# ON A NON-LINEAR TOTAL DIFFERENTIAL EQUATION IN NORMED LINEAR SPACES.

By

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## Index.



**Introduction**. The special functions of classical analysis derive much of their interest and importance from the properties of the numerical differential equations that characterize them. Since the publication of Fréchet's famous 1906 thesis (Palermo Rendiconti), an enormous amount of significant contributions have been made to the study of very general classes of functions in many general spaces — topological spaces, normed linear spaces, and many others — and the study of the spaces themselves has received a considerable amount of attention. Since the independent variables of the functions considered lie in function spaces and other infinitely dimensional spaces, and in other general spaces, it is clear that the special functions of general analysis could not in general, from their very nature, be characterized by ordinary numerical differential equations. It appears to the author that the characterization problem for special functions in general analysis will have to be sought amongst functional equations in Fréchet differentials or in other differentials of general analysis.

In this paper we investigate the properties and solutions of a special differential system in normed linear spaces

$$\delta y(x) = T(y(x), \delta x, y(x)), y(0) = y_0, \tag{1.1}$$

in which the unknown function $y(x)$ occurs quadratically in the function $T$ and $\delta y(x)$ is the Fréchet differential of $y(x)$. In Theorem 1 we prove that under some natural assumptions, the differential system has a unique entire analytic solution in accordance with the theory of analytic functions in Banach spaces.[1] This special function $y(x)$, characterized by the given differential system, is given explicitly by (3.1) as an abstract power series that involves certain iterations of the function $T$. The $i$th successive Fréchet differential of $y(x)$ with equal increments is given by formula (4.2). The differential system (1.1) is shown in section 5 (Theorem 2) to define a continuous transformation group in an abstract parameter. Use is made of the Michal-Paxson-Elconin generalized Lie differential equations.[2]

Some special instances are first taken up in section 6. The Banach spaces are taken to be spaces of functions whose values are in normed linear rings.[3] The particular case in which the normed linear ring is the space of square matrices — normed in any one of several equivalent ways — and the function $T(y_1, x, y_2)$ is taken as in (6.1), is of considerable interest since *in this case the system* (1.1) *characterizes the matrizant functional provided* $y_0$ *is the unit matrix.* The matrizant it will be rembered is the functional expansion occurring in the

[1] The theory of analytic functions in (real and complex) normed linear spaces was initiated by the author in collaboration with ROBERT S. MARTIN in the author's seminar at the California Institute of Technology during the year 1931—1932. Fréchet's pioneer work on abstract polynomials (Journal Math. Pures et Appl., 1929) was naturally a source of inspiration and encouragement in pointing the way to abstract power series.

[2] MICHAL, A. D. and PAXSON, E. W., »Maps of Abstract Topological Spaces in Banach Spaces», Bull. of Amer. Math. Soc., vol. 42 (1936), pp. 529—534; »Addendum», Bull. of Amer. Math. Soc., vol. 43 (1937), p. 888. MICHAL, A. D. and ELCONIN, V., »Differential Properties of Abstract Transformation Groups with Abstract Parameters», Amer. Journ. of Math. vol. 59 (1937), pp. 129—143. See also MICHAL, A. D., HIGHBERG, I. E., and TAYLOR, A. E., »Abstract Euclidean Spaces with Independently Postulated Analytical and Geometrical Metrics», Annali di Pisa, vol. VI (1937), pp. 117—148.

[3] MICHAL, A. D., »The Total Differential Equation for the Exponential Function in Non-Commutative Normed Linear Rings», Proc. of the National Academy of Sciences (U. S. A.) vol. 31 (1945), pp. 315—317. See also MICHAL, A. D. and MARTIN, R. S., »Some Expansions in Vector Space», Journal de Mathématiques Pures et Appliquées, vol. 13 (1934) pp. 69—91.

Peano solution of a system of $n$ linear differential equations in $n$ unknown functions with variable coefficients. As an instance of our general theory then, we have solved the characterization problem for the matrizant. We believe this result on the matrizant to be new and that other problems — not discussed in this paper — in the theory of special functionals and their applications to geometry can now be attacked with a good chance for success.

The theory of the system (1.1) and the methods of the first five sections are instrumental in obtaining definitive results on the solution of an »ordinary» linear differential equation (7.1) in Banach spaces as an entire analytic functional of the one-parameter (numerical) linear transformation on the right hand side of (7.1) — see Theorem 3 and Theorem 4. The solution is characterized by the completely integrable *linear* differential system (7.6) in Fréchet differentials.

Theorem 6 gives the generalized Taylor's series expansion of the solution of (1.1) in an increasing order of Fréchet differentials. The Corollary to Theorem 6 gives the generalized Taylor's series expansion for the well known matrizant functional — thus giving another new property of the matrizant.

Theorem 3, Theorem 4 and Theorem 6 with its Corollary are used in the proof of Theorem 7 of section 9. In this theorem the solutions of a system (9.1) of numerical linear differential equations as functionals of the coefficients are shown to be characterized by the completely integrable linear differential system (9.3) in Fréchet differentials. The generalized Taylor's series expansions for the solution functionals are given by the expansions (9.4) and (9.5). *There are obvious but important applications of Theorem 7 to the approximate solutions of systems of linear differential equations.*

The paper closes with two sections on related topics. Section 10 gives an existence and uniqueness theorem for the non-linear system (10.1), while section 11 shows briefly how to develop the subject matter of this paper in complex Banach spaces (complete normed linear spaces with complex number multipliers).

It should be stated here at the outset that the known existence and uniqueness theorems on completely integrable differential equations in Fréchet differentials in Banach spaces (Michal and Elconin, Acta mathematica, vol. 68 (1937), pp. 71—107) are not strong enough to be used in obtaining some of the results of the present paper.

1. **The Differential Equation.** Let $B_1$ and $B_2$ be Banach spaces (complete normed linear spaces with real number multipliers) and let $T(y_1, x, y_2)$ be a tri-

linear function (additive and continuous in each of the three variables) on $B_1 B_2 B_1$ to $B_1$. Consider the differential system

$$\delta y(x) = T(y(x), \delta x, y(x)), \; y(0) = y_0, \tag{1.1}$$

where $\delta y(x)$ is the Fréchet differential of $y(x)$ at $x = x$ with increment $\delta x$, and $y_0$ is any chosen element of $B_1$.

We shall need to make some assumptions on the trilinear function $T(y_1, x, y_2)$. We shall often write $T(y_1, x, y_2)$ simply as $y_1 \cdot x \cdot y_2$.

**Assumption 1.**

$$(y \cdot x_2 \cdot y) \cdot x_1 \cdot y = y \cdot x_2 \cdot (y \cdot x_1 \cdot y) \tag{1.2}$$

*for all* $y \in B_1$ *and* $x_1, x_2 \in B_2$.

Define the linear function of $y_2$,

$$T^i(y_1, x, y_2) \tag{1.3}$$

with $y_1$ and $x$ as parameters, as the *ith iteration of the linear function* $T(y_1, x, y_2)$ *of* $y_2$.

**Assumption 2.**

$$T^i(y, x_1, y) \cdot x_2 \cdot T^j(y, x_1, y) = T^i(y, x_1, y \cdot x_2 \cdot T^j(y, x_1, y)) \tag{1.4}$$

*for all positive integers* $i$ *and* $j$.

**Assumption 3.** *There exists a positive* $M(y)$ *such that*

$$\| T^i(y, x, y) \| \leq \frac{\| y \| M^i(y)}{i!} \| x \|^i \qquad (i = 1, 2, \ldots). \tag{1.5}$$

The condition (1.2) in Assumption 1 implies that the »condition of complete integrability[1]» for the total differential equation (1.1) is satisfied.

Although the restrictions imposed by the three assumptions are rather strong, the author's more interesting instances of the total differential system (1.1) satisfy all the three assumptions. The reader is referred to sections 6, 7 and 9 for a brief discussion of some of these instances.

2. **Some Consequences of the Assumptions.** To prove our existence and uniqueness theorem for entire analytic solutions of the differential system (1.1)

[1] Michal, A. D. and Elconin, V., »Completely Integrable Differential Equations in Abstract Spaces», Acta mathematica, vol. 68 (1937), pp. 71—107. The condition of complete integrability for (1.1) is $(y \cdot x_2 \cdot y) \cdot x_1 \cdot y + y \cdot x_1 \cdot (y \cdot x_2 \cdot y) = (y \cdot x_1 \cdot y) \cdot x_2 \cdot y + y \cdot x_2 \cdot (y \cdot x_1 \cdot y)$ for all $y \in B_1$ and $x_1, x_2 \in B_2$.

we need the results of the following lemmas, the first two of which are consequences of condition (1.2) and do not use the assumptions 2 and 3.

**Lemma 1.** *If* $x_1, x_2 \in B_2$ *and* $y_1, y_2, y_3 \in B_1$ *are arbitrary elements, then the following identity holds:*

$$P_{y_1 y_2 y_3}(y_1 \cdot x_2 \cdot y_2) \cdot x_1 \cdot y_3 = P_{y_1 y_2 y_3} y_1 \cdot x_2 \cdot (y_2 \cdot x_1 \cdot y_3), \tag{2.1}$$

*where* $P_{y_1 y_2 y_3}$ *denotes the sum of six terms obtained by permuting* $y_1$, $y_2$, *and* $y_3$.

**Lemma 2.**

$$T^2(y, x, y) = T(y, x, y) \cdot x \cdot y \text{ for all } x \in B_2 \text{ and } y \in B_1. \tag{2.2}$$

$$y \cdot x \cdot ((y \cdot x \cdot y) \cdot x \cdot y) = ((y \cdot x \cdot y) \cdot x \cdot y) \cdot x \cdot y \tag{2.3}$$

*for all* $x \in B_2$ *and* $y \in B_1$.

Lemma 1 is proved by two successive Fréchet differentiations in $y$ of condition (1.2). Clearly (2.1) implies condition (1.2). Now (2.2) of Lemma 2 is an immediate consequence of (1.2). To prove the identity (2.3), take $x_2 = x_1 = x$, $y_1 = y_3 = y$, and $y_2 = y \cdot x \cdot y$ in (2.1). Then use (2.2) to obtain the identity (2.3).

In view of (2.2), the identity (2.3) can be written

$$T^3(y, x, y) = T^2(y, x, y) \cdot x \cdot y. \tag{2.4}$$

For later use, we observe that Lemma 2 is equivalent to the following identities:

$$T(y, x, y) \cdot x \cdot y = y \cdot x \cdot T(y, x, y), \tag{2.5}$$

$$T^2(y, x, y) \cdot x \cdot y = y \cdot x \cdot T^2(y, x, y). \tag{2.6}$$

The following lemma makes use of Assumption 1 and part of Assumption 2, but not Assumption 3.

**Lemma 3.** *The following identity holds for all* $x \in B_2$ *and* $y \in B_1$:

$$T^n(y, x, y) \cdot x \cdot y = y \cdot x \cdot T^n(y, x, y) \quad (n = 1, 2, \ldots). \tag{2.7}$$

To prove this lemma, let us observe that (2.7) for $n = 1$ and $n = 2$ holds by virtue of Lemma 2 whose validity depends only on the condition (1.2). We shall give an induction proof of (2.7) for $n \geq 1$. Take $x_2 = x_1 = x$, $y_1 = T^n(y, x, y)$, and $y_2 = y_3 = y$ in identity (2.1), and obtain for $n \geq 1$

$$
(2.8)\quad \begin{aligned} & T^n(y, x, y)\cdot x\cdot y + (y\cdot x\cdot T^n(y, x, y))\cdot x\cdot y + (y\cdot x\cdot y)\cdot x\cdot T^n(y, x, y) \\ & \quad = T^n(y, x, y)\cdot x\cdot (y\cdot x\cdot y) + y\cdot x\cdot (T^n(y, x, y)\cdot x\cdot y) + y\cdot x\cdot (y\cdot x\cdot T^n(y, x, y)). \end{aligned}
$$

On using the induction hypothesis in (2.8) we obtain

$$
(2.9)\quad \begin{aligned} & 2\, T^{n+1}(y, x, y)\cdot x\cdot y + (y\cdot x\cdot y)\cdot x\cdot T^n(y, x, y) \\ & \qquad = 2\, y\cdot x\cdot T^{n+1}(y, x, y) + T^n(y, x, y)\cdot x\cdot (y\cdot x\cdot y). \end{aligned}
$$

But from (1.4) we have

$$
(2.10)\qquad (y\cdot x\cdot y)\cdot x\cdot T^n(y, x, y) = T^n(y, x, y)\cdot x\cdot (y\cdot x\cdot y)
$$

and hence from (2.9)

$$
(2.11)\qquad T^{n+1}(y, x, y)\cdot x\cdot y = y\cdot x\cdot T^{n+1}(y, x, y).
$$

But we know that (2.7) holds for $n = 1, 2$. Hence the lemma follows immediately.

The following lemma makes use only of assumption 3.

**Lemma 4.** *The power series in the Banach variable* $x$

$$
(2.12)\qquad y_0 + \sum_{i=1}^{\infty} T^i(y_0, x, y_0)
$$

*defines an entire analytic function*[1] $y(x)$ *on* $B_2$ *to* $B_1$.

3. **Existence and Uniqueness Theorem for Differential System.** With the aid of the preceding four lemmas we can shorten the proof of the main theorem of the paper concerning the differential system (1.1).

**Theorem 1.** *Under assumptions* 1, 2 *and* 3 *of* § 1, *and under the assumption*[2] 4

$$
T^i(y, x_1, y\cdot x_2\cdot y) = T^i(y, x_1, y)\cdot x_2\cdot y
$$

---

[1] A power series $\sum_{i=0}^{\infty} P_i(x)$ in Banach spaces defines an entire analytic function of $x$ if the real power series $\sum_{i=0}^{\infty} m(P_i)\lambda^i$ converges for all real $\lambda$ and if $m(P_i)$ is the modulus of the homogeneous polynomial $P_i(x)$ of degree $i$ in Banach spaces.

[2] It can be shown that the »associativity» $(i): (y_1\cdot x_2\cdot y_2)\cdot x_1\cdot y_3 = y_1\cdot x_2\cdot (y_2\cdot x_1\cdot y_3)$ for all $y_i \in B_1$ and $x_i \in B_2$ implies that assumptions 1 and 2 of § 1 and the above assumption 4 are satisfied by the trilinear function $T(y_1, x, y_2)$. Hence, the conclusions of Theorem 1 hold under the restrictions that the trilinear function $T(y_1, x, y_2)$ satisfies assumption 3 of § 1 and the identity $(i)$. The special instances considered in this paper satisfy assumption 3 of § 1 and the identity $(i)$.

*for all* $y \in B_1$, $x_1, x_2 \in B_2$ $(i = 2, 3, \ldots)$, *the differential system* (1.1) *has a unique entire analytic solution given by*

$$(3.1) \qquad y(x) = y_0 + \sum_{i=1}^{\infty} \Delta^i(x, y_0), \text{ where } \Delta^i(x, y_0) = T^i(y_0, x, y_0)$$

*so that* $\Delta^i(x, y_0)$ *is the ith iterate, as a function of* $z$, *of* $\Delta(x, z) = y_0 \cdot x \cdot z$ *evaluated for* $z = y_0$.

To prove this theorem, let us then find necessary and sufficient conditions that an entire analytic function

$$(3.2) \qquad y(x) = y_0 + \sum_{i=1}^{\infty} \Omega_i(x)$$

satisfy the differential system (1.1). In (3.2), $\Omega_i(x)$ is a homogeneous polynomial of degree $i$ on $B_2$ to $B_1$, i. e., $\Omega_i(x)$ is a continuous function of $x$ such that

(1) $\Omega_i(\lambda x) = \lambda^i \Omega_i(x)$ for all real $\lambda$ and all $x \in B_2$;

(2) $\Omega_i(x + \lambda z) = \sum_{r=0}^{i} \lambda^r \Phi_r(x, z)$ for all real $\lambda$ and all $x, z \in B_2$.

Robert S. Martin[1] first proved (it was later proved independently by Mazur and Orlicz[2] that a homogeneous polynomial in Banach spaces has a unique polar, i. e., there exists a unique $i$-linear function $\omega_i(x_1, x_2, \ldots, x_i)$ such that $\omega_i(x, x, \ldots, x) = \Omega_i(x)$. The polar is in fact defined by

$$\frac{1}{i!} \Delta^i_{x_1 x_2 \ldots x_i} \Omega_i(0),$$

where $\Delta^i_{x_1 x_2 \ldots x_i}$ is the $i$th successive difference operator with successive increments $x_1, x_2, \ldots, x_i$. Robert S. Martin also proved that the Fréchet differential of a homogeneous polynomial $\Omega_i(x)$ exists and is given by

$$(3.3) \qquad \delta\, \Omega_i(x) = i\, \omega_i(x, x, \ldots, x, \delta x).$$

[1] Martin, R. S., »Contributions to the Theory of Functionals», California Institute of Technology Thesis, June, 1932.

[2] Mazur, S. and Orlicz, W., Studia Mathematica, vol. 5 (1934), pp. 50—68. This paper did not appear until 1936. See also Taylor, A. E. »Additions to the Theory of Polynomials in Normed Linear Spaces». The Tôhoku Math. Jour., vol. 44 (1938) pp. 302—318.

By a theorem of the author[1] on the term by term Fréchet differentiability of a power series in Banach spaces, it follows that the Fréchet differential of the entire analytic function $y(x)$ in (3.2) exists for each $x \in B_2$ and is given by

$$\delta y(x) = \sum_{i=1}^{\infty} i\, \omega_i(x, x, \ldots, x, \delta x). \tag{3.4}$$

If we use (3.4) in the differential system (1.1) we find

$$\Omega_1(x) = A(x, y_0) \tag{3.5}$$

$$\Omega_2(x) = A^2(x, y_0) \tag{3.6}$$

and for $n \geq 2$, the following relations

$$(n+1)\,\omega_{n+1}(x, x, \ldots, x, \delta x) = y_0 \cdot \delta x \cdot \Omega_n(x) + \Omega_n(x) \cdot \delta x \cdot y_0 + \sum_{\substack{i+j=n \\ i,j \geq 1}} \Omega_i(x) \cdot \delta x \cdot \Omega_j(x). \tag{3.7}$$

The result (3.6) was obtained on using (2.2) of Lemma 2.

We see that a necessary condition that (3.7) hold is that for $n \geq 2$

$$\Omega_{n+1}(x) = \frac{1}{n+1}\Big\{y_0 \cdot x \cdot \Omega_n(x) + \Omega_n(x) \cdot x \cdot y_0 + \sum_{\substack{i+j=n \\ i,j \geq 1}} \Omega_i(x) \cdot x \cdot \Omega_j(x)\Big\}. \tag{3.8}$$

If we now use results (3.5) and (3.6), and if we use an induction proof on (3.8), we find on using (2.7) of Lemma 3 and (1.4) (Assumption 2) for $x_2 = x_1 = x$ and $y = y_0$ that

$$\Omega_i(x) = A^i(x, y_0) \qquad (i = 1, 2, \ldots). \tag{3.9}$$

Now the conditions (3.9) are also sufficient that $y(x)$ in (3.2) satisfy the differential system (1.1). In fact an application of (1.2), (1.4) and assumption 4 to the evident formula for the first Fréchet differential $\delta T^{n+1}(y_0, x, y_0)$ $(n \geq 2)$

$$\begin{aligned}\delta T^{n+1}(y_0, x, y_0) = {} & y_0 \cdot \delta x \cdot T^n(y_0, x, y_0) + y_0 \cdot x \cdot (y_0 \cdot \delta x \cdot T^{n-1}(y_0, x, y_0)) \\ & + T^2(y_0, x, y_0 \cdot \delta x \cdot T^{n-2}(y_0, x, y_0)) + \cdots + T^{n-1}(y_0, x, y_0 \cdot \delta x \cdot T(y_0, x, y_0)) \\ & + T^n(y_0, x, y_0 \cdot \delta x \cdot y_0)\end{aligned}$$

shows that (3.7) holds for $n \geq 2$. Hence by Lemma 4, the truth of the theorem follows.

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[1] MICHAL, A. D., »The Fréchet Differentials of Regular Power Series in Normed Linear Spaces» Duke Math. Jour., vol. 13 (1946), pp. 57—59.

4. **Higher Order Differentials of the Solution.** It is possible to give the expression for the $n$th successive Fréchet differential with equal increments of the solution function (3.1) of the differential system (1.1). By a simple calculation the second Fréchet differential with equal increments $\delta x$ given by

$$\delta^2 y(x) = 2!\, T^2(y, \delta x, y). \tag{4.1}$$

We shall prove the following general formula by induction:

$$\delta^i y(x) = i!\, T^i(y, \delta x, y) \textit{ for any positive integer } i. \tag{4.2}$$

Clearly from the induction hypothesis for $i = n$, we have

$$\delta^{n+1} y(x) = n!\, \delta T^n(y, \delta x, y). \tag{4.3}$$

From the elementary theorems on Fréchet differentials and definition of $T^n(y, \delta x, y)$ we find

$$\begin{aligned} \delta T^n(y, \delta x, y) = T(y, \delta x, y) \cdot \delta x \cdot T^{n-1}(y, \delta x, y) \\ + y \cdot \delta x \cdot \delta T^{n-1}(y, \delta x, y). \end{aligned} \tag{4.4}$$

From the induction hypothesis for $i = n - 1, n$ we have

$$\begin{aligned} \delta^{n-1} y(x) &= (n-1)!\, T^{n-1}(y, \delta x, y), \\ \delta^n y(x) &= n!\, T^n(y, \delta x, y). \end{aligned} \tag{4.5}$$

Hence $\delta^n y(x) = (n-1)!\, \delta T^{n-1}(y, \delta x, y)$, which implies

$$\delta T^{n-1}(y, \delta x, y) = n\, T^n(y, \delta x, y). \tag{4.6}$$

On using (4.15), the definition of $T^{n+1}(y, \delta x, y)$, and the identity (1.4) for $i = 1$, $j = n - 1$, we find that (4.4) reduces to

$$\delta T^n(y, \delta x, y) = (n+1)\, T^{n+1}(y, \delta x, y). \tag{4.7}$$

This result inserted in (4.3) gives (4.2) for $i = n + 1$.

The following corollary of the Theorem 1 is now clear.

**Corollary.** *If the increment $\delta x = x$, then the solution of the differential system (1.1) will have the generalized »Maclaurin series expansion»*

$$y(x) = y_0 + \sum_{i=1}^{\infty} \frac{1}{i!}\, \delta^i y(0), \tag{4.8}$$

*where $\delta^i y(0)$ is the $i$th successive Fréchet differential of $y(x)$ at $x = 0$ with all $i$ increments equal to $\delta x = x$.*

5. **A Continuous Transformation Group.** The differential system (1.1) defines a continuous transformation group in the Banach space $B_1$ with a parameter ranging over the Banach space $B_2$. The classical Lie theory of finite and infinite continuous groups naturally does not treat of such generalized groups. The beginnings of a generalized Lie theory, with the differential aspects as the main flavor, was given by Michal and Elconin.[1] The generalized Lie differential equations were also given by Michal and Paxson.[2]

From the existence and uniqueness theorem of section 3 we see that the differential system (1.1) with a slight change in notation)

$$(5.1) \qquad \delta \bar{y}(\alpha) = T(\bar{y}(\alpha), \delta\alpha, \bar{y}(\alpha)), \qquad \bar{y}(0) = y$$

is satisfied by

$$(5.2) \qquad \bar{y} = f(y, \alpha),$$

where

$$(5.3) \qquad f(y, \alpha) = y + \sum_{i=1}^{\infty} T^i(y, \alpha, y).$$

Let us look at (5.2) as a transformation in $B_2$ for each value of the parameter $\alpha \in B_1$. Clearly $\bar{y} = y$ for $\alpha = 0$ so that the identity transformation corresponds to $\alpha = 0$.

It follows from Theorem 2.5 of Michal and Elconin[3] that

$$(5.4) \qquad f(f(y, \alpha), \beta) = f(y, \alpha + \beta).$$

Evidently the inverse transformation to (5.2) corresponds to $-\alpha$, so that the unique solution of the non-linear equation (5.2) is given by

$$(5.5) \qquad y = f(\bar{y}, -\alpha)$$

*i. e., by the entire function of* $\alpha$

$$(5.6) \qquad y = \bar{y} + \sum_{i=1}^{\infty} (-1)^i T^i(\bar{y}, \alpha, \bar{y}).$$

**Theorem 2.** *The differential system* (1.1) *defines an Abelian continuous transformation group in* $B_1$ *with the translation group of* $B_2$ *as its two identical parameter groups.*

[1] Michal, A. D. and Elconin, V., loc. cit.
[2] Michal, A. D. and Paxson, E. W., loc. cit.
[3] Michal, A. D. and Elconin, V. loc. cit.

6. **Some Instances of the Differential System** (1.1). Let $B_1$ be the Banach space of real continuous functions $y(t, s)$ over $a \leq t, s \leq b$, and let $B_2$ be the Banach space of real continuous functions $x(s)$ over $a \leq s \leq b$. The norm in each case is taken in the usual way as the maximum of the absolute value of the function. If we take $T(y_1, x, y_2)$ as

$$\int_r^t y_1(t, s)\, x(s)\, y_2(s, r)\, ds, \tag{6.1}$$

then the differential system (1.1) in this instance becomes

$$\begin{cases} \delta y\,[x(u)/t, r] = \int_r^t y\,[x(u)/t, s]\,\delta x(s)\, y\,[x(u)/s, r]\, ds \\ y\,[0/t, r] = y_0(t, r). \end{cases} \tag{6.2}$$

It is not difficult to show that assumptions 1, 2, 3, and 4 are all satisfied. Hence by the existence and uniqueness theorem of section 3 the unique entire functional solution of (6.2) is given by

$$y(x) = y_0 + \sum_{i=1}^{\infty} \mathrm{I}^{*i}(y_0, x) * y_0 \tag{6.3}$$

where $\mathrm{I}(y_0, x)$ stands for $y_0(t, s)\, x(s)$ and the $*$ denotes integral composition powers and products in accordance with the definition: $y_1 * y_2$ stands for

$$\int_r^t y_1(t, s)\, y_2(s, r)\, ds. \tag{6.4}$$

As we shall see later, the following modification of the previous instance will be of considerable importance. Let $N$ be a complete normed linear ring[1] (not necessarily commutative) with $I$ as a unit element. Let $B_1$ and $B_2$ be Banach spaces defined as in the previous instance with the difference that the *values* of the functions are now in $N$. Hence the norms in $B_1$ and $B_2$ are defined respectively by

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[1] MICHAL, A. D., »The Total Differential Equation for the Exponential Function in Non-Commutative Normed Linear Rings», Proc. of the National Acad. of Sc. (U. S. A.), vol. 31 (1945), pp. 315—317. See also MICHAL, A. D., and MARTIN, R. S. »Some Expansions in Vector Space» Journal de Mathématiques Pures et Appliquées vol. 13 (1934), pp. 69—91.

$$
(6.5) \qquad \begin{cases} \| y \| = \max\limits_{a \leq t, s \leq b} \| y(t, s) \|_N \\ \| x \| = \max\limits_{a \leq s \leq b} \| x(s) \|_N, \end{cases}
$$

where $\| \quad \|_N$ is the norm of $N$, and the ordinary products of functions in (6.1) to (6.4) are to be interpreted as (non-commutative) ring products.

If the initial condition function $y_0(t, r) = I$, the unit of $N$, then the solution of (6.2) becomes

$$
(6.6) \qquad \begin{cases} y[x(u)/t, r] = I + \int_r^t x(u)\,du + \int_r^t x(u)\,du \int_r^u x(v)\,dv \\ \qquad\qquad + \int_r^t x(u)\,du \int_r^u x(v)\,dv \int_r^v x(w)\,dw + \cdots, \end{cases}
$$

where $x(u)x(v)$, $x(u)x(v)x(w)$, etc. are associative but not commutative ring products.

The functional $y[x(u)/t, r]$ has also the following properties:

(1) If $x(u)x(v) = x(v)x(u)$ for all real $u$, $v$ in the interval $(a, b)$, then

$$
y[x(u)/t, r] = e^{\int_r^t x(u)\,du},
$$

where $e^{\lambda}$ is the exponential function in the normed linear ring $N$. — See my paper in Proc. Nat. Acad. of Sciences, 1945, for a characterization of $e^{\lambda}$.

(2) The Fréchet differential of the functional $y[x(u)/t, r]$ commutes with the numerical derivative of $y[x(u)/t, r]$ with respect to the variable $t$. (This second property is an evident consequence of the result that (6.6) satisfies the differential system (6.2).)

$$
(3) \qquad y[x_1(u) + x_2(u)/t, r] = y[x_1(u)/t, r]\, y[x_2(v)/t, r]
$$

if $x_1(u)x_2(v) = x_2(v)x_1(u)$ for all $u$, $v$, in the interval $(a, b)$. In particular, this identity holds if $x_1(u) = I$, the unit of the normed linear ring $N$, and in fact we have the formula $y[I + x(u)/t, r] = e^{t-r}\, y[x(u)/t, r]$, where $e^t$ is the numerical exponential function.

If in particular, the normed linear ring $N$ is that of all square matrices $x = (x^i_j)$ of real numbers $x^i_j$ with $n$ rows and normed, say, as

$$
(6.7) \qquad \| x \|_N = \sqrt{\sum_{i,j=1}^{n} (x^i_j)^2},
$$

we see that (6.6) is the *matrizant*[1] and that *the completely integrable differential system* (6.2) with $y_0(t, r) = I$ *characterizes the matrizant.* Hence the results of the previous sections immediately apply to the matrizant. These theorems on the matrizant are believed to be new. For example, let us write explicitly the $n$th successive Fréchet differential of the matrizant $y_r^t[x(u)]$ as given by (4.2):

$$\text{(6.8)} \quad \begin{cases} \delta^n y_r^t[x] = n! \int_r^t y_s^t[x]\, \delta x(s_1)\, ds_1 \int_r^{s_1} y_{s_2}^{s_1}[x]\, \delta x(s_2)\, y_{s_3}^{s_2}[x]\, ds_2 \\ \qquad \cdots \int_r^{s_{n-1}} y_{s_n}^{s_{n-1}}[x]\, \delta x(s_n)\, y_r^{s_n}[x]\, ds_n \\ (n \geq 1). \end{cases}$$

It is of some interest at this point to inquire into the term by term Fréchet differentiability of the functional expansion (6.6). An application of the author's theorem[2] on the Fréchet differentiability of power series in Banach spaces shows that the Fréchet differential of the functional $y[x(u)/t, r]$ exists for each $x(u)$ of $B_2$ and is given by the alternative expansion

$$\text{(6.9)} \quad \begin{aligned} \delta y[x(u)/t, r] = \int_r^t \delta x(u)\, du + \sum_{i=2}^{\infty} \Bigg[ & \int_r^t \delta x(u_1)\, du_1 \int_r^{u_1} x(u_2)\, du_2 \cdots \int_r^{u_{i-1}} x(u_i)\, du_i \\ & + \cdots + \int_r^t x(u_1)\, du_1 \int_r^{u_1} x(u_2)\, du_2 \cdots \int_r^{u_{i-1}} \delta x(u_i)\, du_i \Bigg]. \end{aligned}$$

If the matric-valued function $y_0(t, r)$ is taken to be an arbitrary matric-valued function in $B_1$ instead of the unit matrix $I$, then we obtain a generalization of the matrizant and the following — by Theorem 2 — will be an infinite continuous transformation group on $B_1$ to $B_1$ with $A(s) \in B_2$ as the variable parameter of the group:

$$\text{(6.10)} \quad \bar{y}_r^t = y_{0r}^{\,t} + \int_r^t y_{0s_1}^{\,t} A(s_1)\, y_{0r}^{\,s_1}\, ds_1 + \int_r^t y_{0s_1}^{\,t} A(s_1)\, ds_1 \int_r^{s_1} y_{0s_2}^{\,s_1} A(s_2)\, y_{0r}^{\,s_2}\, ds_2 + \cdots + \cdots.$$

## 7. Applications to »Ordinary» Differential Equations in Banach Spaces.

The functional expansion (6.6) of the previous instance enters in an essential way in connection with the treatment of an ordinary linear differential equation

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[1] See, for example, expansion (4.8), page 22 of MICHAL, A. D., »Matrix and Tensor Calculus with Applications to Mechanics, Elasticity, and Aeronautics», Galcit series, John Wiley & Sons (New York, 1947).

[2] Duke Math. Journal, loc. cit.

in Banach spaces. Let $a(x, w)$ be a linear function of $w \in B_3$, a Banach space, with values in $B_3$ and depending parametrically on a real variable $x$ for $a \leq x \leq b$. The differential equation

$$\frac{d w(x)}{d x} = a(x, w) \tag{7.1}$$

can be written conveniently as

$$\frac{d w(x)}{d x} = A(x) \cdot w(x), \tag{7.2}$$

where $A \cdot w$ is the bilinear function on $N B_3$ to $B_3$ and $N$ here stands for the wellknown complete normed linear ring of linear transformations on $B_3$ to $B_3$. We shall assume that $A(x)$ is continuous in the interval $(a, b)$. It should be emphasized here that the restriction of continuity on $A(x)$ is merely illustrative. Other well known function spaces of functions $A(x)$ can be considered leading to similar results with the evident changes in interpretations of the notations. If the Banach spaces $B_1$ and $B_2$ are the two function spaces described in the second instance of section 6, and if the space $N$ is taken as the normed linear ring of linear transformations on $B_3$ to $B_3$, then, if we write the expansion (6.6) in the following notation

$$\Omega_\xi^x [A(\eta)] = I + \int_\xi^x A(\eta)\, d\eta + \int_\xi^x A(\eta)\, d\eta \int_\xi^\eta A(\theta)\, d\theta + \cdots, \tag{7.3}$$

we can state the following two theorems.

**Theorem 3.** *The unique solution — obviously continuous — of the differential system*

$$\frac{d w(x)}{d x} = A(x) \cdot w(x), \qquad w(a) = w_0 \qquad (A(x) \textit{ continuous in } (a, b)) \tag{7.4}$$

*is given by*

$$w(x) = \Omega_a^x [A(\eta)] \cdot w_0, \tag{7.5}$$

*where the functional* $\Omega_\xi^x [A(\eta)]$ *is defined by the entire functional expansion* (7.3).

**Theorem 4.** *There exists a unique entire analytic solution of the completely integrable system in Fréchet differentials*

$$\begin{cases} \delta w [A / x] = \int_a^x \Omega_\xi^x [A]\, \delta A(\xi) \cdot w [A / \xi]\, d\xi, \\ w [0 / x] = w_0. \end{cases} \tag{7.6}$$

*where* $\Omega_\xi^x[A]$ *is a given functional on* $B_2$ *to* $B_1$ *and is defined by* (7.3). *It is given by the entire analytic functional* (7.5) *with* $A(\eta)$ *ranging over the Banach space* $B_2$.

The proof of Theorem 3 is similar to that of the classical theorem on numerical systems of linear differential equations while the proof of Theorem 4 uses similar methods to those of the proof of Theorem 1.

Theorem 4 is an existence and uniqueness theorem. However, under the *assumption* that the Fréchet differential $\delta w[A/x]$ exists, it can be shown quite readily that $w[A/x]$ must necessarily satisfy the differential system (7.6). In fact, since $w[A/x]$ satisfies the system (7.4), we see that the Fréchet differential $\delta w[A/x]$ must necessarily satisfy the linear differential system

$$\frac{d}{dx}(\delta w) = A(x)\cdot\delta w + \delta A(x)\cdot w, \quad \{\delta w[A/x]\}_{x=a} = 0.$$

On solving this system for the arbitrarily chosen $A(x)$, we see that $w[A/x]$ as a functional of $A(x)$ must satisfy the differential system (7.6). We cannot go into details in this paper, but more extensive and similar studies can be made for higher order equations and for non-linear equations.

The $n$th successive Fréchet differential with equal increments of the functional $\Omega_a^x[A(\eta)]$ exists for each positive integer $n$ and is given by

$$\delta^n \Omega_a^x[A] = n!\int_a^x \overset{*}{F}{}^n[A, \delta A/x, \xi]\,\Omega_a^\xi[A]\,d\xi \tag{7.7}$$

while the $n$th successive Fréchet differential with equal increments of the solution $w[A/x]$ of (7.4) as a functional of $A(x)$ exists for each positive integer $n$ and is given by

$$\delta^n w[A/x] = n!\int_a^x \overset{*}{F}{}^n[A, \delta A/x, \xi]\cdot w[A/\xi]\,d\xi. \tag{7.8}$$

In (7.7) and (7.8) we understand that

$$F[A, \delta A/x, \xi] = \Omega_\xi^x[A]\,\delta A(\xi), \tag{7.9}$$

and $\overset{*}{F}{}^n[A, \delta A/x, \xi]$ stands for the $n$th combined ring (normed linear ring $N$) and integral composition power with variable limits. For example

$$\overset{*}{F}{}^2[A, \delta A/x, \xi] = \int_a^x F[A, \delta A/x, \eta]\,F[A, \delta A/\eta, \xi]\,d\eta.$$

Theorem 3 and Theorem 4 can also be obtained by suitably generalizing some of the contents of my Proceedings 1945 paper »Differential Equations in Fréchet Differentials Occurring in Integral Equations». We shall speak briefly of this generalization. Consider the functional equation

$$f = y + K \odot y \tag{7.10}$$

with respect to which we make the following assumptions:

(1) $K \odot y$ is a bilinear function whose values and independent variable $y$ are in a Banach space $B$ while the independent variable $K$ ranges over a complete normed linear ring $R$ for which a unit $I$ is not assumed to exist.

(2) $(K_1 K_2) \odot y = K_1 \odot (K_2 \odot y)$ for all $K_1, K_2 \in R$ and $y \in B$.

(3) There exists a positive number $M$ such that

$$\| K^i \| \leq \frac{M^{i-1}}{(i-1)!} \| K \|^i \qquad (i = 2, 3, \ldots).$$

To the differential system (17) of my Proceedings 1945 paper, there will now correspond the differential system

$$\begin{cases} \delta y(K) = -(\delta K + k\,\delta K) \odot y \\ y(0) = f, \end{cases} \tag{7.11}$$

where $k(K) = -K + K^2 - K^3 + \cdots$, while the unique solution of (7.10) is given by

$$y = f + k \odot f. \tag{7.12}$$

It can be shown that the $n$th Fréchet differential of $y(K)$ with equal increments $\delta K$ is given by

$$\delta^n y(K) = (-1)^n n! (\delta K + k\,\delta K)^n \odot y \qquad (n = 1, 2, \ldots). \tag{7.13}$$

This has been found rather useful in the specializations and applications of the general theory — see, for example, expansion (9.5) for the solutions of a system of linear differential equations as functionals of the coefficient functions.

§ 8. **Generalized Taylor's Series Expansions for Solution of Total Differential System.** If $m(p_n)$ is the modulus of a homogeneous polynomial $p_n(x)$ of degree $n$ on a Banach space $E_1$ to a Banach space $E_2$ and if $m(\omega_n)$ is the modulus of the polar $\omega_n(x_1, x_2, \ldots, x_n)$ of $p_n(x)$, then we have the following result due to R. S. Martin (1932), loc. cit.

**Theorem 5.** *The moduli $m(p_n)$ and $m(\omega_n)$ of the homogeneous polynomial $p_n(x)$ and its polar $\omega_n(x_1, x_2, \ldots, x_n)$ respectively satisfy the inequalities*

$$1 \leq \frac{m(\omega_n)}{m(p_n)} \leq \frac{n^n}{n!}. \tag{8.1}$$

**Proof.** That $1 \leq \dfrac{m(\omega_n)}{m(p_n)}$ is clear. To prove the second inequality, it is known that

$$\omega_n(x_1, x_2, \ldots, x_n) = \frac{\Delta^n_{x_1 x_2 \ldots x_n} p_n(0)}{n!}, \tag{8.2}$$

where $\Delta^n_{x_1 x_2 \ldots x_n}$ is the $n$th difference operator with successive increments $x_1, x_2, \ldots, x_n$. Since $\Delta^n_{x_1 x_2 \ldots x_n} p_n(0)$ is a sum of $2^n$ terms of form $p_n\left(\frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} \varepsilon_i x_i\right)$ with $\varepsilon_i = \pm 1$, we obtain from (8.2) with the aid of the triangular inequality and the inequality $\| p_n(x) \| \leq m(p_n) \| x \|^n$, the following inequality

$$\| \omega_n(x_1, \ldots, x_n) \| \leq \frac{2^n}{n!} m(p_n) \left(\frac{n}{2}\right)^n (\max_i \| x_i \|)^n. \tag{8.3}$$

The theorem follows now readily since $m(\omega_n) = l \cdot u \cdot b \cdot \| \omega_n(x_1, \ldots, x_n) \|$ for $\| x_1 \| = \| x_2 \| = \cdots = \| x_n \| = 1$.

With the aid of Theorem 1 and Theorem 5, we can prove the following theorem without much difficulty.

**Theorem 6.** *For any given $x_0 \in B_2$, the entire analytic solution $y(x)$ of* (1.1) *can be expanded in a generalized Taylor's series of successive Fréchet differentials with equal increments $\delta x$ valid for all $\delta x \in B_2$*

$$y(x_0 + \delta x) = y(x_0) + \sum_{i=1}^{\infty} \frac{1}{i!} [\delta^i y(x)]_{x = x_0}. \tag{8.4}$$

If we use (6.8) and (7.7), we obtain the following important new expansions for the matrizant.

**Corollary.** *The following generalized Taylor's series expansions hold for the matrizant for all continuous matrices* $A(s) = (a^i_j(s))$ and $B(s) = (b^i_j(s))$:

$$(8.5)\quad \begin{cases} \Omega_r^t[A(s)+B(s)] = \Omega_r^t[A(s)] + \int_r^t \Omega_{s_1}^t[A]\,B(s_1)\,\Omega_r^{s_1}[A]\,ds_1 \\ \qquad + \sum_{i=2}^{\infty} \int_r^t \Omega_{s_1}^t[A]\,B(s_1)\,ds_1 \int_r^{s_1} \Omega_{s_2}^{s_1}[A]\,B(s_2)\,\Omega_{s_2}^{s_2}[A]\,ds_2 \\ \qquad \cdots \int_r^{s_{i-1}} \Omega_{s_i}^{s_{i-1}}[A]\,B(s_i)\,\Omega_r^{s_i}[A]\,ds_i; \end{cases}$$

$$(8.6)\quad \Omega_r^t[A(s)+B(s)] = \Omega_r^t[A(s)] + \sum_{i=1}^{\infty} \int_r^t \overset{*}{F}{}^i[A,\, B/t,\, \xi]\,\Omega_r^{\xi}[A]\,d\xi,$$

*where* $\overset{*}{F}{}^i[A, B/t, \xi]$ *stands for the ith combined matric and integral composition power of* $\Omega_{\xi}^t[A]\,B(\xi)$ *— see statement following* (7.9). *The expansions* (8.5) *and* (8.6) *are uniformly convergent for all* $t, r$ *in* $t_0 \leq t, r \leq t_1$.

9. **Solutions of Systems of Numerical Linear Differential Equations as Entire Analytic Functionals of their Coefficients.** If we write the numerical linear differential system

$$(9.1)\quad \frac{d\,w^i(x)}{d\,x} = a_j^i(x)\,w^j(x), \quad w^i(a) = w_0^i \quad (a_j^i(x) \text{ continuous in } a \leq x \leq b)$$

as the matric differential system

$$(9.2)\quad \frac{d\,w(x)}{d\,x} = A(x)\,w(x), \quad w(a) = w_0,$$

we can specialize Theorem 3 and Theorem 4, (7.8), and use Theorem 6 with its corollary to obtain the following theorem.

**Theorem 7.** *Let* $\Omega_{\xi}^{\xi}[A(s)]$ *be the matrizant of the matric-valued function* $A(s)$ *and* $w[A(s)/x]$ *the unique solution of* (9.2) *as a functional of* $A(s)$, *then the following results give information as to the dependence of the solutions of* (9.1) *as functionals of the coefficients* $a_j^i(x)$. *There exists a unique entire analytic functional solution of the completely integrable system in Fréchet differentials*

$$(9.3)\quad \begin{cases} \delta\,w[A/x] = \int_a^x \Omega_{\xi}^x[A]\,\delta A(\xi)\,w[A/\xi]\,d\xi, \\ w[0/x] = w_0. \end{cases}$$

*It is given by the entire analytic functional solution of* (9.2) *as a functional of* $A(s)$. *The generalized Taylor's series expansion in Fréchet differentials, valid for all continuous* $A(s)$ *and* $\delta A(s)$,

$$(9.4)\qquad w[A(s)+\delta A(s)/x] = w[A(s)/x] + \sum_{i=1}^{\infty} \frac{1}{i!}\, \delta^i w[A(s)/x],$$

*leads to the following equivalent expansions:*

$$(9.5)\qquad \begin{cases} w[A(s)+\delta A(s)/x] = w[A(s)/x] + \int_a^x \Omega_{s_1}^x [A]\, \delta A(s_1)\, w[A/s_1]\, ds_1 \\ \qquad + \sum_{i=2}^{\infty} \int_a^x \Omega_{s_1}^x [A]\, \delta A(s_1)\, ds_1 \int_a^{s_1} \Omega_{s_2}^{s_1} [A]\, \delta A(s_2)\, \Omega_{s_3}^{s_2} [A]\, ds_2 \\ \qquad \int_a^{s_{i-1}} \Omega_{s_i}^{s_{i-1}} [A]\, \delta A(s_i)\, w[A/s_i]\, ds_i, \end{cases}$$

$$(9.6)\qquad w[A(s)+\delta A(s)/x] = w[A(s)/x] + \sum_{i=1}^{\infty} \int_a^x \overset{*}{F}{}^i [A,\, \delta A/x,\, \xi]\, w[A/\xi]\, d\xi,$$

*where* $\overset{*}{F}{}^i [A,\, \delta A/x,\, \xi]$ *stands for the ith combined matric and integral composition power of* $\Omega_\xi^x [A]\, \delta A(\xi)$.

Aside from its great theoretical interest, we believe that this theorem could be used effectively in obtaining approximate solutions of a large class of systems of linear differential equations with variable coefficients (and more simply with constant coefficients) whenever the solution of only one system of the class is known. The degree and character of the approximation is evident from the definition of a Fréchet differential and of successive Fréchet differentials. The following two corollaries will illustrate some of the applications of Theorem 7.

**Corollary 1.** *If* $w[A/x]$ *is, say, the known solution of the matric differential system* $\frac{dw(x)}{dx} = Aw,\ w(a) = w_0$ *(A, a constant square matrix, and* $a \le x < \infty$ *for any finite a, then for any constant square matrix B, the solution of*

$$(9.7)\qquad \frac{dw(x)}{dx} = (A+B)\,w,\quad w(a) = w_0$$

*is given by*

$$w[A+B/x] = w[A/x] + \int_a^x e^{(x-s)A}\, B\, w[A/s]\, ds$$

*within first order Fréchet differential corrections to* $w[A/x]$.

**Corollary 2.** *Under the hypothesis of Corollary* 1, *the solution of* (9.7) *is given by*

$$w[A+B/x] = w[A/x] + \int_a^x e^{(x-s_1)A}\, B\, w[A/s_1]\, ds_1 + \int_a^x e^{(x-s_1)A}\, B\, ds_1 \int_a^{s_1} e^{(s_1-s_2)A}\, B\, w[A/s_2]\, ds_2$$

*within second order Fréchet differential corrections to* $w[A/x]$.

10. **An Associated Differential System.** There are other significant non-linear differential systems related to (1.1) whose unique solutions are entire analytic functions. Let the trilinear function $T(y, x, z)$ satisfy the assumptions of Theorem 1. Consider the following differential system in the unknown functions $y(x)$ and $z(x)$ and their Fréchet differentials $\delta y(x)$ and $\delta z(x)$ respectively:

$$\text{(10.1)} \qquad \begin{cases} \delta y(x) = T(z(x), \delta x, z(x)) - T(y(x), \delta x, y(x)) \\ \delta z(x) = -T(z(x), \delta x, y(x)) - T(y(x), \delta x, z(x)) \\ y(0) = 0, \; z(0) = z_0. \end{cases}$$

By methods similar to those in the proof of Theorem 1 the following existence and uniqueness theorem can be established.

**Theorem 8.** *Under the hypothesis of Theorem* 1, *the differential system* (10.1) *in normed linear spaces is completely integrable and possesses the following unique entire analytic solution functions*

$$\text{(10.2)} \qquad \begin{cases} y(x) = \sum_{i=1}^{\infty} (-1)^{i-1} T^{2i-1}(z_0, x, z_0), \\ z(x) = z_0 + \sum_{i=1}^{\infty} (-1)^{i} T^{2i}(z_0, x, z_0), \end{cases}$$

*where* $T^n(z_0, x, z_0)$ *is the nth iteration of* $T(z_0, x, z)$ *as a linear function of* $z$ *evaluated for* $z = z_0$.

If the trilinear function $T(y, x, z)$ and the Banach spaces $B_1$ and $B_2$ are taken as in Section 6 with the norm (6.7) as the norm of the normed linear matric ring $N$ with unit $z_0 = I$, the identity matrix, the resultant system (10.1) possesses the unique solution

$$\text{(10.3)} \qquad \begin{cases} y_r^t[x(u)] = \int_r^t x(u)\,du - \int_r^t x(u)\,du \int_r^u x(v)\,dv \int_r^v x(w)\,dw + \cdots, \\ z_r^t[x(u)] = I - \int_r^t x(u)\,du \int_r^u x(v)\,dv + \cdots. \end{cases}$$

11. **The Differential Systems in Complex Banach Spaces.** The results of this paper can be shown to hold with some modifications for the case of complex Banach spaces — complete normed linear spaces with complex number multipliers. In some cases stronger theorems can be proved. For example, take the case of Theorem 1. We can prove

**Theorem 9.** *Let $B_1$ and $B_2$ be complex Banach spaces. Under the assumptions of Theorem* 1 *there exists a unique solution of the differential system* (1.1). *It is given by the entire analytic function* (3.1).

The uniqueness in Theorem 1 was proved within the class of all (single valued) entire analytic functions whereas the unicity of the solution in Theorem 9 is asserted within the class of all (single valued) functions defined throughout the complex Banach space $B_1$.

The proof of Theorem 9 proceeds as the proof of Theorem 1 as soon as it is established that any function that satisfies (1.1) throughout the complex Banach space $B_1$, with values in the complex Banach space $B_2$, is necessarily an entire analytic function. But this follows readily from known results. For, suppose $y(x)$ satisfies (1.1) throughout the complex Banach space $B_1$. Obviously then, the Fréchet differential of $y(x)$ exists everywhere in $B_1$. Hence $y(x)$ is continuous throughout $B_1$ and the Gateaux differential of $y(x)$ exists everywhere in $B_1$. This means that $y(x)$ is analytic in A. E. Taylor's[1] sense in an arbitrary sphere about the origin and hence it follows that $y(x)$ is an entire analytic function in our sense.

In the case of the analogues of the results in Section 9, it is clear that the independent variable $x$ will be a complex variable and that the integrals will be line integrals in the complex plane extended over paths within the Mittag-Leffler star[2] of the coefficients $a_j^i(x)$ of the differential equations (9.1).

[1] TAYLOR, A. E., »Analytic Functions in General Analysis», California Institute of Technology Thesis, June, 1936. A briefer version appeared later in Annali di Pisa, vol. 6 (1937), pp. 277—292.

[2] E. L. INCE, Ordinary Differential Equations, pp. 408—411.

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