AN EXTENSION OF THE SLUTZKY-FRÉCHET THEOREM.

By

J. M. HAMMERSLEY.

Lectureship in the Design and Analysis of Scientific Experiment, University of Oxford.

§ 0. Notation and conventions.

In this paper German capital letters denote Euclidean vector spaces of finite dimensionality. Small German letters denote point sets in these spaces; and $\mathfrak{x} - \mathfrak{x}'$ denotes the (perhaps empty) set of all points which belong to $\mathfrak{x}$ and not to $\mathfrak{x}'$. Script letters denote classes of point sets. Clarendon type denotes points (or vectors) of a Euclidean space. Ordinary italic type is reserved for scalar quantities. The symbol $\Rightarrow$ denotes implication, the arrow pointing from the premiss to the conclusion; and the double-headed arrow $\Leftrightarrow$ means 'implies and is implied by'. Two statements I and II, which together imply a third III, are linked by an ampersand: — 'I & II $\Rightarrow$ III'.

§ 1. Introduction.

Let $y(x)$ be a continuous one-valued function of x, and consider the equations

$$\lim_{\nu \to \infty} x_\nu = x, \tag{1.1}$$

$$\lim_{\nu \to \infty} (x_\nu - x) = 0, \tag{1.2}$$

$$\lim_{\nu \to \infty} \{y(x_\nu) - y(x)\} = 0, \tag{1.3}$$

$$\lim_{\nu \to \infty} y(x_\nu) = y(x). \tag{1.4}$$

When x and x_ν are real variables, it is familiar that

$$(1.1) \Leftrightarrow (1.2) \Rightarrow (1.3) \Leftrightarrow (1.4). \tag{1.5}$$

For random variables, the position is different. Slutzky (**4**) proved

$$(1.2) \Rightarrow (1.3) \tag{1.6}$$

when x_ν is a random variable and x a real variable; while Fréchet (**1**) proved (1.6) in case x_ν and x were both random variables. It is an immediate consequence of the definition of 'lim' for random variables that

$$(1.1) \Leftarrow (1.2) \quad \text{and} \quad (1.3) \Rightarrow (1.4) \tag{1.7}$$

but the converse statements

$$(1.1) \Rightarrow (1.2) \quad \text{and} \quad (1.3) \Leftarrow (1.4) \tag{1.8}$$

are generally false. It is, however, easy to find special cases in which (1.8) is true for certain specific random variables; and then the question naturally arises whether, given *any* random variables satisfying (1.1), we can *always* find at least one special case such that (1.2) is also true. In Theorem 1 I shall give an affirmative answer to this question: so that, combining Theorem 1 with the Slutzky-Fréchet theorem (1.6) and with the second part of (1.7), we shall have established

$$(1.1) \Rightarrow (1.4) \tag{1.9}$$

for random variables. However (1.9) is insufficient for certain practical applications; and I shall prove a generalisation of it in Theorem 3: namely, that (1.6) and (1.9) remain true for almost-certainly-continuous many-valued vector functions of a vector variable.

A number of authors have discussed, in a few special cases, the distribution of the zeros of a random polynomial. I hope to show elsewhere how the extended form of (1.9) provides a general solution to this problem.

§ 2. One-valued random variables and their limits.

Let $\mathfrak{X}$ denote an n-dimensional Euclidean space. A *probability set function* $F[\mathfrak{x}]$ is any one-valued real non-negative completely-additive set function defined for all Borel sets $\mathfrak{x}$ of $\mathfrak{X}$ and satisfying $F[\mathfrak{X}] = 1$. If $\mathfrak{x}$ is the particular set of all points, whose coordinates do not exceed the corresponding coordinates of a given point $\mathbf{x}$ of $\mathfrak{X}$, we write $F[\mathfrak{x}] = F(\mathbf{x})$ and call $F(\mathbf{x})$ a *cumulative distribution function.* Obviously $F[\mathfrak{x}]$ uniquely determines $F(\mathbf{x})$, and the converse is a consequence of Lebesgue's theory of integration. A cumulative distribution function is monotone increasing and everywhere continuous on the right. For the purposes of axiomatic theory it is permissible to identify a *one-valued random variable* $\mathbf{x}^*$ with a probability set function. Asterisks will hereinafter denote random variables. If the functional form of F, either as a probability set function or as a cumulative distribution function, is supposed given we say that F *determines* the random variable $\mathbf{x}^*$ identified with it.

This corresponds to saying that a real variable $\mathbf{x}$ is determined when the numerical values of its coordinates are supposed given. A *random constant* $\mathbf{a}^*$ is the random variable identified with that probability set function $F[\mathfrak{x}]$ which equals 1 or 0 according as the fixed point $\mathbf{a}$ belongs to $\mathfrak{x}$ or not.

Let $\mathfrak{X}_i$, $i = 1, 2, \ldots, m$, be an n_i-dimensional Euclidean space in which $\mathfrak{x}_i$ is a typical Borel set. Let $\mathbf{x}_i^*$ be a one-valued random variable in $\mathfrak{X}_i$ determined by $F_i[\mathfrak{x}_i]$. In the direct product space $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_m$ any probability set function G is called a *joint determination* of $\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_m^*$ if it satisfies

$$G[\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_{i-1} \times \mathfrak{x}_i \times \mathfrak{X}_{i+1} \times \cdots \times \mathfrak{X}_m] = F_i[\mathfrak{x}_i] \tag{2.1}$$

for all values of i and all Borel sets $\mathfrak{x}_i$ of $\mathfrak{X}_i$. The random variable identified with G is written $\mathbf{x}_1^* \times \mathbf{x}_2^* \times \cdots \times \mathbf{x}_m^*$. We say that the $\mathbf{x}_i^*$ are *independently distributed* if a stronger form of (2.1) holds, namely

$$G[\mathfrak{x}_1 \times \mathfrak{x}_2 \times \cdots \times \mathfrak{x}_m] = F_1[\mathfrak{x}_1] F_2[\mathfrak{x}_2] \ldots F_m[\mathfrak{x}_m] \tag{2.2}$$

for all Borel sets $\mathfrak{x}_i \subseteq \mathfrak{X}_i$.

Let G in (2.1) be a joint determination of $\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_m^*$. Let $\mathbf{y} = \mathbf{y}(\mathbf{x}_1, \mathbf{x}_2 \ldots, \mathbf{x}_m)$ be a one-valued Borel-measurable mapping of $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_m$ into a Euclidean space $\mathfrak{Y}$. Let $\mathfrak{y}$ be a Borel set of $\mathfrak{Y}$, and let $\mathfrak{x}(\mathfrak{y})$ be the set of all points $(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_m)$ in $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_m$ for which $\mathbf{y}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) \in \mathfrak{y}$. Since $\mathbf{y}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ is a Borel-measurable function, $\mathfrak{x}(\mathfrak{y})$ is a Borel set. The *function of several jointly determined random variables*

$$\mathbf{y}^* = \mathbf{y}(\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_m^*) = \mathbf{y}(\mathbf{x}_1^* \times \mathbf{x}_2^* \times \cdots \times \mathbf{x}_m^*)$$

is defined to be the random variable identified with

$$H[\mathfrak{y}] = G[\mathfrak{x}(\mathfrak{y})],$$

it being easy to verify that $H[\mathfrak{y}]$ so defined is a probability set function. Indeed this is a particular case of some more general theory discussed by Hammersley (**3**).

In a Euclidean space $\mathfrak{X}$, let $\mathbf{x}^*$ be a random variable determined by the cumulative distribution function $F(\mathbf{x})$ and let $\mathbf{x}_\nu^*$, $\nu = 1, 2, \ldots$, be a sequence of random variables respectively determined by $F_\nu(\mathbf{x})$. If, as $\nu \to \infty$, $F_\nu(\mathbf{x})$ tends to $F(\mathbf{x})$ at every point of continuity of $F(\mathbf{x})$, we say that $\mathbf{x}_\nu^*$ *converges in distribution* to $\mathbf{x}^*$, and we write

$$\operatorname*{dlim}_{\nu \to \infty} \mathbf{x}_\nu^* = \mathbf{x}^*. \tag{2.3}$$

Let $\mathbf{a}$ be a constant vector, and let $\mathfrak{x}(\mathbf{a})$ denote the set of points $\mathbf{x}$ satisfying

$|\mathbf{x} - \mathbf{a}| < \delta$, where $\delta > 0$ is any prescribed positive number. If to every prescribed pair of positive numbers $\delta > 0$ and $\varepsilon > 0$ we can find a positive integer $\nu_0 = \nu_0(\delta, \varepsilon)$ such that the probability set functions $F_\nu[\mathfrak{x}]$ of $\mathbf{x}_\nu^*$ satisfy

$$F_\nu[\mathfrak{x}(\mathbf{a})] > 1 - \varepsilon, \qquad \nu \geq \nu_0(\delta, \varepsilon)$$

we say that $\mathbf{x}_\nu^*$ *converges in probability* to $\mathbf{a}$, and we write

$$\operatorname*{plim}_{\nu\to\infty} \mathbf{x}_\nu^* = \mathbf{a}. \tag{2.4}$$

It is not difficult to see that

$$\operatorname*{plim}_{\nu\to\infty} \mathbf{x}_\nu^* = \mathbf{a} \Leftrightarrow \operatorname{dlim} \mathbf{x}_\nu^* = \mathbf{a}^*. \tag{2.5}$$

If, for each value of ν, $\mathbf{x}_\nu^*$ and $\mathbf{x}^*$ are jointly determined by some given G_ν, and if the function $\mathbf{x}_\nu^* - \mathbf{x}^*$ of such a pair of jointly determined random variables converges in probability to the zero vector as $\nu \to \infty$, we say that $\mathbf{x}_\nu^*$ converges in probability to $\mathbf{x}^*$, and write

$$\operatorname*{plim}_{\nu\to\infty} \mathbf{x}_\nu^* = \mathbf{x}^*. \tag{2.6}$$

Thus

$$\operatorname*{plim}_{\nu\to\infty} (\mathbf{x}_\nu^* - \mathbf{x}^*) = \mathbf{0} \Leftrightarrow \operatorname*{plim}_{\nu\to\infty} \mathbf{x}_\nu^* = \mathbf{x}^* \Leftrightarrow \operatorname*{dlim}_{\nu\to\infty} (\mathbf{x}_\nu^* - \mathbf{x}^*) = \mathbf{0}^* \tag{2.7}$$

when $\mathbf{x}_\nu^*$ and $\mathbf{x}^*$ are jointly determined; and it is quite simple to show that $(2.7) \Rightarrow (2.3)$. This is a fuller explanation of the first part of (1.7). On the other hand, the truth of '$(2.3) \Rightarrow (2.7)$' depends upon the form of the joint determination of $\mathbf{x}^*$ and $\mathbf{x}_\nu^*$. We shall now prove in Theorem 1 that, amongst the class of all joint determinations of any given pair of individually determined random variables $\mathbf{x}^*$ and $\mathbf{x}_\nu^*$, there is always at least one joint determination such that $(2.3) \Rightarrow (2.7)$.

Theorem 1. *If* $\mathbf{x}^*$ *is a given one-valued random variable, and if* $\mathbf{x}_\nu^*$, $\nu = 1, 2, \ldots$, *is a sequence of given one-valued random variables satisfying*

$$\operatorname*{dlim}_{\nu\to\infty} \mathbf{x}_\nu^* = \mathbf{x}^*, \tag{2.8}$$

then, for each value of ν, *there exists a joint determination of* $\mathbf{x}^*$ *and* $\mathbf{x}_\nu^*$ *such that*

$$\operatorname*{dlim}_{\nu\to\infty} (\mathbf{x}_\nu^* - \mathbf{x}^*) = \mathbf{0}^*. \tag{2.9}$$

Take $\mathfrak{X}$ to be the Euclidean space in which $\mathbf{x}^*$ is defined; and write $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ for a typical point of $\mathfrak{X}$, and $\mathfrak{x}$ for a typical Borel set of $\mathfrak{X}$. Suppose that $F[\mathfrak{x}]$ and $F_\nu[\mathfrak{x}]$ are the given probability set functions which determine

$\mathbf{x}^*$ and $\mathbf{x}_\nu^*$ respectively, and that $F(\mathbf{x})$ and $F_\nu(\mathbf{x})$ are the corresponding cumulative distribution functions. Let $\delta > 0$ and $\varepsilon > 0$ be any pair of prescribed positive numbers.

We can find a finite number $U = U(\varepsilon) > 0$ such that

(i) $F(\mathbf{x})$ is continuous on each of the hyperplanes $\mathfrak{h}_{ij}$, $(i = 1, 2, \ldots, n;\ j = 1, 2)$, where $\mathfrak{h}_{i1}$ is the hyperplane $x_i = +U$ and $\mathfrak{h}_{i2}$ is the hyperplane $x_i = -U$; and

(ii) $F[\mathfrak{x}_0] < \frac{1}{2}\varepsilon$, where $\mathfrak{x}_0$ is the set of all points which violate at least one of the n inequalities $-U < x_i \leq +U$, $i = 1, 2, \ldots, n$.

We can now find a finite sequence of numbers u_k, $k = 1, 2, \ldots m$, where $m = m(\delta, \varepsilon)$, such that

(iii) $-U = u_1 < u_2 < \cdots < u_m = +U$; and

(iv) $u_{k+1} - u_k < \delta / \sqrt{n}$, $k = 1, 2, \ldots, m-1$; and

(v) $F(\mathbf{x})$ is continuous on the hyperplanes $\mathfrak{h}^{ik}$, $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m)$, where $\mathfrak{h}^{ik}$ is the hyperplane $x_i = u_k$.

Write $M = M(\delta, \varepsilon) = (m-1)^n$; and let $\mathfrak{x}_p$, $p = 1, 2, \ldots, M$, denote the half-open finite intervals in $\mathfrak{X}$

$$u_{k(i)} < x_i \leq u_{k(i)+1}, \quad i = 1, 2, \ldots, n \tag{2.10}$$

enumerated in some specific order, where $k(i)$ denotes an integer (depending upon i) selected from the integers $1, 2, \ldots, m-1$. Consider the non-negative numbers

$$a_p = F[\mathfrak{x}_p], \quad b_p = F_\nu[\mathfrak{x}_p], \quad p = 0, 1, 2, \ldots, M, \tag{2.11}$$

where b_p is a function of ν. Since $\mathfrak{x}_0, \mathfrak{x}_1, \ldots, \mathfrak{x}_M$ are mutually disjoint and cover $\mathfrak{X}$ completely

$$\sum_p a_p = \sum_p b_p = 1. \tag{2.12}$$

Let δ_{pq} denote the Kronecker delta ($\delta_{pq} = 1$ or 0 according as $p = q$ or $p \neq q$); and let $\Lambda(\theta) = \theta$ if $\theta \neq 0$ while $\Lambda(0) = 1$. Define for $p, q = 0, 1, 2, \ldots, M$

$$c_{pq} = \frac{(a_p + b_p - |a_p - b_p|)\,\delta_{pq}}{2} + \frac{(|a_p - b_p| + a_p - b_p)(|a_q - b_q| - a_q + b_q)}{2\,\Lambda(\sum_p |a_p - b_p|)}. \tag{2.13}$$

In view of (2.12) and $a_p \geq 0$, $b_p \geq 0$, we find without difficulty

$$c_{pq} \geq 0, \quad \sum_p c_{pq} = b_q, \quad \sum_q c_{pq} = a_p, \quad \sum_p c_{pp} = 1 - \tfrac{1}{2} \sum_p |a_p - b_p|. \tag{2.14}$$

Let $\mathfrak{Z}$ denote the $2n$-dimensional space $\mathfrak{X} \times \mathfrak{X}$, and let $\mathfrak{z}'$ denote any Borel set of $\mathfrak{Z}$ which can be expressed in the form

$$\mathfrak{z}' = \mathfrak{x}' \times \mathfrak{x}'', \quad \mathfrak{x}' \subseteq \mathfrak{x}_p \text{ and } \mathfrak{x}'' \subseteq \mathfrak{x}_q \text{ for some } p, q, \tag{2.15}$$

$\mathfrak{x}'$ and $\mathfrak{x}''$ being Borel sets. Define G', depending upon ν, by

$$G'[\mathfrak{z}'] = F[\mathfrak{x}']F_\nu[\mathfrak{x}'']c_{pq}/\Lambda(a_p b_q), \tag{2.16}$$

where the values of p and q are those appearing in (2.15). It is easy to see that $G'[\mathfrak{z}]$ is a non-negative completely-additive set function for all sets $\mathfrak{z}'$ satisfying (2.15) for any fixed pair p, q. Now the intervals $\mathfrak{x}_p \times \mathfrak{x}_q$ are mutually disjoint and cover $\mathfrak{Z}$ completely; and any Borel set of $\mathfrak{Z}$ can be built up from an enumerable number of sets of the form $\mathfrak{x}' \times \mathfrak{x}''$. Therefore we may uniquely define $G[\mathfrak{z}]$ as that non-negative completely-additive set function of Borel sets $\mathfrak{z} \subseteq \mathfrak{Z}$ such that $G[\mathfrak{z}'] = G'[\mathfrak{z}']$ for all sets of the type $\mathfrak{z}'$. Let $\mathfrak{x}$ be any Borel set of $\mathfrak{X}$. There is a unique decomposition

$$\mathfrak{x} = \sum_p \mathfrak{x}^p, \quad \mathfrak{x}^p \subseteq \mathfrak{x}_p,$$

namely $\mathfrak{x}^p = \mathfrak{x} \cdot \mathfrak{x}_p$. Now

$$a_p = 0 \Rightarrow F[\mathfrak{x}^p] = 0; \quad b_q = 0 \Rightarrow c_{pq} = 0;$$

and so (2.14) establishes

$$\begin{aligned} G[\mathfrak{x} \times \mathfrak{X}] &= \sum_{pq} F[\mathfrak{x}^p]F_\nu[\mathfrak{x}_q]c_{pq}/\Lambda(a_p b_q) \\ &= \sum_p \frac{F[\mathfrak{x}^p]}{\Lambda(a_p)} \sum_q \frac{c_{pq} b_q}{\Lambda(b_q)} = \sum_p \frac{F[\mathfrak{x}^p]a_p}{\Lambda(a_p)} = \sum_p F[\mathfrak{x}^p] = F[\mathfrak{x}]. \end{aligned} \tag{2.17}$$

Similarly

$$G[\mathfrak{X} \times \mathfrak{x}] = F_\nu[\mathfrak{x}]. \tag{2.18}$$

Hence

$$G[\mathfrak{Z}] = G[\mathfrak{X} \times \mathfrak{X}] = F[\mathfrak{X}] = 1;$$

so that G is a probability set function. Whereupon (2.17) and (2.18) show that G jointly determines $\mathbf{x}^*$ and $\mathbf{x}_\nu^*$.

Now write $\mathbf{z} = \{z_1, z_2, \ldots, z_{2n}\}$ for a typical point of $\mathfrak{Z}$, and let $\mathfrak{z}_0$ be the set of all points $\mathbf{z}$ which satisfy all the inequalities

$$|z_i - z_{n+i}| < \delta/\sqrt{n}, \quad i = 1, 2, \ldots, n.$$

From (iv) and (2.10)

$$\mathfrak{z}_0 \supseteq \sum_{p=1}^{M} \mathfrak{x}_p \times \mathfrak{x}_p;$$

and therefore by (2.14)

$$\begin{aligned} G[\mathfrak{z}_0] &\geq G[\sum_{p=1}^{M} \mathfrak{x}_p \times \mathfrak{x}_p] = \sum_{p=1}^{M} G[\mathfrak{x}_p \times \mathfrak{x}_p] = \sum_{p=1}^{M} c_{pp} = 1 - c_{00} - \tfrac{1}{2}\sum_{p=0}^{M} |a_p - b_p| \\ &\geq 1 - a_0 - \tfrac{1}{2}\sum_{p=0}^{M} |a_p - b_p| > 1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{2}\sum_{p=0}^{M} |a_p - b_p|, \end{aligned} \tag{2.19}$$

where in the final step we have employed condition (ii). Now each of the numbers a_p and b_p can be expressed as the sum or difference of 2^n quantities of the form $F(\mathbf{x})$ or $F_\nu(\mathbf{x})$ where $\mathbf{x}$ is an intersection of fixed hyperplanes $\mathfrak{h}^{ik}$. Consequently (2.8) and condition (v) show that we can determine $\nu_0 = \nu_0(\delta, \varepsilon)$ such that, for each p, $|a_p - b_p| < \varepsilon/(M+1)$, $\nu \geq \nu_0$. On substitution into (2.19) we get

$$G[\mathfrak{z}_0] > 1 - \varepsilon, \quad \nu \geq \nu_0(\delta, \varepsilon),$$

which establishes (2.9) and completes the proof.

§ 3. Almost-certainly-continuous many-valued vector functions.

Suppose that, to each point $\mathbf{x}$ of an n-dimensional Euclidean space $\mathfrak{X}$, there corresponds a system $\mathbf{y}(\mathbf{x})$ of p points (not necessarily distinct) in a q-dimensional Euclidean space $\mathfrak{Y}$. We call $\mathbf{y}(\mathbf{x})$ a *p-valued q-dimensional vector function* of $\mathbf{x}$. If there are defined a system of p one-valued functions of $\mathbf{x}$

$$\mathbf{y}_1(\mathbf{x}), \mathbf{y}_2(\mathbf{x}), \ldots, \mathbf{y}_p(\mathbf{x}) \tag{3.1}$$

such that, having due regard to multiple points, the points (3.1) coincide with the points $\mathbf{y}(\mathbf{x})$ for each $\mathbf{x}$ in $\mathfrak{X}$, then we call the functions (3.1) an *indexing* of $\mathbf{y}(\mathbf{x})$. If $\mathbf{y}(\mathbf{x})$ possesses at least one indexing (3.1) such that $\mathbf{y}_j(\mathbf{x})$ is a Borel-measurable function for each fixed $j = 1, 2, \ldots, p$, then $\mathbf{y}(\mathbf{x})$ is called a *many-valued Borel-measurable function*. In this paper we shall only be concerned with Borel-measurable $\mathbf{y}(\mathbf{x})$; and we shall therefore assume that (3.1) is an indexing for which $\mathbf{y}_j(\mathbf{x})$ is Borel-measurable for each fixed j.

We say that $\mathbf{y}(\mathbf{x})$ is *continuous in a Borel set* $\mathfrak{x}_0$ if, for every prescribed $\varepsilon > 0$ and all points $\mathbf{x} \in \mathfrak{x}_0$, there exists $\eta = \eta(\varepsilon, \mathbf{x}) > 0$ and at least one permutation $1'$, $2', \ldots, p'$ (possibly depending on ε, $\mathbf{x}$, $\mathbf{x}'$) of the integers $1, 2, \ldots, p$ such that

$$\mathbf{x} \in \mathfrak{x}_0 \;\&\; |\mathbf{x} - \mathbf{x}'| < \eta \Rightarrow |\mathbf{y}_j(\mathbf{x}) - \mathbf{y}_{j'}(\mathbf{x}')| < \varepsilon, \; j = 1, 2, \ldots, p. \tag{3.2}$$

If further $F[\mathfrak{x}_0] = 1$, where F determines a random variable $\mathbf{x}^*$, we say that $\mathbf{y}(\mathbf{x})$ is *almost-certainly-continuous with respect to* $\mathbf{x}^*$.

Theorem 2. *If $\varepsilon > 0$ and $\theta > 0$ are prescribed, and if $F[\mathfrak{x}]$ is a probability set function, and if $\mathbf{y}(\mathbf{x})$ is a p-valued Borel-measurable vector function, continuous in a Borel set $\mathfrak{x}_0$, then we can find a Borel set $\hat{\mathfrak{x}}$, satisfying $F[\hat{\mathfrak{x}}] \geq (1-\theta) F[\mathfrak{x}_0]$, and a number $\eta = \eta(\varepsilon, \theta)$, independent of $\mathbf{x}$, such that*

$$\mathbf{x} \in \hat{\mathfrak{x}} \;\&\; |\mathbf{x} - \mathbf{x}'| < \eta \Rightarrow |\mathbf{y}_j(\mathbf{x}) - \mathbf{y}'_j(\mathbf{x}, \mathbf{x}')| < \varepsilon, \; j = 1, 2, \ldots, p$$

where $\mathbf{y}'_j(\mathbf{x}, \mathbf{x}')$ *is a Borel-measurable function of* $\mathbf{x}$ *and* $\mathbf{x}'$ *for each fixed* $j = 1, 2, \ldots, p$, *and the set* $\mathbf{y}'_1(\mathbf{x}, \mathbf{x}'), \mathbf{y}'_2(\mathbf{x}, \mathbf{x}'), \ldots, \mathbf{y}'_p(\mathbf{x}, \mathbf{x}')$ *is a permutation (depending on* $\mathbf{x}$*) of the set* $\mathbf{y}_1(\mathbf{x}'), \mathbf{y}_2(\mathbf{x}'), \ldots, \mathbf{y}_p(\mathbf{x}')$.

When $p = 1$ this theorem reduces to one on uniform continuity over the 'non-trivial' part of a probability set. Surprisingly enough, the standard textbooks on topological measure theory do not mention even this special case, which therefore seems new.[1]

Let π_k, $k = 1, 2, \ldots, p!$, denote the permutations of p objects, and let $(1k)$, $(2k), \ldots, (pk)$ denote the result of applying π_k to the integers $1, 2, \ldots, p$. Let m be a positive integer. Let C_m denote the class (containing at least the empty Borel set) of all Borel sets $\mathfrak{x}$ of $\mathfrak{X}$ with the property

$$\mathbf{x} \in \mathfrak{x} \,\&\, |\mathbf{x} - \mathbf{x}'| < m^{-1} \Rightarrow |\mathbf{y}_j(\mathbf{x}) - \mathbf{y}_{(jk)}(\mathbf{x}')| < \varepsilon; \; j = 1, 2, \ldots, p; \text{ some } k; \tag{3.3}$$

where 'some k' means that there is at least one value of k (possibly depending on $\mathbf{x}$ and $\mathbf{x}'$) such that (3.3) holds for all j with this fixed k. We notice first that C_m, $m = 1, 2, \ldots$, is a monotone increasing collection of σ-rings [Halmos (**2**)]: that is to say

$$\mathfrak{x} \in C_m \,\&\, \mathfrak{x}' \in C_m \Rightarrow \mathfrak{x} - \mathfrak{x}' \in C_m; \tag{3.4}$$

$$\mathfrak{x}_s \in C_m, \; s = 1, 2, \ldots \Rightarrow \sum_{s=1}^{\infty} \mathfrak{x}_s \in C_m; \tag{3.5}$$

$$m < m' \Rightarrow C_m \subseteq C_{m'}. \tag{3.6}$$

Since F is a probability set function there exists M_m, the least upper bound of $F[\mathfrak{x}]$ for $\mathfrak{x} \in C_m$. We have

$$\mathfrak{x} \in C_m \Rightarrow F[\mathfrak{x}] \leq M_m. \tag{3.7}$$

Moreover we can find $\mathfrak{x}_{m\alpha} \in C_m$, $\alpha = 1, 2, \ldots$, such that $F[\mathfrak{x}_{m\alpha}] \geq M_m - \alpha^{-1}$. Write $\mathfrak{x}_m = \sum_{\alpha=1}^{\infty} \mathfrak{x}_{m\alpha}$ and notice that $\sum_{\alpha=1}^{\beta} \mathfrak{x}_{m\alpha}$, $\beta = 1, 2, \ldots$, is a monotone increasing sequence of Borel sets. Then

$$F[\mathfrak{x}_m] = F[\lim_{\beta\to\infty} \sum_{\alpha=1}^{\beta} \mathfrak{x}_{m\alpha}] = \lim_{\beta\to\infty} F[\sum_{\alpha=1}^{\beta} \mathfrak{x}_{m\alpha}] \geq \sup_{\beta} F[\mathfrak{x}_{m\beta}] \geq \sup_{\beta} (M_m - \beta^{-1}) = M_m. \tag{3.8}$$

But (3.5), (3.7), and (3.8) now show

$$\mathfrak{x}_m \in C_m, \; F[\mathfrak{x}_m] = M_m. \tag{3.9}$$

When we have thus found $\mathfrak{x}_m$ to satisfy (3.9) for each $m = 1, 2, \ldots$ we define

$$\mathfrak{x}^m = \sum_{\mu=1}^{m} \mathfrak{x}_\mu, \; m = 1, 2, \ldots, \infty. \tag{3.10}$$

[1] Professor Kac has remarked to me in conversation that the special case $p = 1$ can be deduced from Lusin's theorem.

Now (3.5), (3.6), and (3.7) show $F[\mathfrak{x}^m] \leq M_m$; and (3.9) and (3.10) show $F[\mathfrak{x}^m] \geq M_m$. So we have

$$\mathfrak{x}^m \in C_m, \; F[\mathfrak{x}^m] = M_m. \tag{3.11}$$

The definition of M_m, the fact that F is a probability set function, and (3.6) demonstrate

$$M_m \leq M_{m'} \leq 1, \quad m < m'; \tag{3.12}$$

so $M = \lim_{m\to\infty} M_m$ exists. Further $\mathfrak{x}^m$, $m = 1, 2, \ldots$, is a monotone increasing sequence of Borel sets. Thus

$$F[\mathfrak{x}^\infty] = F[\lim_{m\to\infty} \mathfrak{x}^m] = \lim_{m\to\infty} F[\mathfrak{x}^m] = \lim_{m\to\infty} M_m = M. \tag{3.13}$$

Since $\mathfrak{x}_0$ is a Borel set by hypothesis, $\mathfrak{x}_0 - \mathfrak{x}^\infty$ is a Borel set (perhaps empty). We shall show that the supposition

$$F[\mathfrak{x}_0 - \mathfrak{x}^\infty] > 0 \tag{3.14}$$

leads to a contradiction.

In the n-dimensional space $\mathfrak{X}$, in which $F[\mathfrak{x}]$ is defined, a bounded half-open set of all points $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ which satisfy all the inequalities

$$a_i - h < x_i \leq a_i + h, \; a_i \text{ and } h \text{ finite}, \; i = 1, 2, \ldots, n \tag{3.15}$$

is called a hypercube. Given a hypercube (3.15), the set of all points satisfying, for each value of i, one or other (but not both) of the inequalities

$$a_i - h < x_i \leq a_i \quad \text{or} \quad a_i < x_i \leq a_i + h$$

is called a first hyperquadrant of the hypercube (3.15). We then inductively define a $(q+1)$th hyperquadrant of (3.15) as a first hyperquadrant of a qth hyperquadrant of (3.15). The unqualified term 'hyperquadrant' will mean a qth hyperquadrant for some unspecified positive integer q.

If (3.14) holds, we can find a hypercube $\mathfrak{h}_0$ such that

$$F[(\mathfrak{x}_0 - \mathfrak{x}^\infty)\cdot \mathfrak{h}_0] > 0, \tag{3.16}$$

because F is a probability set function. Let $\mathfrak{h}'$ denote the union of all hyperquadrants $\mathfrak{h}$ of $\mathfrak{h}_0$ which satisfy

$$F[(\mathfrak{x}_0 - \mathfrak{x}^\infty)\cdot \mathfrak{h}] = 0. \tag{3.17}$$

The set of hyperquadrants $\mathfrak{h}$ satisfying (3.17) is at most enumerable, because it is a subset of the enumerable set of all hyperquadrants of $\mathfrak{h}_0$. Hence $\mathfrak{h}'$ is a Borel set, and

$$F[(\mathfrak{x}_0 - \mathfrak{x}^\infty)\cdot \mathfrak{h}'] = 0.$$

Consequently, from (3.16)

$$F[(\mathfrak{x}_0-\mathfrak{x}^\infty)\cdot(\mathfrak{h}_0-\mathfrak{h}')]>0. \qquad (3.18)$$

Now (3.18) implies that $(\mathfrak{x}_0-\mathfrak{x}^\infty)\cdot(\mathfrak{h}_0-\mathfrak{h}')$ is not empty. So we can choose a point $\mathbf{x}_0$ (hereafter fixed) such that

$$\mathbf{x}_0\in(\mathfrak{x}_0-\mathfrak{x}^\infty)\cdot(\mathfrak{h}_0-\mathfrak{h}'). \qquad (3.19)$$

Since (3.19) implies $\mathbf{x}_0\in\mathfrak{h}_0$, we may define $\mathfrak{h}_q$ to be the qth hyperquadrant of $\mathfrak{h}_0$ such that $\mathbf{x}_0\in\mathfrak{h}_q$. This definition is unique, because, for each fixed q, the several qth hyperquadrants of $\mathfrak{h}_0$ are mutually disjoint. Further

$$F[(\mathfrak{x}_0-\mathfrak{x}^\infty)\cdot\mathfrak{h}_q]>0,\quad q=1,2,\ldots, \qquad (3.20)$$

for otherwise $\mathbf{x}_0\in\mathfrak{h}_q\subseteq\mathfrak{h}'$ in contradiction to (3.19).

Next (3.19) implies $\mathbf{x}_0\in\mathfrak{x}_0$; so that $\mathbf{y}(\mathbf{x})$ is continuous at $\mathbf{x}_0$ by hypothesis. Therefore, $\mathbf{x}_0$ being fixed, we can find a positive integer $r=r(\mathbf{x}_0,\varepsilon)=r(\varepsilon)$ such that

$$|\mathbf{x}-\mathbf{x}_0|<2r^{-1}\rightarrow|\mathbf{y}_j(\mathbf{x}_0)-\mathbf{y}_{(jk)}(\mathbf{x})|<\tfrac{1}{2}\varepsilon;\quad j=1,2,\ldots,p;\ \text{some } k. \qquad (3.21)$$

Let $\mathfrak{s}$ denote the set of all points $\mathbf{x}$ satisfying $|\mathbf{x}-\mathbf{x}_0|<r^{-1}$. If the value of h [see (3.15)] for $\mathfrak{h}_q$ is h_q, $h_q=2^{-q}h_0\rightarrow 0$ as $q\rightarrow\infty$ because h_0 is finite. Hence we can choose a value of q, say $q=t$, so that $\mathfrak{h}_t\subseteq\mathfrak{s}$. Now let $\mathbf{x}$ and $\mathbf{x}'$ be any two points satisfying

$$\mathbf{x}\in(\mathfrak{x}_0-\mathfrak{x}^\infty)\cdot\mathfrak{h}_t \text{ and } |\mathbf{x}-\mathbf{x}'|<r^{-1}. \qquad (3.22)$$

Then

$$\mathbf{x}\in\mathfrak{h}_t\subseteq\mathfrak{s}\Rightarrow|\mathbf{x}-\mathbf{x}_0|<r^{-1}\Rightarrow|\mathbf{x}'-\mathbf{x}_0|<2r^{-1}.$$

A fortiori

$$|\mathbf{x}-\mathbf{x}_0|<2r^{-1} \text{ and } |\mathbf{x}'-\mathbf{x}_0|>2r^{-1};$$

so that (3.21) shows that there exist integers k' and k'' with $1\le k', k''\le p!$ such that

$$|\mathbf{y}_j(\mathbf{x}_0)-\mathbf{y}_{(jk')}(\mathbf{x})|<\tfrac{1}{2}\varepsilon,\ |\mathbf{y}_j(\mathbf{x}_0)-\mathbf{y}_{(jk'')}(\mathbf{x}')|<\tfrac{1}{2}\varepsilon,\quad j=1,2,\ldots,p.$$

Whereupon

$$|\mathbf{y}_{(jk')}(\mathbf{x})-\mathbf{y}_{(jk'')}(\mathbf{x}')|<\varepsilon,\quad j=1,2,\ldots,p.$$

Now apply the inverse permutation $\pi_{k'}^{-1}$ to these last inequalities, and there results

$$|\mathbf{y}_j(\mathbf{x})-\mathbf{y}_{(jk)}(\mathbf{x}')|<\varepsilon;\quad j=1,2,\ldots,p;\ \text{some } k. \qquad (3.23)$$

Since (3.22)$\Rightarrow$(3.23), we have from (3.3)

$$(\mathfrak{x}_0-\mathfrak{x}^\infty)\cdot\mathfrak{h}_t\in C_r;$$

and therefore by (3.5)

$$\mathfrak{x}^r + (\mathfrak{x}_0 - \mathfrak{x}^\infty)\cdot \mathfrak{h}_t \in C_r. \tag{3.24}$$

Now $\mathfrak{x}^r \subseteq \mathfrak{x}^\infty$; so $\mathfrak{x}^r$ and $(\mathfrak{x}_0 - \mathfrak{x}^\infty)\cdot \mathfrak{h}_t$ are mutually disjoint. Therefore, by (3.24), (3.7), (3.20), and (3.11)

$$M_r \geq F[\mathfrak{x}^r + (\mathfrak{x}_0 - \mathfrak{x}^\infty)\cdot \mathfrak{h}_t] = F[\mathfrak{x}^r] + F[(\mathfrak{x}_0 - \mathfrak{x}^\infty)\cdot \mathfrak{h}_t] > F[\mathfrak{x}^r] = M_r,$$

which is the required contradiction. So we must abandon the supposition (3.14); and there only remains the possibility

$$F[\mathfrak{x}_0 - \mathfrak{x}^\infty] = 0.$$

Consequently

$$F[\mathfrak{x}_0] \leq F[\mathfrak{x}^\infty] = M = \lim_{m\to\infty} F[\mathfrak{x}^m].$$

This last equation shows that we can find an integer s such that

$$(1-\theta)F[\mathfrak{x}_0] \leq F[\mathfrak{x}^s]. \tag{3.25}$$

We now choose $\eta = \eta(\varepsilon, \theta)$ to satisfy $0 < \eta < s^{-1}$, and put $\hat{\mathfrak{x}} = \mathfrak{x}^s$. We have

$$F[\hat{\mathfrak{x}}] \geq (1-\theta) F[\mathfrak{x}_0], \tag{3.26}$$

and, as a stronger case of (3.3)

$$\mathbf{x} \in \hat{\mathfrak{x}} \;\&\; |\mathbf{x} - \mathbf{x}'| < \eta \Rightarrow |\mathbf{y}_j(\mathbf{x}) - \mathbf{y}_{(jk)}(\mathbf{x}')| < \varepsilon; \;\; j = 1, 2, \ldots, p; \;\; \text{some } k. \tag{3.27}$$

Let $\mathfrak{Z} = \mathfrak{X} \times \mathfrak{X}'$ denote the space of points $\mathbf{z} = (\mathbf{x}, \mathbf{x}')$; and let $\mathbf{R}_\lambda$, $\lambda = 1, 2, \ldots$, be an enumeration of the rational points of $\mathfrak{Y}$. For each value of $k = 1, 2, \ldots, p!$, let $\mathfrak{z}_k$ denote the (possibly empty) set of points $\mathbf{z} = (\mathbf{x}, \mathbf{x}')$ which satisfy all the inequalities

$$|\mathbf{y}_j(\mathbf{x}) - \mathbf{y}_{(jk)}(\mathbf{x}')| < \varepsilon, \;\; j = 1, 2, \ldots, p; \tag{3.28}$$

and let $\mathfrak{z}$ denote the set of points for which (3.28) holds for some k (perhaps depending on $\mathbf{x}$ and $\mathbf{x}'$). Then

$$\mathfrak{z} = \sum_{k=1}^{p!} \mathfrak{z}_k.$$

The set of points $\mathbf{x}$, satisfying $|\mathbf{y}_j(\mathbf{x}) - \mathbf{R}_\lambda| < \frac{1}{2}\varepsilon$ for fixed j and fixed λ, is a Borel set $\mathfrak{x}_{j\lambda}$ since $\mathbf{y}_j(\mathbf{x})$ is a Borel-measurable function of $\mathbf{x}$. Similarly the set of points $\mathbf{x}'$, satisfying $|\mathbf{y}_{(jk)}(\mathbf{x}') - \mathbf{R}_\lambda| < \frac{1}{2}\varepsilon$ for fixed j and fixed k and fixed λ, is a Borel set $\mathfrak{x}'_{(jk)\lambda}$. Consequently

$$\mathfrak{z}_k = \prod_{j=1}^{p} \{\sum_{\lambda=1}^{\infty} (\mathfrak{x}_{j\lambda} \times \mathfrak{x}'_{(jk)\lambda})\}$$

is a Borel set. We define

$$\mathfrak{z}^{\varkappa} = \mathfrak{z}_{\varkappa} - \sum_{k=1}^{\varkappa-1} \mathfrak{z}_k, \quad \varkappa = 2, 3, \ldots, p!, \quad \mathfrak{z}^1 = \mathfrak{Z} - \sum_{\varkappa=2}^{p!} \mathfrak{z}^{\varkappa}.$$

Then $\mathfrak{z}^{\varkappa}$, $\varkappa = 1, 2, \ldots, p!$, are mutually disjoint Borel sets covering $\mathfrak{Z}$; so that their characteristic functions

$$\chi_{\varkappa}(\mathbf{x}, \mathbf{x}') = \begin{cases} 1 \text{ if } \mathbf{z} = (\mathbf{x}, \mathbf{x}') \in \mathfrak{z}^{\varkappa} \\ 0 \text{ if } \mathbf{z} = (\mathbf{x}, \mathbf{x}') \notin \mathfrak{z}^{\varkappa} \end{cases}$$

are Borel-measurable functions of $\mathbf{x}$ and $\mathbf{x}'$. The theorem is now proved by taking

$$y_j'(\mathbf{x}, \mathbf{x}') = \sum_{\varkappa=1}^{p!} \chi_{\varkappa}(\mathbf{x}, \mathbf{x}') \mathbf{y}_{(j\varkappa)}(\mathbf{x}'), \quad j = 1, 2, \ldots, p. \tag{3.29}$$

The case of Theorem 2 which will interest us in this paper arises when $\mathbf{y}(\mathbf{x})$ is almost-certainly-continuous, and we have $F[\hat{\mathfrak{x}}] \geq 1 - \theta$. The counter-example

$$y(x) = x^{-1}, \quad F(x) = \begin{cases} 0 \text{ if } x < 0 \\ x \text{ if } 0 \leq x \leq 1 \\ 1 \text{ if } x > 1 \end{cases}$$

shows that Theorem 2 would be false were $\theta = 0$ permitted.

§ 4. Many-valued random variables.

An unordered set of p probability set functions determines a *p-valued random variable*, $\mathbf{x}^*$. Suppose that, for any given Borel set $\mathfrak{x}$, these p probability set functions are arranged in an arbitrary order and then denoted by $F_j[\mathfrak{x}]$, $j = 1, 2, \ldots, p$. This indexing may depend in general upon the set $\mathfrak{x}$ chosen; but the symmetric sum

$$F[\mathfrak{x}] = p^{-1} \sum_{j=1}^{p} F_j[\mathfrak{x}]$$

is evidently independent of the indexing. It is moreover easy to verify that $F[\mathfrak{x}]$ is a probability set function, which we call the *condensed probability set function of* $\mathbf{x}^*$. The corresponding *condensed cumulative distribution function* is

$$F(\mathbf{x}) = p^{-1} \sum_{j=1}^{p} F_j(\mathbf{x})$$

where $F_j(\mathbf{x})$ are specified by an indexing of the p cumulative distribution functions of $\mathbf{x}^*$. Further, $F[\mathfrak{x}]$ and $F(\mathbf{x})$ determine the same one-valued random variable, which we call the *condensation* of $\mathbf{x}^*$ and denote by ${}^c\mathbf{x}^*$.

If $\mathbf{x}^*$ is a one-valued random variable, and $\mathbf{y}(\mathbf{x})$ is a p-valued Borel-measurable function with an indexing (3.1) of one-valued Borel-measurable functions, then

$$\mathbf{y}_j^* = \mathbf{y}_j(\mathbf{x}^*), \quad j = 1, 2, \ldots, p$$

will be p one-valued random variables determined by $H_j[\mathfrak{y}]$ say. Whereupon

$$H[\mathfrak{y}] = p^{-1}\sum_{j=1}^{p} H_j[\mathfrak{y}]$$

will be the condensed probability set function of the many-valued random variable $\mathbf{y}(\mathbf{x}^*)$, and will determine a condensation denoted by ${}^c\mathbf{y}(\mathbf{x}^*)$.

Theorem 3. *If* $\mathbf{x}^*$ *and* $\mathbf{x}_\nu^*$, $\nu = 1, 2, \ldots$, *are one-valued random variables satisfying*

$$\operatorname*{dlim}_{\nu\to\infty} \mathbf{x}_\nu^* = \mathbf{x}^*, \tag{4.1}$$

and if $\mathbf{y}(\mathbf{x})$ *is a many-valued Borel-measurable function which is almost-certainly-continuous with respect to* $\mathbf{x}^*$, *then*

$$\operatorname*{dlim}_{\nu\to\infty} {}^c\mathbf{y}(\mathbf{x}_\nu^*) = {}^c\mathbf{y}(\mathbf{x}^*). \tag{4.2}$$

It will be noticed that $\mathbf{y}(\mathbf{x})$ need not be almost-certainly-continuous with respect to $\mathbf{x}_\nu^*$ for any value of ν at all.

Suppose that $F[\mathfrak{x}]$ and $F_\nu[\mathfrak{x}]$ are the probability set functions determining $\mathbf{x}^*$ and $\mathbf{x}_\nu^*$ respectively. Let $\varepsilon > 0$ and $\theta > 0$ be any pair of prescribed positive numbers. Since $\mathbf{y}(\mathbf{x})$ is almost-certainly-continuous with respect to $\mathbf{x}^*$ we can find a Borel set $\hat{\mathfrak{x}}$, satisfying

$$F[\hat{\mathfrak{x}}] \geq 1 - \tfrac{1}{2}\theta, \tag{4.3}$$

and a number $\eta = \eta(\varepsilon, \theta)$ such that

$$\mathbf{x} \in \hat{\mathfrak{x}} \;\&\; |\mathbf{x} - \mathbf{x}'| < \eta \Rightarrow |\mathbf{y}_j(\mathbf{x}) - \mathbf{y}_j'(\mathbf{x}, \mathbf{x}')| < \varepsilon, \quad j = 1, 2, \ldots, p \tag{4.4}$$

where $\mathbf{y}_j'(\mathbf{x}, \mathbf{x}')$ is defined by (3.29).

Let $\mathfrak{Z} = \mathfrak{X} \times \mathfrak{X}'$ be the space of points $\mathbf{z} = (\mathbf{x}, \mathbf{x}')$. By (4.1), we can take the probability set function $G[\mathfrak{z}]$ defined in Theorem 1 to be a joint determination of $\mathbf{x}^*$ and $\mathbf{x}_\nu^*$. We put the quantity δ of Theorem 1 equal to $\eta(\varepsilon, \theta)$, and the quantity ε of Theorem 1 equal to the quantity $\tfrac{1}{2}\theta$ of the present theorem. Then, with $\mathfrak{z}_0$ denoting the set of points $\mathbf{z} = (\mathbf{x}, \mathbf{x}')$ satisfying $|\mathbf{x} - \mathbf{x}'| < \eta$, we have from Theorem 1

$$G[\mathfrak{x} \times \mathfrak{X}] = F[\mathfrak{x}]; \quad G[\mathfrak{X} \times \mathfrak{x}'] = F_\nu[\mathfrak{x}']; \quad G[\mathfrak{z}_0] > 1 - \tfrac{1}{2}\theta, \ \nu \geq \nu_0'(\varepsilon, \theta); \tag{4.5}$$

where $\nu_0'(\varepsilon, \theta) = \nu_0\{\eta(\varepsilon, \theta), \frac{1}{2}\theta\}$. Then, with $\hat{\mathfrak{z}} = \hat{\mathfrak{x}} \times \mathfrak{X}'$ we have

$$G[\hat{\mathfrak{z}}] = F[\hat{\mathfrak{x}}] \geq 1 - \tfrac{1}{2}\theta. \tag{4.6}$$

Then (4.5) and (4.6) show that

$$\begin{aligned} G[\mathfrak{z}_0 \cdot \hat{\mathfrak{z}}] &= 1 - G[\mathfrak{Z} - (\mathfrak{z}_0 \cdot \hat{\mathfrak{z}})] = 1 - G[(\mathfrak{Z} - \mathfrak{z}_0) + (\mathfrak{Z} - \hat{\mathfrak{z}})] \\ &\geq 1 - G[\mathfrak{Z} - \mathfrak{z}_0] - G[\mathfrak{Z} - \hat{\mathfrak{z}}] = -1 + G[\mathfrak{z}_0] + G[\hat{\mathfrak{z}}] > 1 - \theta, \end{aligned} \tag{4.7}$$

while (4.4) becomes

$$\mathbf{z} = (\mathbf{x}, \mathbf{x}') \in \mathfrak{z}_0 \cdot \hat{\mathfrak{z}} \Rightarrow |\mathbf{y}_j(\mathbf{x}) - \mathbf{y}_j'(\mathbf{x}, \mathbf{x}')| < \varepsilon, \quad j = 1, 2, \ldots, p. \tag{4.8}$$

Now, since ε and θ are arbitrary, (4.7) and (4.8) imply

$$\operatorname*{plim}_{\nu \to \infty} \{\mathbf{y}_j(\mathbf{x}^*) - \mathbf{y}_j'(\mathbf{x}^*, \mathbf{x}_\nu^*)\} = \mathbf{0}, \quad j = 1, 2, \ldots, p$$

and, since (2.7) ⇒ (2.3), we have

$$\operatorname*{dlim}_{\nu \to \infty} \mathbf{y}_j'(\mathbf{x}^*, \mathbf{x}_\nu^*) = \mathbf{y}_j(\mathbf{x}^*), \quad j = 1, 2, \ldots, p.$$

Summing the corresponding cumulative distribution functions over all values of j, and remembering that a distribution function has at most an enumerable number of discontinuities, we deduce without difficulty

$$\operatorname*{dlim}_{\nu \to \infty} {}^c\mathbf{y}'(\mathbf{x}^*, \mathbf{x}_\nu^*) = {}^c\mathbf{y}(\mathbf{x}^*). \tag{4.9}$$

We complete the proof by showing that $\mathbf{y}'(\mathbf{x}^*, \mathbf{x}_\nu^*)$ has the same condensed probability set function as $\mathbf{y}(\mathbf{x}_\nu^*)$. Let $\mathfrak{y}$ denote a typical Borel set of the space $\mathfrak{Y}$ of points $\mathbf{y}$. Let $\mathfrak{z}_j'(\mathfrak{y})$ denote the set of all points $\mathbf{z} = (\mathbf{x}, \mathbf{x}')$ such that $\mathbf{y}_j'(\mathbf{x}, \mathbf{x}') \in \mathfrak{y}$; and let $\mathfrak{z}_j(\mathfrak{y})$ denote the set of all $\mathbf{z}$ such that $\mathbf{y}_j(\mathbf{x}') \in \mathfrak{y}$. Since $\mathbf{y}_j(\mathbf{x}')$ and $\mathbf{y}_j'(\mathbf{x}, \mathbf{x}')$ are Borel-measurable functions $\mathfrak{z}_j(\mathfrak{y})$ and $\mathfrak{z}_j'(\mathfrak{y})$ are Borel sets. The condensed probability set function of $\mathbf{y}'(\mathbf{x}^*, \mathbf{x}_\nu^*)$ is

$$p^{-1}\sum_{j=1}^{p} G[\mathfrak{z}_j'(\mathfrak{y})] = p^{-1}\sum_{j=1}^{p}\sum_{\varkappa=1}^{p!} G[\mathfrak{z}^\varkappa \cdot \mathfrak{z}_j'(\mathfrak{y})], \tag{4.10}$$

where $\mathfrak{z}^\varkappa$, $\varkappa = 1, 2, \ldots, p!$, are the disjoint Borel sets covering $\mathfrak{Z}$ defined in Theorem 2. If $\mathbf{z} \in \mathfrak{z}^\varkappa \cdot \mathfrak{z}_j'(\mathfrak{y})$, (3.29) shows $\mathbf{y}_j'(\mathbf{x}, \mathbf{x}') = \mathbf{y}_{(j\varkappa)}(\mathbf{x}')$; and so

$$\mathfrak{z}^\varkappa \cdot \mathfrak{z}_j'(\mathfrak{y}) = \mathfrak{z}^\varkappa \cdot \mathfrak{z}_{(j\varkappa)}(\mathfrak{y}).$$

Substituting into (4.10), we get

$$p^{-1}\sum_{j=1}^{p} G[\mathfrak{z}'_j(\mathfrak{y})] = p^{-1}\sum_{\varkappa=1}^{p!}\sum_{j=1}^{p} G[\mathfrak{z}^{\varkappa}\cdot\mathfrak{z}_{(j\varkappa)}(\mathfrak{y})] = p^{-1}\sum_{\varkappa=1}^{p!}\sum_{j=1}^{p} G[\mathfrak{z}^{\varkappa}\cdot\mathfrak{z}_j(\mathfrak{y})]$$

$$= p^{-1}\sum_{j=1}^{p}\sum_{\varkappa=1}^{p!} G[\mathfrak{z}^{\varkappa}\cdot\mathfrak{z}_j(\mathfrak{y})] = p^{-1}\sum_{j=1}^{p} G[\mathfrak{z}_j(\mathfrak{y})],$$

which is the condensed probability set function of $\mathbf{y}(\mathbf{x}_\nu^*)$.

§ 5. References.

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Lectureship in the Design and Analysis of Scientific Experiment, University of Oxford.