THE PROBLEM OF UNITARY EQUIVALENCE.[1]

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The question of equivalence of matrices under the group G of unitary transformations has received attention from several writers [1, 2, 3, 4]. Fundamental in most investigations is the theorem of Schur [5] that any matrix A of complex numbers can be transformed by some unitary matrix into triangle form: (a_{ij}), $a_{ij} = 0$ whenever $i > j$. A short proof of Schur's theorem appears in Murnaghan's book [6].

This theorem alone is not enough to settle the equivalence question; two matrices may be in triangle form, equivalent under G, and yet not equal. An example is given by the matrices $\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

If a matrix is in triangle form, the diagonal elements are the characteristic roots. Schur proves further that it is possible to find a unitary matrix U such that UAU^* is in triangle form and has its characteristic roots arranged in any order along the main diagonal. In order that two matrices be equivalent under G it is clearly necessary that they have the same characteristic roots; this condition is by no means sufficient.

This article investigates the question of equivalence under G:

$$A_1, B_1 \text{ given; } X \text{ to be found so that}$$

$$X A_1 X^* = B_1, \quad X X^* = 1. \tag{1}$$

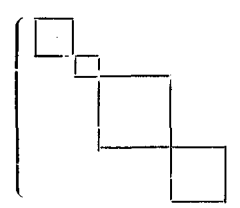
To solve problem (1), we follow a standard procedure: A_1 is transformed into a unique canonical form C_1. This canonical form will have the properties ordinarily ascribed to canonical forms. The definition of canonical form will be determinative; the canonical form will be unique; and the definition will be so arranged that two matrices equivalent under G have the same canonical form.

[1] The solution herewith presented was completed in 1948, but not published until now.

It seems unwise to us to apply Schur's theorem to A_1 directly. For we do not know how to find easily what unitary matrices will transform a triangle matrix into another triangle matrix with the same main diagonal. For this reason we prefer to consider $A_1 A_1^*$, a positive semi-definite hermitian matrix. By Schur's theorem, there is a unitary matrix V such that $D_1 = V(A_1 A_1^*) V^*$ is triangular. Hence D_1 is diagonal, and the diagonal elements are all real and non-negative. Let it be assumed moreover that equal elements are grouped along the main diagonal, with groups arranged in decreasing order of value. Set $F_1 = V A_1 V^*$; clearly $F_1 F_1^* = D_1$, and from now on we deal with F_1 instead of with A_1; the two are equivalent under G and will have the same canonical form. There are three cases to consider. First, if F_1 is scalar, define $C_1 = F_1 = A_1$. The centralizer of F_1 is G. Second, if $F_1 F_1^*$ is scalar but F_1 is not scalar, set $F_1 F_1^* = \operatorname{diag}(r, r, \ldots, r)$, $r > 0$. By a corollary of Schur's theorem[1], F_1 can be transformed into diagonal form; $W F_1 W^* = C_1 = \operatorname{diag}(e^{i\theta(1)}\sqrt{r}, e^{i\theta(2)}\sqrt{r}, \ldots, e^{i\theta(n)}\sqrt{r})$, $0 \leqq \theta(1) \leqq \cdots \leqq \theta(n) < 2\pi$, $W W^* = 1$. In C_1, the equal diagonal elements are grouped.

Hence the centralizer of C_1 is the subgroup of G which consists of matrices which are of the form

(2)



Here the first box has dimension $t_1 \times t_1$, where t_1 is the number of diagonal elements of C_1 which are equal to $e^{i\theta(1)}\sqrt{r}$; the second box has dimension $t_2 \times t_2$, where t_2 is the number of elements in the second group of equal elements on the diagonal of C_1, etc.

In the third case $F_1 F_1^*$ is diagonal but not scalar. Here, the definition of canonical form for F_1 presents some difficulty. The subgroup G' of G which centralizes $F_1 F_1^*$ consists of matrices like the picture (2), where the dimensions of the boxes are arranged to conform with the structure of $F_1 F_1^*$. (We have assumed that equal elements are grouped along the main diagonal of $F_1 F_1^*$.) The logical thing to do is to define a canonical form C_1 as a certain transform of F_1 not under G, but under the subgroup of G which centralizes $F_1 F_1^*$. The

[1] See lemma 3 of this article.

definition is actually carried out by induction. In substance, F_1 is first transformed so that a certain submatrix takes on specified form; at this stage it is necessary to know the subgroup of G' which leaves unaltered the specified submatrix of F_1. The definition is so arranged that this last subgroup consists of matrices of the form (2), with some modifications. The subgroups of G' which are involved are called generalized diagonal unitary groups, and are defined below.

Before proceeding to the general inductive definition, which is necessarily abstract, we prefer to study special cases. For the first case, consider the possibility that the characteristic roots of $F_1 F_1^*$ are distinct. It is not excluded that F_1 be singular, so that one of the characteristic roots of $F_1 F_1^*$ is zero. The case of distinct characteristic roots is the one which will occur "in general".

Here, $F_1 F_1^* = \operatorname{diag}(r_1, r_2, \ldots, r_n)$, $r_1 > r_2 > \cdots > r_n$. The subgroup of G which centralizes $F_1 F_1^*$ is the set of diagonal unitary matrices $\operatorname{diag}(e^{i\theta(1)}, e^{i\theta(2)}, \ldots, e^{i\theta(n)})$. Let us see how F_1 is transformed by such a matrix; for simplicity, let

$$F_1 = \begin{pmatrix} f_{11}, & f_{12}, & f_{13} \\ f_{21}, & f_{22}, & f_{23} \\ f_{31}, & f_{32}, & f_{33} \end{pmatrix}$$

Then $\operatorname{diag}(e^{i\theta}, e^{i\varphi}, e^{i\psi}) \cdot F_1 \cdot \operatorname{diag}(e^{-i\theta}, e^{-i\varphi}, e^{-i\psi})$

$$= \begin{pmatrix} f_{11} & f_{12}\, e^{i\varphi - i\theta} & f_{13}\, e^{i\psi - i\theta} \\ f_{21}\, e^{i\theta - i\varphi} & f_{22} & f_{23}\, e^{i\psi - i\varphi} \\ f_{31}\, e^{i\theta - i\psi} & f_{32}\, e^{i\varphi - i\psi} & f_{33} \end{pmatrix}$$

If each of the numbers $f_{12}, f_{13}, f_{21}, f_{31}$ is zero, then F_1 is transformed essentially by the $(n-1)\times(n-1)$ matrix $\operatorname{diag}(e^{i\varphi}, e^{i\psi})$, and the definition of canonical form may be expected to proceed by induction. If one of the elements in the first row or column is not zero, then we choose the transforming matrix so that the first of $f_{12}\, e^{i\varphi - i\theta}$, $f_{13}\, e^{i\psi - i\theta}$, $f_{21}\, e^{i\theta - i\varphi}$, $f_{31}\, e^{i\theta - i\psi}$ which is not zero shall be real and positive. To fix the ideas, let $f_{12} = 0$, $f_{13} \neq 0$. The transformed matrix K_1 is not exactly the canonical form eventually to be defined; however the $(1, 3)$th element of K_1 is positive, and the same fact is true concerning all transforms of K_1 by the group $\operatorname{diag}(e^{i\theta}, e^{i\varphi}, e^{i\theta})$. This last group involves fewer parameters than do any groups mentioned before, so that the problem of defining a canonical form has been pushed one step ahead and it may be expected again that induction would succeed. Indeed the canonical form is completely determined by the additional specification that the first of the numbers $f_{21}\, e^{i\theta - i\varphi}$, $f_{23}\, e^{i\theta - i\varphi}$, $f_{32}\, e^{i\varphi - i\theta}$

which is not zero shall be real and positive. If all three of f_{21}, f_{23}, f_{32}, are zero, then the last group leaves unaltered not only the (1, 3)th element of K_1, but also every element of K_1, so that K_1 has been reduced to canonical form in any case. Next we reduce

$$F_1 = \begin{pmatrix} \frac{1-i}{2} & \frac{-1+i}{\sqrt{3}} & \frac{-3-5i}{2\sqrt{3}} & 0 & 0 \\ \frac{3+i}{\sqrt{3}} & \frac{2}{3} & \frac{-1+i}{3} & 0 & 0 \\ \frac{1-i}{2\sqrt{3}} & \frac{-1+5i}{3} & \frac{5+3i}{6} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

to canonical form. Note that $F_1 F_1^* = \operatorname{diag}(4, 4, 4, 1, 1)$. If a hermitian matrix is known to have characteristic roots 4, 4, 4, 1, 1, it is of course possible to transform the matrix into $F_1 F_1^*$ by a unitary matrix, and it is not hard to show that a suitable transforming matrix can be found rationally in a finite number of steps. The unitary matrices which centralize $F_1 F_1^*$ can be written in the form

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} \\ 0 & 0 & 0 & x_{54} & x_{55} \end{pmatrix} = \begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix},$$

where X^{11} is the principal 3×3 minor, and X^{22} is the lower right 2×2 minor, etc. Let F_1 be divided into sub-matrices in a conformal manner:

$$F_1 = \begin{pmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{pmatrix}.$$

The transform $X F_1 X^*$ is equal to

$$\begin{pmatrix} X^{11} F^{11} X^{11*} & F^{12} \\ F^{21} & X^{22} F^{22} X^{22*} \end{pmatrix}$$

where $F^{21} = 0$, $F^{12} = 0$. If we wish to reduce F_1 to canonical form by transforming by a matrix from the centralizer of $F_1 F_1^*$, it is necessary to use induction, that is, to assume that the equivalence problem has been solved in the 2×2 and also in the 3×3 case. Let us recall briefly what the solution was, as regards $X^{11} F^{11} F^{11*}$. Since $F^{11} F^{11*}$ is scalar, but F^{11} is not scalar, we have to deal with case 2; F^{11} can be transformed into diagonal form. In order actually to carry out this transformation, it is only necessary to know the characteristic roots of F^{11}. They are $2, 2i, -2i$, so that the canonical form for F^{11} is $\operatorname{diag}(2, 2i, -2i)$ and the canonical form for F_1 is $\operatorname{diag}(2, 2i, -2i, 1, -1)$.

The example

$$F_1 = \begin{Vmatrix} \frac{1-i}{2} & \frac{-1+i}{\sqrt{3}} & 0 & \frac{-3-5i}{2\sqrt{3}} & 0 \\ \frac{3+i}{\sqrt{3}} & \frac{2}{3} & 0 & \frac{-1+i}{3} & 0 \\ \frac{1-i}{2\sqrt{3}} & \frac{-1+5i}{3} & 0 & \frac{5+3i}{6} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{Vmatrix}$$

presents a more general problem than the preceding examples.

We have $F_1 F_1^* = \operatorname{diag}(4, 4, 4, 1, 1)$, so that the centralizer of $F_1 F_1^*$ is the same as in the preceding example. The matrices $F^{11}, F^{12}, F^{21}, F^{22}$ which have to be considered are different in this case, so that it is no longer true that $F^{11} F^{11*}$ is scalar. It is necessary to find a matrix X^{11} so that $X^{11} F^{11} F^{11*} X^{11*}$ is diagonal, and then to transform $X^{11} F^{11} X^{11*}$ by a group with fewer parameters, namely the centralizer of $X^{11} F^{11} F^{11*} X^{11*}$, etc. If the characteristic roots of $F^{11} F^{11*}$ are determined, all these steps can be performed in a finite number of steps and rationally.

The canonical form of an arbitrary matrix is defined inductively according to a plan suggested by the preceding example. In order to have a successful inductive definition, it is necessary to consider not only matrix equations such as $X^{11} F^{11} X^{11*} = L^{11}$, which occur in the main stream of the induction, but also equations such as $X^{11} F^{12} X^{22*} = L^{12}$. The occurrence of this last equation is illustrated in the first example, where however all the matrices have dimension 1: $e^{i\theta} f_{12} e^{-i\varphi} = l_{12}$. As each partial reduction is carried out, it is necessary to give a simple rule for determining what is the group which leaves the transformed matrix unaltered in the spots which have been picked out for improvement.

The definition as given below is finite, rational, and constructive at all steps, except that it is necessary to find the characteristic roots of a matrix F_1, or of a submatrix F^{11}, or of an auxiliary matrix $F^{12} F^{12*}$ *of lower dimension* than the matrix F_1. In many accepted solutions of equivalence problems, it is necessary to find the characteristic roots of a matrix. I do not know whether the present problem could be solved without the resolution of algebraic equations of degree n, where $n = \dim F_1$. On the other hand, a direct attack on the equation $U F_1 U = L_1$ leads to equations of degree much greater than n.

(3) Let $X = \begin{pmatrix} X^{11} & 0 \\ 0 & X^{22} \end{pmatrix}$ be a unitary matrix, and let F_1 be such that $F_1 F_1^*$ is the direct product $\begin{pmatrix} \boldsymbol{r} & 0 \\ 0 & \boldsymbol{s} \end{pmatrix}$ of two scalar matrices, the dimensions of which are $\dim X^{11}$ and $\dim X^{22}$ respectively. The unitary matrices which fix $F_1 F_1^*$ are those which have the form of X. Let $F_1 = \begin{pmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{pmatrix}$ be the division of F_1 into submatrices conformal with the above division of X into submatrices. Then

(4) $$X F_1 X^* = \begin{pmatrix} X^{11} F^{11} X^{11*} & X^{11} F^{12} X^{22*} \\ X^{22} F^{21} X^{22*} & X^{22} F^{22} X^{22*} \end{pmatrix}$$

This last equation makes it clear that in order to make a definition of canonical form by induction, it will be necessary to consider the matrix equation $P A Q = B$, where A, B are rectangular, and P, Q are unitary. The necessary facts are stated in the following lemma.

Lemma 1. Let $A_{1\beta}$ be a $k(1) \times k(\beta)$ matrix. There are two unitary matrices $U^{k(1)}$ and $U^{k(\beta)}$ of dimensions $k(1) \times k(1)$ and $k(\beta) \times k(\beta)$ respectively such that $U^{k(1)} A_{1\beta} U^{k(\beta)}$ has the form

(5) $\operatorname{diag}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{m-1})$, bordered by zeros, where $r_1, r_2, \ldots, r_{m-1}$ are real and positive: $r_1 > r_2 > \cdots > r_{m-1} > 0$. [The border of zeros may consist of any number of rows, any number of columns, and in particular the border may be absent.]

Let us return now to equation (4). If by chance F^{11} and F^{22} were themselves both scalar, equation (4) would read

(6) $$X F_1 X^* = \begin{pmatrix} F^{11}, & X^{11} F^{12} X^{22*} \\ X^{22} F^{21} X^{22*}, & F^{22} \end{pmatrix}$$

so that we should seek a canonical form for $X^{11} F^{12} X^{22*}$ (lemma 1). Next it is necessary to know which matrices of the form (5) leave the matrix (6) unaltered in the upper right-hand box. Lemma 2 describes these matrices.

Lemma 2. Let $A_{1\beta}$ have the form (5). Then the equation $A_{1\beta} = U^{k(1)} A_{1\beta} U^{k(\beta)}$ holds if and only if $U^{k(1)}$ has the form diag $(X^{K(1)}, X^{K(2)}, \ldots, X^{K(m-1)}, X^{K(m)})$, and at the same time $U^{k(\beta)}$ is equal to

$$\text{diag}\,(X^{K(1)}, X^{K(2)}, \ldots, X^{K(m-1)}, X^{K(m+1)}).$$

This means that $U^{k(1)}$ and $U^{k(\beta)}$ are equal element for element in the first $K(1) + K(2) + \cdots + K(m-1)$ rows and columns; the superscript $K(j)$ denotes a matrix of dimension dim $\boldsymbol{r}_j$. The symbol $X^{K(m)}$ is to be expunged above if no row of (5) is zeros, and the symbol $X^{K(m+1)}$ is to be expunged if no column of (5) is zeros. We prove now

Lemma 3. Let A_1 be an $n \times n$ matrix such that $A_1 A_1^*$ is scalar. Then A_1 can be transformed into diagonal form by some unitary matrix. In particular, then, any unitary matrix can be transformed into diagonal form by some unitary matrix.

Proof: By Schur's theorem, A_1 can be transformed into triangular form B_1 by a unitary matrix, and thus B_1 is a triangular matrix such that $B_1 B_1^*$ is scalar. If $B_1 B_1^* = 0$, it is clear that $B_1 = 0$. If on the other hand $B_1 B_1^* = \boldsymbol{r}$, where $r \neq 0$, then r is real and positive, and $\frac{1}{\sqrt{r}} B_1$ is both unitary and triangular, hence diagonal.

We give now a general definition of the groups which appear at successive stages in the inductive definition of the canonical form of A_1. For each value of k, v satisfying the inequalities below, let $X^{k,v} = (x_{ij}^{k,v})$ be a $k \times k$ matrix of indeterminates,

$$1 \leqq k \leqq n, \quad 1 \leqq v \leqq [n/k]; \text{ let } 1 \leqq k(t) \leqq n,$$
$$1 \leqq v(t) \leqq [n/k(t)]\,(t = 1, \ldots, s); \text{ let } \Sigma_{t=1}^{s} k(t) = n.$$

The symbol

$$X = \text{diag}\,(X^{k(1), v(1)}, X^{k(2), v(2)}, \ldots, X^{k(s), v(s)})$$

is called an $n\,x\,n$ matrix formula. In the symbol $X^{k(t), v(t)}$, the first superscript $k(t)$ is equal to the dimension of the matrix $X^{k(t), v(t)}$, and the second superscript $v(t)$ is a counting superscript; if $k(t) = k(u)$, $v(t) = v(u)$, then $X^{k(t), v(t)}$ and $X^{k(u), v(u)}$ are necessarily equal.

Examples are the following:

$$\operatorname{diag}(X^{1,1},\ X^{1,2},\ X^{1,3}) \equiv \begin{pmatrix} x^{1,1} & 0 & 0 \\ 0 & x^{1,2} & 0 \\ 0 & 0 & x^{1,3} \end{pmatrix};$$

$$\operatorname{diag}(X^{1,1},\ X^{1,2},\ X^{1,1}) \equiv \begin{pmatrix} x^{1,1} & 0 & 0 \\ 0 & x^{1,2} & 0 \\ 0 & 0 & x^{1,1} \end{pmatrix};$$

$$\operatorname{diag}(X^{2,1},\ X^{2,2}) \equiv \begin{pmatrix} x^{21}_{11} & x^{21}_{12} & 0 & 0 \\ x^{21}_{21} & x^{21}_{22} & 0 & 0 \\ 0 & 0 & x^{22}_{11} & x^{22}_{12} \\ 0 & 0 & x^{22}_{21} & x^{22}_{22} \end{pmatrix}$$

which differs only in notation from

$$\begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & x_{43} & x_{44} \end{pmatrix};$$

and $\operatorname{diag}(X^{2,1},\ X^{2,1})$, which differs only in notation from the formula

$$\begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & x_{11} & x_{12} \\ 0 & 0 & x_{21} & x_{22}. \end{pmatrix}$$

If the indeterminates in the symbol X be replaced by complex numbers so chosen that X becomes a unitary matrix after the substitution, each submatrix $X^{k(j),\,v(j)}$ becomes a unitary matrix also, and conversely if each submatrix $X^{k(j),\,v(j)}$ is unitary, then so is X. The notation $k(j)$ is more easily printed than is k_j; otherwise we would use the latter in place of the former.

The class of $n \times n$ unitary matrices which can be obtained by substitution in a single matrix formula fill out a group G; G is called a *generalized diagonal unitary* group and is generated by the formula.

To give another example, a formula for the group of unitary matrices in the form

$$\begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ 0 & x_2 & x_3 & 0 & 0 \\ 0 & x_4 & x_5 & 0 & 0 \\ 0 & 0 & 0 & x_6 & 0 \\ 0 & 0 & 0 & 0 & x_1 \end{pmatrix}$$

would be diag $(X^{11}, X^{21}, X^{12}, X^{11})$.

It is true that for every g.d. group, there is a matrix which is commutative whith the matrices of this group and with no others. Moreover if the centralizer of a matrix is a gen. diag. unitary group, it is easy to tell (by inspection) what this group is, and to write down its generating formula.

Rather than find a canonical form for matrices under transformation only by the full unitary group, we prefer to find a canonical form under transformation by any gen. diag. unitary group. The canonical form will have the property that its centralizer is some gen. diag. unitary group. In view of the large number of such groups, it is to be expected that there will be a large number of canonical forms; too many to be described in any such pictorial language, as Jordan's canonical form, for example. On the other hand the description of canonical form, while not pictorial, is straightforward and exact, and the method for obtaining it is perfectly workable in any practical case.

The advantage of describing a canonical form for a matrix under *any* gen. diag. group is this, that an ordered pair of matrices A_1, A_2 can be reduced simultaneously to a canonical pair C_1, C_2. This is done by first transforming A_1, A_2 to the pair (C_1, D_2), and then transforming D_2 to canonical form by matrices of the gen. diag. group which centralizes C_1.

In order to make a definition of canonical form for A_1 under group G, we shall assume, as a hypothesis for the inductive definition, that a canonical form has been defined for all matrices A under all gen. diag. groups G_0 whenever either

$$(\delta)\ \dim A_0 = \dim G_0 < n, \text{ or}$$

$$(\varepsilon)\ \dim A_0 = \dim G_0 = n,\ G_0 \subset G; \text{ and}$$

that moreover, the centralizer of the canonical form for A_0 is a gen. diag. group (of dimension dim A_0). We shall give a definition of a canonical form for A_1 under G by carrying out a simple construction and then calling on the induction hypothesis. We shall prove at the same time that the centralizer of this canonical form is a gen. diag. unitary group. As we have already remarked, it is then easy to describe this centralizer and find the formula which generates it.

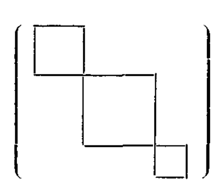
Many definitions not ordinarily so described in the literature are in fact inductive definitions in the sense of the preceding paragraph. The Peano definition of number, the Jordan canonical form for a matrix, and almost all finite and transfinite constructions are really of the above type. The definition is like a recursion formula, in that both allow actual computations to be carried through, in a finite, possibly very large number of steps. Definitions which are not of this type are those based on existence theorems of one sort or another.

The form of the definition to be given is different, depending on the form of the gen. diag. group G; there are three cases. Under each case are subcases, depending on the character of the matrix A_1. It should be observed that the inductive method of definition is effective not only for the reasons given above, but also because when $\dim G = 1$, the canonical form of A_1 is A_1, and the centralizer of this canonical form is G, so that the inductive definition may begin.

Case I. The case when the transforming group is the entire unitary group has already been discussed. There are three subcases: I a, A_1 is scalar; I b, $A_1 A_1^*$ is scalar but A_1 is not scalar; I c, $A A_1^*$ is not scalar. In case I a, A_1 is commutative with every element of G, and so is its own canonical form. The centralizer of A_1 is a gen. diag. unitary group, namely G. In case I b, the canonical form for A_1 is diagonal:

$$C_1 = \operatorname{diag}\,(e^{i\theta(1)}\sqrt{r},\ e^{i\theta(2)}\sqrt{r},\ \ldots,\ e^{i\theta(n)}\sqrt{r}),\quad 0 \leqq \theta(1) \leqq \cdots \leqq \theta(n) < 2\pi.$$

The diagonal of C_1 has $k(1)$ equal elements, followed by $k(2)$ elements equal to each other but different from any of the preceding ones, etc. The centralizer of C_1 is the group consisting of unitary matrices of the form



where the first box is $k(1) \times k(1)$, the second box is $k(2) \times k(2)$, etc. This group is generated by the matrix formula $\operatorname{diag}(X^{k(1),1}, X^{k(2),2}, \ldots)$ and is a gen. diag. unitary group.

It is in case I c that the induction hypotheses (δ), (ε) are used. There is a matrix W such that $W(A_1 A_1^*) W^*$ is diagonal, indeed with diagonal elements real and positive and written in decreasing order of magnitude; say $k(1)$ equal elements, followed by $k(2)$ equal elements, each less than the preceding, etc. Set $F_1 = W A_1 W^*$. Then $F_1 F_1^* = W A_1 A_1^* W^*$. The matrices which transform $F_1 F_1^*$ into itself are precisely the matrices of the gen. diag. group diag $(X^{k(1),1}, X^{k(2),2}, \ldots)$. Moreover, if K_1 is a transform of F_1 by any matrix of this group, then $K_1 K_1^* = F_1 F_1^*$. The canonical form for A_1 shall be the (already defined) canonical form for F_1 under this last group. Actually W is an arbitrary representative of a certain coset of this group; any other representative of the same coset could take the place of W in the discussion.

Case II. We must now define a canonical form for a matrix under a gen. diag. group which is a proper subgroup of the full unitary group. We prefer to do this in stages; first, suppose the transforming group is $G_2 \equiv \{\text{diag}\, X^{k(1),1}, X^{k(1),1}, \ldots, X^{k(1),1}\}$, that is, suppose each matrix of the group has the same $k(1) \times k(1)$ matrix $X^{k(1),1}$ repeated along the main diagonal a certain number of times. Let A_{ij} be a conformal subdivision of A_1 into $k(1) \times k(1)$ submatrices $(i, j = 1, 2, \ldots, n/k(1))$ There are three subcases. First, if each of the submatrices A_{i1}, A_{1i} is scalar, then the transforming group operates essentially on the minor $(A_{ij})\, i, j > 1$ of A_1, for which a canonical form has been defined, according to the induction hypothesis. By the same token, the centralizer of this canonical form is a gen. diag. unitary group.

Second, let each submatrix A_{i1} be scalar; let not all A_{1i} be scalar; let A_{1j} be the first of the A_{1i} $(i = 2, \ldots)$ which is not scalar. The transform of A_1 by the arbitrary matrix diag $(X^{k(1)}, X^{k(1)}, \ldots, X^{k(1)})$ of the transforming group replaces the submatrix A_{1j} of A_1 by the submatrix $X^{k(1)} A_{ij} X^{k(1)*}$. Since A_{1j} has dimension lower than that of A_1, a canonical matrix has been defined among all matrices of the set $X^{k(1)} A_{ij} X^{k(1)*}$, and we choose for $X^{k(1)}$ some matrix $U^{k(1)}$ for which $U^{k(1)} A_{ij} U^{k(1)*}$ is in the (uniquely defined) canonical form.

Again all the matrices which transform A_{ij} as $U^{k(1)}$ does belong to some coset of the centralizer of $U^{k(1)} A_{ij} U^{k(1)*}$. According to the induction hypothesis, this centralizer is a gen. diag. unitary group. Let the general element of this group be Y. Let G_3 be the group generated by the formula which arises when each symbol $X^{k(1),1}$ in the generating formula for G_2 is replaced by Y: $G_3 = \{\text{diag}\,(Y, Y, \ldots, Y)\}$. Clearly G_3 is a gen. diag. group properly contained in G_2. The canonical form for A_1 is defined as the (already defined) canonical form for diag $(U^{k(1)}, U^{k(1)}, \ldots, U^{k(1)})\, A_1$ diag $(U^{k(1)*}, U^{k(1)*}, \ldots, U^{k(1)*})$ under G_3.

In the third subcase, not all the A_{i1} are scalar, so that some transform of A_1 under G_2 does differ from A_1 in one of the first $k(1)$ columns.

Here the reasoning proceeds as in the second subcase; we choose j so that A_{j1} is the first of the A_{i1} which is not scalar, etc.

Case III. In this case we admit for a transforming group a more general type of proper subgroup of the full unitary group, namely the group $G_2 = \text{diag}\,(X^{k(1), v(1)}, X^{k(2), v(2)}, \ldots, X^{k(s), v(s)})$, and in order that cases II and III be mutually exclusive, we insist that at least two of the X's in the above formula be different: for at least one pair of numbers u, w, $(k(u), v(u)) \neq (k(w), v(w))$. The construction is similar to that in case II. As in that case, so here we have to notice that $X A_{1i} X^* = A_{1i}$ for all x only if A_{1i} is scalar; we must further notice that $X A_{1j} Y^* = A_{1j}$ [or $Y A_{j1} X^* = A_{j1}$] for all X, Y only if $A_{1j} = \mathbf{0}$ [$A_{j1} = \mathbf{0}$]. Then we depend on lemmas 1 and 2 instead of on the induction hypothesis (as in case II). In the end the transforming group is a gen. diag. unitary group because of lemma 2.

The inductive definition is now complete.

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