THE INHOMOGENEOUS MINIMUM OF A TERNARY QUADRATIC FORM

BY

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1. Let \( Q(x, y, z) \) be an indefinite ternary quadratic form with real coefficients and determinant \( D \neq 0 \). Davenport \([4]\) has shown that, given any real numbers \( x_0, y_0, z_0 \), there exist \( x, y, z \) congruent (modulo 1) to \( x_0, y_0, z_0 \) satisfying

\[
|Q(x, y, z)| \leq \left( \frac{17}{16} |D| \right) \text{;} \tag{1.1}
\]

the equality sign can hold if and only if \( Q \) is equivalent (under integral unimodular transformation of the variables) to a multiple of the form

\[
Q_1(x, y, z) = x^2 + 5y^2 - z^2 + 5yz + zx.
\]

The main weapon used in the proof was a generalization of Minkowski’s result on the inhomogeneous minimum of a binary quadratic form, namely:

If \( f(x, y) \) is a binary quadratic form with real coefficients and discriminant \( \Delta^2 \), where \( \Delta > 0 \), and \( \mu > 0 \), \( \nu > 0 \), \( \mu \nu \geq \frac{1}{4\nu} \), then, for any real numbers \( x_0, y_0 \), there exist \( x, y \equiv x_0, y_0 \) (mod 1) satisfying

\[
-\nu \Delta \leq f(x, y) \leq \mu \Delta. \tag{1.2}
\]

By obtaining an ‘isolation’ of this inequality when \( \nu \) is approximately \( 2\mu \), Davenport was able to show that the result (1.1) is isolated: that is to say, there exists a positive constant \( \delta \) such that the inequality

\[
|Q(x, y, z)| \leq (1-\delta) \left( \frac{17}{16} |D| \right) \text{;} \tag{1.3}
\]

can be satisfied whenever \( Q \) is not equivalent to a multiple of the special form \( Q_1 \).
Recently Swinnerton-Dyer and I [3] made a detailed investigation of results of the type (1.2) and developed a technique for obtaining best possible results for any value of the ratio \( r/\mu \). I use this technique here, together with Davenport’s general method of attack on the problem, to find the best possible value of \( \delta \) in (1.3).

The proof leads naturally to a stronger assertion than (1.3) and shows that the result (1.1) is isolated not only in respect of the form \( Q_1 \) but also in respect of the values \( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \) (mod 1) of \( x_0, y_0, z_0 \). To make this statement precise we introduce the following notation:

If \( Q=Q(x, y, z) \) is any indefinite ternary quadratic form and \( x_0, y_0, z_0 \) any real numbers, we set
\[
M(Q; x_0, y_0, z_0) = \min \{ Q(x, y, z) \mid x \equiv x_0, y \equiv y_0, z \equiv z_0 \mod 1 \},
\]
where the lower bound is taken over all sets \( x, y, z \equiv x_0, y_0, z_0 \) (mod 1). We then write
\[
M(Q) = \max_{x_0, y_0, z_0} M(Q; x_0, y_0, z_0),
\]
where the upper bound is taken over all real \( x_0, y_0, z_0 \); we call \( M(Q) \) the inhomogeneous minimum of \( Q \).

Clearly (1.1) implies that always
\[
M(Q) \leq (\frac{Q_0}{\delta} | D |)^{\frac{1}{2}}.
\]

Now if \( T \) is any \( 3 \times 3 \) matrix with integral elements and determinant \( \pm 1 \) and we make the transformation of the variables expressed in vector notation by
\[
X = TX,
\]
then \( Q(x, y, z) \) becomes, say, \( Q'(X, Y, Z) \), and the forms \( Q, Q' \) are said to be equivalent. If also we define
\[
X_0 = TX_0,
\]
then it is clear that
\[
M(Q'; X_0, Y_0, Z_0) = M(Q; x_0, y_0, z_0).
\]
Further, since \( X_0, Y_0, Z_0 \) run through all real numbers when \( x_0, y_0, z_0 \) do, we have
\[
M(Q') = M(Q).
\]

It will always be understood, when we pass to an equivalent form by a transformation (1.6), that any particular values of \( x_0, y_0, z_0 \) under consideration are subjected to the corresponding transformation (1.7).

The complete statement of the results we shall obtain is given, in the above notation, by
Theorem 1. (i) If $Q(x, y, z)$ is not equivalent to a multiple of either of the forms

$$Q_1(x, y, z) = x^2 - y^2 - z^2 + xy - 7yz + zx$$

$$Q_2(x, y, z) = 2x^2 - y^2 + 15z^2,$$

then

$$M(Q) < (\frac{1}{4} |D|)^4.$$  \hspace{1cm} (1.12)

(ii) For the special forms $Q_1, Q_2$ we have

$$M(Q_i; x_0, y_0, z_0) < (\frac{1}{4} |D|)^4 \quad (i = 1, 2)$$  \hspace{1cm} (1.13)

unless $x_0, y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \pmod{1}$; further,

$$M(Q_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = (\frac{1}{4} |D|)^4 = M(Q_1),$$

$$M(Q_2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = (\frac{1}{4} |D|)^4 = M(Q_2).$$  \hspace{1cm} (1.14, 1.15)

In the course of the proof we shall use the following lemmas:

Lemma 1. If $Q(x, y, z)$ is indefinite and has determinant $D < 0$ then there exist integers $x_1, y_1, z_1$ satisfying

$$0 < Q(x_1, y_1, z_1) \leq (4 |D|)^4.$$  \hspace{1cm} (1.16)

This is Theorem 2 of Davenport [5].

Lemma 2. Let $\beta, B$ be real numbers with $B > \frac{1}{2}$. Then for any real $x_0$ there exists an $x$ satisfying

$$x \equiv x_0 \pmod{1}, \quad |x^2 - \beta^2| < B,$$

provided that

$$\beta^2 < B^2 + \frac{1}{4} \quad \text{if } B \text{ is integral},$$

$$\beta^2 < B + \frac{1}{2} \lceil 2B \rceil^2 \quad \text{if } B \text{ is not integral}.$$  \hspace{1cm} (1.17)

This result is contained in Davenport [4], Lemma 5.

Lemma 3. Let $T$ be an integral $2 \times 2$ matrix of infinite order and of determinant $\pm 1$, and let $\mathcal{R}$ be a bounded point set in the Cartesian plane. Suppose that, for some point $A$ with integral coordinates, any point $P$ of $\mathcal{R}$ has the property that either $T(P) - A$ belongs to $\mathcal{R}$ or $T(P)$ is not congruent (mod 1) to a point of $\mathcal{R}$.

Then, if $P$ is a point such that $T^n(P)$ is congruent (mod 1) to a point of $\mathcal{R}$ for all integral $n \geq 0$, $P$ is the unique point $F$ of $\mathcal{R}$ defined by

$$T(F) - A = F.$$  \hspace{1cm} (1.18)

This result is due to Cassels, and is quoted by Bambah [1]; an alternative proof is given in Barnes and Swinnerton-Dyer [2], Theorem D; (the region $\mathcal{R}^*$ ap-
pearing in this theorem may be taken as the set of all points of the plane which are not congruent (mod 1) to a point of \( R \).

2. The results stated in Theorem 1 for \( M(Q_1; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) and \( M(Q_2; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) are easily established by congruence considerations, and it is convenient to dispose of these at once.

(i) We have

\[
4Q_1 = (2x + y + z)^2 - 5(y + 3z)^2 + 40z^2.
\]

If \( x, y, z \equiv 1, \frac{1}{3}, \frac{1}{3} \), then \( 2x, 2y, 2z \) are odd integers; we may therefore write

\[
4Q_1 = X^2 - 5Y^2 + 10Z^2,
\]

where \( X, Y, Z \) are integral, \( Z = 2z \) is odd and \( X - Y = 2x - 2z \) is even. We then have

\[
4Q_1 = 2 \pmod{4}, \quad 4Q_1 = 0, \quad \pm 1 \pmod{5},
\]

whence \( |4Q_1| \geq 6 \). We have thus shown that

\[
|Q_1(x, y, z)| \geq \frac{1}{2} \quad \text{whenever } x, y, z \equiv 1, \frac{1}{3}, \frac{1}{3}.
\]

Since

\[
|Q_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})| = \frac{1}{2}, \quad D(Q_1) = -\frac{125}{2},
\]

it follows that

\[
M(Q_1; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{2}{5} = (\frac{25}{2})D
\]

as required.

(ii) If \( x, y, z \equiv 1, \frac{1}{3}, \frac{1}{3} \), then

\[
4Q_2 = 2X^2 - Y^2 + 15Z^2,
\]

where \( X, Y, Z \) are odd integers. Hence

\[
4Q_2 = 0 \pmod{8},
\]

and it is easy to see, by considering congruences mod 3, that \( 4Q_2 = 0 \). We therefore have

\[
|Q_2(x, y, z)| \geq 2 \quad \text{whenever } x, y, z \equiv 1, \frac{1}{3}, \frac{1}{3}.
\]

Since

\[
|Q_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})| = 2, \quad D(Q_2) = -30,
\]

it follows that

\[
M(Q_2; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 2 = (\frac{10}{3})D
\]

as required.

To complete the proof of Theorem 1 we have therefore to establish

**Theorem 2. The inequality**

\[
M(Q; x_0, y_0, z_0) < (\frac{1}{15}D)^{\frac{1}{4}} \quad (2.1)
\]

*holds unless Q is equivalent to a multiple of Q1 or Q2 with x_0, y_0, z_0 \equiv 1, \frac{1}{3}, \frac{1}{3} \pmod{1}.*
For the proof of Theorem 2, we first observe that there is no loss of generality
in supposing that \( D < 0 \) (since we may consider \( -Q \) in place of \( Q \) if necessary). Let
\( a = Q(x_1, y_1, z_1) \) be any value assumed by \( Q \) for coprime integers \( x_1, y_1, z_1 \) satisfying
(1.16), so that
\[
0 < a \leq (4|D|)^{1/2}. \tag{2.2}
\]
Making an appropriate equivalence transformation, we see that \( 1/a \) \( Q(x, y, z) \) is equi-
valent to a form
\[
f(x, y, z) = (x + hy + gz)^2 - \phi(y, z) \tag{2.3}
\]
where \( h, g \) are real and \( \phi(y, z) \) is an indefinite quadratic form of discriminant
\[
\Delta^2 = \frac{4|D|}{a^2} \geq 1. \tag{2.4}
\]
Then (2.1) is equivalent to the assertion that
\[
M(f; x_0, y_0, z_0) < \left( \frac{1}{a} \Delta^2 \right)^{1/2}. \tag{2.5}
\]
The first step in the proof of (2.5) is the consideration of the possible forms of
\( \phi(y, z) \). In this section we prove

**Theorem 3.** If \( f(x, y, z) \) is given by (2.3), (2.4), then (2.5) holds unless either
\[
(i) \quad \phi(y, z) = \frac{1}{4}(y^2 + 8yz + z^2), \quad y_0, z_0 = \frac{1}{2}, \frac{1}{2}, \tag{2.6}
\]
or
\[
(ii) \quad \phi(y, z) = 2y^2 + 12yz + 3z^2, \quad y_0, z_0 = \frac{1}{2}, \frac{1}{2}, \tag{2.7}
\]
or
\[
(iii) \quad \phi(y, z) = k(y^2 + 6yz + z^2), \quad y_0, z_0 = \frac{1}{2}, \frac{1}{2}\tag{2.8}
\]
where
\[
0.9006 < k < 1.0063 \tag{2.9}
\]
(or equivalent forms).

It is convenient to set
\[
d = \left( \frac{1}{a} \Delta^2 \right)^{1/2}, \tag{2.10}
\]
so that, by (2.4),
\[
d \geq \left( \frac{1}{a} \right)^{1/4} > \frac{1}{2}. \tag{2.11}
\]

**Lemma 4.** Let \( \mu > 0, \nu > 0 \) be defined by
\[
\mu \Delta = \frac{1}{2} d - \frac{1}{4} \tag{2.12}
\]
\[
\nu \Delta = \left\{ \begin{array}{ll}
\frac{1}{2} d + \frac{1}{4} (d^2 + 1) & \text{if } d \text{ is not integral} \\
\frac{1}{2} (d^2 + 1) & \text{if } d \text{ is integral}.
\end{array} \right. \tag{2.13}
\]
Suppose that there exist \( y, z \equiv y_0, z_0 \pmod{1} \) with

\[ -\mu \Delta < \phi(y, z) < \nu \Delta. \]

Then for any \( x_0 \)

\[ M(f; x_0, y_0, z_0) < \frac{1}{\Delta} d = (\frac{1}{\Delta})^4. \]

Proof. If in (2.14) we have \( \phi(y, z) \leq 0 \), then, for any \( x_0 \), we can choose \( x \equiv x_0 \) with \( |x + hy + gz| \leq \frac{1}{2} \). For this choice of \( x, y, z \) we have

\[ 0 \leq f(x, y, z) < \frac{1}{2} + \mu \Delta = \frac{1}{2} d. \]

If, however, \( \phi(y, z) > 0 \), we have

\[ 0 < \phi(y, z) < \nu \Delta; \]

applying Lemma 2 with \( \beta^2 = \phi(y, z) \), \( B = \frac{1}{2} d \) (noting that then \( B > \frac{1}{2} \) by (2.11)), we see that for any \( x_0 \) we can choose \( x \equiv x_0 \) with

\[ |f(x, y, z)| < \frac{1}{2} d. \]

The required result (2.15) follows immediately.

In the notation of Barnes and Swinnerton-Dyer [3] we denote by \( \mathbb{R}_m \) the set of points of the \( \xi, \eta \)-plane defined by

\[ -1 \leq \xi \eta \leq m. \]

An inhomogeneous lattice \( \mathcal{L} \) is a set of points

\[ \xi = \alpha x + \beta y, \]
\[ \eta = \gamma x + \delta y, \]

where \( x, y \) run through all numbers congruent \( \pmod{1} \) to \( x_0, y_0 \) respectively, and

\[ \Delta = \Delta(\mathcal{L}) = |\alpha \delta - \beta \gamma| + 0 \]

is the determinant of \( \mathcal{L} \). \( \mathcal{L} \) is admissible for \( \mathbb{R}_m \) if it has no point in the interior of \( \mathbb{R}_m \). The critical determinant \( D_m \) of \( \mathbb{R}_m \) is defined to be the lower bound of \( \Delta(\mathcal{L}) \) over all admissible lattices \( \mathcal{L} \). We now have

**Lemma 5.** For all \( m \geq 1 \),

\[ D_m \geq 4 \sqrt{m}. \]

This result is equivalent to Davenport's result quoted in § 1 (Davenport [4], Lemma 3). A less direct proof is given in Barnes and Swinnerton-Dyer [3].
Now since $\phi(y, z)$ has discriminant $\Delta^2$, it may be expressed as the product of two linear forms of determinant $\Delta$. Thus the form

$$\frac{1}{\mu} \Delta \phi(y, z)$$

with $y, z = y_0, z_0$ runs over the values of $\xi \eta$ corresponding to a lattice $C$ of determinant $\frac{1}{\mu}$. From the definition of $D_m$ it is therefore clear that (2.10) is certainly soluble, for any $y_0, z_0$, if

$$\frac{1}{\mu} < D_m,$$

where $m = \frac{v}{\mu}$.

Combining this result with Lemma 3, we have

**Lemma 6.** If $\mu, v$ are defined as in Lemma 4 and

$$m = \frac{v}{\mu},$$

(2.17)

then the inequality (2.5) certainly holds unless

$$\frac{1}{\mu} \geq D_m.$$  

(2.18)

As a first step towards the elimination of possible values of $d$, we use (2.18) with the estimate (2.16) for $D_m$.

**Lemma 7.** If (2.5) does not hold, then $d$ satisfies either

\[ d = 2, \]

or

\[ 2.969 < d \leq 3, \]

or

\[ 3.975 < d \leq 4, \]

or

\[ 4.994 < d \leq 5. \]

(2.19)  

(2.20)  

(2.21)  

(2.22)

**Proof.** By Lemma 6 and (2.12) we have

$$\frac{1}{\mu} \geq 4 \sqrt{m},$$

i.e.

$$16 \mu v \leq 1.$$ 

Substituting for $\mu, v$ and noting that, by (2.10),

$$8 \Delta^2 = 15d^2,$$
this inequality becomes

$$8(2d-1)(2d+|d|^2) \leq 15d^a$$ if $d$ is not integral, \hspace{1cm} (2.23)

$$8(2d-1)(d^2 + 1) \leq 15d^a$$ if $d$ is integral. \hspace{1cm} (2.24)

Now (2.24) may be written in the form

$$(d-2)(d^2 - 6d + 4) \leq 0,$$

and this inequality is easily seen to be false if $d \geq 6$ or if $\frac{1}{2} < d \leq 1$. Thus (2.24) can hold for integral $d > \frac{1}{2}$ only if $d = 2, 3, 4$ or $5$. Further, since $[d] > d - 1$, $2d + |d|^2 > d^2 + 1$. Hence (2.23) cannot hold if $d$ satisfies $d \geq 6$ or $\frac{1}{2} < d \leq 1$.

It remains for us to consider non-integral $d$ satisfying (2.19) and $1 < d < 6$.

(i) If $[d] = 1$, (2.23) is

$$15d^a - 32d^2 + 8 \geq 0;$$

the l.h.s. takes its greatest values at the end-points of the interval $1 < d < 2$ and is negative for $d = 1$ and $d = 2$. Hence (2.23) is never satisfied.

(ii) If $[d] = 2$, (2.23) is

$$15d^a - 32d^2 - 48d + 32 \geq 0;$$

the l.h.s. increases with $d$ for $d \geq 2$ and is negative when $d = 2.969$; hence $d$ satisfies (2.20).

(iii) If $[d] = 3$, (2.23) is

$$15d^a - 32d^2 - 128d + 72 \geq 0;$$

the l.h.s. increases with $d$ for $d \geq 3$ and is negative when $d = 3.975$; hence $d$ satisfies (2.21).

(iv) If $[d] = 4$, (2.23) is

$$15d^a - 32d^2 - 240d + 128 \geq 0;$$

the l.h.s. increases with $d$ for $d \geq 4$ and is negative when $d = 4.994$; hence $d$ satisfies (2.22).

(v) If $[d] = 5$, (2.23) is

$$15d^a - 32d^2 - 384d + 200 \geq 0;$$

the l.h.s. increases with $d$ for $d \geq 5$ and is negative when $d = 6$; hence (2.23) does not hold.

This completes the proof of the lemma.
Corresponding to the values of $d$ allowed by Lemma 6, we find the following values of $m=\frac{\nu}{\mu}$:

\begin{align}
 m &= \frac{1}{2}, \\
 2 \leq m &< 2.0126, \\
 \frac{17}{7} \leq m &< 2.4389, \\
 \frac{26}{9} &\leq m < 2.8915.
\end{align}

Now the estimate (2.16) is known to be best possible if and only if $m$ is of the form
\[ m = 1 + \frac{2}{r} \quad (r = 1, 2, 3, \ldots) \]
or $m = 1$; in particular $D_m = 4\sqrt{m}$ for the values $m = \frac{1}{2}, m = 2$. However, for the remaining values of $m$ given in (2.19')-(2.22'), strict inequality holds in (2.16). The results we shall need are given in the following four lemmas:

**Lemma 8.** If $m = \frac{1}{2}$ and $\mathcal{L}$ is admissible for $R_m$, then either $\Delta(\mathcal{L}) > 4\sqrt{\frac{3}{2}}$ or $\mathcal{L}$ is given by
\[ \xi \eta = \frac{2}{3}(x^2 + 8xy + y^2), \quad x, y \equiv \frac{1}{2}, \frac{1}{2} \pmod{1}. \]

**Lemma 9.** If $m \geq 2$ and $\mathcal{L}$ is admissible for $R_m$, then either $\Delta(\mathcal{L}) \geq \sqrt{33}$ or $\mathcal{L}$ is given by
\[ \xi \eta = k(x^2 + 6xy + y^2), \quad x, y \equiv \frac{1}{2}, \frac{1}{2} \pmod{1}, \quad k \geq \frac{1}{2} m. \]

**Lemma 10.** If $m \geq \frac{17}{7}$ and $\mathcal{L}$ is admissible for $R_m$, then either $\Delta(\mathcal{L}) > \sqrt{\frac{30}{17}}$ or $\mathcal{L}$ is given by
\[ \xi \eta = k(2x^2 + 12xy + 3y^2), \quad x, y \equiv \frac{1}{2}, \frac{1}{2} \pmod{1}, \quad k \geq \frac{4}{5} m. \]

**Lemma 11.** If $m \geq \frac{26}{9}$ and $\mathcal{L}$ is admissible for $R_m$, then $\Delta(\mathcal{L}) \geq 4\sqrt{3}$.

In order to avoid interrupting the main argument, we defer the discussion of these results until § 4.

Now suppose that (2.19) holds, so that $d = 2, m = \frac{3}{2}$. Then
\[ \mu \Delta = \frac{1}{2}d - \frac{1}{2} = \frac{3}{2}, \]
\[ \Delta^2 - \frac{1}{2}d^2 = 15, \]
\[ \frac{1}{\mu} = 4\sqrt{\frac{3}{2}}. \]
Hence, by Lemmas 3 and 8, (2.5) holds unless
\[ \phi(y, z) = \frac{1}{2}(y^2 + 8yz + z^2), \quad y, z \equiv \frac{1}{2}, \frac{1}{1}; \]
this is (i) of Theorem 3.

Next suppose that (2.20) holds, so that \( m \) satisfies (2.20'). Then, since
\[ \frac{1}{\mu} = \frac{4\Delta}{2d-1} = \frac{4}{2d-1} \sqrt{\frac{15d^2}{8}}, \]
it is easily verified that \( \frac{1}{\mu} < \sqrt{33} \). By Lemmas 3 and 9 it follows that (2.5) holds unless \( \phi(y, z) \) is equivalent to a positive multiple of \( y^2 + 6yz + z^2 \) with \( y, z \equiv \frac{1}{2}, \frac{1}{1} \) (mod 1).

This shows that (2.8) of Theorem 3 holds for some \( k > 0 \). Also, since then \( \Delta^2 = 50k^2 \), we have
\[ \Delta^2 = 50k^2 = \frac{1}{\mu^2} d^2. \]
The bounds (2.9) for \( k \) now follow from the bounds for \( d \) given in (2.20). Thus \( \phi(y, z) \) satisfies Theorem 3 (iii).

Next suppose that (2.21) holds, so that \( m \) satisfies (2.21'). Then the inequality
\[ \frac{1}{\mu} \geq \frac{1}{2} \sqrt{30} \]
cannot hold unless
\[ m = \frac{7}{2}, \quad d = 4, \quad \frac{1}{\mu} = \frac{1}{2} \sqrt{30}. \]
For (2.25) is equivalent to
\[ \frac{1}{2} \sqrt{30} \leq \frac{\Delta}{2d-1} = \frac{1}{2d-1} \sqrt{\frac{15d^2}{8}}, \]
which reduces to
\[ 49d^2 - 64(2d-1)^2 \geq 0; \]
since \( 3.975 < d \leq 4 \), this is true only if \( d = 4 \) and the sign of equality holds: this gives (2.26). It now follows at once from Lemmas 3 and 10 that (2.5) holds unless \( \phi(y, z) \) is equivalent to a positive multiple of
\[ 2y^2 + 12yz + 3z^2, \quad y, z \equiv \frac{1}{2}, \frac{1}{1} \quad \text{and} \quad d = 4. \]
Since then
\[ \Delta^2 = \frac{1}{\mu^2} d^2 = 120, \]
we see that (ii) of Theorem 3 holds.

Suppose finally that (2.22) holds, so that \( m \) satisfies (2.22'). Then, by Lemmas 3 and 11, (2.5) holds unless
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\[
\frac{1}{\mu} \geq 4\sqrt{3}.
\]

But this inequality is equivalent to

\[
\sqrt{3} \leq \frac{1}{4\mu} = \frac{\Delta}{2d - 1},
\]

\[
3(2d - 1)^2 \leq \Delta^2 = \frac{15}{8} d^2,
\]

and it is easily verified that this is false for \(d\) satisfying (2.18).

This completes the proof of Theorem 3.

3. The next step in the proof of Theorem 1 is to decide what values of \(x_0, h, \) and \(g\) are allowable in (2.3) if (2.5) is not satisfied and \(\phi(y, z)\) is given by one of the forms in Theorem 3.

Lemma 12. If \(f(x, y, z)\) is given by (2.3), where \(\phi(y, z)\) is given by (2.6), then (2.5) holds unless \(f\) is equivalent to

\[
f_1(x, y, z) = x^2 - \frac{1}{4}(y^2 + 8yz + z^2)
\]

and

\[
x_0, y_0, z_0 = \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \pmod{1}.
\]

Proof. We have

\[
f(x, y, z) = (x + hy + gz)^2 = \frac{1}{4}(y^2 + 8yz + z^2)
\]

with \(x, y, z \equiv x_0, \frac{1}{2}, \frac{1}{2} \pmod{1} \). Since \(\Delta^2 = 15\),

\[
(\frac{1}{15} \Delta^2)^k = 1.
\]

Now

\[
f(\pm x, \frac{1}{2}, \frac{1}{2}) = (\pm x + \frac{1}{2}h - \frac{1}{2} g)^2 + \frac{1}{4},
\]

\[
f(\pm x, \frac{1}{2}, \frac{1}{2}) = (\pm x + \frac{1}{2}h + \frac{1}{2} g)^2 - \frac{1}{4}.
\]

For any real \(x_o\), we can choose \(x \equiv x_0\) with

\[
|x + \frac{1}{2}h - \frac{1}{2} g| \leq \frac{1}{6};
\]

hence, by (3.3),

\[
M(f; x_0, y_0, z_0) \leq 1 = (\frac{1}{15} \Delta^2)^k;
\]

and the sign of inequality holds unless

\[
x_o + \frac{1}{2}h - \frac{1}{2} g \equiv \frac{1}{6} \pmod{1}.
\]

In the same way, taking the lower sign in (3.3), we see that (2.5) holds unless

\[
-x_o + \frac{1}{2}h - \frac{1}{2} g \equiv \frac{1}{6} \pmod{1}.
\]
Similarly, choosing \( x = x_0 \) with
\[
\frac{1}{2} \leq \left| \pm x + \frac{1}{2} h + \frac{1}{2} g \right| \leq \frac{3}{2},
\]
we see that (2.5) holds unless
\[
x_0 + \frac{1}{2} h + \frac{1}{2} g \equiv \frac{1}{4} \pmod{1},
\]
\[
-x_0 + \frac{1}{2} h + \frac{1}{2} g \equiv \frac{1}{4} \pmod{1}.
\]

Since the above four congruences imply that
\[
x_0 \equiv \frac{1}{4}, \quad h \equiv g \equiv 0 \pmod{1},
\]
the lemma follows at once.

**Lemma 13.** If \( f(x, y, z) \) is given by (2.3), where \( \phi(y, z) \) is given by (2.7), then (2.5) holds unless \( f \) is equivalent to
\[
f_\Delta(x, y, z) = x^2 - (2y^2 + 12yz + 3z^2)
\]
and
\[
x_0, y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2} \pmod{1}.
\]

**Proof.** We have
\[
f(x, y, z) = (x + hy + gz)^2 - (2y^2 + 12yz + 3z^2),
\]
with \( x, y, z \equiv x_0, \frac{1}{2}, \frac{1}{2} \pmod{1} \). Since \( \Delta^2 = 120 \),
\[
\left(\frac{\Delta}{\Delta^2}\right)^{\frac{1}{2}} = 2.
\]

Now
\[
f(\pm x, \frac{1}{2} - h, \pm g) = (\pm x + \frac{1}{2} h - \frac{1}{2} g)^2 + \frac{1}{4},
\]
\[
f(\pm x, \frac{1}{2} + h, \pm g) = (\pm x + \frac{1}{2} h + \frac{1}{2} g)^2 - \frac{1}{4}.
\]
Choosing \( x = x_0 \) to satisfy any one of
\[
\left| \pm x + \frac{1}{2} h - \frac{1}{2} g \right| \leq \frac{3}{2},
\]
\[
\frac{3}{2} \leq \left| \pm x + \frac{1}{2} h + \frac{1}{2} g \right| \leq \frac{5}{2},
\]
we see, precisely as in Lemma 12, that (2.5) holds unless
\[
x_0 \equiv \frac{1}{4}, \quad h \equiv g \equiv 0 \pmod{1}.
\]

This gives the result of the lemma.

For the case (iii) of Theorem 3, we want to show that (2.5) holds unless
\( x_0 = \frac{1}{2}, \quad h = \frac{1}{2}, \quad g = \frac{1}{2} \pmod{1} \) and \( k = 1 \). For this, the simple argument used in Lemmas 12 and 13 is not sufficient. However, the complete result will follow by a consideration of the automorphs of \( f(x, y, z) \) and an application of Lemma 4. The proof divides naturally into two stages, given in the following two lemmas.
Lemma 14. If \( f(x, y, z) \) is given by (2.3), where \( \phi(y, z) \) satisfies (2.8), (2.9), then if (2.5) does not hold we have
\[
h \equiv g \equiv \frac{1}{2} \pmod{1};
\]
(3.6)
further, in the form equivalent to \( f(x, y, z) \) with \( h = g = \frac{1}{2} \),
\[
|x_0 - \frac{1}{2}| < .016 \pmod{1}.
\]
(3.7)

Proof. There is clearly no loss of generality in supposing that
\[
0 \leq h, g < 1
\]
in (2.3). We then have to prove that \( h = g = \frac{1}{2} \) and that (3.7) holds under the assumption that (2.5) is false for some \( x_0 \).

Since \( y_0, x_0 \equiv \frac{1}{2} \pmod{1} \) and \( \Delta^2 = 50k^2 \), (2.5) holds unless, for some \( x_0 \),
\[
M(f ; x_0, \frac{1}{2}, \frac{1}{2}) \geq (\frac{10}{9}k)^2 \cdot (\frac{4}{3} \times .9814)^2 > 1.484.
\]
(3.9)

Now
\[
f(\pm x, \frac{1}{2}, \frac{1}{2}) = (\pm x + \frac{1}{2}h + \frac{1}{2}g)^2 - \frac{1}{2}k,
\]
\[
f(\pm x, \frac{1}{2}, -\frac{1}{2}) = (\pm x + \frac{1}{2}h - \frac{1}{2}g)^2 + \frac{1}{2}k.
\]
Hence, for any \( x_0 \) for which (3.9) holds, we have
\[
|p \pm x_0 + \frac{1}{2}h \pm \frac{1}{2}g| > 1.484,
\]
(3.10)
\[
(p \pm p_0 + \frac{1}{2}h - \frac{1}{2}g)^2 + \frac{1}{2}k > 1.484
\]
(3.11)
for all integral \( p \) and any choice of sign. In (3.10) we choose \( p \) so that
\[
1 \leq |p \pm x_0 + \frac{1}{2}h \pm \frac{1}{2}g| = \alpha < 2,
\]
and in (3.11) we choose \( p \) so that
\[
\beta = |p \pm x_0 + \frac{1}{2}h - \frac{1}{2}g| \leq \frac{1}{2}.
\]
We then have, from (3.10), either
\[
\alpha^2 > \frac{1}{2}k + 1.484 > 3.6005, \quad \alpha > 1.99,
\]
or
\[
\alpha^2 < \frac{1}{2}k - 1.484 < 1.032, \quad \alpha < 1.016;
\]
it follows that
\[
-.016 < \pm x_0 + \frac{1}{2}h + \frac{1}{2}g < .016 \pmod{1}.
\]
(3.12)
Similarly, from (3.11) we deduce that
\[
\beta^2 > 1.484 - \frac{1}{2}k > .2261, \quad \beta > .475,
\]
whence
\[ \frac{1}{2} - .025 < x_0 + \frac{1}{2} h - \frac{1}{3} g < \frac{1}{2} + .025 \pmod{1}. \] (3.13)

Adding (3.12), (3.13) with suitable choices of sign we find that
\[ \frac{1}{2} - .041 < h, g < \frac{1}{2} + .041 \pmod{1}, \] (3.14)
\[ -.032 < 2x_0 < .032 \pmod{1}, \]
whence either \( |x_0| < .016 \pmod{1} \) or \( |x_0 - \frac{1}{2}| < .016 \pmod{1} \). If \( h, g \) satisfy (3.8) it is clear from (3.12) that the second alternative must hold, i.e. that \( x_0 \) satisfies (3.7).

If we apply the integral unimodular transformation \( x = X, y = -Z, z = Y + 6Z \) to \( f(x, y, z) \) we find that
\[ f(x, y, z) = (X + h_1 Y + g_1 Z)^2 - \frac{1}{2} k(Y^2 + 6YZ + Z^2) \]
where

\[ \begin{align*}
    h_1 &= g, \\
    g_1 &= 6g - h
\end{align*} \]
and
\[ X, Y, Z = x_0 \frac{1}{2}, \frac{1}{2} \pmod{1}. \]

It follows that (3.14) must still hold if \( h, g \) are replaced by \( h_1, g_1 \). Similarly, using the inverse transformation \( x = X, y = 6Y + Z, z = -Y \), we see that (3.14) must still hold if \( h, g \) are replaced by
\[ h^{-1} = 6h - g, \quad g^{-1} = h. \]

Let now \( \mathcal{R} \) be the region of the \( h, g \)-plane defined by
\[ \frac{1}{2} - .041 < h, g < \frac{1}{2} + .041, \] (3.15)
and let \( T \) be the matrix
\[ T = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix} \]
(which is clearly of infinite order). Then, if \( P \) is the point \( (h, g) \), we have
\[ P_1 = \begin{pmatrix} h_1 \\ g_1 \end{pmatrix} = T(P), \]
\[ P^{-1} = \begin{pmatrix} h^{-1} \\ g^{-1} \end{pmatrix} = T^{-1}(P), \]

Since \( 0 \leq h, g < 1 \), (3.14) shows that \( P \in \mathcal{R} \). Also, by what has been proved above, \( T(P) \) and \( T^{-1}(P) \) are congruent \( \pmod{1} \) to a point of \( \mathcal{R} \); and since \( P \) satisfies (3.15) it is clear that in fact
\[ T(P) = (0, 2) \in \mathcal{R}, \quad T^{-1}(P) = (2, 0) \in \mathcal{R}. \]

Finally, the argument shows that the point \( T^n(P) \) must satisfy (3.14), i.e. must be congruent to a point of \( \mathcal{R} \), if (3.9) holds.
It now follows from Lemma 4 that this is possible only if \( P \) satisfies
\[
T(P) = P + (0, 2),
\]
i.e. if \( P = (h, g) = (\frac{1}{2}, \frac{1}{2}) \).
This completes the proof of the lemma.

**Lemma 15.** Suppose that
\[
f(x, y, z) = (x + \frac{1}{2} y + \frac{1}{2} z)^2 - \frac{2}{3} k(y^2 + 6yz + z^2),
\]
where \( k \) satisfies (2.9), and suppose that (2.5) is false with \( y_0, z_0 \equiv \frac{1}{2} \) (mod 1) and \( x_0 \) satisfying (3.7). Then
\[
k = 1, \quad x_0 \equiv \frac{1}{2} \text{ (mod 1)}. \tag{3.17}
\]

**Proof.** Since \( f \) has determinant \( D = -\frac{2}{3} k^2 \) and \( k \) satisfies (2.9), it is quickly verified that (2.2) holds, i.e.
\[
0 < a \leq (4|D|)^{\frac{1}{8}}
\]
with
\[
a = f(1, 1, 0) = \frac{1}{2} - \frac{1}{3} k.
\]
If we make the equivalence transformation
\[
x = X + Z, \quad y = Y, \quad z = X
\]
we find that
\[
f(x, y, z) = a F(X, Y, Z)
\]
so that
\[
F(X, Y, Z) = \left[ X - \frac{1}{4a} (15k - 3) Y + \frac{3}{2a} Z \right]^2 - \Phi(Y, Z)
\]
with
\[
X, Y, Z \equiv \frac{1}{2}, \frac{1}{2}, x_0 - \frac{1}{2} \text{ (mod 1)}. \tag{3.20}
\]
Now the form (3.19) is of the original type (2.3) and we are supposing that (2.4) is false, i.e. that
\[
M(F; \frac{1}{2}, \frac{1}{2}, x_0 - \frac{1}{2}) \geq (12 \Delta)^{\frac{1}{4}}
\]
(where here \( \Delta^2 \) is the discriminant of \( \Phi(Y, Z) \)). By Theorem 3 it follows that we can apply an equivalence transformation to \( Y, Z \), say \( Y = \alpha Y' + \beta Z', \ Z = \gamma Y' + \delta Z' \) so that \( \Phi(Y, Z) \) is transformed into one of (2.6), (2.7), (2.8) (with \( Y', Z' \) for \( y, z \)), and that then
\[
Y', Z' \equiv \frac{1}{2}, \frac{1}{2} \text{ (mod 1)}. \tag{3.20}
\]
Since \( \alpha, \beta, \gamma, \delta \) are integers, we deduce that each of \( Y \) and \( Z \) must be congruent to 0 or \( \frac{1}{2} \) (mod 1); hence, by (3.20),
and so, by (3.7), \(x_0 \equiv \frac{1}{3} \pmod{1}\) as required.

Further, by Lemmas 12, 13 and 14, we see that each of the coefficients 
\(\frac{3}{2a} (15k-3)\) and \(\frac{3}{2a}\) must be congruent to either 0 or \(\frac{1}{3}\) \((\pmod{1})\). Since \(a = \frac{2}{3}k\) and \(k\) satisfies (2.9), it is easy to see that this can hold only if

\[
\frac{3}{2a} = \frac{3}{2}, \quad \frac{1}{4a} (15k-3) = 3,
\]

whence \(a = 1, k = 1\). This proves the lemma.

By Theorem 3 and Lemmas 12-15, we have now shown that (2.5) holds unless

\[
f(x, y, z) \text{ is equivalent to one of}
\]

\[
f_1(x, y, z) = x^2 - \frac{1}{3}(y^2 + 8yz + z^2), \quad x_0, y_0, z_0 = \frac{1}{3}, \frac{1}{3}, \frac{1}{3};
\]

\[
f_2(x, y, z) = x^2 - (2y^2 + 12yz + 3z^2), \quad x_0, y_0, z_0 = \frac{1}{3}, \frac{1}{3}, \frac{1}{3};
\]

\[
f_3(x, y, z) = (x + \frac{1}{2}y + \frac{1}{2}z)^2 - \frac{1}{9}(y^2 + 6yz + z^2), \quad x_0, y_0, z_0 = \frac{1}{3}, \frac{1}{3}, \frac{1}{3}.
\]

To complete the proof of Theorem 2 (and hence of Theorem 1) we have only to observe that

\[
f_4(x, y, z) = Q_1(x, y, z);
\]

\[
2f_1(x, y, z) = 2x^2 - (y + 4z)^2 + 15z^2
\]

\(~Q_2(x, y, z) \text{ with } x_0, y_0, z_0 = \frac{1}{3}, \frac{1}{3}, \frac{1}{3};\)

\[
f_5(x, y, z) = 2(x - y - 3z)^2 - (x - 2y + 6z)^2 + 15z^2
\]

\(~Q_2(x, y, z) \text{ with } x_0, y_0, z_0 = \frac{1}{3}, \frac{1}{3}, \frac{1}{3}.
\)

4. Proof of Lemmas 8-11. For the proofs of Lemmas 8-11 we must appeal to the general theory of two-dimensional inhomogeneous lattices developed in Barnes and Swinnerton-Dyer [3]. For the convenience of the reader we state briefly the particular results we shall need.

We denote by \([b_1, b_2, b_3, \ldots]\) the continued fraction

\[
b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots}},
\]

where \(b_i\) is integral and \(|b_i| \geq 2\). If \(b_i > 0\) for all \(i\) and \(b_i \geq 4\) for some arbitrarily large \(i\), we have
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\[ [b_1, b_2, \ldots, b_n, \ldots] < [b_1, b_2, \ldots, b_n, b'_n, \ldots] \tag{4.1} \]

provided only that \( b_n+1 < b'_n+1 \), in particular

\[ [b_1, b_2, \ldots, b_n - 1] < [b_1, b_2, \ldots, b_n, \ldots] < [b_1, b_2, \ldots, b_n]. \tag{4.2} \]

Let \( \{a_n\} (\infty < n < \infty) \) be a chain of positive even integers for which the inequality \( a_n \geq 4 \) holds for some arbitrarily large \( n \) of each sign. For each \( n \) we define

\[ \theta_n = [a_n, a_{n-1}, a_{n-2}, \ldots] \]
\[ \phi_n = [a_{n+1}, a_{n+2}, a_{n+3}, \ldots] \]

so that, by (4.2), \( \theta_n > 1, \phi_n > 1 \). For any real \( \lambda, \mu \) with \( \lambda \mu > 0 \), the inhomogeneous lattice \( \mathcal{L} \) defined by

\[ \xi = \lambda \left( \theta_n (u - \frac{1}{2}) + (v - \frac{1}{2}) \right) \]
\[ \eta = \mu \left( (u - \frac{1}{2}) + \phi_n (v - \frac{1}{2}) \right), \]

where \( u, v \) run through all integral values, is called a symmetrical lattice corresponding to the chain \( \{a_n\} \). If \( \mathcal{L} \) has determinant \( \Delta \), we have \( \Delta = \lambda \mu (\theta_n \phi_n - 1) \), so that, for points of \( \mathcal{L} \),

\[ \xi \eta = \frac{\Delta}{\theta_n \phi_n - 1} (\theta_n x + y)(x + \phi_n y), \quad x, y = \frac{1}{2}, \frac{1}{2} \mod 1. \tag{4.3} \]

A symmetrical lattice \( \mathcal{L} \) is admissible for \( \mathcal{R}_m: -1 \leq \xi \eta \leq m \) (\( m > 1 \)) if and only if the inequalities

\[ \frac{\Delta}{m} = \frac{4(\theta_n \phi_n - 1)}{(\theta_n + 1)(\phi_n + 1)} = \Delta^+_n \tag{4.4} \]
\[ \Delta \geq \frac{4(\theta_n \phi_n - 1)}{2(\theta_n - 1)(\phi_n - 1)} = \Delta^-_n \tag{4.5} \]

hold for all \( n \).

For any \( m > 1 \), all critical lattices of \( \mathcal{R}_m \) are symmetrical. Moreover, if \( 1 < m \leq 3 \), the inequality

\[ \Delta(\mathcal{L}) \geq 2(m + 1) \tag{4.6} \]

holds for any \( \mathcal{R}_m \)-admissible \( \mathcal{L} \) which is not symmetrical.

Finally, if \( 0 < D < 2(k + 1) \) and, for any \( n \),

\[ \Delta^+_n \leq D, \quad \Delta^-_n \leq \frac{D}{k}, \tag{4.7} \]

then the inequality
holds with $\alpha = \theta_n$ or $\alpha = \phi_n$.

**Proof of Lemma 8.** Let $m = \frac{5}{3}$ and suppose that $\mathcal{L}$ is $\mathcal{R}_m$-admissible and has $\Delta(\mathcal{L}) \leq 4\sqrt[3]{\frac{5}{3}} = 4\sqrt[3]{m}$. Since $2(m+1) > 4\sqrt[3]{m}$, $\mathcal{L}$ must be symmetrical; and, by (4.4), (4.5) we require

$$2(k-1) - D < 2(k+1) - D$$

for all $n$. Thus (4.7) holds with $D = 4\sqrt[3]{\frac{5}{3}}$, $k = \frac{5}{3}$; since now $D^2 = 16k$, (4.8) shows that for all $n$

$$\theta_n = \phi_n = \frac{2(k-1)}{2(k+1) - D} = 4 + \sqrt{15} = [8, 8, 8, \ldots].$$

Hence $\{a_n\}$ is the periodic chain $\{8\}$ and, by (4.3),

$$\xi \eta = \frac{\Delta}{2 \sqrt{15}} (x^2 + 8xy + y^2), \quad x, y = \frac{1}{4}, \frac{1}{2} \pmod{1}.$$ 

Finally, since now $\Delta_n = 4\sqrt[3]{\frac{5}{3}}$ for all $n$, we require

$$\Delta_n \geq 4\sqrt[3]{\frac{5}{3}}, \quad \Delta_n \leq 4\sqrt[3]{\frac{5}{3}},$$

whence

$$\Delta = 4\sqrt[3]{\frac{5}{3}}, \quad \Delta = 4\sqrt[3]{\frac{5}{3}} = \frac{5}{3}.$$ 

**Proof of Lemma 9.** It is shown in [3], Theorem 9, that if $m \geq 2$ and $\mathcal{L}$ is admissible for $\mathcal{R}_m$, then either $\Delta(\mathcal{L}) \geq \sqrt{33}$ or $\mathcal{L}$ is a symmetrical lattice corresponding to the chain $\{6\}$. Lemma 9 follows at once from this, on observing that, for the chain $\{6\}$ we have

$$\theta_n = \phi_n = [6] = 3 + 2\sqrt{2}$$

for all $n$,

$$\xi \eta = \frac{\Delta}{4\sqrt{2}} (x^2 + 6xy + y^2), \quad x, y = \frac{1}{4}, \frac{1}{2} \pmod{1},$$

where

$$\Delta \geq m \Delta^* = \frac{4m}{\sqrt{2}}, \quad \frac{\Delta}{4\sqrt{2}} \geq \frac{1}{m}.$$ 

**Proof of Lemma 10.** Suppose first that the inequalities

$$\Delta_n \leq \frac{1}{2}, \quad \Delta_n \leq \frac{8}{3}, \quad \frac{1}{2} \Delta_n \leq \frac{1}{30}$$

(4.9)
hold for all $n$. We show that then $\{a_n\}$ is the periodic chain $\{6, 4\}$. For (4.7) holds with 
$$D = \frac{5}{2}\sqrt{30}, \quad k = \frac{1}{2},$$
and so (4.8) gives, for all $n$,
$$\left| x - \frac{5}{12 - 2\sqrt{30}} \right| \leq \frac{1}{12 - 2\sqrt{30}}$$
$$\frac{2}{6 - \sqrt{30}} \leq x \leq \frac{3}{6 - \sqrt{30}},$$
i.e.
$$[4, 6] \leq x \leq [6, 4], \quad (4.10)$$
where $x = \theta_n$ or $x = \phi_n$. Using (4.1), (4.2), we see that $a_n - 1 < \theta_n = [a_n, a_{n-1}, \ldots] < a_n$, and so (4.10) shows that $a_n = 4$ or 6. If $a_n = 4$, (4.10) with $x = \theta_n$ and $x = \phi_n$ gives
$$[4, a_{n-1}, \ldots] \geq [4, 6, 4, \ldots],$$
$$[4, a_{n+1}, \ldots] \geq [4, 6, 4, \ldots],$$
whence $a_{n-1} \geq 6$, $a_{n+1} \geq 6$, so that $a_{n-1} = a_{n+1} = 6$. Similarly, if $a_n = 6$, (4.10) shows that $a_{n-1} = a_{n+1} = 4$. It follows that $\{a_n\}$ is $\{6, 4\}$, as required.

Now if $\mathcal{L}$ is symmetrical and admissible for $\mathcal{R}_m$ with $m \geq \frac{1}{2}$, either (4.9) holds for all $n$ or, by (4.4) and (4.5),
$$\Delta > \min \left\{ \frac{5}{30}, \frac{8m}{17} \right\} = \frac{5}{30};$$
while if $\mathcal{L}$ is not symmetrical, (4.6) gives
$$\Delta \geq 2(m + 1) = \frac{5}{2} > \frac{5}{30}.$$

It follows that if $\mathcal{L}$ is $\mathcal{R}_m$-admissible, with $m \geq \frac{1}{2}$ and $\Delta (\mathcal{L}) \leq \frac{5}{30}$, then $\mathcal{L}$ is a symmetrical lattice corresponding to the chain $\{6, 4\}$. For this chain, $\theta_n$ and $\phi_n$ are $\frac{5}{6}(6 + \sqrt{30})$, $\frac{5}{6}(6 + \sqrt{30})$ in some order, for each $n$, whence
$$\xi \eta = \frac{\Delta}{2\sqrt{30}} = \frac{2x^3 + 12xy + 3y^2}{6}, \quad x, y \equiv \frac{1}{2}, \frac{1}{2} \pmod{1},$$
$$\frac{\Delta}{2\sqrt{30}} \geq \frac{4m}{17}.$$

Proof of Lemma 11. For $m \geq \frac{5}{6}$ we have
$$2(m + 1) > 4\sqrt{3};$$
it is therefore sufficient to show that there exists no symmetrical lattice satisfying

$$\Delta_n < 4\sqrt{3}, \quad \frac{\Delta^*}{\eta} < 4\sqrt{3}$$  \hspace{1cm} (4.11)$$

for all \( n \).

Now if (4.11) holds, then (4.7) holds with \( D = 4\sqrt{3}, \ k = \frac{\eta}{\theta} \). Hence, by (4.8),

$$\left| \frac{17}{35 - 18\sqrt{3}} \right| < \frac{6}{35 - 18\sqrt{3}},$$

$$\left\{ \frac{11}{35 - 18\sqrt{3}} \right\} < x < \frac{23}{35 - 18\sqrt{3}},$$

with \( x = \theta_n \) or \( x = \phi_n \). Thus for all \( n \) we have

$$\theta_n > \frac{11}{35 - 18\sqrt{3}} > \frac{11}{3.8231} > 2.87,$$

so that \( a_n \geq 4 \) for all \( n \).

If now \( a_n \geq 6 \) for some \( n \), we have, using (4.1),

$$\theta_n \geq [6, 4, 4, 4, ...] = 4 + \sqrt{3}, \quad \phi_n \geq [4, 4, 4, ...] = 2 + \sqrt{3},$$

whence

$$\frac{1}{4} \Delta_n = \frac{10 + 6\sqrt{3}}{(5 + \sqrt{3})(3 + \sqrt{3})} = \frac{9 + 7\sqrt{3}}{33} = 0.64...,$$

whereas (4.11) gives

$$\frac{1}{4} \Delta^* < \frac{9\sqrt{3}}{26} < 0.6.$$

It follows that \( a_n = 4 \) for all \( n \). But then \( \Delta_n = 4\sqrt{3} \), contradicting (4.11). Thus (4.11) cannot hold for all \( n \).

5. It is not difficult, using the same methods, to show that Theorem 1 remains true if in (1.12) and (1.13) we replace \( 4/15 \) by a slightly smaller constant. The ranges of \( d \) given in Lemma 7 are then slightly increased, but Theorem 3 still holds with the forms (2.6), (2.7) replaced by \( \frac{1}{4} k (y^2 + 8yz + z^2) \), \( k (2y^2 + 12yz + 3z^2) \), where \( k \) is nearly 1. (For the proof of this, we need stronger versions of Lemmas 8 and 10, but these are easily obtained.) We may then show, just as in Lemmas 14 and 15, that in each case \( k \) must be 1. I have not given the details, to avoid complicating the main lines of the proof.

Thus the 'second minimum' \( (\Delta_n |D|)^{1/4} \) is isolated, and the problem remains to find the third and any further minima. Since Davenport [4] has given a (zero) form with \( M(Q) = (\frac{1}{4} |D|)^{1/4} \), the third minimum is at least \( (\frac{1}{4} |D|)^{1/4} \).
I think it likely that the methods of this paper will not prove adequate for a complete analysis of the problem. It is easy to see that, in particular, the method will break down if there are uncountably many distinct lattices admissible for $R_m$ with determinant not exceeding $1/\mu$; and this situation does in fact arise if one attempts to find the forms $Q$ with $M(Q) \geq (|D|)^t$.

However, a complete answer to the problem may be obtainable by the use of 'local' methods on the chain $\{a_n\}$ associated with $\phi(y, z)$ in the form (2.3). I hope to investigate this attack in the near future.

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References