ON NORMALIZABLE TRANSFORMATIONS IN HILBERT SPACE.

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1. **Introduction.** Zaanen [4] has recently extended the theory of normal transformations in Hilbert space, developed by Rellich [2], to normalizable transformations, and has applied his results to certain special systems of Fredholm integral equations

$$(1) \qquad \lambda \psi_i(x) = \sum_{j=1}^{n} \int_{\Delta} K_{ij}(x, y)\, \psi_j(y)\, dy \qquad (i = 1, \ldots, n).$$

Now, if the adjoint of a normalizable transformation can be defined in the space (as is always possible in a complete Hilbert space), then we shall show that the existence Theorem 10 of Zaanen [4] can be extended. If, in addition, the normalizable transformation is completely continuous (as is the case for the kernels of (1) above), then a further extension will be obtained, and in this case one of the hypotheses of Theorem 12 of [4] may be omitted. This result will then be applied in § 3 to integral systems (1) which are definitely self-conjugate adjoint or J-definite according to [5], which is a generalization of the definite systems treated by Reid [1] and Wilkins [3].

The notation of Zaanen [4] will be employed. Spaces will be denoted by capital German letters and transformations by capital Roman letters. If the adjoint transformation of K exists it will be denoted by K^*.

2. **Existence theorems.** Let $\mathfrak{R}$ be a Hilbert space with elements $f, g, h, \phi, \ldots$ and inner product (f, g). Let H denote a bounded, positive, self-adjoint transformation on $\mathfrak{R}$ to $\mathfrak{R}$; i.e., a bounded linear transformation satisfying $(Hf, g) = (f, Hg)$ and $(Hf, f) \geq 0$ for arbitrary f and g of $\mathfrak{R}$. The set of all elements h for which $Hh \equiv 0$ shall be designated by $[\mathfrak{L}]$, while $[\mathfrak{M}]$ shall denote the orthogonal complement of $[\mathfrak{L}]$. We shall assume that every $f \in \mathfrak{R}$ is expressible

in the form $f = g + h$, where $g \in [\mathfrak{M}]$ and $h \in [\mathfrak{L}]$, the projection of f on $[\mathfrak{M}]$ being denoted by $Ef = g$. The bounded linear transformations K and $\tilde{K}$ are termed *H-adjoint* by Zaanen [4] when $(HKf, g) = (Hf, \tilde{K}g)$ for arbitrary f and g of $\mathfrak{R}$. If, in addition, $HK\tilde{K} = H\tilde{K}K$, the transformation K is defined to be *normalizable*. It is to be noted that when $\mathfrak{R}$ is a complete Hilbert space, the above decomposition of its elements always exists and, moreover, the condition for H-adjointness is equivalent to $HK = \tilde{K}^*H$ or $H\tilde{K} = K^*H$.

Theorem 2.1. *Let $\mathfrak{R}$ be a Hilbert space in which the adjoint K^* of a bounded, linear, normalizable transformation K is defined. If $HK \neq 0$ and the transformation EK is completely continuous, then K^* has at least one characteristic value $\neq 0$.*

Under the above hypotheses it follows from Theorem 10 of Zaanen [4] that there exists a $\mu_1 \neq 0$ with a corresponding nonzero element ϕ_1 such that $\mu_1 \phi_1 = EK\phi_1$, $\bar{\mu}_1 \phi_1 = E\tilde{K}\phi_1$. Consequently, $\bar{\mu}_1 H\phi_1 = HE\tilde{K}\phi_1 = H\tilde{K}\phi_1 = K^*H\phi_1$. Moreover, $H\phi_1 \not\equiv 0$ as $EK\phi_1 \in [\mathfrak{M}]$, and, therefore, $\phi_1 \in [\mathfrak{M}]$. Thus the transformation K^* possesses the characteristic value $\bar{\mu}_1 \neq 0$ with corresponding characteristic element $H\phi_1$. Furthermore, if $\tilde{K}^*$ exists, it readily follows that this transformation $\tilde{K}^*$ possesses the characteristic value $\mu_1 \neq 0$ with corresponding characteristic element $H\phi_1$.

Corollary 1. *Let $\mathfrak{R}$ be a complete Hilbert space in which the bounded linear transformation K is normalizable. If $HK \neq 0$ and EK is completely continuous, then K^* has at least one characteristic value $\neq 0$.*

Corollary 2. *Let $\mathfrak{R}$ be a complete Hilbert space in which the bounded linear transformation K is completely continuous and normalizable. If $HK \neq 0$ then K possesses at least one characteristic value $\neq 0$.*

Furthermore, it is to be noted that Theorems 5 and 11 of Zaanen [4] may be extended in an entirely analogous manner to yield existence theorems similar to those above for the transformations K and $\tilde{K}$ rather than for EK and $E\tilde{K}$.

3. **Definite integral systems.**[1] Following the notation of Zaanen [4, § 14] let $\Delta \equiv [a_1, b_1; \ldots; a_m, b_m]$ be an interval in m-dimensional Euclidean space, the point $(x_1, \ldots, x_m)$ be denoted by x and the complete Hilbert space of all func-

[1] The results of this section have also been independently derived by W. T. Reid in a recent study of completely continuous symmetrizable transformations in Hilbert space.

tions $f(x)$ for which $|f(x)|^2$ is Lebesgue integrable by $L_2^{(m)}$. Let $\Delta \times \Delta$ denote the interval $[a_1, b_1; \ldots; a_m, b_m : a_1, b_1; \ldots; a_m, b_m]$ in $2m$-dimensional Euclidean space, and let $L_2^{(2m)}$ represent the space of all functions $f(x, y)\,(x, y \in \Delta)$, for which $|f(x, y)|^2$ is Lebesgue integrable over $\Delta \times \Delta$. The space under consideration in this section is the space $[L_2^{(m)}]^{(n)}$, the complete Hilbert space of all elements $\{f\} \equiv \{f_1, f_2, \ldots, f_n\}$, $f_i \in L_2^{(m)}\,(i = 1, \ldots, n)$.

For the integral systems (1) with matrix kernel $K(x, y) \equiv \| K_{ij}(x, y) \|$ it will be assumed that *there exists a matrix* $S(x) \equiv \| S_{ij}(x) \|$ *each of whose elements is bounded and measurable in* Δ, *such that*

(H_0) $\sum_{j=1}^{n} S_{ij}(x)\psi_j(x) \not\equiv 0$ *for all* i ($i = 1, \ldots, n$) *for any non-trivial set of solutions* $\psi_1(x), \ldots, \psi_n(x)$ *of* (1).

(H_1) $S(x)$ *is a hermitian matrix almost everywhere in* Δ.

(H_2) $K_1(x, y) \equiv S(x) K(x, y)$ *is a hermitian matrix almost everywhere in* $\Delta \times \Delta$.

An integral system (1) is termed *definitely self-conjugate adjoint* with respect to a matrix $S(x)$ in case (H_0), (H_1), (H_2) and the following condition are satisfied with this matrix (see [5, § 3]):

(H_3) $S(x)$ *is of positive type in* Δ.

Now, for a definitely self-conjugate adjoint integral system (1) the transformation K in $[L_2^{(m)}]^{(n)}$ defined by $\{g\} = K\{f\}$, where

$$g_i(x) = \sum_{j=1}^{n} \int_{\Delta} K_{ij}(x, y) f_j(y)\, dy \qquad (i = 1, \ldots, n),$$

is completely continuous and normalizable relative to the transformation H defined by $\{h\} = H\{f\}$, where $h_i(x) = \sum_{j=1}^{n} S_{ij}(x) f_j(x)$ ($i = 1, \ldots, n$). Zaanen [4, p. 239] has obtained an existence theorem for such definite integral systems (1) for which the matrix kernel is of the form $K(x, y) \equiv A(x, y) S(y)$. However, from Corollary 2 to Theorem 2.1 above it follows that the condition $H\{f\} = 0$ implying that $K\{f\} = 0$, which automatically holds if $K(x, y) \equiv A(x, y) S(y)$, need no longer be assumed for general kernels of definitely self-conjugate adjoint systems (1).

Theorem 3.1. *If* $K_{1_{ij}}(x, y) \neq 0$ *on a set of positive measure in* $\Delta \times \Delta$ *for at least one of the elements of the matrix* $K_1(x, y)$ *of a definitely self-conjugate adjoint* *integral system* (1), *then this system possesses at least one real characteristic value* $\neq 0$.

The reality of the characteristic value in the above theorem, as well as in the result below, follows either from the properties of such definite integral systems, or by the extension of Theorem 11 of Zaanen [4] previously mentioned.

On the other hand, the system (1) is termed *J-definite* (see [5, § 3]), in case it satisfies (H_0), (H_1), (H_2) and the following condition with a matrix $S(x)$:

$(H_{3,J})$ *the matrix* $K_1(x, y)$ *is of positive type in* $\Delta \times \Delta$.

Clearly, the transformation K induced by the kernel $K(x, y)$ of a J-definite integral system (1) is completely continuous and normalizable relative to the transformation H defined by $\{k\} = H\{f\}$, where $k_i(x) = \sum_{j=1}^{n} \int_{\Delta} K_{1_{ij}}(x, y) f_j(y)\, dy$. However, the condition that $H\{f\} = 0$ implies $K\{f\} = 0$ is no longer satisfied, in general, even for the important case where $K(x, y) \equiv A(x, y) S(y)$. Nevertheless, Corollary 2 to Theorem 2.1 above yields the desired result.

Theorem 3.2. *If* $\sum_{k=1}^{n} \int_{\Delta} K_{1_{ik}}(x, t) K_{kj}(t, y)\, dt \neq 0$ *on a set of positive measure in* $\Delta \times \Delta$ *for at least one of the elements of the matrix* $\int_{\Delta} K_1(x, t) K(t, y)\, dt$ *of a J-definite integral system* (1), *then this system possesses at least one real characteristic value* $\neq 0$.

September, 1950.

Bibliography.

1. W. T. Reid, *Expansion problems associated with a system of integral equations*, Transactions of the American Mathematical Society, vol. 33 (1931), pp. 475—485.
2. F. Rellich, *Spektraltheorie in nicht-separablen Räumen*, Math. Annalen, vol. 110 (1934), pp. 342—356.
3. J. E. Wilkins, Jr., *Definitely self-conjugate adjoint integral equations*, Duke Math. Journal, vol. 11 (1944), pp. 155—166.
4. A. C. Zaanen, *Nonmalisable transformations in Hilbert space and systems of linear integral equations*, this Journal, vol. 83 (1950), pp. 197—248.
5. H. J. Zimmerberg, *Definite integral systems*, Duke Math. Journal, vol. 15 (1948), pp. 371—388.