THE INHOMOGENEOUS MINIMUM OF A TERNARY QUADRATIC FORM

BY

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1. Let $Q(x, y, z)$ be an indefinite ternary quadratic form with real coefficients and determinant $D \neq 0$. Davenport [4] has shown that, given any real numbers x_0, y_0, z_0, there exist x, y, z congruent (modulo 1) to x_0, y_0, z_0 satisfying

$$|Q(x, y, z)| \leq (\tfrac{27}{100}|D|)^{\frac{1}{3}}; \tag{1.1}$$

the equality sign can hold if and only if Q is equivalent (under integral unimodular transformation of the variables) to a multiple of the form

$$Q_1(x, y, z) = x^2 + 5y^2 - z^2 + 5yz + zx.$$

The main weapon used in the proof was a generalization of Minkowski's result on the inhomogeneous minimum of a binary quadratic form, namely:

If $f(x, y)$ is a binary quadratic form with real coefficients and discriminant Δ^2, where $\Delta > 0$, and $\mu > 0$, $\nu > 0$, $\mu\nu \geq \frac{1}{16}$, then, for any real numbers x_0, y_0, there exist $x, y \equiv x_0, y_0 \pmod 1$ satisfying

$$-\nu\Delta \leq f(x, y) \leq \mu\Delta. \tag{1.2}$$

By obtaining an 'isolation' of this inequality when ν is approximately 2μ, Davenport was able to show that the result (1.1) is isolated: that is to say, there exists a positive constant δ such that the inequality

$$|Q(x, y, z)| \leq (1-\delta)(\tfrac{27}{100}|D|)^{\frac{1}{3}} \tag{1.3}$$

can be satisfied whenever Q is not equivalent to a multiple of the special form Q_1.

Recently Swinnerton-Dyer and I [3] made a detailed investigation of results of the type (1.2) and developed a technique for obtaining best possible results for any value of the ratio ν/μ. I use this technique here, together with Davenport's general method of attack on the problem, to find the best possible value of δ in (1.3).

The proof leads naturally to a stronger assertion than (1.3) and shows that the result (1.1) is isolated not only in respect of the form Q_1 but also in respect of the values $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ (mod 1) of x_0, y_0, z_0. To make this statement precise we introduce the following notation:

If $Q = Q(x, y, z)$ is any indefinite ternary quadratic form and x_0, y_0, z_0 any real numbers, we set

$$M(Q; x_0, y_0, z_0) = \text{g.l.b.}\ |Q(x, y, z)|, \tag{1.4}$$

where the lower bound is taken over all sets $x, y, z \equiv x_0, y_0, z_0$ (mod 1). We then write

$$M(Q) = \text{l.u.b.}\ M(Q; x_0, y_0, z_0), \tag{1.5}$$

where the upper bound is taken over all real x_0, y_0, z_0; we call $M(Q)$ the *inhomogeneous minimum* of Q.

Clearly (1.1) implies that always

$$M(Q) \leq (\tfrac{27}{100}|D|)^{\frac{1}{3}}.$$

Now if T is any 3×3 matrix with integral elements and determinant ± 1 and we make the transformation of the variables expressed in vector notation by

$$\underset{\sim}{X} = T\underset{\sim}{x}, \tag{1.6}$$

then $Q(x, y, z)$ becomes, say, $Q'(X, Y, Z)$, and the forms Q, Q' are said to be equivalent. If also we define

$$\underset{\sim}{X}_0 = T\underset{\sim}{x}_0, \tag{1.7}$$

then it is clear that

$$M(Q'; X_0, Y_0, Z_0) = M(Q; x_0, y_0, z_0). \tag{1.8}$$

Further, since X_0, Y_0, Z_0 run through all real numbers when x_0, y_0, z_0 do, we have

$$M(Q') = M(Q). \tag{1.9}$$

It will always be understood, when we pass to an equivalent form by a transformation (1.6), that any particular values of x_0, y_0, z_0 under consideration are subjected to the corresponding transformation (1.7).

The complete statement of the results we shall obtain is given, in the above notation, by

Theorem 1. (i) *If* $Q(x, y, z)$ *is not equivalent to a multiple of either of the forms*

$$Q_1(x, y, z) = x^2 - y^2 - z^2 + xy - 7yz + zx \tag{1.10}$$

$$Q_2(x, y, z) = 2x^2 - y^2 + 15z^2, \tag{1.11}$$

then

$$M(Q) < (\tfrac{4}{15}|D|)^{\frac{1}{3}}. \tag{1.12}$$

(ii) *For the special forms* Q_1, Q_2 *we have*

$$M(Q_i; x_0, y_0, z_0) < (\tfrac{4}{15}|D|)^{\frac{1}{3}} \qquad (i = 1, 2) \tag{1.13}$$

unless $x_0, y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ (mod 1); *further,*

$$M(Q_1; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{27}{100}|D|)^{\frac{1}{3}} = M(Q_1), \tag{1.14}$$

$$M(Q_2; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{4}{15}|D|)^{\frac{1}{3}} = M(Q_2). \tag{1.15}$$

In the course of the proof we shall use the following lemmas:

Lemma 1. *If* $Q(x, y, z)$ *is indefinite and has determinant* $D < 0$ *then there exist integers* x_1, y_1, z_1 *satisfying*

$$0 < Q(x_1, y_1, z_1) \le (4|D|)^{\frac{1}{3}}. \tag{1.16}$$

This is Theorem 2 of Davenport [5].

Lemma 2. *Let* β, B *be real numbers with* $B > \frac{1}{4}$. *Then for any real* x_0 *there exists an* x *satisfying*

$$x \equiv x_0 \pmod 1, \qquad |x^2 - \beta^2| < B,$$

provided that

$$\beta^2 < B^2 + \tfrac{1}{4} \qquad \textit{if } B \textit{ is integral},$$

$$\beta^2 < B + \tfrac{1}{4}[2B]^2 \qquad \textit{if } B \textit{ is not integral}.$$

This result is contained in Davenport [4], Lemma 5.

Lemma 3. *Let* T *be an integral* 2×2 *matrix of infinite order and of determinant* ± 1, *and let* $\mathcal{R}$ *be a bounded point set in the Cartesian plane. Suppose that, for some point* A *with integral coordinates, any point* P *of* $\mathcal{R}$ *has the property that either* $T(P) - A$ *belongs to* $\mathcal{R}$ *or* $T(P)$ *is not congruent* (mod 1) *to a point of* $\mathcal{R}$.

Then, if P *is a point such that* $T^n(P)$ *is congruent* (mod 1) *to a point of* $\mathcal{R}$ *for all integral* $n \gtreqless 0$, P *is the unique point* F *of* $\mathcal{R}$ *defined by*

$$T(F) - A = F.$$

This result is due to Cassels, and is quoted by Bambah [1]; an alternative proof is given in Barnes and Swinnerton-Dyer [2], Theorem D; (the region $\mathcal{R}^*$ appearing in this theorem may be taken as the set of all points of the plane which are not congruent (mod 1) to a point of $\mathcal{R}$).

2. The results stated in Theorem 1 for $M(Q_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $M(Q_2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ are easily established by congruence considerations, and it is convenient to dispose of these at once.

(i) We have

$$4Q_1 = (2x + y + z)^2 - 5(y + 3z)^2 + 40z^2.$$

If $x, y, z \equiv \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$, then $2x$, $2y$, $2z$ are odd integers; we may therefore write

$$4Q_1 = X^2 - 5Y^2 + 10Z^2,$$

where X, Y, Z are integral, $Z = 2z$ is odd and $X - Y = 2x - 2z$ is even. We then have

$$4Q_1 \equiv 2 \pmod 4, \quad 4Q_1 \equiv 0, \ \pm 1 \pmod 5,$$

whence $|4Q_1| \geq 6$. We have thus shown that

$$|Q_1(x, y, z)| \geq \tfrac{3}{2} \quad \text{whenever } x, y, z \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}.$$

Since

$$|Q_1(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})| = \tfrac{3}{2}, \quad D(Q_1) = -\tfrac{25}{2},$$

it follows that

$$M(Q_1; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{3}{2} = (\tfrac{27}{100}|D|)^{\frac{1}{3}},$$

as required.

(ii) If $x, y, z \equiv \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$, then

$$4Q_2 = 2X^2 - Y^2 + 15Z^2,$$

where X, Y, Z are odd integers. Hence

$$4Q_2 \equiv 0 \pmod 8,$$

and it is easy to see, by considering congruences mod 3, that $4Q_2 \neq 0$. We therefore have

$$|Q_2(x, y, z)| \geq 2 \quad \text{whenever } x, y, z \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}.$$

Since

$$|Q_2(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{1}{2})| = 2, \quad D(Q_2) = -30,$$

it follows that

$$M(Q_2; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = 2 = (\tfrac{4}{15}|D|)^{\frac{1}{3}}$$

as required.

To complete the proof of Theorem 1 we have therefore to establish

Theorem 2. *The inequality*

$$M(Q; x_0, y_0, z_0) < (\tfrac{4}{15}|D|)^{\frac{1}{3}} \tag{2.1}$$

holds unless Q is equivalent to a multiple of Q_1 or Q_2 with $x_0, y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ (mod 1).

For the proof of Theorem 2, we first observe that there is no loss of generality in supposing that $D<0$ (since we may consider $-Q$ in place of Q if necessary). Let $a=Q(x_1, y_1, z_1)$ be any value assumed by Q for coprime integers x_1, y_1, z_1 satisfying (1.16), so that

$$0<a\leq(4|D|)^{\frac{1}{3}}. \tag{2.2}$$

Making an appropriate equivalence transformation, we see that $\frac{1}{a}Q(x, y, z)$ is equivalent to a form

$$f(x, y, z)=(x+hy+gz)^2-\phi(y, z) \tag{2.3}$$

where h, g are real and $\phi(y, z)$ is an indefinite quadratic form of discriminant

$$\Delta^2=\frac{4|D|}{a^3}\geq 1. \tag{2.4}$$

Then (2.1) is equivalent to the assertion that

$$M(f; x_0, y_0, z_0)<(\tfrac{1}{15}\Delta^2)^{\frac{1}{3}}. \tag{2.5}$$

The first step in the proof of (2.5) is the consideration of the possible forms of $\phi(y, z)$. In this section we prove

Theorem 3. *If $f(x, y, z)$ is given by (2.3), (2.4), then (2.5) holds unless either*

$$\text{(i)}\quad \phi(y, z)=\tfrac{1}{2}(y^2+8yz+z^2),\quad y_0, z_0\equiv\tfrac{1}{2}, \tfrac{1}{2}, \tag{2.6}$$

or

$$\text{(ii)}\quad \phi(y, z)=2y^2+12yz+3z^2,\quad y_0, z_0\equiv\tfrac{1}{2}, \tfrac{1}{2}, \tag{2.7}$$

or

$$\text{(iii)}\quad \phi(y, z)=\tfrac{5}{4}k(y^2+6yz+z^2),\quad y_0, z_0\equiv\tfrac{1}{2}, \tfrac{1}{2}, \tag{2.8}$$

where

$$\cdot 9906<k<1.0063 \tag{2.9}$$

(or equivalent forms).

It is convenient to set

$$d=(\tfrac{8}{15}\Delta^2)^{\frac{1}{3}}, \tag{2.10}$$

so that, by (2.4),

$$d\geq(\tfrac{8}{15})^{\frac{1}{3}}>\tfrac{4}{5}. \tag{2.11}$$

Lemma 4. *Let $\mu>0$, $\nu>0$ be defined by*

$$\mu\Delta=\tfrac{1}{2}d-\tfrac{1}{4} \tag{2.12}$$

$$\nu\Delta=\begin{cases}\tfrac{1}{2}d+\tfrac{1}{4}[d]^2 & \textit{if } d \textit{ is not integral}\\ \tfrac{1}{4}(d^2+1) & \textit{if } d \textit{ is integral.}\end{cases} \tag{2.13}$$

Suppose that there exist $y, z \equiv y_0, z_0 \pmod 1$ *with*

$$-\mu\Delta < \phi(y, z) < \nu\Delta. \tag{2.14}$$

Then for any x_0

$$M(f; x_0, y_0, z_0) < \tfrac{1}{2} d = (\tfrac{1}{15}\Delta^2)^{\frac{1}{3}}. \tag{2.15}$$

Proof. If in (2.14) we have $\phi(y,z) \leq 0$, then, for any x_0, we can choose $x \equiv x_0$ with $|x + hy + gz| \leq \frac{1}{2}$. For this choice of x, y, z we have

$$0 \leq f(x, y, z) < \tfrac{1}{4} + \mu\Delta = \tfrac{1}{2} d.$$

If, however, $\phi(y, z) > 0$, we have

$$0 < \phi(y, z) < \nu\Delta;$$

applying Lemma 2 with $\beta^2 = \phi(y, z)$, $B = \frac{1}{2} d$ (noting that then $B > \frac{1}{4}$ by (2.11)), we see that for any x_0 we can choose $x \equiv x_0$ with

$$|f(x, y, z)| < \tfrac{1}{2} d.$$

The required result (2.15) follows immediately.

In the notation of Barnes and Swinnerton-Dyer [3] we denote by $\mathcal{R}_m$ the set of points of the ξ, η-plane defined by

$$-1 \leq \xi\eta \leq m.$$

An inhomogeneous lattice $\mathcal{L}$ is a set of points

$$\xi = \alpha x + \beta y,$$
$$\eta = \gamma x + \delta y,$$

where x, y run through all numbers congruent (mod 1) to x_0, y_0 respectively, and

$$\Delta = \Delta(\mathcal{L}) = |\alpha\delta - \beta\gamma| \neq 0$$

is the determinant of $\mathcal{L}$. $\mathcal{L}$ is *admissible* for $\mathcal{R}_m$ if it has no point in the interior of $\mathcal{R}_m$. The critical determinant D_m of $\mathcal{R}_m$ is defined to be the lower bound of $\Delta(\mathcal{L})$ over all admissible lattices $\mathcal{L}$. We now have

Lemma 5. *For all* $m \geq 1$,

$$D_m \geq 4\sqrt{m}. \tag{2.16}$$

This result is equivalent to Davenport's result quoted in § 1 (Davenport [4], Lemma 3). A less direct proof is given in Barnes and Swinnerton-Dyer [3].

Now since $\phi(y, z)$ has discriminant Δ^2, it may be expressed as the product of two linear forms of determinant Δ. Thus the form

$$\frac{1}{\mu\Delta}\phi(y, z)$$

with $y, z \equiv y_0, z_0$ runs over the values of $\xi\eta$ corresponding to a lattice $\mathfrak{L}$ of determinant $\frac{1}{\mu}$. From the definition of D_m it is therefore clear that (2.10) is certainly soluble, for any y_0, z_0, if

$$\frac{1}{\mu} < D_m, \quad \text{where} \quad m = \frac{\nu}{\mu}.$$

Combining this result with Lemma 3, we have

Lemma 6. *If* μ, ν *are defined as in* Lemma 4 *and*

$$m = \frac{\nu}{\mu}, \tag{2.17}$$

then the inequality (2.5) *certainly holds unless*

$$\frac{1}{\mu} \geq D_m. \tag{2.18}$$

As a first step towards the elimination of possible values of d, we use (2.18) with the estimate (2.16) for D_m.

Lemma 7. *If* (2.5) *does not hold, then* d *satisfies either*

$$d = 2, \tag{2.19}$$

or

$$2.969 < d \leq 3, \tag{2.20}$$

or

$$3.975 < d \leq 4, \tag{2.21}$$

or

$$4.994 < d \leq 5. \tag{2.22}$$

Proof. By Lemma 6 and (2.12) we have

$$\frac{1}{\mu} \geq 4\sqrt{m},$$

i.e.

$$16\,\mu\nu \leq 1.$$

Substituting for μ, ν and noting that, by (2.10),

$$8\Delta^2 = 15 d^3,$$

this inequality becomes

$$8(2d-1)(2d+[d]^2) \leq 15d^3 \quad \text{if } d \text{ is not integral,} \tag{2.23}$$

$$8(2d-1)(d^2+1) \leq 15d^3 \quad \text{if } d \text{ is integral.} \tag{2.24}$$

Now (2.24) may be written in the form

$$(d-2)(d^2-6d+4) \leq 0,$$

and this inequality is easily seen to be false if $d \geq 6$ or if $\frac{4}{5} < d \leq 1$. Thus (2.24) can hold for integral $d > \frac{4}{5}$ only if $d = 2, 3, 4$ or 5. Further, since $[d] > d-1$, $2d + [d]^2 > d^2 + 1$. Hence (2.23) cannot hold if d satisfies $d \geq 6$ or $\frac{4}{5} < d \leq 1$.

It remains for us to consider non-integral d satisfying (2.19) and $1 < d < 6$.

(i) If $[d] = 1$, (2.23) is

$$15d^3 - 32d^2 + 8 \geq 0;$$

the l.h.s. takes its greatest values at the end-points of the interval $1 < d < 2$ and is negative for $d = 1$ and $d = 2$. Hence (2.23) is never satisfied.

(ii) If $[d] = 2$, (2.23) is

$$15d^3 - 32d^2 - 48d + 32 \geq 0;$$

the l.h.s. increases with d for $d \geq 2$ and is negative when $d = 2.969$; hence d satisfies (2.20).

(iii) If $[d] = 3$, (2.23) is

$$15d^3 - 32d^2 - 128d + 72 \geq 0;$$

the l.h.s. increases with d for $d \geq 3$ and is negative when $d = 3.975$; hence d satisfies (2.21).

(iv) If $[d] = 4$, (2.23) is

$$15d^3 - 32d^2 - 240d + 128 \geq 0;$$

the l.h.s. increases with d for $d \geq 4$ and is negative when $d = 4.994$; hence d satisfies (2.22).

(v) If $[d] = 5$, (2.23) is

$$15d^3 - 32d^2 - 384d + 200 \geq 0;$$

the l.h.s. increases with d for $d \geq 5$ and is negative when $d = 6$; hence (2.23) does not hold.

This completes the proof of the lemma.

Corresponding to the values of d allowed by Lemma 6, we find the following values of $m=\frac{\nu}{\mu}$:

$$m=\tfrac{5}{3}, \tag{2.19'}$$

$$2\leq m<2.0126, \tag{2.20'}$$

$$\tfrac{17}{7}\leq m<2.4389, \tag{2.21'}$$

$$\tfrac{26}{9}\leq m<2.8915. \tag{2.22'}$$

Now the estimate (2.16) is known to be best possible if and only if m is of the form

$$m=1+\frac{2}{r} \qquad (r=1,\,2,\,3,\,\ldots)$$

or $m=1$; in particular $D_m=4\sqrt{m}$ for the values $m=\frac{5}{3}$, $m=2$. However, for the remaining values of m given in (2.19′) – (2.22′), strict inequality holds in (2.16). The results we shall need are given in the following four lemmas:

Lemma 8. *If* $m=\frac{5}{3}$ *and* $\mathfrak{L}$ *is admissible for* $\mathcal{R}_m$, *then either* $\Delta(\mathfrak{L})>4\sqrt{\tfrac{5}{3}}$ *or* $\mathfrak{L}$ *is given by*

$$\xi\eta=\frac{2}{3}(x^2+8xy+y^2), \quad x,\,y\equiv\tfrac{1}{2},\,\tfrac{1}{2}\ (\mathrm{mod}\ 1).$$

Lemma 9. *If* $m\geq 2$ *and* $\mathfrak{L}$ *is admissible for* $\mathcal{R}_m$, *then either* $\Delta(\mathfrak{L})\geq\sqrt{33}$ *or* $\mathfrak{L}$ *is given by*

$$\xi\eta=k(x^2+6xy+y^2), \quad x,\,y\equiv\tfrac{1}{2},\,\tfrac{1}{2}\ (\mathrm{mod}\ 1), \quad k\geq\tfrac{1}{2}m.$$

Lemma 10. *If* $m\geq\frac{17}{7}$ *and* $\mathfrak{L}$ *is admissible for* $\mathcal{R}_m$, *then either* $\Delta(\mathfrak{L})>\frac{8}{7}\sqrt{30}$ *or* $\mathfrak{L}$ *is given by*

$$\xi\eta=k(2x^2+12xy+3y^2), \quad x,\,y\equiv\tfrac{1}{2},\,\tfrac{1}{2}\ (\mathrm{mod}\ 1), \quad k\geq\tfrac{4}{17}m.$$

Lemma 11. *If* $m\geq\frac{26}{9}$ *and* $\mathfrak{L}$ *is admissible for* $\mathcal{R}_m$, *then* $\Delta(\mathfrak{L})\geq 4\sqrt{3}$.

In order to avoid interrupting the main argument, we defer the discussion of these results until § 4.

Now suppose that (2.19) holds, so that $d=2$, $m=\frac{5}{3}$. Then

$$\mu\Delta=\tfrac{1}{2}d-\tfrac{1}{4}=\tfrac{3}{4},$$

$$\Delta^2=\tfrac{15}{8}d^3=15,$$

$$\frac{1}{\mu}=4\sqrt{\tfrac{5}{3}}.$$

Hence, by Lemmas 3 and 8, (2.5) holds unless

$$\phi(y, z) = \tfrac{1}{2}(y^2 + 8yz + z^2), \quad y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2};$$

this is (i) of Theorem 3.

Next suppose that (2.20) holds, so that m satisfies (2.20′). Then, since

$$\frac{1}{\mu} = \frac{4\Delta}{2d-1} = \frac{4}{2d-1}\sqrt{\frac{15d^3}{8}},$$

it is easily verified that $\frac{1}{\mu} < \sqrt{33}$. By Lemmas 3 and 9 it follows that (2.5) holds unless $\phi(y, z)$ is equivalent to a positive multiple of $y^2 + 6yz + z^2$ with $y, z \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$. This shows that (2.8) of Theorem 3 holds for some $k > 0$. Also, since then $\Delta^2 = 50k^2$, we have

$$\Delta^2 = 50k^2 = \tfrac{15}{8}d^3.$$

The bounds (2.9) for k now follow from the bounds for d given in (2.20). Thus $\phi(y, z)$ satisfies Theorem 3 (iii).

Next suppose that (2.21) holds, so that m satisfies (2.21′). Then the inequality

$$\frac{1}{\mu} \geq \tfrac{8}{7}\sqrt{30} \tag{2.25}$$

cannot hold unless

$$m = \tfrac{17}{7}, \quad d = 4, \quad \frac{1}{\mu} = \tfrac{8}{7}\sqrt{30}. \tag{2.26}$$

For (2.25) is equivalent to

$$\tfrac{2}{7}\sqrt{30} \leq \frac{\Delta}{2d-1} = \frac{1}{2d-1}\sqrt{\frac{15d^3}{8}},$$

$$49d^3 - 64(2d-1)^2 \geq 0,$$

which reduces to

$$(d-4)(49d^2 - 60d + 16) \geq 0;$$

since $3.975 < d \leq 4$, this is true only if $d = 4$ and the sign of equality holds: this gives (2.26). It now follows at once from Lemmas 3 and 10 that (2.5) holds unless $\phi(y, z)$ is equivalent to a positive multiple of

$$2y^2 + 12yz + 3y^2, \quad y, z \equiv \tfrac{1}{2}, \tfrac{1}{2} \text{ and } d = 4. \text{ Since then}$$

$$\Delta^2 = \tfrac{15}{8}d^3 = 120,$$

we see that (ii) of Theorem 3 holds.

Suppose finally shat (2.22) holds, so that m satisfies (2.22′). Then, by Lemmas 3 and 11, (2.5) holds unless

$$\frac{1}{\mu} \geq 4\sqrt{3}.$$

But this inequality is equivalent to

$$\sqrt{3} \leq \frac{1}{4\mu} = \frac{\Delta}{2d-1},$$

$$3(2d-1)^2 \leq \Delta^2 = \tfrac{15}{8} d^3,$$

and it is easily verified that this is false for d satisfying (2.18).

This completes the proof of Theorem 3.

3. The next step in the proof of Theorem 1 is to decide what values of x_0, h and g are allowable in (2.3) if (2.5) is not satisfied and $\phi(y, z)$ is given by one of the forms in Theorem 3.

Lemma 12. *If $f(x, y, z)$ is given by* (2.3), *where $\phi(y, z)$ is given by* (2.6), *then* (2.5) *holds unless f is equivalent to*

$$f_1(x, y, z) = x^2 - \tfrac{1}{2}(y^2 + 8yz + z^2) \tag{3.1}$$

and

$$x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1.$$

Proof. We have

$$f(x, y, z) = (x + hy + gz)^2 - \tfrac{1}{2}(y^2 + 8yz + z^2)$$

with $x, y, z \equiv x_0, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$. Since $\Delta^2 = 15$,

$$(\tfrac{1}{15}\Delta^2)^{\frac{1}{3}} = 1. \tag{3.2}$$

Now

$$f(\pm x, \tfrac{1}{2}, -\tfrac{1}{2}) = (\pm x + \tfrac{1}{2}h - \tfrac{1}{2}g)^2 + \tfrac{3}{4}, \tag{3.3}$$

$$f(\pm x, \tfrac{1}{2}, \tfrac{1}{2}) = (\pm x + \tfrac{1}{2}h + \tfrac{1}{2}g)^2 - \tfrac{5}{4}. \tag{3.4}$$

For any real x_0, we can choose $x \equiv x_0$ with

$$|x + \tfrac{1}{2}h - \tfrac{1}{2}g| \leq \tfrac{1}{2};$$

hence, by (3.3),

$$M(f; x_0, y_0, z_0) \leq 1 = (\tfrac{1}{15}\Delta^2)^{\frac{1}{3}};$$

and the sign of inequality holds unless

$$x_0 + \tfrac{1}{2}h - \tfrac{1}{2}g \equiv \tfrac{1}{2} \pmod 1.$$

In the same way, taking the lower sign in (3.3), we see that (2.5) holds unless

$$-x_0 + \tfrac{1}{2}h - \tfrac{1}{2}g \equiv \tfrac{1}{2} \pmod 1.$$

Similarly, choosing $x \equiv x_0$ with

$$\tfrac{1}{2} \le | \pm x + \tfrac{1}{2}h + \tfrac{1}{2}g | \le \tfrac{3}{2},$$

we see that (2.5) holds unless

$$x_0 + \tfrac{1}{2}h + \tfrac{1}{2}g \equiv \tfrac{1}{2} \pmod 1,$$

$$-x_0 + \tfrac{1}{2}h + \tfrac{1}{2}g \equiv \tfrac{1}{2} \pmod 1.$$

Since the above four congruences imply that

$$x_0 \equiv \tfrac{1}{2}, \quad h \equiv g \equiv 0 \pmod 1,$$

the lemma follows at once.

Lemma 13. *If $f(x, y, z)$ is given by (2.3), where $\phi(y, z)$ is given by (2.7), then (2.5) holds unless f is equivalent to*

$$f_2(x, y, z) = x^2 - (2y^2 + 12yz + 3z^2) \tag{3.5}$$

and

$$x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1.$$

Proof. We have

$$f(x, y, z) = (x + hy + gz)^2 - (2y^2 + 12yz + 3z^2),$$

with $x, y, z \equiv x_0, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$. Since $\Delta^2 = 120$,

$$(\tfrac{1}{15}\Delta^2)^{\frac{1}{3}} = 2.$$

Now

$$f(\pm x, \tfrac{1}{2}, -\tfrac{1}{2}) = (\pm x + \tfrac{1}{2}h - \tfrac{1}{2}g)^2 + \tfrac{7}{4},$$

$$f(\pm x, \tfrac{1}{2}, \tfrac{1}{2}) = (\pm x + \tfrac{1}{2}h + \tfrac{1}{2}g)^2 - \tfrac{17}{4}.$$

Choosing $x \equiv x_0$ to satisfy any one of

$$| \pm x + \tfrac{1}{2}h - \tfrac{1}{2}g | \le \tfrac{1}{2},$$

$$\tfrac{3}{2} \le | \pm x + \tfrac{1}{2}h + \tfrac{1}{2}g | \le \tfrac{5}{2},$$

we see, precisely as in Lemma 12, that (2.5) holds unless

$$x_0 \equiv \tfrac{1}{2}, \quad h \equiv g \equiv 0 \pmod 1.$$

This gives the result of the lemma.

For the case (iii) of Theorem 3, we want to show that (2.5) holds unless $x_0 \equiv \tfrac{1}{2}$, $h \equiv \tfrac{1}{2}$, $g \equiv \tfrac{1}{2} \pmod 1$ and $k = 1$. For this, the simple argument used in Lemmas 12 and 13 is not sufficient. However, the complete result will follow by a consideration of the automorphs of $f(x, y, z)$ and an application of Lemma 4. The proof divides naturally into two stages, given in the following two lemmas.

Lemma 14. *If $f(x, y, z)$ is given by (2.3), where] $\phi(y, z)$ satisfies (2.8), (2.9), then if (2.5) does not hold we have*

$$h \equiv g \equiv \tfrac{1}{2} \pmod 1; \tag{3.6}$$

further, in the form equivalent to $f(x, y, z)$ with $h = g = \frac{1}{2}$,

$$|x_0 - \tfrac{1}{2}| < .016 \pmod 1. \tag{3.7}$$

Proof. There is clearly no loss of generality in supposing that

$$0 \leq h, g < 1 \tag{3.8}$$

in (2.3). We than have to prove that $h = g = \frac{1}{2}$ and that (3.7) holds under the assumption that (2.5) is false for some x_0.

Since $y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2} \pmod 1$ and $\Delta^2 = 50k^2$, (2.5) holds unless, for some x_0,

$$M(f; x_0, \tfrac{1}{2}, \tfrac{1}{2}) \geq (\tfrac{10}{3} k^2)^{\frac{1}{3}} > (\tfrac{10}{3} \times .9814)^{\frac{1}{3}} > 1.484. \tag{3.9}$$

Now

$$f(\pm x, \tfrac{1}{2}, \tfrac{1}{2}) = (\pm x + \tfrac{1}{2}h + \tfrac{1}{2}g)^2 - \tfrac{5}{2}k,$$

$$f(\pm x, \tfrac{1}{2}, -\tfrac{1}{2}) = (\pm x + \tfrac{1}{2}h - \tfrac{1}{2}g)^2 + \tfrac{5}{4}k.$$

Hence, for any x_0 for which (3.9) holds, we have

$$|(p \pm x_0 + \tfrac{1}{2}h + \tfrac{1}{2}g)^2 - \tfrac{5}{2}k| > 1.484, \tag{3.10}$$

$$(p \pm p_0 + \tfrac{1}{2}h - \tfrac{1}{2}g)^2 + \tfrac{5}{4}k > 1.484 \tag{3.11}$$

for all integral p and any choice of sign. In (3.10) we choose p so that

$$1 \leq |p \pm x_0 + \tfrac{1}{2}h + \tfrac{1}{2}g| = \alpha < 2,$$

and in (3.11) we choose p so that

$$\beta = |p \pm x_0 + \tfrac{1}{2}h - \tfrac{1}{2}g| \leq \tfrac{1}{2}.$$

We then have, from (3.10), *either*

$$\alpha^2 > \tfrac{5}{2}k + 1.484 > 3.9605, \quad \alpha > 1.99,$$

or

$$\alpha^2 < \tfrac{5}{2}k - 1.484 < 1.032, \quad \alpha < 1.016;$$

it follows that

$$-.016 < \pm x_0 + \tfrac{1}{2}h + \tfrac{1}{7}g < .016 \pmod 1. \tag{3.12}$$

Similarly, from (3.11) we deduce that

$$\beta^2 > 1.484 - \tfrac{5}{4}k > .2261, \quad \beta > .475,$$

whence

$$\tfrac{1}{2} - .025 < \pm x_0 + \tfrac{1}{2}h - \tfrac{1}{2}g < \tfrac{1}{2} + .025 \pmod 1. \tag{3.13}$$

Adding (3.12), (3.13) with suitable choices of sign we find that

$$\tfrac{1}{2} - .041 < h, g < \tfrac{1}{2} + .041 \pmod 1, \tag{3.14}$$

$$-.032 < 2x_0 < .032 \qquad \pmod 1,$$

whence *either* $|x_0| < .016 \pmod 1$ *or* $|x_0 - \tfrac{1}{2}| < .016 \pmod 1$. If h, g satisfy (3.8) it is clear from (3.12) that the second alternative must hold, i.e. that x_0 satisfies (3.7).

If we apply the integral unimodular transformation $x = X$, $y = -Z$, $z = Y + 6Z$ to $f(x, y, z)$ we find that

$$f(x, y, z) = (X + h_1 Y + g_1 Z)^2 - \tfrac{5}{4}k(Y^2 + 6YZ + Z^2)$$

where

$$h_1 = g, \quad g_1 = 6g - h$$

and

$$X, Y, Z \equiv x_0, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1.$$

It follows that (3.14) must still hold if h, g are replaced by h_1, g_1. Similarly, using the inverse transformation $x = X$, $y = 6Y + Z$, $z = -Y$, we see that (3.14) must still hold if h, g are replaced by

$$h_{-1} = 6h - g, \quad g_{-1} = h.$$

Let now $\mathcal{R}$ be the region of the h, g-plane defined by

$$\tfrac{1}{2} - .041 < h, g < \tfrac{1}{2} + .041, \tag{3.15}$$

and let T be the matrix

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix}$$

(which is clearly of infinite order). Then, if P is the point (h, g), we have

$$P_1 = \begin{pmatrix} h_1 \\ g_1 \end{pmatrix} = T(P),$$

$$P_{-1} = \begin{pmatrix} h_{-1} \\ g_{-1} \end{pmatrix} = T^{-1}(P),$$

Since $0 \le h, g < 1$, (3.14) shows that $P \varepsilon \mathcal{R}$. Also, by what has been proved above, $T(P)$ and $T^{-1}(P)$ are congruent (mod 1) to a point of $\mathcal{R}$; and since P satisfies (3.15) it is clear that in fact

$$T(P) - (0, 2) \,\varepsilon\, \mathcal{R}, \quad T^{-1}(P) - (2, 0) \,\varepsilon\, \mathcal{R}.$$

Finally, the argument shows that the point $T^n(P)$ must satisfy (3.14), i.e. must be congruent to a point of $\mathcal{R}$, if (3.9) holds.

It now follows from Lemma 4 that this is possible only if P satisfies

$$T(P) = P + (0, 2),$$

i.e. if $P = (h, g) = (\frac{1}{2}, \frac{1}{2})$.

This completes the proof of the lemma.

Lemma 15. *Suppose that*

$$f(x, y, z) = (x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - \tfrac{5}{4}k(y^2 + 6yz + z^2), \tag{3.16}$$

where k satisfies (2.9), *and suppose that* (2.5) *is false with* $y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2}$ (mod 1) *and* x_0 *satisfying* (3.7). *Then*

$$k = 1, \quad x_0 \equiv \tfrac{1}{2} \pmod{1}. \tag{3.17}$$

Proof. Since f has determinant $D = -\frac{25}{2}k^2$ and k satisfies (2.9), it is quickly verified that (2.2) holds, i.e.

$$0 < a \le (4|D|)^{\frac{1}{3}}$$

with

$$a = f(1, 1, 0) = \tfrac{9}{4} - \tfrac{5}{4}k.$$

If we make the equivalence transformation

$$x = X + Z, \quad y = Y, \quad z = X \tag{3.18}$$

we find that

$$\begin{aligned} f(x, y, z) &= aF(X, Y, Z) \\ &= aX^2 + (\tfrac{1}{4} - \tfrac{5}{4}k)Y^2 + Z^2 - (\tfrac{15}{2}k - \tfrac{3}{2})XY + 3XZ + YZ, \end{aligned} \tag{3.19}$$

so that

$$F(X, Y, Z) = \left\{X - \frac{1}{4a}(15k - 3)Y + \frac{3}{2a}Z\right\}^2 - \Phi(Y, Z)$$

with

$$X, Y, Z \equiv \tfrac{1}{2}, \tfrac{1}{2}, x_0 - \tfrac{1}{2} \pmod{1}. \tag{3.20}$$

Now the form (3.19) is of the original type (2.3) and we are supposing that (2.4) is false, i.e. that

$$M(F; \tfrac{1}{2}, \tfrac{1}{2}, x_0 - \tfrac{1}{2}) \ge (\tfrac{1}{15}\Delta^2)^{\frac{1}{3}}$$

(where here Δ^2 is the discriminant of $\Phi(Y, Z)$). By Theorem 3 it follows that we can apply an equivalence transformation to Y, Z, say $Y = \alpha Y' + \beta Z'$, $Z = \gamma Y' + \delta Z'$ so that $\Phi(Y, Z)$ is transformed into one of (2.6), (2.7), (2.8) (with Y', Z' for y, z), and that then

$$Y', Z' \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod{1}.$$

Since $\alpha, \beta, \gamma, \delta$ are integers, we deduce that each of Y and Z must be congruent to 0 or $\frac{1}{2}$ (mod 1); hence, by (3.20),

$$x_0 \equiv 0 \quad \text{or} \quad \tfrac{1}{2} \pmod 1,$$

and so, by (3.7), $x_0 \equiv \frac{1}{2} \pmod 1$ as required.

Further, by Lemmas 12, 13 and 14, we see that each of the coefficients $-\dfrac{1}{4a}(15k-3)$ and $\dfrac{3}{2a}$ must be congruent to either 0 or $\frac{1}{2}$ (mod 1). Since $a = \frac{9}{4} - \frac{5}{4}k$ and k satisfies (2.9), it is easy to see that this can hold only if

$$\frac{3}{2a} = \tfrac{3}{2}, \quad \frac{1}{4a}(15k-3) = 3,$$

whence $a=1$, $k=1$. This proves the lemma.

By Theorem 3 and Lemmas 12–15, we have now shown that (2.5) holds unless $f(x, y, z)$ is equivalent to one of

$$f_1(x, y, z) = x^2 - \tfrac{1}{2}(y^2 + 8yz + z^2), \quad x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2},$$

$$f_2(x, y, z) = x^2 - (2y^2 + 12yz + 3z^2), \quad x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}$$

$$f_3(x, y, z) = (x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - \tfrac{5}{4}(y^2 + 6yz + z^2), \quad x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}.$$

To complete the proof of Theorem 2 (and hence of Theorem 1) we have only to observe that

$$\begin{aligned} f_3(x, y, z) &= Q_1(x, y, z); \\ 2f_1(x, y, z) &= 2x^2 - (y+4z)^2 + 15z^2 \\ &\sim Q_2(x, y, z) \quad \text{with} \quad x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; \\ f_2(x, y, z) &= 2(x - y - 3z)^2 - (x - 2y + 6z)^2 + 15z^2 \\ &\sim Q_2(x, y, z) \quad \text{with} \quad x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}. \end{aligned}$$

4. Proof of Lemmas 8–11. For the proofs of Lemmas 8–11 we must appeal to the general theory of two-dimensional inhomogeneous lattices developed in Barnes and Swinnerton-Dyer [3]. For the convenience of the reader we state briefly the particular results we shall need.

We denote by $[b_1, b_2, b_3, \ldots]$ the continued fraction

$$b_1 - \frac{1}{b_2 -}\,\frac{1}{b_3 -}\,\cdots,$$

where b_i is integral and $|b_i| \geq 2$. If $b_i > 0$ for all i and $b_i \geq 4$ for some arbitrarily large i, we have

$$[b_1, b_2, \ldots, b_n, b_{n+1}, \ldots] < [b_1, b_2, \ldots, b_n, b'_{n+1}, b'_{n+2}, \ldots] \tag{4.1}$$

provided only that $b_{n+1} < b'_{n+1}$, in particular

$$[b_1, b_2, \ldots, b_n - 1] < [b_1, b_2, \ldots, b_n, \ldots] < [b_1, b_2, \ldots, b_n]. \tag{4.2}$$

Let $\{a_n\}\,(-\infty < n < \infty)$ be a chain of positive even integers for which the inequality $a_n \geq 4$ holds for some arbitrarily large n of each sign. For each n we define

$$\theta_n = [a_n,\ a_{n-1},\ a_{n-2}, \ldots]$$

$$\phi_n = [a_{n+1},\ a_{n+2},\ a_{n+3}, \ldots],$$

so that, by (4.2), $\theta_n > 1$, $\phi_n > 1$. For any real λ, μ with $\lambda\mu > 0$, the inhomogeneous lattice $\mathfrak{L}$ defined by

$$\xi = \lambda\{\theta_n(u - \tfrac{1}{2}) + (v - \tfrac{1}{2})\}$$

$$\eta = \mu\{(u - \tfrac{1}{2}) + \phi_n(v - \tfrac{1}{2})\},$$

where u, v run through all integral values, is called a symmetrical lattice corresponding to the chain $\{a_n\}$. If $\mathfrak{L}$ has determinant Δ, we have $\Delta = \lambda\mu(\theta_n\phi_n - 1)$, so that, for points of $\mathfrak{L}$,

$$\xi\eta = \frac{\Delta}{\theta_n\phi_n - 1}(\theta_n x + y)(x + \phi_n y), \quad x, y \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1. \tag{4.3}$$

A symmetrical lattice $\mathfrak{L}$ is admissible for $\mathcal{R}_m$: $-1 \leq \xi\eta \leq m\ (m > 1)$ if and only if the inequalities

$$\frac{\Delta}{m} \geq \frac{4(\theta_n\phi_n - 1)}{(\theta_n + 1)(\phi_n + 1)} = \Delta_n^+, \tag{4.4}$$

$$\Delta \geq \frac{4(\theta_n\phi_n - 1)}{(\theta_n - 1)(\phi_n - 1)} = \Delta_n^- \tag{4.5}$$

hold for all n.

For any $m > 1$, all critical lattices of $\mathcal{R}_m$ are symmetrical. Moreover, if $1 < m \leq 3$, the inequality

$$\Delta(\mathfrak{L}) \geq 2(m+1) \tag{4.6}$$

holds for any $\mathcal{R}_m$-admissible $\mathfrak{L}$ which is not symmetrical.

Finally, if $0 < D < 2(k+1)$ and, for any n,

$$\Delta_n^- \leq D, \quad \Delta_n^+ \leq \frac{D}{k}, \tag{4.7}$$

then the inequality

$$\left|\alpha-\frac{2(k-1)}{2(k+1)-D}\right|\leq\frac{\sqrt{D^2-16k}}{2(k+1)-D} \tag{4.8}$$

holds with $\alpha=\theta_n$ or $\alpha=\phi_n$.

Proof of Lemma 8. Let $m=\frac{5}{3}$ and suppose that $\mathfrak{L}$ is $\mathcal{R}_m$-admissible and has $\Delta(\mathfrak{L})\leq 4\sqrt{\frac{5}{3}}=4\sqrt{m}$. Since $2(m+1)>4\sqrt{m}$, $\mathfrak{L}$ must be symmetrical; and, by (4.4), (4.5) we require

$$\Delta_n^-\leq 4\sqrt{\tfrac{5}{3}},\quad \Delta_n^+\leq 4\sqrt{\tfrac{3}{5}}$$

for all n. Thus (4.7) holds with $D=4\sqrt{\frac{5}{3}}$, $k=\frac{5}{3}$; since now $D^2=16k$, (4.8) shows that for all n

$$\theta_n=\phi_n=\frac{2(k-1)}{2(k+1)-D}=4+\sqrt{15}=[8,\,8,\,8,\ldots].$$

Hence $\{a_n\}$ is the periodic chain $\{\overset{\times}{8}\}$ and, by (4.3),

$$\xi\eta=\frac{\Delta}{2\sqrt{15}}(x^2+8xy+y^2),\quad x,y\equiv\tfrac{1}{2},\tfrac{1}{2}\pmod 1.$$

Finally, since now $\Delta_n^-=4\sqrt{\frac{5}{3}}$ for all n, we require

$$\Delta\geq 4\sqrt{\tfrac{5}{3}},\quad \Delta\leq 4\sqrt{\tfrac{5}{3}},$$

whence

$$\Delta=4\sqrt{\tfrac{5}{3}},\quad \frac{\Delta}{2\sqrt{15}}=\tfrac{2}{3}.$$

Proof of Lemma 9. It is shown in [3], Theorem 9, that if $m\geq 2$ and $\mathfrak{L}$ is admissible for $\mathcal{R}_m$, then either $\Delta(\mathfrak{L})\geq\sqrt{33}$ or $\mathfrak{L}$ is a symmetrical lattice corresponding to the chain $\{\overset{\times}{6}\}$. Lemma 9 follows at once from this, on observing that, for the chain $\{\overset{\times}{6}\}$ we have

$$\theta_n=\phi_n=[\overset{\times}{6}]=3+2\sqrt{2}$$

for all n,

$$\xi\eta=\frac{\Delta}{4\sqrt{2}}(x^2+6xy+y^2),\quad x,y\equiv\tfrac{1}{2},\tfrac{1}{2}\pmod 1,$$

where

$$\Delta\geq m\,\Delta_n^+=\frac{4m}{\sqrt{2}},\quad \frac{\Delta}{4\sqrt{2}}\geq\tfrac{1}{2}m.$$

Proof of Lemma 10. Suppose first that the inequalities

$$\Delta_n^-\leq\tfrac{8}{7}\sqrt{30},\quad \Delta_n^+\leq\tfrac{8}{17}\sqrt{30} \tag{4.9}$$

hold for all n. We show that then $\{a_n\}$ is the periodic chain $\{\overset{\times}{6}, \overset{\times}{4}\}$. For (4.7) holds with

$$D = \tfrac{8}{7}\sqrt{30}, \quad k = \tfrac{17}{7},$$

and so (4.8) gives, for all n,

$$\left|\alpha - \frac{5}{12 - 2\sqrt{30}}\right| \leq \frac{1}{12 - 2\sqrt{30}}$$

$$\frac{2}{6 - \sqrt{30}} \leq \alpha \leq \frac{3}{6 - \sqrt{30}},$$

i.e.

$$[\overset{\times}{4}, \overset{\times}{6}] \leq \alpha \leq [\overset{\times}{6}, \overset{\times}{4}], \tag{4.10}$$

where $\alpha = \theta_n$ or $\alpha = \phi_n$. Using (4.1), (4.2), we see that $a_n - 1 < \theta_n = [a_n, a_{n-1}, \dots] < a_n$, and so (4.10) shows that $a_n = 4$ or 6. If $a_n = 4$, (4.10) with $\alpha = \theta_n$ and $\alpha = \phi_{n-1}$ gives

$$[4, a_{n-1}, \dots] \geq [4, 6, 4, \dots],$$

$$[4, a_{n+1}, \dots] \geq [4, 6, 4, \dots],$$

whence $a_{n-1} \geq 6$, $a_{n+1} \geq 6$, so that $a_{n-1} = a_{n+1} = 6$. Similarly, if $a_n = 6$, (4.10) shows that $a_{n-1} = a_{n+1} = 4$. It follows that $\{a_n\}$ is $\{\overset{\times}{6}, \overset{\times}{4}\}$, as required.

Now if $\mathcal{L}$ is symmetrical and admissible for $\mathcal{R}_m$ with $m \geq \frac{17}{7}$, either (4.9) holds for all n or, by (4.4) and (4.5),

$$\Delta > \min\left\{\tfrac{8}{7}\sqrt{30}, \ \frac{8m}{17}\sqrt{30}\right\} = \tfrac{8}{7}\sqrt{30};$$

while if $\mathcal{L}$ is not symmetrical, (4.6) gives

$$\Delta \geq 2(m+1) = \tfrac{48}{7} > \tfrac{8}{17}\sqrt{30}.$$

It follows that if $\mathcal{L}$ is $\mathcal{R}_m$-admissible, with $m \geq \frac{17}{7}$ and $\Delta(\mathcal{L}) \leq \frac{8}{7}\sqrt{30}$, then $\mathcal{L}$ is a symmetrical lattice corresponding to the chain $\{\overset{\times}{6}, \overset{\times}{4}\}$. For this chain, θ_n and ϕ_n are $\frac{1}{3}(6 + \sqrt{30})$, $\frac{1}{2}(6 + \sqrt{30})$ in some order, for each n, whence

$$\xi\eta = \frac{\Delta}{2\sqrt{30}}(2x^2 + 12xy + 3y^2), \quad x, y \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1,$$

$$\frac{\Delta}{2\sqrt{30}} \geq \frac{m\,\Delta_n^+}{2\sqrt{30}} = \frac{4m}{17}.$$

Proof of Lemma 11. For $m \geq \frac{26}{9}$ we have

$$2(m+1) > 4\sqrt{3};$$

it is therefore sufficient to show that there exists no symmetrical lattice satisfying

$$\Delta_n^- < 4\sqrt{3}, \quad \tfrac{26}{9}\Delta_n^+ < 4\sqrt{3} \tag{4.11}$$

for all n.

Now if (4.11) holds, then (4.7) holds with $D = 4\sqrt{3}$, $k = \frac{26}{9}$. Hence, by (4.8),

$$\left|\alpha - \frac{17}{35 - 18\sqrt{3}}\right| < \frac{6}{35 - 18\sqrt{3}},$$

$$\frac{11}{35 - 18\sqrt{3}} < \alpha < \frac{23}{35 - 18\sqrt{3}},$$

with $\alpha = \theta_n$ or $\alpha = \phi_n$. Thus for all n we have

$$\theta_n > \frac{11}{35 - 18\sqrt{3}} > \frac{11}{3.8231} > 2.87,$$

so that $a_n \geq 4$ for all n.

If now $a_n \geq 6$ for some n, we have, using (4.1),

$$\theta_n \geq [6, 4, 4, 4, \dots] = 4 + \sqrt{3}, \quad \phi_n \geq [4, 4, 4, \dots] = 2 + \sqrt{3},$$

whence

$$\tfrac{1}{4}\Delta_n^+ \geq \frac{10 + 6\sqrt{3}}{(5 + \sqrt{3})(3 + \sqrt{3})} = \frac{9 + 7\sqrt{3}}{33} = 0.64\dots,$$

whereas (4.11) gives

$$\tfrac{1}{4}\Delta_n^+ < \frac{9\sqrt{3}}{26} < 0.6.$$

It follows that $a_n = 4$ for all n. But then $\Delta_n^- = 4\sqrt{3}$, contradicting (4.11). Thus (4.11) cannot hold for all n.

5. It is not difficult, using the same methods, to show that Theorem 1 remains true if in (1.12) and (1.13) we replace $4/15$ by a slightly smaller constant. The ranges of d given in Lemma 7 are then slightly increased, but Theorem 3 still holds with the forms (2.6), (2.7) replaced by $\frac{1}{2}k(y^2 + 8yz + z^2)$, $k(2y^2 + 12yz + 3z^2)$, where k is nearly 1. (For the proof of this, we need stronger versions of Lemmas 8 and 10, but these are easily obtained.) We may then show, just as in Lemmas 14 and 15, that in each case k must be 1. I have not given the details, to avoid complicating the main lines of the proof.

Thus the 'second minimum' $(\frac{4}{15}|D|)^{\frac{1}{3}}$ is isolated, and the problem remains to find the third and any further minima. Since Davenport [4] has given a (zero) form with $M(Q) = (\frac{1}{4}|D|)^{\frac{1}{3}}$, the third minimum is at least $(\frac{1}{4}|D|)^{\frac{1}{3}}$.

I think it likely that the methods of this paper will not prove adequate for a complete analysis of the problem. It is easy to see that, in particular, the method will break down if there are uncountably many distinct lattices admissible for $\mathcal{R}_m$ with determinant not exceeding $1/\mu$; and this situation does in fact arise if one attempts to find the forms Q with $M(Q) \geq (\frac{1}{4}|D|)^{\frac{1}{3}}$.

However, a complete answer to the problem may be obtainable by the use of 'local' methods on the chain $\{a_n\}$ associated with $\phi(y, z)$ in the form (2.3). I hope to investigate this attack in the near future.

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