NON-HOMOGENEOUS BINARY QUADRATIC FORMS.[1]

I. Two Theorems of Varnavides.

By

R. P. BAMBAH

ST. JOHN'S COLLEGE, CAMBRIDGE.

Introduction.

1. Let $f(x, y)$ be an indefinite binary quadratic form $a x^2 + b x y + c y^2$, with positive discriminant $d = b^2 - 4ac$. A well-known theorem of Minkowski states that, for any real numbers x_0, y_0, there exist integers x, y such that

$$|f(x + x_0, y + y_0)| \leq \frac{1}{4}\sqrt{d},$$

the sign of equality being necessary if and only if $f(x, y)$ is equivalent to a multiple of xy.

Heinhold [1], Davenport [1], Varnavides [1] and Barnes [1] have found better estimates for the minimum for non-critical f.

Recently Davenport [2, 3, 4] studied the special forms $x^2 + xy - y^2$ and $5x^2 - 11xy - 5y^2$ and obtained interesting results about their minima. Varnavides [2, 3, 4] applied Davenport's method to the forms $x^2 - 2y^2$, $x^2 - 7y^2$, and $x^2 - 11y^2$. In this note we give straight-forward geometrical proofs of Varnavides' results about the forms $x^2 - 7y^2$ and $x^2 - 11y^2$.

The results we prove can be stated as

Theorem 1: *Let $f(x, y) = x^2 - 7y^2$. Then given any two real numbers x_0, y_0 we can find x, y such that*

$$x \equiv x_0 \ (\text{mod. } 1), \quad y \equiv y_0 \ (\text{mod. } 1) \tag{1.1}$$

and

$$|f(x, y)| \leq \frac{9}{14}. \tag{1.2}$$

[1] This note forms a part of author's thesis: Some Results in the Geometry of Numbers: approved for the degree of Ph.D. at the University of Cambridge.

The equality sign in (1.2) *is necessary if and only if*

(1.3) $$x_0 \equiv \frac{1}{2} \pmod{1},\ y_0 \equiv \pm \frac{5}{14} \pmod{1}.$$

If x_0, y_0 *do not satisfy* (1.3) *we can replace* (1.2) *by*

(1.4)[1] $$|f(x, y)| < \frac{1}{1.56}.$$

Theorem 2: *Given* x_0, y_0, *any two real numbers, we can find* (x, y) *such that*

(2.1) $$x \equiv x_0 \pmod{1},\ \ y \equiv y_0 \pmod{1}$$

and

(2.2) $$|x^2 - 11\,y^2| \leq \frac{19}{22}.$$

The sign of equality in (2.2) *is necessary if and only if*

(2.3) $$x_0 \equiv \frac{1}{2} \pmod{1},\ y_0 \equiv \pm \frac{7}{22} \pmod{1}.$$

For all x_0, y_0, *not satisfying* (2.3), *we can replace* (2.2) *by*

(2.4)[1] $$|x^2 - 11\,y^2| < \frac{1}{1.16}.$$

Proof of Theorem 1.

2. We first prove

Lemma 1.1: Let $(x_0, y_0) \equiv \left(\frac{1}{2},\ \pm \frac{5}{14}\right) \pmod{1}$. Then for all $(x, y) \equiv (x_0, y_0) \pmod{1}$, $|x^2 - 7\,y^2| \geq \frac{9}{14}$. For some of these (x, y), for example $\left(\frac{1}{2},\ \pm \frac{5}{14}\right)$, the result holds with the equality sign.

Proof: All $(x, y) \equiv (x_0, y_0) \pmod{1}$ are given by $x = a + \frac{1}{2}$, $y = b \pm \frac{5}{14}$, where a and b are integers.

For these x, y we have

(1.5) $$|x^2 - 7\,y^2| = \left|\left(a + \frac{1}{2}\right)^2 - 7\left(b \pm \frac{5}{14}\right)^2\right| = \left|a^2 + a - 7\,b^2 \mp 5\,b - \frac{9}{14}\right| \geq \frac{9}{14},$$

---

[1] These results are slightly stronger than those of Varnavides in that we do not have the sign of equality in (1.4) and (2.4).

since $a^2 + a - 7b^2 \mp 5b$ is an even integer for all the a and b. The sign of equality in (1.5) arises when, for example, $a = b = 0$. This completes the proof of the lemma.

3. Suppose now x_0, y_0 is a pair of real numbers such that

$$(1.6) \qquad \text{For all } (x, y) \equiv (x_0, y_0) \pmod{1}, \ |x^2 - 7y^2| \geq \frac{1}{1.56}.$$

After Lemma 1.1, it will suffice for the proof of the theorem to show that x_0, y_0 *must satisfy the relation* (1.3).

The rest of the proof will, therefore, be concerned with the proof of the above.

Let y_1 be the unique number for which

$$-\frac{1}{2} < y_1 \leq \frac{1}{2}, \ y_1 \equiv y_0 \pmod{1}.$$

Consider the values of x satisfying the relation

$$x^2 - 7y_1^2 < \frac{1}{1.56} \leq (x+1)^2 - 7y_1^2.$$

The above is equivalent to

$$x^2 < \frac{1}{1.56} + 7y_1^2 = b^2 \text{ (say)} \leq (x+1)^2,$$

i.e.

$$\left.\begin{aligned} &-b < x < b, \\ &\text{and} \\ &\textit{either } x + 1 \geq b, \textit{ or } x + 1 \leq -b \end{aligned}\right\} \cdot (*)$$

Since it is impossible for x to be simultaneously less than $-b-1$ and greater than $-b$, we must have $x \geq b - 1$.

Now $b - 1 > -b$, since $b > \frac{1}{2}$ Therefore (*) is satisfied if and only if

$$b - 1 \leq x < b,$$

i.e. the values of x form a half-open interval of length 1. This interval contains a unique number $x_1 \equiv x_0 \pmod{1}$. Therefore there exists one and only one pair x_1, y_1, such that

(1.7) $$\begin{cases} (x_1, y_1) \equiv (x_0, y_0) \pmod{1}, \quad -\frac{1}{2} < y_1 \le \frac{1}{2}, \text{ and} \\ x_1^2 - 7y_1^2 < \frac{1}{1.56} \le (x_1 + 1)^2 - 7y_1^2. \end{cases}$$

Similarly there exist unique numbers x_2, y_2 such that

(1.8) $$\begin{cases} x_2, y_2 \equiv (x_0, y_0) \pmod{1}, \quad -\frac{1}{2} < y_2 \le \frac{1}{2}, \text{ and} \\ x_2^2 - 7y_2^2 < \frac{1}{1.56} \le (x_2 - 1)^2 - 7y_2^2. \end{cases}$$

Clearly $y_1 = y_2$. We suppose

(1.9) $$0 \le y_1 = y_2 \le \frac{1}{2}.$$

The procedure for negative y_1 is similar.

By (1.6), (1.7) and (1.8) we must have

(1.10) $$x_1^2 - 7y_1^2 \le -\frac{1}{1.56}, \quad x_2^2 - 7y_2^2 \le -\frac{1}{1.56}.$$

4. We now introduce a few definitions.

Definition 1: A point $P(x, y)$ in the x—y plane will be said to be "congruent" to the point $Q(x', y')$ if we have

$$(x, y) \equiv (x', y') \pmod{1}.$$

We will then write $P \equiv Q$.

Definition 2: We shall call two regions R and S in the x—y plane "congruent" regions, if a translation through integer distances parallel to the axes changes R into S and vice versa. We, then, write $R \equiv S$.

Obviously if $R \equiv S$, every point in R has a congruent point in S and vice versa.

Definition 3: A translation $\mathcal{T}_{m,n}$ will mean the translation through a distance m parallel to x-axis and n parallel to the axis of y.

5. Now, let us represent a pair x, y of real numbers by the point P in the x, y plane with co-ordinates (x, y). Then we have only to prove:

"*Let $P_0(x_0, y_0)$ be a point such that no point congruent to it lies in the region $\mathcal{T}$: defined by the inequality*

$$|x^2 - 7y^2| < \frac{1}{1.56}.$$

Then P_0 must be congruent to one of the two points $\left(\frac{1}{2}, \pm\frac{5}{14}\right)$".

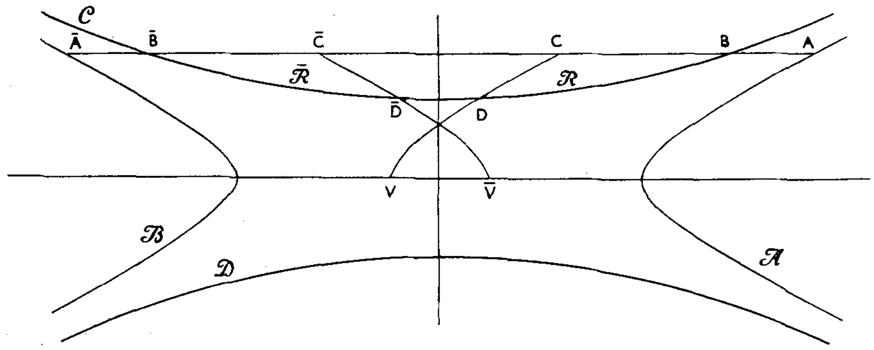

Fig. 1

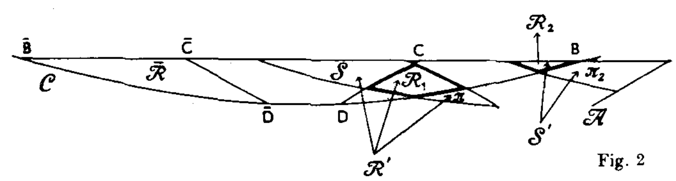

Fig. 2

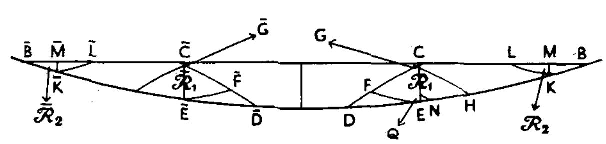

Fig. 3

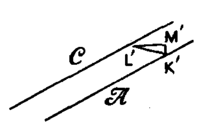

Fig. 4

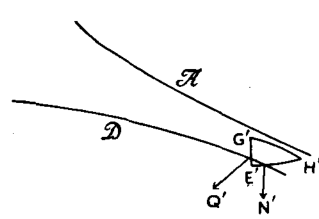

Fig. 5

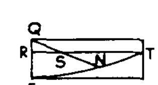

Fig. 6

Figs. 1—6.

6. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ be the arcs of the hyperbolas $x^2 - 7y^2 = \pm\frac{1}{1.56}$. Then $\mathscr{T}$ is the open region included between these arcs. (See fig. 1).

Let the line $y = \frac{1}{2}$ meet these arcs in the points A, B, $\bar{B}$ and $\bar{A}$ as shown in the figure. Move the part of $\mathscr{A}$, lying between $A\bar{A}$ and the x-axis, through a distance -1 parallel to the x-axis. Let it take up the position CDV, with the points C, D and V on $A\bar{A}$, $\mathscr{C}$ and the x-axis respectively. Clearly the equation of CDV is $(x+1)^2 - 7y^2 = \frac{1}{1.56}$.

Also move the part of $\mathscr{B}$, between $A\bar{A}$ and the x-axis, through distance 1 parallel to the x-axis to take up the position $\bar{C}\bar{D}\bar{V}$, as shown in figure 1. The equation of $\bar{C}\bar{D}\bar{V}$ is $(x-1)^2 - 7y^2 = \frac{1}{1.56}$.

Denote the closed curvilinear triangles BCD and $\bar{B}\bar{C}\bar{D}$ by $\mathscr{R}$ and $\bar{\mathscr{R}}$.

Then the relations (1.7), (1.9) and (1.10) mean that there exists a unique point $P_1 \equiv P_0$ in $\mathscr{R}$, while relations (1.8), (1.9) and (1.10) mean that there is just one point P_2 congruent to P_0 and lying in $\bar{\mathscr{R}}$.

Clearly it will suffice for our theorem to show that "P_1 must coincide with $\left(\frac{1}{2}, \frac{5}{14}\right)$".

For convenience of reference, we tabulate below the co-ordinates of the vertices of $\mathscr{R}$ and $\bar{\mathscr{R}}$. $\bar{\mathscr{R}}$ is obviously the image of $\mathscr{R}$ in the y-axis.

Table I.

Point	Curves on which it lies	Co-ordinates
B	$y = \frac{1}{2},\ x^2 - 7y^2 = -\frac{1}{1.56}$	$\left(\sqrt{\frac{173}{156}}, \frac{1}{2}\right) = (1.053\ldots, 0.5)$.
C	$y = \frac{1}{2},\ (x+1)^2 - 7y^2 = \frac{1}{1.56}$	$\left(\sqrt{\frac{373}{156}} - 1, \frac{1}{2}\right) = (0.546\ldots, 0.5)$.
D	$x^2 - 7y^2 = -\frac{1}{1.56},\ (x+1)^2 - 7y^2 = \frac{1}{1.56}$	$\left(\frac{11}{78}, y'\right)$ (the value of y' unimportant)
	Image in y-axis of	
$\bar{B}$	B	$(-1.053\ldots, 0.5)$.
$\bar{C}$	C	$(-0.546\ldots, 0.5)$.
$\bar{D}$	D	$\left(-\frac{11}{78}, y'\right)$.

7. Let the translation $\mathcal{T}_{1,0}$ change $\bar{\mathcal{R}}$ into $\mathcal{R}'$. Then we assert that $\mathcal{R}'$ consists of three parts (see fig. 2)

i) π, which lies in $\mathcal{T}$,

ii) the closed curvilinear quadrilateral $\mathcal{R}_1$, which lies in $\mathcal{R}$, and

iii) the region $\mathcal{S}$, which lies outside $\mathcal{T}$ as well as $\mathcal{R}$. The above assertion will clearly follow if we can show that

1. the upper vertices of $\mathcal{R}'$ lie to the left of C,
2. the lower vertex of $\mathcal{R}'$ lies inside $\mathcal{T}$, and
3. the hyperbolic arcs in the boundaries of $\mathcal{R}$ and $\mathcal{R}'$ meet each other in single points.

We first observe

Lemma A: Let $\mathcal{C}$ be an infinite arc of a hyperbola. Let AB be a finite arc of a hyperbole $\mathcal{D}$, whose asymptotes are parallel to those of $\mathcal{C}$. Then, if A and B lie on opposite sides of $\mathcal{C}$, AB intersects $\mathcal{C}$ in a single point; but, if A and B lie on the same side of $\mathcal{C}$, AB meets $\mathcal{C}$ in two points or none.

Proof: Obvious, since $\mathcal{C}$ and $\mathcal{D}$ cannot intersect in more than two finite points.

After Lemma A, 3 is a direct consequence of 1 and 2.

The condition 2 is obviously satisfied since the lower vertex of $\mathcal{R}'$, i.e. the new position of $\bar{D}$, lies on the line $D\bar{D}$: $y = y'$, at a distance less than 1 to the right of D, and since the arc DC is at distance 1 from $\mathcal{A}$.

The condition 1, too, is easily verified, since the co-ordinates of the upper vertices of $\mathcal{R}'$, obtained by adding $(1, 0)$ to those of $\bar{B}$ and $\bar{C}$, are $(-.053\ldots, .5)$ and $(.453\ldots, .5)$ while those of C are $(.546\ldots, .5)$.

Consequently our assertion about $\mathcal{R}'$ is true and fig. 2 is correct.

Now $\mathcal{R}'$ is congruent to $\bar{\mathcal{R}}$. Therefore $\mathcal{R}'$ contains a point $Q \equiv P_2 \equiv P_0$. As Q cannot lie in $\mathcal{T}$, it must lie either in $\mathcal{R}_1$ or in $\mathcal{S}$; we include the common boundary in $\mathcal{R}_1$ only.

Now, let the translation $\mathcal{T}_{1,0}$ change $\mathcal{S}$ into $\mathcal{S}'$. We assert that $\mathcal{S}'$ will consist of two parts i) π_2 lying in $\mathcal{T}$ and ii) the closed curvilinear triangle $\mathcal{R}_2$ lying in $\mathcal{R}$. The assertion will clearly be justified if we can show

1. The upper vertices of $\mathcal{S}'$ are situated relative to B and C as shown in fig. 2, and

2. The points of intersection between the boundaries of $\mathcal{R}$ and $\mathcal{S}'$ are as shown in fig. 2.

After Lemma A, 2 is a direct consequence of 1, which in its turn follows from the fact that the co-ordinates of the upper vertices are (.946 ..., .5) and (1.453 ..., .5) while those of C and B are (.546 ..., .5) and (1.053 ..., .5) respectively.

Therefore the figure is verified.

Now if the point Q lies in $\mathcal{S}$, a point $Q' \equiv Q \equiv P_0$ will lie in $\mathcal{S}'$. As Q' cannot lie in $\mathcal{T}$, it will lie in $\mathcal{R}_2$. So that we conclude that a point $\equiv P_0$ lies in $\mathcal{R}_1$, or $\mathcal{R}_2$.

As both $\mathcal{R}_1$ and $\mathcal{R}_2$ lie in $\mathcal{R}$ and $\mathcal{R}$ contains just one point $P_1 \equiv P_0$, P_1 must obviously lie in $\mathcal{R}_1$ or $\mathcal{R}_2$.

Similarly, we can prove that P_2, the point in $\bar{\mathcal{R}}$ congruent to P_0, must lie in $\bar{\mathcal{R}}_1$, $\bar{\mathcal{R}}_2$, respective images in y-axis of $\mathcal{R}_1$ and $\mathcal{R}_2$.

By considering the equations of boundary arcs of $\mathcal{R}_1$, $\bar{\mathcal{R}}_1$, $\mathcal{R}_2$ and $\bar{\mathcal{R}}_2$ or by simple symmetry considerations, it is easily seen that $\mathcal{R}_1 \equiv \bar{\mathcal{R}}_1$ and $\mathcal{R}_2 \equiv \bar{\mathcal{R}}_2$.

Let the vertices of $\mathcal{R}_1$ be E, F, G, and H, those of $\bar{\mathcal{R}}_1$ be $\bar{E}$, $\bar{F}$, $\bar{G}$, and $\bar{H}$, of $\mathcal{R}_2$ be K, L and B while those of $\bar{\mathcal{R}}_2$ be $\bar{K}$, $\bar{L}$ and $\bar{B}$, as shown in fig. 3.

Join EG and $\bar{E}\bar{G}$. Draw the lines KM, $\bar{K}\bar{M}$ parallel to y-axis to meet $B\bar{B}$ in M and $\bar{M}$. These lines divide $\mathcal{R}_1$, $\mathcal{R}_2$, $\bar{\mathcal{R}}_1$, $\bar{\mathcal{R}}_2$ in two parts each. These parts have various congruence and symmetry relations e.g.

1. $\bar{E}\bar{F}\bar{G}$ is congruent to EHG and symmetric with respect to y-axis to EFG.

2. $\bar{K}\bar{B}\bar{M}$ is congruent to KLM and symmetric with respect to y-axis to KBM.

In view of these relations it will suffice for the proof of Theorem 1 to prove statement A viz. "*Every point, except* $\left(\frac{1}{2}, \frac{5}{14}\right)$, *of* KLM *and* EGH *has a congruent point inside* $\mathcal{T}$". For, if so, because of congruence, every point, except $\left(-\frac{1}{2}, \frac{5}{14}\right)$, in $\bar{K}\bar{B}\bar{M}$ and $\bar{E}\bar{F}\bar{G}$ will have a congruent point in $\mathcal{T}$, and, then by symmetry, every point, except $\left(\frac{1}{2}, \frac{5}{14}\right)$, of KBM and EFG will have a congruent point in $\mathcal{T}$. Combined with the statement A this will mean that every point except $\left(\frac{1}{2}, \frac{5}{14}\right)$ in $\mathcal{R}_1$ or $\mathcal{R}_2$ has a congruent point inside $\mathcal{T}$, so that P_1, which lies in $\mathcal{R}_1$ or $\mathcal{R}_2$ and has no congruent point in $\mathcal{T}$, will have to coincide with the point $\left(\frac{1}{2}, \frac{5}{14}\right)$ and the Theorem will follow.

Table II.

Point	Curves through it	Equations of the Curves	Coordinates of the point
E	EF, EH	$(x-1)^2 - 7y^2 = -\frac{1}{1.56}$, $x^2 - 7y^2 = -\frac{1}{1.56}$	$\left(\frac{1}{2}, \sqrt{\frac{139}{1092}}\right) = (.5, .3567\ldots)$
H	HG, EH	$(x-2)^2 - 7y^2 = \frac{1}{1.56}$, $x^2 - 7y^2 = -\frac{1}{1.56}$	$\left(\frac{53}{78}, \frac{1}{78}\sqrt{\frac{6709}{7}}\right) = (.679\ldots, .3969\ldots)$
G	HG, FG	$(x-2)^2 - 7y^2 = \frac{1}{1.56}$, $(x+1)^2 - 7y^2 = \frac{1}{1.56}$	$\left(\frac{1}{2}, \sqrt{\frac{251}{1092}}\right) = (.5, .4794\ldots)$
K	KL, KB	$(x-2)^2 - 7y^2 = -\frac{1}{1.56}$, $x^2 - 7y^2 = -\frac{1}{1.56}$	$\left(1, \sqrt{\frac{64}{273}}\right) = (1, .484\ldots)$
M	MK, MB	$x = 1$, $y = \frac{1}{2}$	$\left(1, \frac{1}{2}\right) = (1, .5)$
L	KL, LB	$(x-2)^2 - 7y^2 = -\frac{1}{1.56}$, $y = \frac{1}{2}$	$\left(2 - \sqrt{\frac{173}{156}}, \frac{1}{2}\right) = (.946\ldots, .5)$.

We now prove statement A in the Lemmas 1.2—1.4 below. For convenience for reference we tabulate the co-ordinates of some points.

8. **Lemma 1.2**: Every point in KLM has a congruent point in $\mathcal{T}$.

Proof: Let the translation $\mathcal{J}_{3,1}$ change KLM into $K'L'M'$. (See fig. 4). Then

$$K' = K + (3,\ 1)[1] = (4,\ 1.484\ldots) \text{ lies in } \mathcal{T},$$

since

$$|4^2 - 7(1.484\ldots)^2|^1 < .58\ldots < \frac{1}{1.56} = .641\ldots;$$

also,

$$L' = L + (3,\ 1) = (3.946\ldots,\ 1.5) \text{ lies in } \mathcal{T},$$

since

$$|(3.946\ldots)^2 - 7(1.5)^2| < 7(1.5)^2 - (3.9)^2 = .54 < \frac{1}{1.56}.$$

As the triangle $K'L'M'$ lies entirely within the rectangle formed by the lines through K' and L' parallel to the axes, the above implies that $K'L'M'$ lies inside $\mathcal{T}$ and the lemma follows.

Lemma 1.3: Every point in EHG excluding a closed curvilinear triangle ENQ has a congruent point in $\mathcal{T}$.

Proof: Let the translation $\mathcal{J}_{1,-1}$ change EHG into $E'H'G'$. (See fig. 5). Then,

$$H' = H + (1,\ -1) = (1.679\ldots,\ -.603\ldots) \text{ lies inside } \mathcal{T},$$

since

$$|(1.679\ldots)^2 - 7(.603\ldots)^2| < 2.89 - 2.52 = .37 < \frac{1}{1.56};$$

$$G' = G + (1,\ -1) = (1.5,\ -.5205\ldots) \text{ lies inside } \mathcal{T},$$

since

$$|(1.5)^2 - 7(.5205\ldots)^2| < 2.25 - 1.75 = .5 < \frac{1}{1.56};$$

$$E' = E + (1,\ -1) = \left(1.5,\ \sqrt{\frac{139}{1092}} - 1\right)$$ lies below the boundary $\mathcal{D}$ i.e. in that part of $x^2 - 7y^2 < -\frac{1}{1.56}$, where y is negative, since

$$(1.5)^2 - 7\left(\sqrt{\frac{139}{1092}} - 1\right)^2 < -.646\ldots < -\frac{1}{1.56} = -.641\ldots$$

This proves that the position of the points E', G' and H' is as shown in fig. 5.

[1] $1.484\ldots = 1 + \sqrt{\frac{64}{273}}$.

We also observe that $E'H'$ and $H'G'$ are arcs of hyperbolas with asymptotes parallel to $x \pm \sqrt{7}\,y = 0$.

By Lemma A, $E'H'$ meets $\mathcal{D}$ in a single point, N' (say). As $E'G'$ is a line parallel to the y-axis, it intersects $\mathcal{D}$ in one point, Q' (say). The arc $G'H'$ arises from $\mathcal{B}$ by a translation $\mathcal{T}_{3,-1} = \mathcal{T}_{1,0} + \mathcal{T}_{1,0} + \mathcal{T}_{1,-1}$. Therefore its equation is

$$(1.11) \qquad (x-3)^2 - 7(y+1)^2 = \frac{1}{1.56}.$$

The equation of $\mathcal{D}$ is

$$x^2 - 7y^2 = -\frac{1}{1.56}.$$

Therefore, eliminating y between (1.11) and the above, we find that the points of intersection, if any, of $G'H'$ and $\mathcal{D}$ satisfy the equation

$$0 = -\frac{175}{39} - 7x^2 + \left(3x - \frac{14}{39}\right)^2 = 2x^2 - \frac{84}{39}x - \frac{175 \times 39 - 196}{39^2}.$$

This has a negative root. Therefore, as all the points on $G'H'$ have a positive abscissa, there is at most one point of intersection of $G'H'$ and $\mathcal{D}$. But by lemma A, the points of intersection of $G'H'$ and $\mathcal{D}$ are two or none. Therefore $G'H'$ does not intersect $\mathcal{D}$.

The equation of $\mathcal{A}$ is

$$x^2 - 7y^2 = \frac{1}{1.56}.$$

Eliminating y between (1.11) and above, we see that the points of intersection, if any, of $G'H'$ and $\mathcal{A}$ satisfy the equation

$$0 = (1-3x)^2 - 7x^2 + \frac{7}{1.56} = 2x^2 - 6x + \left(1 + \frac{7}{1.56}\right).$$

As this equation has no real root, $G'H'$ does not intersect $\mathcal{A}$. Hence the position of $E'G'H'$ is as shown in fig. 5. The translation $\mathcal{T}_{-1,1}$, i.e. the translation inverse to $\mathcal{T}_{1,-1}$, changes $E'N'Q'$ into ENQ of the lemma.

9. Now we give an easy lemma which we shall apply later.

Lemma B: Let $a > 0$, $r > 1$, $a_0 \leq a$, be any three real numbers. Then, if $N \leq a$ be a positive number, we can find an integer $n \geq 1$, such that

$$a_0 < N r^n \leq a r.$$

Proof: Obvious, since we can find $n \geq 1$ such that

$$a < N r^n \leq a r.$$

Lemma 1.4: Every point, except $\left(\frac{1}{2}, \frac{5}{12}\right)$, in the closed triangle ENQ, has a congruent point in $\mathcal{T}$.

Proof: The equations of the boundary arcs of ENQ are

EN: $$x^2 - 7y^2 = -\frac{1}{1.56}$$

NQ: $$(x+1)^2 - 7(y-1)^2 = -\frac{1}{1.56}$$

EQ: $$x = \frac{1}{2}, \text{ (by definition).}$$

Therefore the co-ordinates of Q are

$$x = \frac{1}{2}, \quad y = 1 - \left[\frac{1}{7}\left(\frac{9}{4} + \frac{25}{39}\right)\right]^{1/2} = \left(\frac{1}{2}, \cdot 35734\ldots\right).$$

As the co-ordinates of E are $\left(\frac{1}{2}, .35677\ldots\right)$, R, the point with co-ordinates $\left(\frac{1}{2}, \frac{5}{14}\right) = \left(\frac{1}{2}, \cdot 35714\ldots\right)$ lies between E and Q on the line EQ.

Let RST, the line $y = \frac{5}{14}$ through R, meet NQ and EN in S and T respectively.

Then the co-ordinates of S are $\left(\sqrt{\frac{2459}{1092}} - 1, \frac{5}{14}\right) = \left(.5006\ldots, \frac{5}{14}\right)$, and those of T are $\left(\sqrt{\frac{375}{1092}}, \frac{5}{14}\right) = \left(.5018\ldots, \frac{5}{14}\right)$. Therefore the position is as shown in fig. 6.

Consequently every point of ENQ has co-ordinates $\left(\frac{1}{2} + \alpha, \frac{5}{14} + \beta\right)$ where

$$0 \leq \alpha < .0019, \quad -.0004 < \beta < .00021.$$

Therefore the points of ENQ, excluding $\left(\frac{1}{2}, \frac{5}{14}\right)$ form a subset of the set Σ consisting of points $\left(\frac{1}{2} + \alpha, \frac{5}{14} + \beta\right)$, where

(1.12) $$0 \leq \alpha < .0019, \quad |\beta| < .0004, \quad (\alpha, \beta) \neq (0, 0).$$

It will consequently suffice to show that every point of Σ has a congruent point in $\mathcal{T}$.

Suppose it is not so. Then let $(x', y') = \left(\frac{1}{2} + \alpha_1,\ \frac{5}{14} + \beta_1\right)$ be a point of Σ such that all points congruent to it lie outside $\mathcal{T}$.

Then, we have

$$0 < \alpha_1 + |\beta_1|\sqrt{7} < .0019 + (.0004)(2.7) < .003, \tag{1.13}$$

and, for all rational integers x, y,

$$\left|\left(x + \alpha_1 + \frac{1}{2}\right)^2 - 7\left(y + \beta_1 + \frac{5}{14}\right)^2\right| \geq \frac{1}{1.56}. \tag{1.14}$$

First suppose $\beta_1 \geq 0$.

The relation (1.14) implies that, for all rational integers x, y, we have

$$\left|\left(x - \frac{3}{2} + \alpha_1\right)^2 - 7\left(y - \frac{9}{14} + \beta_1\right)^2\right| \geq \frac{1}{1.56}, \text{ i.e.}$$

$$\left|\left\{\xi - \left(\frac{3}{2} + \frac{9}{14}\sqrt{7}\right) + (\alpha_1 + \beta_1\sqrt{7})\right\} \cdot \left\{\xi' - \left(\frac{3}{2} - \frac{9}{14}\sqrt{7}\right) + (\alpha_1 - \beta_1\sqrt{7})\right\}\right| \geq \frac{1}{1.56} \tag{1.15}$$

for all integers $\xi = x + y\sqrt{7}$ and their conjugates $\xi' = x - y\sqrt{7}$ in the field $k(\sqrt{7})$.

Write τ for the fundamental unit $8 + 3\sqrt{7} = 15.93\ldots$ of $(k\sqrt{7})$.

Take ξ defined by the relation

$$\xi - \left(\frac{3}{2} + \frac{9}{14}\sqrt{7}\right) = -\left(\frac{3}{2} + \frac{9}{14}\sqrt{7}\right)\tau'^{2n}, \quad n \text{ an integer.}$$

Obviously

$$\xi = \left(\frac{3}{2} + \frac{9}{14}\sqrt{7}\right)(1 - \tau'^{2n}) = \tau'^{n}(9 + 3\sqrt{7})(\tau^{n} - \tau'^{n})\frac{1}{2\sqrt{7}}$$

is an integer since $\tau^n - \tau'^n \equiv 0 \pmod{2\sqrt{7}}$.

Dividing (1.15) by $\left|\left\{\xi - \left(\frac{3}{2} + \frac{9}{14}\sqrt{7}\right)\right\}\left\{\xi' - \left(\frac{3}{2} - \frac{9}{14}\sqrt{7}\right)\right\}\right|$

$$= \left|\left(\frac{3}{2} + \frac{9}{14}\sqrt{7}\right)\left(\frac{3}{2} - \frac{9}{14}\sqrt{7}\right)\right| = \frac{9}{14},$$

we have

$$(1.16)\qquad \left|1-\frac{\alpha_1+\beta_1\sqrt{7}}{\frac{3}{2}+\frac{9}{14}\sqrt{7}}\tau^{2n}\right|\left|1+\frac{\alpha_1-\beta_1\sqrt{7}}{\frac{9}{14}\sqrt{7}-\frac{3}{2}}\cdot\frac{1}{\tau^{2n}}\right|\geq\frac{1}{1.56}\cdot\frac{14}{9}=\frac{14}{14.04}$$

for all n.

Writing $\dfrac{\alpha_1+\beta_1\sqrt{7}}{\frac{3}{2}+\frac{9}{14}\sqrt{7}}=\varrho$ and $\dfrac{\alpha_1-\beta_1\sqrt{7}}{\frac{3}{2}-\frac{9}{14}\sqrt{7}}=\varrho'$, (1.16) becomes

$$(1.16')\qquad |1-\varrho\tau^{2n}|\,|1-\varrho'\tau'^{2n}|\geq\frac{14}{14.04}$$

for all n.

Now, as $\tau>1$, from (1.12), we have, for all $n\geq 1$,

$$(1.17)\qquad |1-\varrho'\tau'^{2n}|\leq 1+\frac{\alpha_1+\beta_1\sqrt{7}}{\left(\frac{9}{14}\sqrt{7}-\frac{3}{2}\right)}\cdot\frac{1}{\tau^2}<1+\frac{.003}{250(.17)}<1\frac{1}{14,000}=m\,(\text{say}).$$

We show now that in Lemma B, we can take

$$r=\tau^2,\ a=\frac{1}{r},\ a_0=1-\frac{14}{14.04\,m}\ \text{and}\ N=\varrho.$$

For, $r>1$, and $0<N\leq a$ follows from

$$0<Nr=\frac{\alpha_1+\beta_1\sqrt{7}}{\frac{3}{2}+\frac{9}{14}\sqrt{7}}\tau^2<(.003)(256)<1=ar.$$

Also $a_0<a$, since

$$\frac{ar}{a_0}=\frac{1}{1-\frac{14}{14.04\,m}}=\frac{14.04(14001)}{14.04(14001)-14(14000)}>\frac{196,000}{575}>300>\tau^2.$$

Therefore, all the conditions of the lemma are satisfied so that we can find an $n\geq 1$ such that

$$1-\frac{14}{14.04\,m}<\varrho\tau^{2n}\leq 1.$$

Therefore

$$(1.18)\qquad |1-\varrho\tau^{2n}|<\frac{14}{14.04\,m}.$$

Multiplying by (1.17), we have

$$|1-\varrho\tau^{2n}|\,|1-\varrho'\tau'^{2n}| < m\frac{14}{14.04\,m} = \frac{14}{14.04},$$

which contradicts (1.16′).

Therefore, $\beta_1 < 0$.

Let $\beta_1 = -\beta_1'$, so that $\beta_1' > 0$ and $\alpha_1 + \beta_1'\sqrt{7} > 0$.

The relation (1.14) implies that, for all rational integers x, y,

$$(1.19)\qquad \left|\left(x-\frac{1}{2}+\alpha_1\right)^2 - 7\left(y+\frac{5}{14}-\beta_1'\right)^2\right| \geq \frac{1}{1.56},$$

$$(1.19')\quad \text{i.e.}\ \left|\left\{\xi+\left(\frac{5}{14}\sqrt{7}-\frac{1}{2}\right)+(\alpha_1-\beta_1'\sqrt{7})\right\}\cdot\left\{\xi'-\left(\frac{5}{14}\sqrt{7}+\frac{1}{2}\right)+(\alpha_1+\beta_1'\sqrt{7})\right\}\right| \geq \frac{1}{1.56}$$

for all integers ξ of $k(\sqrt{7})$.

Taking $\xi' - \frac{7+5\sqrt{7}}{14} = -\frac{7+5\sqrt{7}}{14}\tau^{2n}$, we obtain a contradiction as before. Thus there is no point (x', y') in Σ which does not have a congruent point in $\mathcal{T}$. This proves the lemma.

Combining Lemmas 1.2, 1.3 and 1.4 we get statement A and hence the theorem.

Proof of Theorem II.

10. As in the proof of Theorem I we first prove

Lemma 2.1. Let $(x_0, y_0) \equiv \left(\frac{1}{2}, \pm\frac{7}{22}\right)$ (mod. 1). Then for all $(x, y) \equiv (x_0, y_0)$ (mod. 1), $|x^2 - 11y^2| \geq \frac{19}{22}$. For some of these x, y, for example $\left(\frac{1}{2}, \pm\frac{7}{22}\right)$, this result holds with the sign of equality.

Proof: All $(x, y) \equiv (x_0, y_0)$ (mod. 1) are given by $x = a + \frac{1}{2}$, $y = b \pm \frac{7}{22}$, where a and b are integers.

For these x, y, we have

$$(2.5)\quad |x^2 - 11y^2| = \left|\left(a+\frac{1}{2}\right)^2 - 11\left(b\pm\frac{7}{22}\right)^2\right| = \left|a^2 + a - 11b^2 \mp 7b - \frac{19}{22}\right| \geq \frac{19}{22},$$

since $a^2 + a - 11b^2 \mp 7b$ is an even integer.

The sign of equality in (2.5) is necessary when, for example, $a = b = 0$.

II. Let $\mathcal{T}$ be the open region bounded by the arcs of the hyperbolas $x^2 - 11\,y^2 = \pm \frac{1}{1.16}$. *Let* $P_0(x_0, y_0)$ *be a point such that no point congruent to it lies in* $\mathcal{T}$. Then, as in Theorem I, we have only to show that P_0 must be congruent to one of the two points $\left(\frac{1}{2}, \pm \frac{7}{22}\right)$.

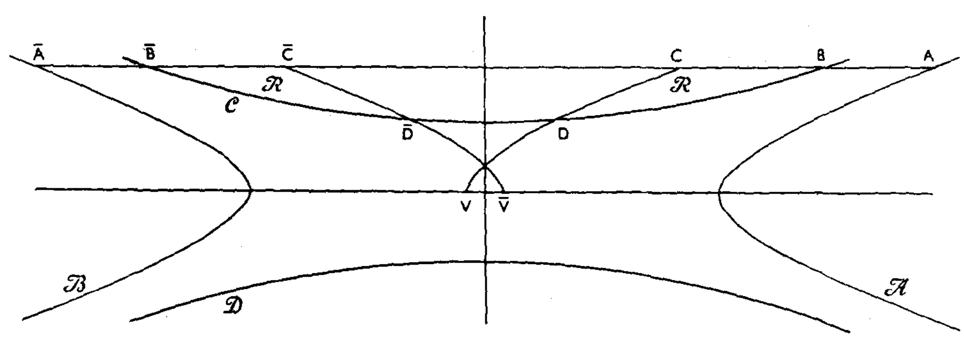

Fig. 7

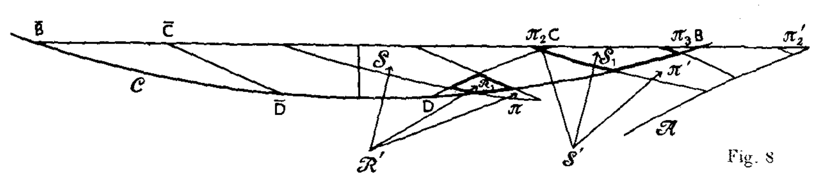

Fig. 8

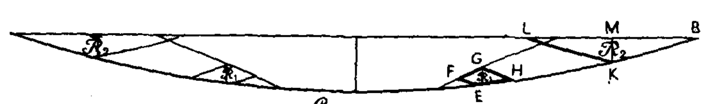

Fig. 9

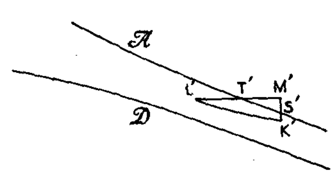
Fig. 10

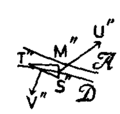
Fig. 11

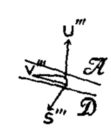
Fig. 12

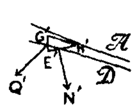
Fig. 13

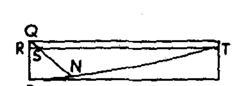

Fig. 14

Figs. 7—14.

12. Let $\mathcal{A}$ and $\mathcal{B}$ be the arcs of the hyperbola $x^2 - 11y^2 = \frac{1}{1.16}$ and $\mathcal{C}$ and $\mathcal{D}$ those of $x^2 - 11y^2 = -\frac{1}{1.16}$, so that $\mathcal{T}$ is the open region enclosed by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. (See fig. 7).

Let the line $y = \frac{1}{2}$ meet these arcs in the points A, B, $\bar{B}$ and $\bar{A}$ as shown in the figure. Now move the part of $\mathcal{A}$ lying between $A\bar{A}$ and the x-axis through a distance -1 parallel to the x-axis. Let it take up the position CDV, with the points C, D and V on $A\bar{A}$, $\mathcal{C}$ and the x-axis respectively. The equation of CDV is $(x+1)^2 - 11y^2 = \frac{1}{1.16}$.

Similarly move the part of $\mathcal{B}$ between $A\bar{A}$ and the x-axis through a distance 1 parallel to the x-axis, to take up the position $\bar{C}\bar{D}\bar{V}$ as shown in the figure. The equation of $\bar{C}\bar{D}\bar{V}$ is $(x-1)^2 - 7y^2 = \frac{1}{1.16}$.

Denote the closed curvilinear triangles BCD, $\bar{B}\bar{C}\bar{D}$ by $\mathcal{R}$ and $\bar{\mathcal{R}}$ respectively.

Now suppose that the unique $y_1 \equiv y_0$ in the interval $-\frac{1}{2} < y_1 \leq \frac{1}{2}$ is non-negative.

Then, as in Theorem I, it is easily seen that both $\mathcal{R}$ and $\bar{\mathcal{R}}$ contain unique points P_1 and P_2 congruent to P_0.

Then it will suffice to prove $P_1 = \left(\frac{1}{2}, \frac{7}{22}\right)$. For, if y_1 were negative, similar argument would give $P_1 = \left(\frac{1}{2}, -\frac{7}{22}\right)$; so that all $P_0(x_0, y_0)$, incongruent to points of $\mathcal{T}$, are congruent to $\left(\frac{1}{2}, \pm\frac{7}{22}\right)$. This is clearly equivalent to the theorem.

For convenience of reference, we tabulate now the co-ordinates of the vertices of $\mathcal{R}$ and $\bar{\mathcal{R}}$. $\bar{\mathcal{R}}$ is obviously the image of $\mathcal{R}$ in the y-axis.

13. Let the translation $\mathcal{T}_{1,0}$ change $\bar{\mathcal{R}}$ into $\mathcal{R}'$. Then we assert that $\mathcal{R}'$ consists of three parts (see fig. 8),

i) π, which lies in $\mathcal{T}$,
ii) the closed curvilinear quadrilateral $\mathcal{R}_1$, which lies in $\mathcal{R}$, and
iii) the region $\mathcal{S}$, which lies outside $\mathcal{T}$ as well as $\mathcal{R}$.

Table III.

<table>
<tr><th>Point</th><th>Curves on which it lies</th><th>Co-ordinates</th></tr>
<tr><td>B</td><td>$y = \frac{1}{2}$, $x^2 - 11y^2 = -\frac{1}{1.16}$</td><td>$\left(\sqrt{\frac{219}{116}}, \frac{1}{2}\right) = (1.374\ldots, .5)$</td></tr>
<tr><td>C</td><td>$y = \frac{1}{2}$, $(x+1)^2 - 11y^2 = \frac{1}{1.16}$</td><td>$\left(\sqrt{\frac{419}{116}} - 1, \frac{1}{2}\right) = (.90\ldots, .5)$</td></tr>
<tr><td>D</td><td>$x^2 - 11y^2 = -\frac{1}{1.16}$, $(x+1)^2 - 11y^2 = \frac{1}{1.16}$</td><td>$\left(\frac{21}{58}, y'\right)$ (the value of y' unimportant)</td></tr>
<tr><td></td><td>Image in y-axis of</td><td></td></tr>
<tr><td>$\bar{B}$</td><td>B</td><td>$(-1.374\ldots, .5)$</td></tr>
<tr><td>$\bar{C}$</td><td>C</td><td>$(-.90\ldots, .5)$</td></tr>
<tr><td>$\bar{D}$</td><td>D</td><td>$\left(-\frac{21}{58}, y'\right)$</td></tr>
</table>

To prove the assertion we have only to show

1. the upper vertices of $\mathcal{R}'$ lie to the left of C,
2. the lower vertex of $\mathcal{R}'$ lies in $\mathcal{T}$, and
3. the hyperbolic arcs in the boundaries of $\mathcal{R}$, and $\mathcal{R}'$ meet each other in single points.

The condition 3 is, by Lemma A, an immediate consequence of 1 and 2.

The condition 2 is obviously satisfied since the lower vertex of $\mathcal{R}'$, i.e. the new position of $\bar{D}$, lies on the line $\bar{D}D$: $y = y'$, at a distance less than 1 to the right of D and since arc DC is at distance 1 from $\mathcal{A}$.

The condition 1, too, is easily verified, since the co-ordinates of the upper vertices of $\mathcal{R}'$, obtained by adding $(1, 0)$ to those of $\bar{B}$ and $\bar{C}$, are $(-.374\ldots, .5)$ and $(.09\ldots, .5)$ while those of C are $(.90\ldots, .5)$.

Consequently our assertion about $\mathcal{R}'$ is true and the position is as shown in fig. 8.

Now $\mathcal{R}'$ is congruent to $\bar{\mathcal{R}}$. Therefore $\mathcal{R}'$ contains a point $Q \equiv P_2 \equiv P_0$. As Q cannot lie in $\mathcal{T}$, it must lie either in $\mathcal{R}_1$ or in $\mathcal{S}$; we include the common boundary of $\mathcal{R}_1$ and $\mathcal{S}$ in $\mathcal{R}_1$ only.

Now, let $\mathcal{T}_{1,0}$ change $\mathcal{S}$ into $\mathcal{S}'$. We assert that $\mathcal{S}'$ is situated as shown in fig. 8. Because of Lemma A, we have only to verify that the positions of the vertices of $\mathcal{S}'$ are as shown. Now the lower vertices of $\mathcal{S}$ lay on CD. Therefore the lower vertices of $\mathcal{S}'$ must lie on $\mathcal{A}$. Also the co-ordinates of the upper vertices obtained by adding $(1, 0)$ to those of upper vertices of $\mathcal{S}$, are $(.625 \ldots, .5)$ and $(1.09 \ldots, .5)$, while those of C and D are $(.90 \ldots, .5)$ and $(1.374 \ldots, .5)$. Therefore the positions of the upper vertices, too, are easily seen to be correctly shown.

Consequently $\mathcal{S}'$ consists of

i) π' lying in $\mathcal{T}$,

ii) the closed curvilinear pentagon $\mathcal{S}_1$, and

iii) π_2 lying neither in $\mathcal{T}$ nor in $\mathcal{R}$; the boundary arcs of $\mathcal{S}_1$ common with π' or π_2 are included in $\mathcal{S}_1$ alone.

Now if the point Q lies in $\mathcal{S}$, a point $Q' \equiv Q \equiv P_0$ will lie in $\mathcal{S}'$. As Q' cannot lie in $\mathcal{T}$, it will lie either in $\mathcal{S}_1$ or in π_2.

The translation $\mathcal{T}_{1,0}$ changes the lower vertex and one of the upper vertices of π_2 to points on $\mathcal{A}$, while the other upper vertex becomes $(1.625 \ldots, .5)$ as shown. Thus the translation $\mathcal{T}_{1,0}$ changes π_2 into π_2' lying entirely in $\mathcal{T}$.

Now $\pi_2' \equiv \pi_2$. Therefore if Q' lay in π_2, a point $Q'' \equiv Q' \equiv P_0$ would lie in $\mathcal{T}$, which is impossible. Therefore Q' cannot lie in π_2.

Consequently a point congruent to P_0 is seen to lie in $\mathcal{R}_1$ or $\mathcal{S}_1$. As both $\mathcal{R}_1$ and $\mathcal{S}_1$ lie in $\mathcal{R}$, and $\mathcal{R}$ contains only one point, namely P_1, $\equiv P_0$, we conclude that *P_1 must lie either in $\mathcal{R}_1$ or in $\mathcal{S}_1$.*

Let $\mathcal{R}_2$ be the closed curvilinear triangle containing π_2, $\mathcal{S}_1$ and the region π_3, shown in fig. 8. *Then we can say that P_1 lies in $\mathcal{R}_1$ or $\mathcal{R}_2$.*

Let the vertices of $\mathcal{R}_1$ and $\mathcal{R}_2$ be E, F, G, H, K, B, and L as shown in fig. 9. Join EG and draw KM parallel to y-axis to meet $A\bar{A}$ at M.

Then, as in Theorem I, it will suffice to show that

"*Every point, except* $\left(\frac{1}{2}, \frac{7}{22}\right)$, *of KLM and EGH has a congruent point inside $\mathcal{T}$*"

This we shall prove in the rest of the paper.

For convenience of reference we tabulate the co-ordinates of some points together with the equations of the curves on which they lie.

Point	Curves through it	Equations of the curves	Co-ordinates of the point
E	EF, EH	$(x-1)^2 - 11y^2 = -\frac{1}{1.16}$, $x^2 - 11y^2 = -\frac{1}{1.16}$	$\left(\frac{1}{2}, \sqrt{\frac{129}{1276}}\right) = (.5, .317\ldots)$
H	HG, EH	$(x-2)^2 - 11y^2 = \frac{1}{1.16}$, $x^2 - 11y^2 = -\frac{1}{1.16}$	$\left(\frac{33}{58}, \frac{1}{58}\sqrt{\frac{3989}{11}}\right) = (.57\ldots, .328\ldots)$
G	HG, GF	$(x-2)^2 - 11y^2 = \frac{1}{1.16}$, $(x+1)^2 - 11y^2 = \frac{1}{1.16}$	$\left(\frac{1}{2}, \sqrt{\frac{161}{1276}}\right) = (.5, .355\ldots)$
K	KL, KB	$(x-2)^2 - 11y^2 = -\frac{1}{1.16}$, $x^2 - 11y^2 = -\frac{1}{1.16}$	$\left(1, \sqrt{\frac{54}{319}}\right) = (1, .411\ldots)$
L	KL, LB	$(x-2)^2 - 11y^2 = -\frac{1}{1.16}$, $y = \frac{1}{2}$	$\left(2 - \sqrt{\frac{219}{116}}, \frac{1}{2}\right) = (.625\ldots, .5)$
M	MK, ML	$x = 1$, $y = \frac{1}{2}$.	$\left(1, \frac{1}{2}\right) = (1. .5)$.

14. **Lemma 2.2:** Every point in KLM has a congruent point in $\mathcal{T}$.

Proof: If not, suppose there is a point P in KLM, such that no point congruent to it lies in $\mathcal{T}$.

Then we shall obtain a contradiction in three stages (i), (ii) and (iii) below.

i) Let the translation $\mathcal{J}_{1,-1}$ change KLM into $K'L'M'$. (See fig. 10). Then,

$$K' = K + (1, -1) = (2, -.588\ldots) \text{ lies in } \mathcal{T},$$

since

$$|2^2 - 11(.588\ldots)^2| < 4 - 11(.58)^2 = .2996 < \frac{1}{1.16};$$

$$L' = L + (1, -1) = (1.625\ldots, .5) \text{ lies in } \mathcal{T},$$

since

$$|(1.625\ldots)^2 - 11(.5)^2| < 2.75 - 2.56 < \frac{1}{1.16};$$

$M' = M + (1, -1) = (2, -.5)$ lies above $\mathcal{A}$ i.e. in that part of

$$x^2 - 11y^2 > \frac{1}{1.16},$$

where x is positive, since

$$2^2 - 11(.5)^2 = 1.25 > \frac{1}{1.16}.$$

Therefore the position of the points K', L' and M' is as shown in fig. 10.

The lines $K'M'$ and $M'L'$, being parallel to the axes, meet $\mathcal{A}$ in single points S' and T' say.

Now $K'L'$ arises from KL: $(x-2)^2 - 11y^2 = -\frac{1}{1.16}$: by translation $\mathcal{J}_{1,-1}$. Therefore, its equation is

$$(x-3)^2 - 11(y+1)^2 = -\frac{1}{1.16}. \tag{2.6}$$

The equation of $\mathcal{D}$ is

$$x^2 - 11y^2 = -\frac{1}{1.16}.$$

Therefore, on eliminating y between (2.6) and the above, we find that the points of intersection, if any, of $K'L'$ and $\mathcal{D}$ satisfy the relation

$$0 = 11x^2 - (3x+1)^2 + \frac{275}{29} = 2x^2 - 6x + \frac{246}{29}. \tag{2.7}$$

This equation has no real roots. Therefore $K'L'$ does not intersect $\mathcal{D}$.

The equation of $\mathcal{A}$ is $x^2 - 11y^2 = \frac{1}{1.16}$.

Therefore, by (2.6) and the above, the points of intersection, if any, of $K'L'$ and $\mathcal{A}$ satisfy

$$(2.8) \qquad 0 = 11x^2 - \left(3x + \frac{4}{29}\right)^2 - \frac{275}{29} = 2x^2 - \frac{24}{29}x - \left(\frac{16}{29^2} + \frac{275}{29}\right).$$

Obviously (2.8) has a negative root. As the x-co-ordinates of all points of $K'L'$ are positive, $K'L'$ and $\mathcal{A}$ cannot intersect in two points. Therefore, by Lemma A, $K'L'$ has no point common with $\mathcal{A}$.

Consequently we see that the situation is as shown in fig. 10, i.e. $K'L'M'$ consist of two parts, i) the curvilinear region $K'L'S'T'$ lying in $\mathcal{T}$, and ii) the closed curvilinear triangle $S'T'M'$ lying outside $\mathcal{T}$.

Since $K'L'M' \equiv KLM$, it contains a point $P' \equiv P$. As P' cannot lie in $\mathcal{T}$, P' lies in the curvilinear triangle $S'T'M'$. The co-ordinates of S' and T' are

$$T' = \left(\sqrt{\frac{419}{116}},\ -\frac{1}{2}\right) = (1.90\ldots,\ -.5),$$

$$S' = \left(2,\ -\sqrt{\frac{91}{319}}\right) = (2,\ -.534\ldots).$$

ii) Let now the translation $\mathcal{J}_{3,-1}$ change $S'M'T'$ into $S''M''T''$. (See fig. 11). Then

$$M'' = M' + (3,\ -1) = (5,\ -1.5) \text{ lies in } \mathcal{T},$$

since

$$|(5)^2 - 11(-1.5)^2| = .25 < \frac{1}{1.16};$$

$$T'' = T' + (3,\ -1) = (4.90\ldots,\ -1.5) \text{ lies in } \mathcal{T},$$

since

$$|(4.90\ldots)^2 - 11(1.5)^2| < .74 < \frac{1}{1.16};$$

$$S'' = S' + (3,\ -1) = (5,\ -1.534\ldots) \text{ lies below } \mathcal{D},$$

since

$$(5)^2 - 11(-1.534\ldots)^2 < -.88 < -\frac{1}{1.16}.$$

This shows that the points S'', M'' and T'' are situated as shown. As $S''M''$ and $M''T''$ are parallel to the axes, $M''T''$ intersects neither $\mathcal{A}$ nor $\mathcal{D}$, $S''M''$

intersects $\mathfrak{D}$ at a single point, U'', say. The hyperbolic arc $S''T''$ does not intersect $\mathcal{A}$ because of the situation of these arcs relative to the lines $S''M''$, $M''T''$. Again, by Lemma A, $S''T''$ intersects $\mathfrak{D}$ at one point V''. In short, the position of $S''M''T''$ is as shown in the figure.

Now $S''M''T''$ is congruent to $S'M'T'$ and, therefore, contains a point $P'' \equiv P' \equiv P$. *As P'' cannot lie in $\mathcal{T}$, it lies in the curvilinear triangle $S''U''V''$.*

The point U'' has co-ordinates $\left(5, \ -\sqrt{\frac{750}{319}}\right) = (5, \ -1.533\ldots)$. Also we note that the abscissa of V'' is greater than that of T'', which is greater than 4. Therefore, the abscissa of any point on $S''V''$ or $U''V''$ lies between 4 and 5.

iii) Now let $\mathcal{T}_{20,-6}$ change $S''U''V''$ into $S'''U'''V'''$. (See fig. 12).

Then

$$S''' = S'' + (20, \ -6) = \left(25, \ -7 - \sqrt{\frac{91}{319}}\right) \text{ lies in } \mathcal{T},$$

since

$$\left|(25)^2 - 11\left(7 + \sqrt{\frac{91}{319}}\right)^2\right|$$

$$= \left|625 - 539 - \frac{91}{29} - \sqrt{\frac{91 \times 154 \times 14}{29}}\right|$$

$$= 625 - 539 - 3.13\ldots - 82.2\ldots < .67 < \frac{1}{1.16};$$

$$U''' = U'' + (20, \ -6) = \left(25, \ -6 - \sqrt{\frac{750}{319}}\right) \text{ lies in } \mathcal{T},$$

since

$$\left|(25)^2 - 11\left(6 + \sqrt{\frac{750}{319}}\right)^2\right| = |625 - 396 - 25.86\ldots - 202.39\ldots| < .75 < \frac{1}{1.16}.$$

The translation $\mathcal{T}_{20,-1}$ does not change the relative position of $S''V''$ and $U''V''$ i.e. $S'''V'''$ lies below $U'''V'''$. Therefore, in order to show that $S'''U'''V'''$ lies inside $\mathcal{T}$, it will suffice to show that (a) $U'''V'''$ does not intersect $\mathcal{A}$, and (b) $S'''V'''$ does not intersect $\mathfrak{D}$.

(a) The arc $U'''V'''$ arises from $U''V''$: $x^2 - 11y^2 = -\frac{1}{1.16}$ by $\mathcal{T}_{20,-6}$.

Therefore

1. the x-co-ordinate of any point on $U'''V'''$ lies between 24 and 25.

2. the equation of $U''' V'''$ is

$$(x-20)^2 - 11(y+6)^2 = -\frac{1}{1.16}. \tag{2.9}$$

The equation of $\mathcal{A}$ is $x^2 - 11y^2 = \frac{1}{1.16}$. Therefore, on eliminating y between (2.9) and the equation of $\mathcal{A}$, we find that the points of intersection, if any, of $U''' V'''$ and $\mathcal{A}$ satisfy the equation

$$\begin{aligned} 0 &= \frac{396}{1.16} - 396x^2 + \left(-20x + \frac{83}{29}\right)^2 \\ &= 4x^2 - \frac{3320}{29}x + \left\{\left(\frac{83}{29}\right)^2 + \frac{9900}{29}\right\} = f(x) \text{ (say)}. \end{aligned} \tag{2.10}$$

Now $f(0) > 0$, $f(4) < 0$, $f(25) < 0$, and $f(\infty) > 0$. Therefore, there is no root of (2.10) in the interval (24, 25). Consequently, $U''' V'''$ does not intersect $\mathcal{A}$, i.e (a) is verified.

(b) The arc $S''' V'''$ arises from $S'' V''$: $(x-3)^2 - 11(y+1)^2 = \frac{1}{1.16}$ by $\mathcal{T}_{20,-6}$. Therefore,

1. The x-co-ordinate of any point on $S''' V'''$ lies between 24 and 25.
2. The equation of $S''' V'''$ is

$$(x-23)^2 - 11(y+7)^2 = \frac{1}{1.16}. \tag{2.11}$$

The equation of $\mathcal{D}$ is $x^2 - 11y^2 = -\frac{1}{1.16}$. Therefore, eliminating y, we find that the common points, if any, of $S''' V'''$ and $\mathcal{D}$ satisfy

$$\begin{aligned} 0 &= 539x^2 - \left(23x + \frac{170}{29}\right)^2 + \frac{(539)25}{29} \\ &= 10x^2 - \frac{7820}{29}x + \frac{361875}{841} = f(x) \text{ (say)}. \end{aligned} \tag{2.12}$$

Now $f(0) > 0$, $f(2) < 0$, $f(25) < 0$ and $f(\infty) > 0$. Therefore, (2.12) has no root between 24 and 25. And so $U''' V'''$ and $\mathcal{D}$ have no common points, i.e. (b) is true.

Consequently $U''' S''' V'''$ lies entirely in $\mathcal{T}$. Now $U''' S''' V'''$ is congruent to $U'' S'' V''$. Therefore a point $P''' \equiv P'' \equiv P$ lies in $U''' S''' V'''$ and hence in $\mathcal{T}$. This gives the required contradiction and the lemma is established, i.e. every point in KLM has a congruent point in $\mathcal{T}$.

Lemma 2.3: Every point in EHG, *excluding* a closed curvilinear triangle ENQ, defined in the proof, has a congruent point in $\mathcal{T}$.

Proof: Let the translation $\mathcal{T}_{5,-2}$ change EHG into $E'H'G'$. (See fig. 13). The point $H' = H + (5, \; -2) = (5.57 \ldots, \; -1.671 \ldots)$ lies in $\mathcal{T}$, since

$$|(5.57\ldots)^2 - 11\,(1.671\ldots)^2| < (5.58)^2 - 11\,(1.67)^2 < .5 < \frac{1}{1.16};$$

$$G' = G + (5, \; -2) = (5.5, \; -1.644\ldots) \text{ lies in } \mathcal{T}$$

since

$$|(5.5)^2 - 11\,(1.644\ldots)^2| < 30.25 - 11\,(1.64)^2 < .67 < \frac{1}{1.16};$$

$$E' = E + (5, \; -2) = \left(5.5, \; \sqrt{\frac{129}{1276}} - 2\right) \text{ lies below } \mathcal{D}$$

since

$$(5.5)^2 - 11\left(\sqrt{\frac{129}{1276}} - 2\right)^2 = 30.25 - 44 - 1.112\ldots$$
$$+ 13.990\ldots < -.87 < -\frac{1}{1.16}.$$

This shows that the position of the points E', G' and H' is as shown in the figure.

As $E'G'$ is a line parallel to the y-axis, it intersects $\mathcal{D}$ in one point, Q' say. Again, by Lemma A, $E'H'$ meets $\mathcal{D}$ in one point, N' say.

The arc $G'H'$ arises from $\mathcal{B}$ by a translation $\mathcal{T}_{2,0} + \mathcal{T}_{5,-2} = \mathcal{T}_{7,-2}$. Therefore, its equation is

$$(x - 7)^2 - 11\,(y + 2)^2 = \frac{1}{1.16}. \tag{2.13}$$

The equation of $\mathcal{D}$ is $x^2 - 11\,y^2 = -\dfrac{1}{1.16}$.

Therefore, eliminating y between (2.13) and the above, we find that points of intersection, if any, of $G'H'$ and $\mathcal{D}$ satisfy the equation

$$0 = -176\,x^2 + \left(14x - \frac{95}{29}\right)^2 - \frac{176\,(25)}{29}$$
$$= 20\,x^2 - \frac{(190)\,14}{29}\,x - \frac{1}{841}\left\{176\,(725) - 95^2\right\}.$$

This has a negative root. Therefore, as all the points on $G'H'$ have a positive abscissa, there is at most one point of intersection of $G'H'$ and $\mathcal{D}$. But by

Lemma A, the points common to $G' H'$ and $\mathcal{D}$ are two or none. Therefore $G' H'$ does not intersect $\mathcal{D}$.

The equation of $\mathcal{A}$ is $x^2 - 11 y^2 = \frac{1}{1.16}$.

Eliminating y between (2.13) and the above, we see that the points of intersection, if any, of $G' H'$ and $\mathcal{A}$ satisfy the equation

$$0 = 176 x^2 + (14 x - 5)^2 + \frac{176(25)}{29} = 20 x^2 - 140 x + 176.72 \ldots$$

As $f(0) > 0$, $f(2) < 0$ and $f(5.5) > 0$, the roots of this equation lie in the open intervals (0, 2) and (2, 5.5). But the x-co-ordinate of every point on $H' G'$ is greater than 5.5, the x-co-ordinate of G'. Therefore $H' G'$ does not intersect $\mathcal{A}$ either. Consequently the position of $E' G' H'$ is as shown in the figure.

The translation $\mathcal{T}_{-5,2}$, i.e. the translation inverse to $\mathcal{T}_{5,-2}$, changes $E' N' Q'$ into ENQ of the lemma.

15. **Lemma 2.4:** Every point, except $\left(\frac{1}{2}, \frac{7}{22}\right)$, in the closed triangle ENQ has a congruent point in $\mathcal{C}$.

Proof: The equations of the boundary arcs of ENQ are

EN: $$x^2 - 11 y^2 = -\frac{1}{1.16}$$

NQ: $$(x+5)^2 - 11(y-2)^2 = -\frac{1}{1.16}$$

EQ: $$x = \frac{1}{2}.$$

Therefore, the co-ordinates of Q are

$$x = \frac{1}{2},\ y = 2 - \sqrt{\frac{3609}{1276}} = (.5,\ .31822\ldots).$$

Also the co-ordinates of E are $\left(\frac{1}{2}, \sqrt{\frac{129}{1276}}\right) = (.5,\ .31795\ldots)$.

Therefore, R, the point with co-ordinates $\left(\frac{1}{2}, \frac{7}{22}\right) = (.5,\ .31818\ldots)$ lies between E and Q on the line EQ.

Let RST, the line $y = \frac{7}{22}$ through R meet NQ and EN in S and T respectively.

Then the co-ordinates of S are $\left(\sqrt{\frac{38601}{1276}}-5,\ \frac{7}{22}\right)=\left(.5001\ldots,\ \frac{7}{22}\right)$ and those of the T are $\left(\sqrt{\frac{321}{1276}},\ \frac{7}{22}\right)=\left(.5015\ldots,\ \frac{7}{22}\right)$.

Thus the position is as shown in fig. 14.

Consequently every point of ENQ has co-ordinates $\left(\frac{1}{2}+\alpha,\ \frac{7}{22}+\beta\right)$, where $0\leq\alpha<.0016,\ -.00024<\beta<.00005$.

Therefore the points of ENQ, excluding $\left(\frac{1}{2},\ \frac{7}{22}\right)$ form a subset of the set Σ: consisting of points $\left(\frac{1}{2}+\alpha,\ \frac{7}{22}+\beta\right)$, where

$$0\leq\alpha<.0016,\ |\beta|<.00024,\ (\alpha,\beta)\neq(0,0). \tag{2.14}$$

It will consequently suffice to show that every point of Σ has a congruent point in $\mathcal{T}$.

Suppose it is not so. Then let $(x',y')=\left(\frac{1}{2}+\alpha_1,\ \frac{7}{22}+\beta_1\right)$ be a point of Σ such that all points congruent to it lie outside $\mathcal{T}$.

Then, we have

$$0<\alpha_1+|\beta_1|\sqrt{11}<.0016+(.00024)\sqrt{11}<.0024, \tag{2.15}$$

and, for all rational integers x, y,

$$\left|\left(x+\frac{1}{2}+\alpha_1\right)^2-11\left(y+\frac{7}{22}+\beta_1\right)^2\right|\geq\frac{1}{1.16}. \tag{2.16}$$

Let $\beta_1\geq 0$.

The relation (2.16) implies that

$$\left|\left\{\xi-\left(\frac{11}{2}+\frac{37\sqrt{11}}{22}\right)+(\alpha_1+\beta_1\sqrt{11})\right\}\cdot\left\{\xi'-\left(\frac{11}{2}-\frac{37\sqrt{11}}{22}\right)+(\alpha_1-\beta_1\sqrt{11})\right\}\right|\geq\frac{1}{1.16} \tag{2.17}$$

for all integers $\xi=x+y\sqrt{11}$ and their conjugates $\xi'=x-y\sqrt{11}$ in the field $k(\sqrt{11})$.

Write τ for the fundamental unit $10+3\sqrt{11}$ of $k(\sqrt{11})$ and τ' for the conjugate of τ.

Then, as in Theorem 1, Lemma 1.4, we get a contradiction by taking ξ defined by the relation:

$$\xi - \left(\frac{11}{2} + \frac{37\sqrt{11}}{22}\right) = -\left(\frac{11}{2} + \frac{37\sqrt{11}}{22}\right)\tau'^{2n}.$$

If $\beta_1 = -\beta_2$, $\beta_2 > 0$, we first deduce from (2.16) that

$$(2.18) \quad \left|\left\{\xi' - \left(\frac{1}{2} - \frac{7\sqrt{11}}{22}\right) + (\alpha_1 - \beta_2\sqrt{11})\right\} \cdot \left\{\xi - \left(\frac{1}{2} + \frac{7\sqrt{11}}{22}\right) + (\alpha_1 + \beta_2\sqrt{11})\right\}\right| \geq \frac{1}{1.16}$$

for all integers ξ of $k(\sqrt{11})$.

Then we get a contradiction by taking ξ defined by

$$\xi - \left(\frac{1}{2} + \frac{7\sqrt{11}}{22}\right) = -\left(\frac{1}{2} + \frac{7\sqrt{11}}{22}\right)\tau'^{2n}.$$

This shows that there is no point (x', y') in Σ which does not have a congruent point in $\mathcal{T}$. This establishes the lemma.

Combining Lemmas 2.1, 2.2, 2.3 and 2.4 we obtain theorem II.

In the end I would like to express my deep sense of gratitude to Prof. L. J. Mordell, F.R.S., for help in preparing this note. I am also grateful to the Royal Commission of 1851 Exhibition for the award of an Overseas Science Research Scholarship which enabled me to carry out this work at the University of Cambridge.

References.

Barnes, E. S. 1. Non-homogeneous binary quadratic forms. Quar. Jour. Math. Oxford, Second Series, *1* (1950), 199—211.

Davenport, H. 1. Non-homogeneous binary quadratic forms I. Konink. Neder. Akad. Wet. *49* (1946), 815—21.

——, 2. Non-homogeneous binary quadratic forms II. Konink. Neder. Akad. Wet. *50* (1947), 378—89.

——, 3. Non-homogeneous binary quadratic forms III. Konink. Neder. Akad. Wet. *50* (1947), 484—91.

——, 4. Non-homogeneous binary quadratic forms IV. Konink. Neder. Akad. Wet. *50* (1947), 909—17.

Heinhold, J. 1. Verallgemeinerung und Verschärfung eines Minkowskischen Satzes, Math. Zeit., *44* (1939), 659—88.

VARNAVIDES, P. 1. Note on non-homogeneous quadratic forms. Quar. Jour. Math. Oxford Ser. *19* (1948), 54—58.

——, 2. Non-homogeneous binary quadratic forms, I, II. Konink. Neder. Akad. Wet. *51* (1948), 396—404, 470—81.

——, 3. On the quadratic form $x^2 - 7y^2$. Proc. Roy. Soc. A, *197* (1949), 256—68.

——, 4. Arithmetical properties of indefinite quadratic forms, Ph.D. thesis, 1948, London University.