b}\right) = 0.$$

But

$$\varphi\left(\frac{\varepsilon}{b}\right) = \lambda \int\limits_0^{\varepsilon} \varphi(y)\, dy = \varphi(1) - \lambda \int\limits_{\varepsilon}^{1} \varphi(y)\, dy.$$

In order to express the last integral in $\varphi(1)$ only, we shall treat it by integration by parts:

$$\lambda \int_{\varepsilon}^{1} \varphi(y)\,dy = \lambda \int_{\varepsilon}^{1} \varphi(y)\,d(y-\varepsilon) = \lambda(1-\varepsilon)\,\varphi(1) - \lambda \int_{\varepsilon}^{1} (y-\varepsilon)\,\varphi'(y)\,dy.$$

In the new integral put $\varphi'(x)$ from (7.2), substitute t for by and integrate by parts again:

$$\lambda \int_{\varepsilon}^{1} (y-\varepsilon)\,\varphi'(y)\,dy = \lambda^2 b \int_{\varepsilon}^{\frac{1}{b}} (y-\varepsilon)\,\varphi(by)\,dy = \lambda^2 \int_{b\varepsilon}^{1} \left(\frac{t}{b}-\varepsilon\right)\varphi(t)\,dt =$$

$$= \lambda^2 b \int_{b\varepsilon}^{1} \varphi(t)\,d\,\frac{\left(\frac{t}{b}-\varepsilon\right)^2}{2!} = \frac{\lambda^2}{2!}\,b\left(\frac{1}{b}-\varepsilon\right)^2 \varphi(1) - \lambda^2 b \int_{b\varepsilon}^{1} \frac{\left(\frac{t}{b}-\varepsilon\right)^2}{2!}\,\varphi'(t)\,dt.$$

Repeat the process again:

$$\lambda^2 b \int_{b\varepsilon}^{1} \frac{\left(\frac{t}{b}-\varepsilon\right)^2}{2!}\,\varphi'(t)\,dt = \lambda^3 b^{1+1} \int_{b\varepsilon}^{\frac{1}{b}} \frac{\left(\frac{t}{b}-\varepsilon\right)^2}{2!}\,\varphi(bt)\,dt =$$

$$= \lambda^3 b \int_{b^2\varepsilon}^{1} \frac{\left(\frac{t}{b^2}-\varepsilon\right)^2}{2!}\,\varphi(t)\,dt = \lambda^3 b^{1+2} \int_{b^2\varepsilon}^{1} \varphi(y)\,d\,\frac{\left(\frac{y}{b^2}-\varepsilon\right)^3}{3!} =$$

$$= \frac{\lambda^3}{3!}\,b^{1+2}\left(\frac{1}{b^2}-\varepsilon\right)^3 \varphi(1) - \lambda^3 b^{1+2} \int_{b^2\varepsilon}^{1} \frac{\left(\frac{y}{b^2}-\varepsilon\right)^3}{3!}\,\varphi'(y)\,dy.$$

Determine $n = n(\varepsilon)$ by the inequality:

$$\frac{1}{b^n} \leq \varepsilon < \frac{1}{b^{n-1}}.$$

Then the integral vanishes after n steps and the formula becomes:

$$(7.6)\qquad \varphi\left(\frac{\varepsilon}{b}\right) = \varphi(1)\left[1 - \lambda(1-\varepsilon) + \frac{\lambda^2}{2!}\,b\left(\frac{1}{b}-\varepsilon\right)^2 - \frac{\lambda^3}{3!}\,b^{1+2}\left(\frac{1}{b^2}-\varepsilon\right)^3 + \right.$$

$$\left. + \cdots + \frac{(-\lambda)^n}{n!}\,b^{\frac{n(n-1)}{2}}\left(\frac{1}{b^{n-1}}-\varepsilon\right)^n\right].$$

To prove (7. 5) comes to the same thing as to show that the square bracket of (7. 6) tends to 0 when n tends to ∞ (ε tends to 0). Hence we shall show:

$$\lim_{\varepsilon\to 0}\sum_{\nu=0}^{n(\varepsilon)}\frac{(-\lambda)^\nu}{\nu!}b^{\frac{\nu(\nu-1)}{2}}\left(\frac{1}{b^{\nu-1}}-\varepsilon\right)^\nu=0.$$

To see this, write the sum:

$$\sum_{\nu=0}^{n(\varepsilon)}\frac{(-\lambda)^\nu}{\nu!}b^{\frac{\nu(\nu-1)}{2}}\left(\frac{1}{b^{\nu-1}}-\varepsilon\right)^\nu=\sum_{\nu=0}^{n(\varepsilon)}\frac{(-\lambda)^\nu}{\nu!}\frac{1}{b^{\frac{\nu(\nu-1)}{2}}}+$$

$$+\sum_{\nu=0}^{n(\varepsilon)}\frac{(-\lambda)^\nu}{\nu!}\frac{1}{b^{\frac{\nu(\nu-1)}{2}}}[(1-\varepsilon\, b^{\nu-1})^\nu-1]=S_1+S_2.$$

When ε tends to 0, then n tends to ∞ and S_1 tends to $H_1(\lambda)$, which is 0 because λ satisfies (7. 4). To prove that S_2 tends to 0, replace its terms by their moduli:

$$|S_2|=\left|\sum_{\nu=0}^{n(\varepsilon)}\frac{(-\lambda)^\nu}{\nu!}\frac{1}{b^{\frac{\nu(\nu-1)}{2}}}[(1-\varepsilon\, b^{\nu-1})^\nu-1]\right|\le\sum_{\nu=0}^{n(\varepsilon)}\frac{|\lambda|^\nu}{\nu!}\frac{1}{b^{\frac{\nu(\nu-1)}{2}}}|(1-\varepsilon\, b^{\nu-1})^\nu-1|.$$

Since $0<\varepsilon\, b^{\nu-1}<\frac{1}{b^{n(\varepsilon)-\nu}}\le 1$, $(\nu=0,\ 1,\ 2,\dots,n(\varepsilon))$, we have the inequality:

$$|(1-\varepsilon\, b^{\nu-1})^\nu-1|\le 1-\left(1-\frac{1}{b^{n(\varepsilon)-\nu}}\right)^\nu.$$

Hence:

$$S_2|\le\sum_{\nu=0}^{n(\varepsilon)}\frac{|\lambda|^\nu}{\nu!}\frac{1}{b^{\frac{\nu(\nu-1)}{2}}}\left[1-\left(1-\frac{1}{b^{n(\varepsilon)-\nu}}\right)^\nu\right]=\sum_{\nu=0}^{\frac{n(\varepsilon)}{2}}+\sum_{\nu=\frac{n(\varepsilon)}{2}+1}^{n(\varepsilon)}=S_2'+S_2'';$$ (e. g. assume $n(\varepsilon)$ to be an even number)

$$S_2'|\le\sum_{\nu=0}^{\frac{n(\varepsilon)}{2}}\frac{|\lambda|^\nu}{\nu!}\frac{1}{b^{\frac{\nu(\nu-1)}{2}}}\left[1-\left(1-\frac{1}{b^{\frac{n(\varepsilon)}{2}}}\right)^\nu\right]\le\left[1-\left(1-\frac{1}{b^{\frac{n(\varepsilon)}{2}}}\right)^{\frac{n(\varepsilon)}{2}}\right]\sum_{\nu=0}^{\frac{n(\varepsilon)}{2}}\frac{|\lambda|^\nu}{\nu!}\frac{1}{b^{\frac{\nu(\nu-1)}{2}}};$$

$$|S_2''|\le\sum_{\nu=\frac{n(\varepsilon)}{2}+1}^{n(\varepsilon)}\frac{|\lambda|^\nu}{\nu!}\frac{1}{b^{\frac{\nu(\nu-1)}{2}}}.$$

But $\lim_{n\to\infty}\left[1-\left(1-\frac{1}{b^{\frac{n}{2}}}\right)^{\frac{n}{2}}\right]=0$ and the series $\sum_{\nu=0}^{\infty}\frac{|\lambda|^{\nu}}{\nu!}\frac{1}{b^{\frac{\nu(\nu-1)}{2}}}$ is convergent for every λ. Hence S_2' and S_2'' tend to 0 when ε tends to 0, that is n tends to ∞.

Thus we have proved that the constructed function is continuous at $x=0$, which means that it can be admitted as a characteristic function. We also see that to every zero of $H_1(\lambda)$ corresponds one and only one characteristic function. Note that $H_1(\lambda)$ is an integral function of genus 0. Hence the integral equation has an infinite number of characteristic values.

(β) The case $\frac{1-a}{b}=1$. ($b<1$). The characteristic function is analytic in the whole of $0\leq x\leq 1$, hence we cannot use (5.2) to obtain it directly. Instead consider the associated integral equation:

$$\varphi(x)=\lambda\int_{\frac{x-a}{b},\,1}^{1}\varphi(y)\,dy. \quad \text{(See } \mathbf{6}\text{)}. \tag{7.7}$$

Put $x=0$ in (6.2):

$$\varphi(0)=\varphi(0)\left[\lambda-\frac{\lambda^2}{2!}\frac{(1-a)^2}{b}+\frac{\lambda^3}{3!}\frac{\left(1-a\frac{b^2-1}{b-1}\right)^3}{b^{1+2}}-\right.$$

$$\left.-\cdots+(-1)^{n-1}\frac{\lambda^n}{n!}\frac{\left(1-a\frac{b^{n-1}-1}{b-1}\right)^n}{b^{\frac{n(n-1)}{2}}}\right]+\frac{\lambda^{n+1}}{b^{\frac{n(n+1)}{2}}}\int_{a\frac{b^n-1}{b-1}}^{1}\varphi\left(\frac{y-a\frac{b^n-1}{b-1}}{b^n}\right)\frac{(y-1)^n}{n!}\,dy.$$

Substitute t for $\dfrac{y-a\frac{b^n-1}{b-1}}{b^n}$ in the integral, observing that $a=1-b$.

$$\varphi(0)\left[1-\lambda+\frac{\lambda^2}{2!}b-\frac{\lambda^3}{3!}b^{1+2}+\cdots+\frac{(-\lambda)^n}{n!}b^{\frac{n(n-1)}{2}}\right]+$$

$$+\lambda^{n+1}b^{\frac{n(n+1)}{2}}\int_0^1\varphi(t)\frac{(t-1)^n}{n!}\,dt=0.$$

Assuming the existence of the characteristic function $\varphi(x)$ we conclude as in (α)

that the remainder-term tends to 0 when n tends to ∞. Hence the characteristic values necessarily satisfy:

$$(7.8) \qquad 1 - \lambda + \frac{\lambda^2}{2!} b - \frac{\lambda^3}{3!} b^{1+2} + \cdots + \frac{(-\lambda)^n}{n!} b^{\frac{n(n-1)}{2}} + \cdots = H_2(\lambda) = 0.$$

If λ is a root of (7.8), the corresponding characteristic function is obtained in the following way:
The derivative of (7.7) is:

$$(7.9) \qquad \varphi'(x) = \begin{cases} 0 & , \text{ if } 0 < x < a; \\ -\frac{\lambda}{b}\varphi\left(\frac{x-a}{b}\right), & \text{if } a < x < 1. \end{cases}$$

$\varphi(x)$ is constant for $0 \leq x \leq a$. Proceeding as in (α) we can obtain $\varphi(x)$ successively in the intervals $a \leq x \leq a\frac{b^2-1}{b-1}$, $a\frac{b^2-1}{b-1} \leq x \leq a\frac{b^3-1}{b-1}$, ..., if we integrate (7.9) and determine the constant of integration so as to make $\varphi(x)$ continuous at the points a, $a\frac{b^2-1}{b-1}$, $a\frac{b^3-1}{b-1}$, ... It remains to show the continuity of the so constructed function for $x = 1$. This may be done as in (α).

(γ) Other method. In (α) and (β) we could construct the characteristic functions because we knew an interval where they were constant, and, starting from that interval, could determine them by recursion. But when there is no such possibility we have to proceed otherwise. This is the case with the integral equations:

$$(7.10) \qquad \varphi(x) = \lambda \int_0^{bx+1-b} \varphi(y)\,dy; \left(\frac{1-a}{b} = 1\right)$$

$$(7.11) \qquad \varphi(x) = \lambda \int_{\frac{x}{b}}^{1} \varphi(y)\,dy$$

(the integral equation associated with that corresponding to $a = 0$).

These integral equations have characteristic functions analytic everywhere in $0 \leq x \leq 1$. Hence we can form their Taylor series in a suitable point. We shall do it for (7.10) only; (7.11) can be treated in exactly the same way.

The successive derivatives are:

$$\begin{cases} \varphi'(x) = \lambda b \varphi(bx + 1 - b); \\ \varphi''(x) = \lambda b^{1+1} \varphi'(bx + 1 - b) = \lambda^2 b^{1+2} \varphi(b^2 x + 1 - b^2); \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \varphi^{(n)}(x) = \lambda^n b^{\frac{n(n+1)}{2}} \varphi(b^n x + 1 - b^n); \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{cases}$$

In the point $x = 1$ the n:th derivative is:

$$\varphi^{(n)}(1) = \lambda^n b^{\frac{n(n+1)}{2}} \varphi(1).$$

Hence the Taylor expansion becomes:

$$\varphi(x) = \varphi(1) \cdot \left\{ 1 + (x - 1) \lambda b + \frac{(x-1)^2}{2!} \lambda^2 b^{1+2} + \dots + \frac{(x-1)^n}{n!} \lambda^n b^{\frac{n(n+1)}{2}} + \dots \right\}.$$

We see that $\varphi(1) = 0$ implies $\varphi(x) = 0$ in $0 \le x \le 1$. Noting that

$$\varphi(1) = \lambda \int_0^1 \varphi(y)\, dy,$$

we get (7. 8).

CHAPTER III.

The Line $y = bx + a$, $(0 \le a \le 1,\ a + b \le 1)$.

8. $K(x, y) = 1$.

Consider a straight line crossing the diagonal $y = x$. We suppose that it goes from the side $(0, 0)$, $(0, 1)$ and first that it goes to the side $(1, 0)$, $(1, 1)$ of the square (fig. 3). Let the equation of the line be: $y = bx + a$, $(0 \le a \le 1,\ 0 \le a + b \le 1)$. Supposing that the kernel is 1 below the line and 0 above, the integral equation can be written:

$$\varphi(x) = \lambda \int_0^{bx+a} \varphi(y)\, dy. \tag{8. 1}$$

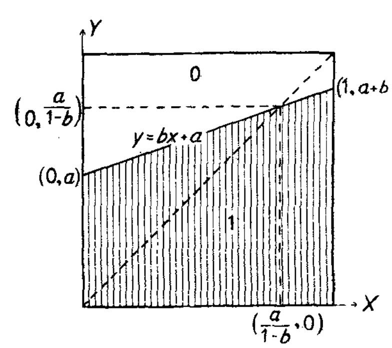

Fig. 3.

The equation (7. 10) is the special case $a + b = 1$ and we can use the method of **7**, (γ), to determine the characteristic values of (8. 1).

The characteristic functions of (8. 1) are analytic in the whole interval $0 \leq x \leq 1$. This is shown by calculating the derivative:

$$\varphi'(x) = \lambda b \varphi(bx + a).$$

For x in the interval $0 \leq x \leq 1$, the point $y = bx + a$ is also in that interval, hence the derivatives of all orders exist and $\varphi(x)$ is analytic. The n:th derivative becomes:

$$\varphi^{(n)}(x) = \lambda^n b^{\frac{n(n+1)}{2}} \varphi\left(b^n x + a \frac{1 - b^n}{1 - b}\right).$$

The invariant point of the transformation $y = bx + a$ is $x = \dfrac{a}{1 - b}$, hence we expand $\varphi(x)$ in its Taylor series in that point:

$$\varphi^{(n)}\left(\frac{a}{1 - b}\right) = \lambda^n b^{\frac{n(n+1)}{2}} \varphi\left(\frac{a}{1 - b}\right);$$

$$\varphi(x) = \varphi\left(\frac{a}{1 - b}\right)\left[1 + \left(x - \frac{a}{1 - b}\right)\lambda b + \frac{\left(x - \frac{a}{1 - b}\right)^2}{2!}\lambda^2 b^{1+2} + \right.$$

$$\left. + \frac{\left(x - \frac{a}{1 - b}\right)^3}{3!}\lambda^3 b^{1+2+3} + \cdots + \frac{\left(x - \frac{a}{1 - b}\right)^n}{n!}\lambda^n b^{\frac{n(n+1)}{2}} + \cdots\right].$$

Note that the conditions which we have imposed on a and b: $0 \leq a \leq 1$, $0 \leq a + b \leq 1$, imply that $|b| \leq 1$, hence the series is convergent.

Putting the series for $\varphi(x)$ into the relation:

$$\varphi\left(\frac{a}{1 - b}\right) = \lambda \int_0^{\frac{a}{1-b}} \varphi(y)\, dy,$$

we obtain the equation for the characteristic values:

$$(8.\ 2) \qquad 1 - \frac{\lambda a}{1 - b} + \frac{1}{2!} b \left(\frac{\lambda a}{1 - b}\right)^2 - \frac{1}{3!} b^{1+2} \left(\frac{\lambda a}{1 - b}\right)^3 + \cdots +$$

$$+ \frac{1}{n!} b^{\frac{n(n-1)}{2}} \left(\frac{-\lambda a}{1 - b}\right)^n + \cdots = H_3(\lambda) = 0.$$

Except for $|b| = 1$ $H_3(\lambda)$ is an integral function of genus 0 having an infinite number of zeros.

If we release the conditions imposed on a and b and only put $0 \leq a \leq 1$, $a + b \leq 1$, the line goes from the side $(0, 0)$, $(0, 1)$ to the side $(1, 0)$, $(1, 1)$ or $(0, 0)$, $(1, 0)$ and $b < 1$ can be less than -1. If $b < -1$, $\varphi(x)$ is no longer analytic and $H_3(\lambda)$ diverges for every λ. These kernels will be treated later on by considering the associated integral equations.

Next we shall apply another method to (8. 1) which will give the denominator of Fredholm. Consider the corresponding non-homogeneous integral equation:

$$\varphi(x) = \lambda \int_0^{bx+a} \varphi(y)\, dy + g(x). \tag{8. 3}$$

(Suppose that $g'(x)$ is integrable.) The derivative of (8. 3) is:

$$\varphi'(x) = \lambda\, b\, \varphi(bx + a) + g'(x).$$

Integrale (8. 3) by parts:

$$\varphi(x) - \lambda(bx + a)\varphi(bx + a) + \lambda \int_0^{bx+a} y\, \varphi'(y)\, dy = g(x).$$

Use the expression for $\varphi'(x)$:

$$\varphi(x) - \lambda(bx + a)\varphi(bx + a) + \lambda^2 b \int_0^{bx+a} \varphi(by + a)\, d\frac{y^2}{2!} = g(x) - \lambda \int_0^{bx+a} y\, g'(y)\, dy.$$

Repeating the process n times, the result is:

$$\begin{aligned} &\varphi(x) - \lambda(bx + a)\cdot\varphi(bx + a) + \frac{\lambda^2}{2!} b(bx + a)^2 \varphi\left(b^2 x + a\frac{1 - b^2}{1 - b}\right) - \\ &- \frac{\lambda^3}{3!} b^{1+2}(bx + a)^3 \varphi\left(b^3 x + a\frac{1 - b^3}{1 - b}\right) + \cdots + \frac{(-\lambda)^n}{n!} b^{\frac{n(n-1)}{2}} (bx + a)^n \varphi\left(b^n x + a\frac{1 - b^n}{1 - b}\right) + \\ &+ (-\lambda)^{n+1} b^{\frac{n(n+1)}{2}} \int_0^{bx+a} \varphi\left(b^n y + a\frac{1 - b^n}{1 - b}\right) \frac{y^n}{n!}\, dy = g(x) - \lambda \int_0^{bx+a} y\, g'(y)\, dy + \\ &+ \frac{\lambda^2}{2!} b^2 \int_0^{bx+a} y^2 g'(by + a)\, dy - \frac{\lambda^3}{3!} b^{2+3} \int_0^{bx+a} y^3 g'\left(b^2 y + a\frac{1 - b^2}{1 - b}\right) dy + \cdots + \\ &+ \frac{(-\lambda)^n}{n!} b^{\frac{n(n+1)}{2} - 1} \int_0^{bx+a} y^n g'\left(b^{n-1} y + a\frac{1 - b^{n-1}}{1 - b}\right) dy. \end{aligned} \tag{8. 4}$$

The present conditions imposed on the line imply that $|b| \leq 1$. Since $\varphi(x)$ is bounded when λ is not a characteristic value, the integral in the left-hand member of (8.4) tends to 0 when n tends to ∞. Put $g(x) = 1$ and $x = \frac{a}{1-b}$ into (8.4) and let n tend to ∞. Dividing both members by $H_3(\lambda)$ of (8.2), the result is:

$$\varphi\left(\frac{a}{1-b}\right) = \frac{1}{H_3(\lambda)}. \tag{8.5}$$

Compare this with the Fredholm expression (notations from E. Goursat: Cours d'Analyse Mathématique, tome III, 4:th edition, Paris 1927, page 368 ff.):

$$\varphi(x) = \lambda \int_0^1 \frac{D\left(\begin{matrix} x \\ y \end{matrix} \middle| \lambda\right)}{D(\lambda)} g(y)\, dy + g(x),$$

or in this case:

$$\varphi\left(\frac{a}{1-b}\right) = \lambda \int_0^1 \frac{D\left(\begin{matrix} \frac{a}{1-b} \\ y \end{matrix} \middle| \lambda\right)}{D(\lambda)}\, dy + 1 = \frac{\lambda \int_0^1 D\left(\begin{matrix} \frac{a}{1-b} \\ y \end{matrix} \middle| \lambda\right) dy + D(\lambda)}{D(\lambda)}. \tag{8.6}$$

The numerator of (8.6) will be shown to be 1. Then (8.5) and (8.6) are identical and we can conclude that $H_3(\lambda) = D(\lambda)$.

We shall make use of the Fredholm identity for $D\left(\begin{matrix} x \\ y \end{matrix} \middle| \lambda\right)$:

$$D\left(\begin{matrix} x \\ y \end{matrix} \middle| \lambda\right) = K(x, y)\, D(\lambda) + \lambda \int_0^1 D\left(\begin{matrix} x \\ s \end{matrix} \middle| \lambda\right) K(s, y)\, ds. \tag{8.7}$$

Put $x = \frac{a}{1-b}$, $y = 0$. Since $K(x, 0) = 1$ we obtain:

$$D\left(\begin{matrix} \frac{a}{1-b} \\ 0 \end{matrix} \middle| \lambda\right) = D(\lambda) + \lambda \int_0^1 D\left(\begin{matrix} \frac{a}{1-b} \\ s \end{matrix} \middle| \lambda\right) ds.$$

Consequently, it is enough to show that

$$D\left(\begin{matrix} \frac{a}{1-b} \\ 0 \end{matrix} \middle| \lambda\right) = 1.$$

The Fredholm expression is:

$$(8.8)\quad D\left(\begin{matrix}x\\y\end{matrix}\middle|\lambda\right) = K(x,y) + \sum_{\nu=1}^{\infty}\frac{(-\lambda)^\nu}{\nu!}\int_0^1\int_0^1\cdots\int_0^1 \begin{vmatrix} K(x,y) & K(x,s_1) & K(x,s_2) & \cdots & K(x,s_\nu)\\ K(s_1,y) & K(s_1,s_1) & K(s_1,s_2) & \cdots & K(s_1,s_\nu)\\ K(s_2,y) & K(s_2,s_1) & K(s_2,s_2) & \cdots & K(s_2,s_\nu)\\ & \cdots & \cdots & \cdots & \\ K(s_\nu,y) & K(s_\nu,s_1) & K(s_\nu,s_2) & \cdots & K(s_\nu,s_\nu)\end{vmatrix} ds_1\, ds_2 \cdots ds_\nu.$$

Thus:

$$D\left(\begin{matrix}\frac{a}{1-b}\\ 0\end{matrix}\middle|\lambda\right) = 1 + \sum_{\nu=1}^{\infty}\frac{(-\lambda)^\nu}{\nu!}\int_0^1\int_0^1\cdots\int_0^1 \begin{vmatrix} 1 & K\left(\frac{a}{1-b},s_1\right) & K\left(\frac{a}{1-b},s_2\right) & \cdots & K\left(\frac{a}{1-b},s_\nu\right)\\ 1 & K(s_1,s_1) & K(s_1,s_2) & \cdots & K(s_1,s_\nu)\\ 1 & K(s_2,s_1) & K(s_2,s_2) & \cdots & K(s_2,s_\nu)\\ & \cdots & \cdots & \cdots & \\ 1 & K(s_\nu,s_1) & K(s_\nu,s_2) & \cdots & K(s_\nu,s_\nu)\end{vmatrix} ds_1\, ds_2 \cdots ds_\nu.$$

The determinants in the right member vanish at every point $s_1, s_2, \ldots, s_\nu$. This can be proved in the following way.

If two or more of the numbers $s_1, s_2, \ldots, s_\nu$ are equal or if any s_μ, $\mu = 1, 2, \ldots, \nu$, equals $\frac{a}{1-b}$, the determinant vanishes because two rows become equal. If not, they are all different and different from $\frac{a}{1-b}$. Let s_p be the greatest and s_q the least of them and treat the two following cases separately:

a) $b \geq 0$. When $s_p > \frac{a}{1-b}$ the $(p+1)$:th column consists of zeros only and thus the determinant vanishes. When $s_p < \frac{a}{1-b}$ all the numbers $s_1, s_2, \ldots, s_\nu$ are less than $\frac{a}{1-b}$. The determinant vanishes because the first and $(q+1)$:th columns are equal (consist of ones only).

b) $b < 0$. When $s_q > \frac{a}{1-b}$ all the numbers $s_1, s_2, \ldots, s_\nu$ are greater than $\frac{a}{1-b}$ and the $(p+1)$:th column contains nothing but zeros. When $s_p < \frac{a}{1-b}$ all the numbers $s_1, s_2, \ldots, s_\nu$ are less than $\frac{a}{1-b}$ and all the elements in the determinant become ones. It remains to examine the determinant when $s_p > \frac{a}{1-b}$

and $s_q < \frac{a}{1-b}$. Note that $a + b s_q$ is the greatest and $a + b s_p$ the least of the numbers $\{a + b s_\mu\}$, $\mu = 1, 2, \ldots, \nu$. If $s_p > a + b s_q$ the determinant vanishes because the $(p + 1)$:th column consists of zeros only. If $s_p \leq a + b s_q$ we deduce:

$$s_q \leq \frac{s_p - a}{b} \leq a + b s_p.$$

(The equality sign only for $b = -1$).

Hence s_q is less than all the numbers $\{a + b s_\mu\}$, $\mu = 1, 2, \ldots, \nu$, and the $(q + 1)$:th column in the determinant consists of nothing but ones. Thus the determinant vanishes, having two equal columns.

We have shown that

$$D\left(\begin{matrix} \frac{a}{1-b} \\ 0 \end{matrix} \middle| \lambda \right) = 1$$

and infer that $D(\lambda) = H_3(\lambda)$.

The integral equation (7. 10) of 7, (γ), is the special case $a + b = 1$. Conversely, if we have treated (7. 10) completely and shown that its denominator of Fredholm is just $H_3(\lambda)$ of (7. 8), we may derive the characteristic values of (8. 1) for $b \geq 0$ in the following way.

For $x \leq \frac{a}{1-b}$ the kernel of the integral equation (8. 1) may be looked upon as having the smaller existence-square of side $\frac{a}{1-b}$ (fig. 3). The characteristic values then satisfy (7. 8) with $\lambda \frac{a}{1-b}$ substituted for λ which gives (8. 2).

For $\frac{a}{1-b} < x \leq 1$ we write the integral equation (8. 1):

$$\varphi(x) = \lambda \int_0^{\frac{1-a}{b}} \varphi(y)\, dy + \lambda \int_{\frac{1-a}{b}}^{bx+a} \varphi(y)\, dy. \tag{8. 9}$$

Assume that λ is a characteristic value and $\varphi(x)$ the corresponding characteristic function of the integral equation in the smaller existence square. For $x > \frac{a}{1-b}$ the first term of the right member of (8. 9) is a constant and hence (8. 9) can be regarded as a non-homogeneous equation of the Volterra type, which always has a unique solution.

The result is that the characteristic values for $0 \leq x \leq 1$ are determined by (8.2).

We can see the same thing immediately by considering the denominator of Fredholm:

$$D(\lambda) = 1 + \sum_{\nu=1}^{\infty} \frac{(-\lambda)^\nu}{\nu!} \int_0^1 \int_0^1 \cdots \int_0^1 \begin{vmatrix} K(s_1, s_1) & K(s_1, s_2) \cdots & K(s_1, s_\nu) \\ K(s_2, s_1) & K(s_2, s_2) \cdots & K(s_2, s_\nu) \\ \cdots & & \cdots \\ K(s_\nu, s_1) & K(s_\nu, s_2) \cdots & K(s_\nu, s_\nu) \end{vmatrix} ds_1\, ds_2 \cdots ds_\nu. \tag{8.10}$$

If any one of the variables $s_1, s_2, \ldots, s_\nu$ is greater than $\frac{a}{1-b}$, the greatest of them, which will be denoted by s_p, is also greater than $\frac{a}{1-b}$. Since we assume that $b > 0$, the p:th column consists of nothing but zeros and the determinant vanishes. Hence the upper limits in the integral may be reduced to $\frac{a}{1-b}$. This means that the denominator of Fredholm of (8.1) is identical with that of the same integral equation considered in the square of side $\frac{a}{1-b}$.

Finally we shall treat the case which we omitted earlier where the line $y = bx + a$ goes from the side $(0, 0)(0, 1)$ to the side $(0, 0)(1, 0)$ in the square: $0 \leq a \leq 1$, $a + b < 0$. We shall deduce the Fredholm denominator from the results just obtained.

The line intersects the x-axis in $-\frac{a}{b}$ and the y-axis in a. First reduce the existence-square to the square of side $\overline{a, -\frac{a}{b}}$ (the kernel is 0 outside this square) (fig. 4). If $-\frac{a}{b} \geq a$, that is $b > -1$, we are back in the previous case and the characteristic values satisfy (8.2), $H_3(\lambda)$ being the denominator of Fredholm.

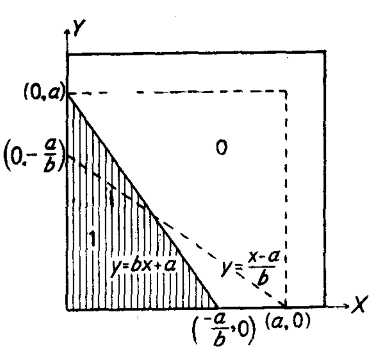

Fig. 4.

When $a > -\frac{a}{b}$ we consider instead the associated integral equation, which has the same characteristic values. Its kernel is 1 below the line $y = \frac{x-a}{b}$ and 0 above it. This line is of the previous type and we get the denominator of Fredholm from (8.2):

$$H_4(\lambda) = 1 - \frac{\lambda a}{1-b} + \frac{1}{2!}\frac{1}{b}\left(\frac{\lambda a}{1-b}\right)^2 - \frac{1}{3!}\frac{1}{b^{1+2}}\left(\frac{\lambda a}{1-b}\right)^3 + \cdots +$$
$$+ \frac{1}{n!}\frac{1}{b^{\frac{n(n-1)}{2}}}\left(\frac{-\lambda a}{1-b}\right)^n + \cdots = 0.$$

CHAPTER IV.

A Symmetric Kernel.

9. $K(x, y) = 1$.

As another example let us use the method of integration by parts to determine the characteristic values of an integral equation whose kernel is 1 in the strip between two lines symmetrical with respect to the diagonals $y = x$ and $y = 1 - x$, and 0 in the rest of the square. This kernel is symmetrical. Hence we know that the characteristic values exist, are infinitely many and real-valued. Only a simple case will be treated here: the lines $y = x \pm a$, $(\frac{1}{2} \leq a < 1)$ (fig. 5).

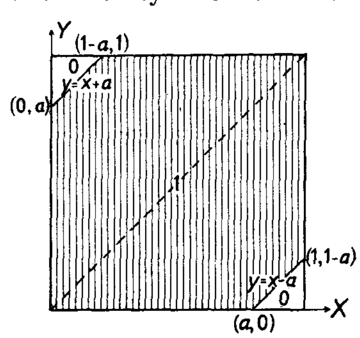

Fig. 5.

The homogeneous integral equation is:

$$(9.\ 1) \qquad \varphi(x) = \lambda \int_{\underline{x-a,\,0}}^{\overline{x+a,\,1}} \varphi(y)\, dy.$$

Differentiating (9. 1) twice in the interval $0 \leq x \leq \leq 1 - a$, we obtain the differential equation $\varphi''(x) + \lambda^2 \varphi(x) = 0$ with the condition $\varphi'(0) = \lambda \varphi(1 - a)$. Hence there is a single characteristic function for every characteristic value.

For $1 - a \leq x \leq a$ $\varphi(x)$ is constant, thus $\varphi(a) = \varphi(1 - a)$. Since the kernel is symmetrical with respect to the diagonal $y = 1 - x$, change x to $1 - x$ in (9. 1):

$$\varphi(1 - x) = \lambda \int_{\underline{1-a-x,\,0}}^{\overline{1+a-x,\,1}} \varphi(y)\, dy.$$

Substitute $1 - t$ for y in the integral:

$$\varphi(1 - x) = \lambda \int_{\underline{x-a,\,0}}^{\overline{x+a,\,1}} \varphi(1 - t)\, dt.$$

Hence $\varphi(1-x)$ is a solution of (9. 1): $\varphi(1-x) = C\varphi(x)$. To determine C observe that $\varphi(a) = \varphi(1-a)$. When $\varphi(a) \neq 0$ this gives $C = 1$. But when $\varphi(a) = 0$ we determine C in the following way

$$\varphi(a) = 0 = \lambda \int_0^1 \varphi(y)\,dy = \lambda \Big[\int_0^{1-a} + \int_{1-a}^{a} + \int_a^1\Big].$$

The second integral in the right-hand member is 0 because $\varphi(x) = \varphi(a) = 0$ in its interval of integration. In the third integral substitute $1-t$ for y:

$$0 = \lambda \int_0^{1-a} \varphi(y)\,dy + \lambda \int_0^{1-a} \varphi(1-y)\,dy = (1 + C)\,\lambda \int_0^{1-a} \varphi(y)\,dy.$$

Hence $1 + C = 0$ or $C = -1$.

We shall have to make a distinction between two cases: $\varphi(a) \neq 0$ and $\varphi(a) = 0$.

***a*) $\varphi(a) \neq 0$.** $\qquad (\varphi(x) = \varphi(1-x))$.

First determine a relation between $\varphi(0)$ and $\varphi(a)$. The integral equation gives:

$$\varphi(0) = \lambda \int_0^a \varphi(y)\,dy = \lambda \int_0^{1-a} \varphi(y)\,dy + \lambda \int_{1-a}^{a} \varphi(y)\,dy.$$

$$\varphi(a) = \lambda \int_0^1 \varphi(y)\,dy = \lambda \int_0^{1-a} \varphi(y)\,dy + \lambda \int_{1-a}^{a} \varphi(y)\,dy + \lambda \int_a^1 \varphi(y)\,dy.$$

But $\varphi(x) = \varphi(1-x)$ implies:

$$\lambda \int_0^{1-a} \varphi(y)\,dy = \lambda \int_a^1 \varphi(y)\,dy.$$

$\varphi(x)$ is constant in the interval $1 - a \leq x \leq a$, hence:

$$\lambda \int_{1-a}^{a} \varphi(y)\,dy = \lambda\,(2a - 1)\,\varphi(a).$$

We derive the following relation between $\varphi(0)$ and $\varphi(a)$:

(9. 2) $$\varphi(0) = \tfrac{1}{2}\,\varphi(a)\big[1 + \lambda(2a - 1)\big].$$

The derivative of (9. 1) is:

$$\varphi'(x) = \begin{cases} \lambda\varphi(x+a), \text{ if } 0 < x < 1-a; \\ 0 \quad , \text{ if } 1-a < x < a; \\ -\lambda\varphi(x-a), \text{ if } a < x < 1. \end{cases}$$

Integrate (9. 1) by parts putting $x = 0$.

$$\varphi(0) = \lambda \int_0^a \varphi(y)\,dy = \lambda a \varphi(a) - \lambda \int_0^a y\,\varphi'(y)\,dy;$$

$$\lambda \int_0^a y\,\varphi'(y)\,dy = \lambda^2 \int_0^{1-a} \varphi(y+a)\,d\frac{y^2}{2!} = \frac{\lambda^2}{2!}(1-a)^2\varphi(1) - \lambda^2 \int_0^{1-a} \frac{y^2}{2!}\varphi'(y+a)\,dy;$$

$$\lambda^2 \int_0^{1-a} \frac{y^2}{2!}\varphi'(y+a)\,dy = -\lambda^3 \int_0^{1-a} \varphi(y)\,d\frac{y^3}{3!} = -\frac{\lambda^3}{3!}(1-a)^3\varphi(1-a) + \lambda^3 \int_0^{1-a} \frac{y^3}{3!}\varphi'(y)\,dy.$$

Generally:

$$\lambda^{2\nu} \int_0^{1-a} \frac{y^{2\nu}}{(2\nu)!}\varphi'(y+a)\,dy = -\lambda^{2\nu+1} \int_0^{1-a} \varphi(y)\,d\frac{y^{2\nu+1}}{(2\nu+1)!} =$$

$$= -\frac{\lambda^{2\nu+1}}{(2\nu+1)!}(1-a)^{2\nu+1}\varphi(1-a) + \lambda^{2\nu+1} \int_0^{1-a} \frac{y^{2\nu+1}}{(2\nu+1)!}\varphi'(y)\,dy;$$

$$\lambda^{2\nu+1} \int_0^{1-a} \frac{y^{2\nu+1}}{(2\nu+1)!}\varphi'(y)\,dy = \lambda^{2\nu+2} \int_0^{1-a} \varphi(y+a)\,d\frac{y^{2\nu+2}}{(2\nu+2)!} =$$

$$= \frac{\lambda^{2\nu+2}}{(2\nu+2)!}(1-a)^{2\nu+2}\varphi(1) - \lambda^{2\nu+2} \int_0^{1-a} \frac{y^{2\nu+2}}{(2\nu+2)!}\varphi'(y+a)\,dy.$$

After $2n + 1$ steps we have:

$$\varphi(0) = \lambda a \varphi(a) - \frac{\lambda^2}{2!}(1-a)^2\varphi(1) - \frac{\lambda^3}{3!}(1-a)^3\varphi(1-a) + \frac{\lambda^4}{4!}(1-a)^4\varphi(1) +$$

$$+ \frac{\lambda^5}{5!}(1-a)^5\varphi(1-a) - \cdots + (-1)^n \frac{\lambda^{2n}}{(2n)!}(1-a)^{2n}\varphi(1) +$$

$$+ (-1)^n \frac{\lambda^{2n+1}}{(2n+1)!}(1-a)^{2n+1}\varphi(1-a) + (-1)^{n+1}\lambda^{2n+2} \int_0^{1-a} \varphi(y+a)\frac{y^{2n+1}}{(2n+1)!}\,dy.$$

Since $\varphi(x)$ is bounded the integral in the right member tends to o when n tends to ∞. Introducing (9. 2) for $\varphi(0)$ we obtain:

$$(9.3)\quad \varphi(a)\Big[1-\lambda+\frac{\lambda^2}{2!}(1-a)^2+\frac{\lambda^3}{3!}(1-a)^2(4a-1)-$$
$$-\frac{\lambda^4}{4!}(1-a)^4-\frac{\lambda^5}{5!}(1-a)^4(8a-3)+\cdots+$$
$$+(-1)^{n-1}\frac{\lambda^{2n}}{(2n)!}(1-a)^{2n}+(-1)^{n-1}\frac{\lambda^{2n+1}}{(2n+1)!}(1-a)^{2n}\left[4an-(2n-1)\right]+\cdots\Big]=$$
$$=\varphi(a)\big[2\left[1-\sin\lambda(1-a)\right]-\cos\lambda(1-a)\left[1+\lambda(2a-1)\right]\big]=$$
$$=\varphi(a)\cdot 2\cos\left(\frac{\pi}{4}+\lambda\frac{1-a}{2}\right)\Big[2\sin\left(\frac{\pi}{4}-\lambda\frac{1-a}{2}\right)-$$
$$-\cos\left(\frac{\pi}{4}-\lambda\frac{1-a}{2}\right)\left[1+\lambda(2a-1)\right]\Big]=\varphi(a)\cdot E(\lambda)=0.$$

The assumption $\varphi(a)\neq 0$ implies $E(\lambda)=0$, hence the characteristic values necessarily satisfy $E(\lambda)=0$, if the corresponding characteristic functions are different from o in a. We shall show that every zero of $E(\lambda)$ is a characteristic value and that $E(\lambda)$ is the Fredholm denominator of the integral equation (9. 1).

$\varphi(x)$ is analytic in $0\leq x\leq 1-a$ and $\varphi(x)=\varphi(1-x)$. Putting $\varphi(1-a)=1$ calculate its Taylor series in the point $x=0$:

$$\begin{array}{ll}
\varphi'(x)=\lambda\,\varphi(x+a); & \varphi''(x)=-\lambda^2\varphi(x);\\
\varphi'''(x)=-\lambda^3\varphi(x+a); & \varphi^{\mathrm{IV}}(x)=\lambda^4\varphi(x);\\
\cdots\quad\cdots\quad\cdots & \cdots\quad\cdots\quad\cdots\\
\varphi^{(2n+1)}(x)=(-1)^n\lambda^{2n+1}\varphi(x+a); & \varphi^{(2n)}(x)=(-1)^n\lambda^{2n}\varphi(x);
\end{array}$$

$$\begin{array}{ll}
\varphi(0)=\tfrac{1}{2}\left[1+\lambda(2a-1)\right]; & \varphi'(0)=\lambda;\\
\varphi''(0)=-\dfrac{\lambda^2}{2}\left[1+\lambda(2a-1)\right]; & \varphi'''(0)=-\lambda^3;\\
\cdots\quad\cdots\quad\cdots & \cdots\quad\cdots\quad\cdots\\
\varphi^{(2n)}(0)=(-1)^n\dfrac{\lambda^{2n}}{2}\left[1+\lambda(2a-1)\right]; & \varphi^{(2n+1)}(0)=(-1)^n\lambda^{2n+1}.
\end{array}$$

The power series becomes:

$$(9.4)\quad \varphi(x)=\tfrac{1}{2}\left[1+\lambda(2a-1)\right]\left\{1-\frac{x^2\lambda^2}{2!}+\frac{x^4\lambda^4}{4!}-\cdots\right\}+$$
$$+\left\{x\lambda-\frac{x^3\lambda^3}{3!}+\frac{x^5\lambda^5}{5!}-\cdots\right\}=\frac{\cos\lambda x}{2}\left[1+\lambda(2a-1)\right]+\sin\lambda x.$$

We have to confirm that (9. 4) satisfies the integral equation. It is sufficient to calculate $\varphi(1-a)-\varphi(0)=1-\frac{1+\lambda(2a-1)}{2}$. The integral equation gives:

$$\varphi(1-a)-\varphi(0)=\lambda\int_0^1\varphi(y)\,dy-\lambda\int_0^a\varphi(y)\,dy=\lambda\int_a^1\varphi(y)\,dy=\lambda\int_0^{1-a}\varphi(y)\,dy.$$

Substituting (9. 4) for $\varphi(x)$ we have to show that:

$$\lambda\int_0^{1-a}\varphi(y)\,dy=\lambda\int_0^{1-a}\left\{\frac{\cos\lambda y}{2}[1+\lambda(2a-1)]+\sin\lambda y\right\}dy=1-\frac{1+\lambda(2a-1)}{2}.$$

If we simplify, this becomes:

$$\frac{1+\lambda(2a-1)}{2}[1+\sin\lambda(1-a)]=\cos\lambda(1-a). \tag{9.5}$$

But (9. 5) is obtained by multiplying $E(\lambda)=0$ by

$$\frac{\cos\lambda(1-a)}{1-\sin\lambda(1-a)}=\operatorname{tg}\left(\frac{\pi}{4}+\lambda\frac{1-a}{2}\right).$$

This factor is finite except for $\lambda=\frac{2}{1-a}\left(\frac{\pi}{4}+n\pi\right)$, $(n=0,\pm1,\pm2,\ldots)$. These exceptional λ-values are zeros of $E(\lambda)$ because $E(\lambda)$ contains the factor $\cos\left(\frac{\pi}{4}+\lambda\frac{1-a}{2}\right)$. Hence $E(\lambda)=0$ implies (9. 5) except for these exceptional λ-values.

We have shown that all the roots of $E(\lambda)=0$ except those coming from the factor $\cos\left(\frac{\pi}{4}+\lambda\frac{1-a}{2}\right)$ are characteristic values, each of them with a single characteristic function, determined by (9. 4). The zeros of $\cos\left(\frac{\pi}{4}+\lambda\frac{1-a}{2}\right)$ give no characteristic functions with $\varphi(a)\neq0$, but it will be shown in a moment that they correspond to the characteristic functions with $\varphi(a)=0$.

***b)* $\varphi(a)=0$.** $(\varphi(x)=-\varphi(1-x))$.

Putting $\varphi(0)=1$, the Taylor series for $\varphi(x)$ becomes:

$$\varphi(x)=1-\frac{x^2\lambda^2}{2!}+\frac{x^4\lambda^4}{4!}-\cdots=\cos x\lambda.$$

To determine λ note that $\varphi(x)$ satisfies the integral equation:

$$\varphi(0) = \lambda \int_0^a \varphi(y)\,dy = \lambda \int_0^{1-a} \cos \lambda y\,dy = \sin \lambda(1-a) = 1;$$

$$\varphi(1-a) = \cos \lambda(1-a) = 0;$$

The characteristic values are thus the common roots of:

$$\begin{cases} \cos \lambda(1-a) = 2 \cos\left(\frac{\pi}{4} + \lambda \frac{1-a}{2}\right) \cos\left(\frac{\pi}{4} - \lambda \frac{1-a}{2}\right) = 0; \\ 1 - \sin \lambda(1-a) = 2 \sin\left(\frac{\pi}{4} - \lambda \frac{1-a}{2}\right) \cos\left(\frac{\pi}{4} + \lambda \frac{1-a}{2}\right) = 0. \end{cases}$$

We infer that these characteristic values are the zeros of $\cos\left(\frac{\pi}{4} + \lambda \frac{1-a}{2}\right)$ which we had to exclude in ***a***).

So far we have proved that the characteristic values of the integral equation are just the roots of $E(\lambda) = 0$, each one of them giving a single characteristic function. All the zeros of $E(\lambda)$ are simple. We shall now turn to the proof that $E(\lambda) = D(\lambda)$, the denominator of Fredholm. Since the integral equation is symmetrical and has a single characteristic function for every characteristic value, the zeros of $D(\lambda)$ are all simple. Hence $D(\lambda)$ and $E(\lambda)$ have the same zeros, all simple. They are both integral functions of genus 1 at most, thus they differ only by an exponential factor $\alpha e^{\beta x}$. Observing that the first two coefficients in their power series are equal, we get: $\alpha = 1$, $\beta = 0$. Hence $D(\lambda)$ and $E(\lambda)$ are identical.

It is also possible to prove that $D(\lambda) = E(\lambda)$ by the later method of 8. Consider the non-homogeneous integral equation:

$$(9.\ 6) \qquad \varphi(x) = \lambda \int_{x-a,\ 0}^{x+a,\ 1} \varphi(y)\,dy + 1.$$

It is obvious that $\varphi(x) = \varphi(1-x)$. (Here there is only one case to be considered). The relation between $\varphi(0)$ and $\varphi(a)$ is:

$$\varphi(0) = \tfrac{1}{2}\varphi(a)\left[1 + \lambda(2a-1)\right] + \tfrac{1}{2}.$$

Integrating (9. 6) by parts as was done with (9. 1) (or observing that $\varphi(x)$ satisfies the differential equation $\varphi''(x) + \lambda^2 \varphi(x) = 0$) we get:

$$(9.\ 7) \qquad \varphi(a) = \frac{\cos \lambda(1-a)}{E(\lambda)}.$$

Compare (9. 7) with the Fredholm expression:

$$\varphi(a)=\lambda\int_0^1\frac{D\left(\begin{matrix}a\\y\end{matrix}\middle|\lambda\right)}{D(\lambda)}dy+1=\frac{\lambda\int_0^1 D\left(\begin{matrix}a\\y\end{matrix}\middle|\lambda\right)dy+D(\lambda)}{D(\lambda)}.$$

Since $\lambda\int_0^1 D\left(\begin{matrix}a\\y\end{matrix}\middle|\lambda\right)dy+D(\lambda)=D\left(\begin{matrix}a\\a\end{matrix}\middle|\lambda\right)$, (8. 7), we have to show that $D\left(\begin{matrix}a\\a\end{matrix}\middle|\lambda\right)=$ $=\cos\lambda(1-a)$, which is possible by calculating the Fredholm determinants.

The numerator and denominator of (9. 7) have the common factor $\cos\left(\frac{\pi}{4}+\lambda\frac{1-a}{2}\right)$. This is due to the fact that the characteristic functions corresponding to the zeros of $\cos\left(\frac{\pi}{4}+\lambda\frac{1-a}{2}\right)$ are orthogonal to 1 in the interval $0\leq x\leq 1$.

The method of integration by parts is also applicable to integral equations of the type (9. 1) generalized in the direction of Chapter V with the curves symmetrical with respect to the diagonals $y=x$ and $y=1-x$.

CHAPTER V.

The Curve $y=f(x)$.

10. The line does not pass through (0, 0) and (1, 1); $K(x,y)=1$.

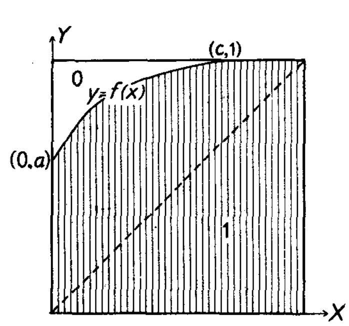

Fig. 6.

Generalize the kernels of 2 or 5 supposing that the line is no longer straight but curved. Let its equation be $y=f(x)$ and suppose that it goes from the point $(0, a=f(0))$ to the point $(c=f^{-1}(1), 1)$ (fig. 6). Impose the following conditions on $f(x)$:

(α): $f(x)$ is non-decreasing and thus has an inverse $f^{-1}(x)$ almost everywhere.

(β): $f'(x)$ exists and is integrable.

(γ): $f(x)>x$, which means that the curve is situated above the diagonal $y=x$.

We shall prove that under these conditions there is only a finite number of characteristic values. For the sake of brevity extend $f(x)$ to be 1 in the interval $f^{-1}(1) \leq x \leq 1$. The integral equation is:

$$\varphi(x) = \lambda \int_0^{f(x),\,1} \varphi(y)\,dy. \tag{10.1}$$

The derivative is:

$$\varphi'(x) = \begin{cases} \lambda f'(x)\,\varphi\left[f(x)\right], & \text{if } 0 < x < f^{-1}(1); \\ 0 & \text{, if } f^{-1}(1) < x < 1. \end{cases}$$

Integrate (10.1) by parts:

$$\varphi(x) = \lambda \Big[y\,\varphi(y)\Big]^{f(x),\,1} - \lambda \int_0^{f(x),\,1} y\,\varphi'(y)\,dy.$$

In the integral introduce the expression for $\varphi'(x)$ and substitute t for $f(y)$:

$$\lambda \int_0^{f(x),\,1} y\,\varphi'(y)\,dy = \lambda^2 \int_0^{f(x),\,f^{-1}(1)} y f'(y)\,\varphi\left[f(y)\right] dy = \lambda^2 \int_{f(0)}^{f^2(x),\,1} f^{-1}(t)\,\varphi(t)\,dt.$$

($f^2(x)$ means $f[f(x)]$, generally $f^n(x)$ is the n:th iterated function $f(x)$).

Using the notation $\int_{f(0)}^{x} f^{-1}(y)\,dy = f_1(x)$, the last integral may be integrated by parts (the expression in square brackets vanishes at the lower limit):

$$\lambda^2 \int_{f(0)}^{f^2(x),\,1} f^{-1}(y)\,\varphi(y)\,dy = \lambda^2 \int_{f(0)}^{f^2(x),\,1} \varphi(y)\,df_1(y) = \lambda^2 \Big[f_1(y)\,\varphi(y)\Big]^{f^2(x),\,1} - \lambda^2 \int_{f(0)}^{f^2(x),\,1} f_1(y)\,\varphi'(y)\,dy.$$

Repeating the process n times and using the notation:

$$f_\nu(x) = \int_{f^\nu(0)}^{x} f_{\nu-1}\left[f^{-1}(y)\right] dy,$$

the result is:

$$\varphi(x) = \lambda \Big[y\,\varphi(y)\Big]^{f(x),\,1} - \lambda^2 \Big[f_1(y)\,\varphi(y)\Big]^{f^2(x),\,1} + \lambda^3 \Big[f_2(y)\,\varphi(y)\Big]^{f^3(x),\,1} - \cdots + \tag{10.2}$$

$$+ (-1)^{n-1} \lambda^n \Big[f_{n-1}(y)\,\varphi(y)\Big]^{f^n(x),\,1} + (-1)^n \lambda^{n+1} \int_{f^n(0)}^{f^{n+1}(x),\,1} f_{n-1}\left[f^{-1}(y)\right] \varphi(y)\,dy.$$

Determine n by the inequality:

$$f^{n-1}(0) < 1 \leq f^{n}(0). \tag{10.3}$$

n is a finite number in consequence of condition (γ). Then the integral in (10. 2) vanishes and this formula is a generalisation of (2. 4) and (5. 2).

In order to determine the characteristic values put $x = 1$ in (10. 2). Since $\varphi(1) = 0$ implies $\varphi(x) \equiv 0$ they necessarily satisfy:

$$1 - \lambda + \lambda^2 f_1(1) - \lambda^3 f_2(1) + \cdots + (-\lambda)^n f_{n-1}(1) = P_4(\lambda) = 0. \tag{10.4}$$

Every root of (10. 4) is a characteristic value because the corresponding characteristic function is determined by (10. 2). The fact is that (10. 2) determines $\varphi(x)$ successively in the intervals $f^{-1}(1) \leq x \leq 1$, $f^{-2}(1) \leq x \leq f^{-1}(1), \ldots, f^{-(n-1)}(1) \leq \leq x \leq f^{-(n-2)}(1)$, $0 \leq x \leq f^{-(n-1)}(1)$, (compare (2. 4')). It is evident that every characteristic value has a single characteristic function.

In **11** we shall prove that $P_4(\lambda)$ is just the Fredholm denominator of the integral equation (10. 1). If we assume for a moment that this is proved, we have not far to look for the following remark. The expression (8. 10) for the denominator of Fredholm as a sum of integrals of determinants does not contain $f'(x)$, neither does the left-hand member of (10. 4). Hence we infer that condition (β) could be omitted and that $f'(x)$ has come into the calculations for formal reasons only. This can also be shown, without having recourse to the denominator of Fredholm, by verifying that (10. 2), which does not contain $f'(x)$, is a consequence of the integral equation (10. 1).

The kernels of **2** and **5** are special cases of the kernel of this section.

Finally notice that there is another simple method which enables us to see immediately that the integral equation (10. 1) has at most n characteristic values. For consider the iterated kernels. First extend the function $f(x)$ defined in the interval $0 \leq x \leq f^{-1}(1)$ to the whole interval $0 \leq x \leq 1$ by making it 1 in $f^{-1}(1) \leq x \leq 1$. We get:

$$K^{(2)}(x, y) = \int_0^1 K(x, s)\, K(s, y)\, ds = \overline{f(x) - f^{-1}(y), 0.}$$

(Here $K(x, y)$ means the kernel of (1. 1) and not $K(x, y)$ of (1. 2)).

$K^{(2)}(x, y)$ is different from (and greater than) 0 for $f(x) > f^{-1}(y)$ only, that is below the line $y = f^2(x)$.

$K^{(3)}(x, y)$ becomes:

$$K^{(3)}(x, y) = \int_0^1 K(x, s) K^{(2)}(s, y)\, ds = \begin{cases} \int_{f^{-2}(y)}^{f(x)} \left\{ f(s) - f^{-1}(y) \right\} ds, & \text{if } f(x) > f^{-2}(y); \\ 0 & \text{, if } f(x) \leq f^{-2}(y). \end{cases}$$

$K^{(3)}(x, y)$ is different from 0 for $y < f^3(x)$ only. In that case it is the sum of one function of x only, one function of y only and one function which is the product of a function of x by a function of y.

Next calculate $K^{(4)}(x, y)$. This function is different from 0 for $y < f^4(x)$ only, in that case it is the sum of one function of x only, one function of y only and two functions which are products of a function of x by a function of y.

Generally $K^{(\nu)}(x, y)$ is different from 0 for $y < f^{\nu}(x)$ only and is then the sum of one function of x, one function of y and $\nu - 2$ functions which are products of a function of x by a function of y.

Defining n by the inequality (10. 3), the n:th iterated kernel $K^{(n)}(x, y)$ is different from 0 in the whole existence-square. Hence $K^{(n)}(x, y)$ is of finite rank and has at most n characteristic functions. We know that the characteristic functions of $K(x, y)$ are among those of $K^{(n)}(x, y)$ and hence conclude that the integral equation (10. 1) can have at most n characteristic values.

11. The line may go through (0, 0) and (1, 1); $K(x, y) = 1$.

When the curve of **10** goes from (0, 0) to (1, 1): $f(0) = 0$, $f(1) = 1$, there is no finite n determined by the inequality (10. 3) and the expansion (10. 2) will not terminate. We shall proceed in another way, at the same time showing that the obtained expression is the denominator of Fredholm.

First assume that conditions (α) and (β) of **10** are fulfilled and (γ) except for $x = 0$ and $x = 1$. Follow the method of **8** considering the non-homogeneous integral equation:

$$\varphi(x) = \lambda \int_0^{f(x)} \varphi(y) + g(x). \tag{11. 1}$$

($g'(x)$ supposed to be integrable.) We shall use the following notations:

$$\begin{cases} f_1(x) = \int\limits_0^x f^{-1}(y)\,dy; \\ f_2(x) = \int\limits_0^x f_1[f^{-1}(y)]\,dy; \\ \cdots \quad \cdots \quad \cdots \\ f_\nu(x) = \int\limits_0^x f_{\nu-1}[f^{-1}(y)]\,dy; \\ \cdots \quad \cdots \quad \cdots \end{cases}$$

Use the relation $\varphi'(x) = \lambda f'(x)\varphi[f(x)] + g'(x)$ and integrate by parts n times:

$$(11.2) \qquad \varphi(x) - \lambda f(x)\varphi[f(x)] + \lambda^2 f_1[f^2(x)]\varphi[f^2(x)] - \lambda^3 f_2[f^3(x)]\varphi[f^3(x)] +$$

$$+ \cdots + (-\lambda)^n f_{n-1}[f^n(x)]\varphi[f^n(x)] + (-\lambda)^{n+1}\int\limits_0^{f^{n+1}(x)} \varphi(y) f_{n-1}[f^{-1}(y)]\,dy =$$

$$= g(x) - \lambda\int\limits_0^{f(x)} y\,g'(y)\,dy + \lambda^2\int\limits_0^{f^2(x)} f_1(y)\,g'(y)\,dy - \lambda^3\int\limits_0^{f^3(x)} f_2(y)\,g'(y)\,dy + \cdots +$$

$$+ (-\lambda)^n \int\limits_0^{f^n(x)} f_{n-1}(y)\,g'(y)\,dy.$$

This expansion does not terminate and therefore we shall examine the order of magnitude of the positive quantities $f_\nu(x)$. From condition (γ): $f(x) \geq x$ follows: $f^{-1}(x) \leq x$. It is evident that $f_\nu(x) \leq \frac{x^{\nu+1}}{(\nu+1)!}$. Hence the integral in the left-hand member of (11.2) tends to 0 when n tends to ∞. Put $g(x) = 1$ and $x = 1$ in (11.2):

$$(11.3) \qquad \varphi(1)[1 - \lambda + \lambda^2 f_1(1) - \lambda^3 f_2(1) + \cdots + (-\lambda)^n f_{n-1}(1) + \cdots] = 1.$$

It is to be expected that the characteristic values will satisfy:

$$H_5(\lambda) = 1 - \lambda + \lambda^2 f_1(1) - \lambda^3 f_2(1) + \cdots + (-\lambda)^n f_{n-1}(1) + \cdots = 0.$$

We shall prove that $H_5(\lambda)$ is the Fredholm denominator of the integral equation (11.1). (11.3) may be written:

$$\varphi(1) = \frac{1}{H_5(\lambda)}.$$

Compare this with the Fredholm expression:

$$\varphi(x) = \lambda \int_0^1 \frac{D\left(\begin{matrix} x \\ y \end{matrix}\middle| \lambda\right)}{D(\lambda)} g(y)\, dy + g(x).$$

We get:

$$\varphi(1) = \frac{\lambda \int_0^1 D\left(\begin{matrix} 1 \\ y \end{matrix}\middle| \lambda\right) dy + D(\lambda)}{D(\lambda)} = \frac{D\left(\begin{matrix} 1 \\ 0 \end{matrix}\middle| \lambda\right)}{D(\lambda)}.$$

The *identity* to be proved is $D\left(\begin{matrix} 1 \\ 0 \end{matrix}\middle| \lambda\right) = 1$. The Fredholm formula for $D\left(\begin{matrix} 1 \\ 0 \end{matrix}\middle| \lambda\right)$ is:

$$D\left(\begin{matrix} 1 \\ 0 \end{matrix}\middle| \lambda\right) = K(1, 0) + \\ + \sum_{\nu=1}^{\infty} \frac{(-\lambda)^\nu}{\nu!} \int_0^1 \int_0^1 \cdots \int_0^1 \begin{vmatrix} K(1, 0) & K(1, s_1) & K(1, s_2) & \cdots & K(1, s_\nu) \\ K(s_1, 0) & K(s_1, s_1) & K(s_1, s_2) & \cdots & K(s_1, s_\nu) \\ K(s_2, 0) & K(s_2, s_1) & K(s_2, s_2) & \cdots & K(s_2, s_\nu) \\ \cdots & & \cdots & & \cdots \\ K(s_\nu, 0) & K(s_\nu, s_1) & K(s_\nu, s_2) & \cdots & K(s_\nu, s_\nu) \end{vmatrix} ds_1\, ds_2 \cdots ds_\nu.$$

$K(x, y) = 1$ when $x > y$. Hence $K(1, 0) = 1$. All the integrals can be shown to be 0 because the determinants vanish in every point $s_1, s_2, \ldots, s_\nu$. Let s_p be the least of these numbers, or one of the least if several of them are equal to each other and less than all the rest. Then the first and $(p + 1)$:th columns are equal, consisting of ones only. Hence the determinants vanish everywhere. We infer that $H_5(\lambda)$ of (11. 3) is the Fredholm denominator of (11. 1).

To obtain an explicit expression for the characteristic function $\varphi(x)$ corresponding to the characteristic value λ we can proceed as follows (the method is the same as that of 7, (α), which gave (7. 6)). Supposing $\varphi(1)$ to be known, $\varphi(x)$ satisfies the following integral equation of the Volterra type:

$$\varphi(x) = -\lambda \int_{f(x)}^{1} \varphi(y)\, dy + \varphi(1).$$

We shall use the notations:

$$\begin{cases} g_1(x, y) = \int\limits_{f(x)}^{y} dy; \\ g_2(x, y) = \int\limits_{f^2(x)}^{y} g_1(x, f^{-1}(y))\, dy; \\ \dots \quad \dots \quad \dots \\ g_\nu(x, y) = \int\limits_{f^\nu(x)}^{y} g_{\nu-1}(x, f^{-1}(y))\, dy; \\ \dots \quad \dots \quad \dots \end{cases}$$

Integration by parts gives:

$$\varphi(x) = \varphi(1) - \lambda \int\limits_{f(x)}^{1} \varphi(y)\, d[y - f(x)] = \varphi(1) - \lambda \int\limits_{f(x)}^{1} \varphi(y)\, d g_1(x, y) =$$

$$= \varphi(1) - \lambda g_1(x, 1) \varphi(1) + \lambda \int\limits_{f(x)}^{1} g_1(x, y) \varphi'(y)\, dy.$$

Since $\varphi'(x) = \lambda f'(x) \varphi[f(x)]$ the new integral can be written, substituting t for $f(y)$:

$$\lambda \int\limits_{f(x)}^{1} g_1(x, y) \varphi'(y)\, dy = \lambda^2 \int\limits_{f(x)}^{1} g_1(x, y) f'(y) \varphi[f(y)]\, dy = \lambda^2 \int\limits_{f^2(x)}^{1} g_1(x, f^{-1}(t)) \varphi(t)\, dt =$$

$$= \lambda^2 \int\limits_{f^2(x)}^{1} \varphi(t)\, d g_2(x, t) = \lambda^2 g_2(x, 1) \varphi(1) - \lambda^2 \int\limits_{f^2(x)}^{1} g_2(x, t) \varphi'(t)\, dt.$$

After n integrations by parts we have:

$$\varphi(x) = \varphi(1)[1 - \lambda g_1(x, 1) + \lambda^2 g_2(x, 1) - \lambda^3 g_3(x, 1) +$$

$$+ \cdots + (-\lambda)^n g_n(x, 1)] + (-\lambda)^{n+1} \int\limits_{f^{n+1}(x)}^{1} g_n(x, f^{-1}(t)) \varphi(t)\, dt.$$

The integral tends to 0 when n tends to ∞. Hence $\varphi(x)$ is determined by:

$$(11.4) \qquad \varphi(x) = \varphi(1)[1 - \lambda g_1(x, 1) + \lambda^2 g_2(x, 1) - \cdots + (-\lambda)^n g_n(x, 1) + \cdots].$$

Note that (11.4) defines a function $\varphi(x)$ for every λ, since a non-homogeneous integral equation of the Volterra type has always a unique solution. But when λ is not a characteristic value we have $\varphi(1) = 0$ which makes $\varphi(x) \equiv 0$.

As in **10** we can show that condition (β) concerning $f(x)$ could be omitted. Modifying the right-hand member of (11.2) by integration by parts to get rid of the derivative $g'(x)$, the result is:

$$g(x)-\lambda f(x)g[f(x)]+\lambda^2 f_1[f^2(x)]g[f^2(x)]-\lambda^3 f_2[f^3(x)]g[f^3(x)]+$$
$$+\cdots+(-\lambda)^n f_{n-1}[f^n(x)]g[f^n(x)]+\lambda\int_0^{f(x)}g(y)\,dy-\lambda^2\int_0^{f^2(x)}g(y)f^{-1}(y)\,dy+$$
$$+\lambda^3\int_0^{f^3(x)}g(y)f_1[f^{-1}(y)]\,dy+\cdots-(-\lambda)^n\int_0^{f^n(x)}g(y)f_{n-2}[f^{-1}(y)]\,dy.$$

Then (11.2) does not contain $f'(x)$ and $g'(x)$. We can directly verify that (11.2) is satisfied by the solutions of (11.1), independently of the existence of $f'(x)$ and $g'(x)$.

When condition (γ) is not fulfilled, that is the line passes across the diagonal $y=x$, the same method can still be used in certain cases, e.g. for kernels similar to those of **8**. When $f(x)=x$ for $x=a$, $(0<a<1)$, $f(x)>x$ for $x<a$ and $f(x)<x$ for $x>a$, we obtain the denominator of Fredholm by considering the kernel in the smaller square $0\le x\le a$, $0\le y\le a$. Also condition (α) can be relaxed in certain cases.

Note that the kernel of **10** could be regarded as a special case of the kernel of this section.

Finally we shall carry through the actual calculation of the Fredholm denominator in a special case, viz. on the supposition that the kernel is 1 below the line $y=x^a$, $(0<a\le 1)$, and 0 above it. We get successively:

$$f^{-1}(x)=x^{\frac{1}{a}},\quad f_1(x)=\frac{x^{1+\frac{1}{a}}}{1+\frac{1}{a}},\quad f_2(x)=\frac{x^{1+\frac{1}{a}+\frac{1}{a^2}}}{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{a}+\frac{1}{a^2}\right)},\ \ldots,$$

$$f_n(x)=\frac{x^{1+\frac{1}{a}+\frac{1}{a^2}+\cdots+\frac{1}{a^n}}}{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{a}+\frac{1}{a^2}\right)\cdots\left(1+\frac{1}{a}+\frac{1}{a^2}+\cdots+\frac{1}{a^n}\right)},\ \ldots;$$

$$D(\lambda)=1-\lambda+\frac{\lambda^2}{1+\frac{1}{a}}-\frac{\lambda^3}{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{a}+\frac{1}{a^2}\right)}+$$
$$+\cdots+\frac{(-\lambda)^n}{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{a}+\frac{1}{a^2}\right)\cdots\left(1+\frac{1}{a}+\frac{1}{a^2}+\cdots+\frac{1}{a^{n-1}}\right)}+\cdots.$$

When $a = 1$ we get $D(\lambda) = e^{-\lambda}$. This is the Fredholm denominator of the Volterra integral equation with kernel 1.

12. $K(x, y) = P(x)\, Q(y)$.

Generalize the kernel of **11** assuming that it is $P(x)\, Q(y)$ below the line $y = f(x)$ and 0 above it (compare **3**). Again consider the non-homogeneous integral equation:

$$\varphi(x) = \lambda\, P(x) \int_0^{f(x)} Q(y)\, \varphi(y)\, dy + g(x). \tag{12. 1}$$

The transformation:

$$\varphi_1(x) = \int_0^{f(x)} Q(y)\, \varphi(y)\, dy$$

gives:

$$\varphi(x) = \lambda\, P(x)\, \varphi_1(x) + g(x).$$

(12. 1) is changed into:

$$\varphi_1(x) = \lambda \int_0^{f(x)} P(y)\, Q(y)\, \varphi_1(y)\, dy + \int_0^{f(x)} Q(y)\, g(y)\, dy. \tag{12. 2}$$

Putting $g_1(x) = \int_0^{f(x)} Q(y)\, g(y)\, dy$ we shall treat (12. 2) by integration by parts in order to obtain its denominator of Fredholm. Finally we shall show that the Fredholm denominators of (12. 1) and (12. 2) are identical.

Introduce the notations:

$$\begin{cases} F_1(x) = \int_0^x P(y)\, Q(y)\, dy; \\ F_2(x) = \int_0^x F_1[f^{-1}(y)]\, P(y)\, Q(y)\, dy; \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ F_\nu(x) = \int_0^x F_{\nu-1}[f^{-1}(y)]\, P(y)\, Q(y)\, dy; \\ \ldots \quad \ldots \quad \ldots \quad \ldots \end{cases}$$

(12. 2) can be written:

$$\varphi_1(x) = \lambda \int_0^{f(x)} \varphi_1(y)\, d\, F_1(y) + g_1(x).$$

Operating as in **11** we obtain:

$$(12.3)\quad \varphi_1(x) - \lambda F_1[f(x)]\varphi_1[f(x)] + \lambda^2 F_2[f^2(x)]\varphi_1[f^2(x)] - \\ - \lambda^3 F_3[f^3(x)]\varphi_1[f^3(x)] + \cdots + (-\lambda)^n F_n[f^n(x)]\varphi_1[f^n(x)] + \\ + (-\lambda)^{n+1}\int_0^{f^{n+1}(x)} F_n[f^{-1}(y)]\,P(y)\,Q(y)\,\varphi_1(y)\,dy = g_1(x) - \lambda\int_0^{f(x)} F_1(y)\,g_1'(y) + \\ + \lambda^2\int_0^{f^2(x)} F_2(y)\,g_1'(y)\,dy - \cdots + (-\lambda)^n\int_0^{f^n(x)} F_n(y)\,g_1'(y)\,dy.$$

Suppose that $P(x)\,Q(x)$ is integrable in $0 \le x \le 1$. Since $0 \le f^{-1}(x) \le x$ we see that the integral in the left-hand member of (12.3) tends to 0 when n tends to ∞. Putting $g_1(x) = 1$ and $x = 1$, (12.3) becomes:

$$\varphi_1(1)[1 - \lambda F_1(1) + \lambda^2 F_2(1) - \lambda^3 F_3(1) + \cdots + (-\lambda)^n F_n(1) + \cdots] = 1.$$

Denoting the expression in square brackets by $H_6(\lambda)$, this can be written:

$$\varphi_1(1) = \frac{1}{H_6(\lambda)}.$$

We shall identify this formula with the Fredholm solution of (12.2):

$$\varphi_1(x) = \lambda\int_0^1 \frac{D_1\left(\begin{matrix}x\\y\end{matrix}\middle|\lambda\right)}{D_1(\lambda)}\,g_1(x)\,dy + g_1(x).$$

$\left(\text{Let } D_1\left(\begin{matrix}x\\y\end{matrix}\middle|\lambda\right) \text{ and } D_1(\lambda) \text{ be the Fredholm functions for (12.2)}\right)$.

For $g_1(x) = 1, x = 1$ this becomes:

$$\varphi_1(1) = \frac{\lambda\int_0^1 D_1\left(\begin{matrix}1\\y\end{matrix}\middle|\lambda\right)dy + D_1(\lambda)}{D_1(\lambda)}.$$

By proving that the numerator is identically 1, we infer that $D_1(\lambda) = H_6(\lambda)$. In the formula (8.7) put $x = 1$, $y = 0$:

$$D_1\left(\begin{matrix}1\\0\end{matrix}\middle|\lambda\right) = P(0)\,Q(0)\left[D_1(\lambda) + \lambda\int_0^1 D_1\left(\begin{matrix}1\\y\end{matrix}\middle|\lambda\right)dy\right].$$

On the other hand the Fredholm expression for $D_1\left(\begin{matrix}1\\0\end{matrix}\middle|\lambda\right)$ is:

$$D_1\left(\begin{matrix}1\\0\end{matrix}\middle|\lambda\right) = K(1,0) + \sum_{\nu=1}^{\infty}\frac{(-\lambda)^\nu}{\nu!}\int_0^1\int_0^1\cdots\int_0^1 \begin{vmatrix} K(1,0) K(1,s_1) K(1,s_2) \cdots K(1,s_\nu) \\ K(s_1,0) K(s_1,s_1) K(s_1,s_2) \cdots K(s_1,s_\nu) \\ K(s_2,0) K(s_2,s_1) K(s_2,s_2) \cdots K(s_2,s_\nu) \\ \cdots \quad \cdots \quad \cdots \\ K(s_\nu,0) K(s_\nu,s_1) K(s_\nu,s_2) \cdots K(s_\nu,s_\nu) \end{vmatrix} ds_1\, ds_2 \cdots ds_\nu =$$

$$= P(0)\,Q(0)\left[1 + \sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu!}\int_0^1\int_0^1\cdots\int_0^1 \begin{vmatrix} 1\; K(1,s_1) K(1,s_2) \cdots K(1,s_\nu) \\ 1\; K(s_1,s_1) K(s_1,s_2) \cdots K(s_1,s_\nu) \\ 1\; K(s_2,s_1) K(s_2,s_2) \cdots K(s_2,s_\nu) \\ \cdots \quad \cdots \quad \cdots \\ 1\; K(s_\nu,s_1) K(s_\nu,s_2) \cdots K(s_\nu,s_\nu) \end{vmatrix} ds_1\, ds_2 \cdots ds_\nu\right].$$

The determinants in the integrals vanish at every point $s_1, s_2, \ldots, s_\nu$ of the interval of integration. $x \geq y$ implies $K(x,y) = P(y)\,Q(y)$. Among the numbers $s_1, s_2, \ldots, s_\nu$ there are always one or more which are the least. Let s_p be one of them. Then the $(p+1)$:th column consists of the numbers $P(s_p)\,Q(s_p)$ only and is therefore proportional to the first column, making the determinant vanish. Thus we have shown:

$$D_1\left(\begin{matrix}1\\0\end{matrix}\middle|\lambda\right) = P(0)\,Q(0) = P(0)\,Q(0)\left[1 + \lambda\int_0^1 D_1\left(\begin{matrix}1\\y\end{matrix}\middle|\lambda\right)dy\right]$$

$$D_1(\lambda) + \lambda\int_0^1 D_1\left(\begin{matrix}1\\y\end{matrix}\middle|\lambda\right)dy = 1 \text{ (even when } P(0)\,Q(0) = 0).$$

We have proved that $H_6(\lambda)$ is the Fredholm denominator of (12.2). It remains to show that (12.1) and (12.2) have the same denominator of Fredholm. Denote that of (12.1) by $D(\lambda)$. Let $E(x,y)$ be the kernel of **11**, 1 below the line $y = f(x)$. The Fredholm formulas are:

$$D(\lambda) = 1 + \sum_{\nu=1}^{\infty}\frac{(-\lambda)^\nu}{\nu!}\int_0^1\int_0^1\cdots\int_0^1 \begin{vmatrix} P(s_1)\,Q(s_1)\,E(s_1,s_1) & P(s_1)\,Q(s_2)\,E(s_1,s_2) & \cdots & P(s_1)\,Q(s_\nu)\,E(s_1,s_\nu) \\ P(s_2)\,Q(s_1)\,E(s_2,s_1) & P(s_2)\,Q(s_2)\,E(s_2,s_2) & \cdots & P(s_2)\,Q(s_\nu)\,E(s_2,s_\nu) \\ \cdots & \cdots & \cdots & \cdots \\ P(s_\nu)\,Q(s_1)\,E(s_\nu,s_1) & P(s_\nu)\,Q(s_2)\,E(s_\nu,s_2) & \cdots & P(s_\nu)\,Q(s_\nu)\,E(s_\nu,s_\nu) \end{vmatrix} ds_1\, ds_2 \cdots ds_\nu$$

$$D_1(\lambda) = 1 + \sum_{\nu=1}^{\infty}\frac{(-\lambda)^\nu}{\nu!}\int_0^1\int_0^1\cdots\int_0^1 \begin{vmatrix} P(s_1)\,Q(s_1)\,E(s_1,s_1) & P(s_2)\,Q(s_2)\,E(s_1,s_2) & \cdots & P(s_\nu)\,Q(s_\nu)\,E(s_1,s_\nu) \\ P(s_1)\,Q(s_1)\,E(s_2,s_1) & P(s_2)\,Q(s_2)\,E(s_2,s_2) & \cdots & P(s_\nu)\,Q(s_\nu)\,E(s_2,s_\nu) \\ \cdots & \cdots & \cdots & \cdots \\ P(s_1)\,Q(s_1)\,E(s_\nu,s_1) & P(s_2)\,Q(s_2)\,E(s_\nu,s_2) & \cdots & P(s_\nu)\,Q(s_\nu)\,E(s_\nu,s_\nu) \end{vmatrix} ds_1\, ds_2 \cdots ds_\nu$$

$D(\lambda)$ and $D_1(\lambda)$ are identical because corresponding integrals may be written:

$$\int_0^1\int_0^1\cdots\int_0^1 \begin{vmatrix} E(s_1, s_1) & E(s_1, s_2) \cdots & E(s_1, s_\nu) \\ E(s_2, s_1) & E(s_2, s_2) \cdots & E(s_2, s_\nu) \\ \cdots & \cdots & \cdots \\ E(s_\nu, s_1) & E(s_\nu, s_2) \cdots & E(s_\nu, s_\nu) \end{vmatrix} \cdot \prod_{m=1}^{\nu} P(s_m)\, Q(s_m)\, ds_m.$$

13. $K(x,y) = \sum_{\nu=1}^{N} P_\nu(x)\, Q_\nu(y)$.

Finally we shall consider a kernel of the form (1. 2) below the line $y = f(x)$ of **11**. Assume that the functions $\{P_\nu(x)\}$ and $\{Q_\nu(x)\}$, $\nu = 1, 2, \ldots, N$, are linearly independent and that the products $P_\nu(x)\, Q_\nu(x)$ are integrable.

The integral equation is:

$$\varphi(x) = \lambda \int_0^{f(x)} \left\{ \sum_{\nu=1}^{N} P_\nu(x)\, Q_\nu(y) \right\} \varphi(y)\, dy. \tag{13. 1}$$

Calculate the resolvent $\Gamma(x, y; \lambda)$ in the point $x = 1$, $y = 0$. The functional equation for $\Gamma(x, y; \lambda)$ can be written:

$$\Gamma(x, y;\, \lambda) = \lambda \int_0^{f(x)} \left\{ \sum_{\nu=1}^{N} P_\nu(x)\, Q_\nu(s) \right\} \Gamma(s, y; \lambda)\; ds + \begin{cases} \sum_{\nu=1}^{N} P_\nu(x)\, Q_\nu(y), & \text{if } 0 \leq y \leq f(x); \\ 0 & , \text{if } f(x) < y \leq 1. \end{cases}$$

Regarding y as a parameter this is an integral equation with $\Gamma(x, y; \lambda)$ as the unknown function. For $y = 0$ it becomes:

$$\Gamma(x, 0; \lambda) = \lambda \int_0^{f(x)} \left\{ \sum_{\nu=1}^{N} P_\nu(x)\, Q_\nu(s) \right\} \Gamma(s, 0; \lambda)\, ds + \sum_{\nu=1}^{N} P_\nu(x)\, Q_\nu(0). \tag{13. 2}$$

Instead of treating (13. 1) directly, we shall consider the following system of integral equations:

$$\varphi_\mu(x) = \lambda \int_0^{f(x)} \left\{ \sum_{\nu=1}^{N} P_\nu(s)\, Q_\mu(s) \varphi_\nu(s) \right\} ds + Q_\mu(0). \qquad (\mu = 1, 2, \ldots, N). \tag{13. 3}$$

$\varphi_\mu(x)$ determined, we obtain $\Gamma(x, 0; \lambda)$ by the formula:

$$\Gamma(x, 0; \lambda) = \sum_{\nu=1}^{N} P_\nu(x)\,\varphi_\nu(x). \tag{13.4}$$

Integrate (13.3) by parts introducing the notations:

$$\begin{cases} F_{\nu\mu}^{(1)}(x) = \int_0^x P_\nu(y)\,Q_\mu(y)\,dy; \\ F_{\nu\mu}^{(2)}(x) = \int_0^x \left\{\sum_{\tau=1}^{N} F_{\tau\mu}^{(1)}[f^{-1}(y)]\,P_\nu(y)\;Q_\tau(y)\right\} dy; \\ \dots \quad \dots \quad \dots \quad \dots \\ F_{\nu\mu}^{(\alpha)}(x) = \int_0^x \left\{\sum_{\tau=1}^{N} F_{\tau\mu}^{(\alpha-1)}[f^{-1}(y)]\,P_\nu(y)\,Q_\tau(y)\right\} dy; \\ \dots \quad \dots \quad \dots \quad \dots \end{cases}$$

The result is:

$$\varphi_\mu(x) - \lambda \sum_{\nu=1}^{N} F_{\nu\mu}^{(1)}[f(x)]\,\varphi_\nu[f(x)] + \lambda^2 \sum_{\nu=1}^{N} F_{\nu\mu}^{(2)}[f^2(x)]\,\varphi_\nu[f^2(x)] - \dots +$$
$$+ (-\lambda)^n \sum_{\nu=1}^{N} F_{\nu\mu}^{(n)}[f^n(x)]\,\varphi_\nu[f^n(x)] + (-\lambda)^{n+1} \int_0^{f^{n+1}(x)} \left\{\sum_{\nu=1}^{N} \varphi_\nu(x)\,d\,F_{\nu\mu}^{(n+1)}(x)\right\} = Q_\mu(0).$$

The integral in the left-hand member tends to 0 when n tends to ∞. Making $x = 1$ we obtain:

$$\varphi_\mu(1) - \lambda \sum_{\nu=1}^{N} F_{\nu\mu}^{(1)}(1)\,\varphi_\nu(1) + \lambda^2 \sum_{\nu=1}^{N} F_{\nu\mu}^{(2)}(1)\,\varphi_\nu(1) - \dots + \tag{13.5}$$
$$+ (-\lambda)^n \sum_{\nu=1}^{N} F_{\nu\mu}^{(n)}(1)\,\varphi_\nu(1) + \dots = Q_\mu(0).$$

Abbreviate:

$$E_{\nu\mu}(\lambda) = \lambda F_{\nu\mu}^{(1)}(1) - \lambda^2 F_{\nu\mu}^{(2)}(1) + \dots + (-1)^{n-1} \lambda^n F_{\nu\mu}^{(n)}(1) + \dots.$$

Since $f(x) \geq x$ it is obvious that $E_{\nu\mu}(\lambda)$ is an integral function of λ. (13.5) can be written:

$$\begin{cases} \varphi_1(1)(1 - E_{11}(\lambda)) - \varphi_2(1)\,E_{21}(\lambda) - \dots - \varphi_N(1)\,E_{N1}(\lambda) = Q_1(0); \\ -\varphi_1(1)\,E_{12}(\lambda) + \varphi_2(1)(1 - E_{22}(\lambda)) - \dots - \varphi_N(1)\,E_{N2}(\lambda) = Q_2(0); \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ -\varphi_1(1)\,E_{1N}(\lambda) - \varphi_2(1)\,E_{2N}(\lambda) - \dots + \varphi_N(1)(1 - E_{NN}(\lambda)) = Q_N(0). \end{cases} \tag{13.6}$$

This is a linear system of equations in $\varphi_1(1), \varphi_2(1), \ldots, \varphi_N(1)$. Putting its solution into (13.4) we get:

$$\Gamma(1,0;\lambda) = \frac{-\begin{vmatrix} 0 & P_1(1) & P_2(1) & \cdots & P_N(1) \\ Q_1(0) & 1 - E_{11}(\lambda) & -E_{21}(\lambda) & \cdots & -E_{N1}(\lambda) \\ Q_2(0) & -E_{12}(\lambda) & 1 - E_{22}(\lambda) & \cdots & -E_{N2}(\lambda) \\ & \cdots & \cdots & & \cdots \\ Q_N(0) & -E_{1N}(\lambda) & -E_{2N}(\lambda) & \cdots & 1 - E_{NN}(\lambda) \end{vmatrix}}{\begin{vmatrix} 1 - E_{11}(\lambda) & -E_{21}(\lambda) & \cdots & -E_{N1}(\lambda) \\ -E_{12}(\lambda) & 1 - E_{22}(\lambda) & \cdots & -E_{N2}(\lambda) \\ \cdots & \cdots & & \cdots \\ -E_{1N}(\lambda) & -E_{2N}(\lambda) & \cdots & 1 - E_{NN}(\lambda) \end{vmatrix}}.$$

The characteristic values are the zeros of the determinant in the denominator. If n denotes the rank of the determinant there are $N - n$ corresponding characteristic functions which can be determined in the following way.

(13.1) corresponds to a homogeneous system of integral equations ((13.3) with $Q_\mu(0) = 0$). From there we can derive a linear and homogeneous system of equations in $\varphi_1(1), \varphi_2(1), \ldots, \varphi_N(1)$ ((13.6) with $Q_\mu(0) = 0$).

Since the determinant has the rank n the system has $N - n$ linearly independent solutions. $\{\varphi_\nu(1)\}$ determined, we obtain $\{\varphi_\nu(x)\}$, $\nu = 1, 2, \ldots, N$, by a method corresponding to that which was used in **11** to derive (11.4).

Finally we get the characteristic functions by the formula:

$$\varphi(x) = \sum_{\nu=1}^{N} P_\nu(x)\, \varphi_\nu(x).$$

14. A line tending to the diagonal $y = x$.

We shall examine the behaviour of the characteristic values of a kernel which is 0 above a line tending to the diagonal $y = x$. When the line consists of the diagonal there are no characteristic values since the integral equation is of the Volterra type. In **2** we saw that all the characteristic values of a certain kernel tended to ∞ when the boundary line tended towards the diagonal.

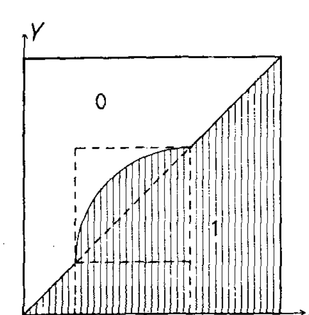

Fig. 7.

Suppose, e. g., that the line consists of the diagonal curved upwards between the two points x_0 and

x_1 (fig. 7) and that it satisfies the conditions of **11** in the square $x_0 \leq x \leq x_1$, $x_0 \leq y \leq x_1$. It is easy to see that the characteristic values are obtained by considering the kernel in the smaller square. To prove this consider the homogeneous integral equation (compare 8)

$$\varphi(x) = \lambda \int_0^1 K(x, y) \varphi(y) \, dy.$$

$$(K(x, y) = 0 \text{ above the line}).$$

When $0 \leq x \leq x_0$ this is a Volterra equation. Hence $\varphi(x) = 0$ for every λ when $x \leq x_0$. When $x_0 \leq x \leq x_1$ regard the smaller square $x_0 \leq x \leq x_1$, $x_0 \leq y \leq x_1$ and calculate the characteristic values of $K(x, y)$ in it.

When $x_1 \leq x \leq 1$ we write the integral equation:

$$\varphi(x) = \lambda \int_{x_0}^{x_1} K(x, y) \varphi(y) \, dy + \lambda \int_{x_1}^{x} K(x, y) \varphi(y) \, dy. \tag{14. 1}$$

Let λ be one of the characteristic values of the kernel in the smaller square. Then the first term of the right-hand side of (14. 1) is a known function. (14. 1) can be regarded as a non-homogeneous integral equation of the Volterra type. Hence it has a unique solution.

We obtain the characteristic values as the zeros of the Fredholm denominator $D_*(\lambda)$ of $K(x, y)$ calculated in the smaller square. The Fredholm denominator $D(\lambda)$ of $K(x, y)$ in the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ has the same zeros as $D_*(\lambda)$. Hence $D_*(\lambda)$ and $D(\lambda)$ differ by a factor $e^{\alpha x}$.

The characteristic values depend on the shape of the line between $x = x_0$ and $x = x_1$ and tend to ∞ when x_1 tends to x_0.

An exemple: let the line consist of the left and upper side of the square $x_0 \leq x \leq x_1$, $x_0 \leq y \leq x_1$ and assume that the kernel is 1 below the line. Denote the area between the line and the diagonal by ε: $\varepsilon = \frac{(x_1 - x)^2}{2}$. This kernel has a single characteristic value $\lambda = \frac{1}{x_1 - x_0}$ tending to infinity when x_1 tends to x_0. We have:

$$\frac{1}{\lambda^2} = 2\varepsilon.$$

More general: for the kernels of **10** and **11**, if $H_5(\lambda)$ is of genus 0, we have:

$$\sum \frac{1}{\lambda_\nu^2} = 2\varepsilon.$$

ε again denotes the area between the line and the diagonal $y = x$.