THE INHOMOGENEOUS MINIMA OF BINARY QUADRATIC FORMS (III)

BY

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1. Introduction

Let $\mathfrak{L}$ be an inhomogeneous lattice of determinant $\Delta = \Delta(\mathfrak{L})$ in the ξ, η-plane, i.e. a set of points given by

$$\begin{aligned} \xi &= \xi_0 + \alpha x + \beta y, \\ \eta &= \eta_0 + \gamma x + \delta y, \end{aligned} \tag{1.1}$$

where $\xi_0, \eta_0, \alpha, \beta, \gamma, \delta$ are real, $\Delta = |\alpha\delta - \beta\gamma| \neq 0$, and x, y take all integral values. In vector notation, $\mathfrak{L}$ is the set of points

$$P = P_0 + xA + yB,$$

where the lattice vectors $A = (\alpha, \gamma)$ and $B = (\beta, \delta)$ are said to generate $\mathfrak{L}$. It is clear that $\mathfrak{L}$ has infinitely many pairs A, B of generators. Corresponding to any such pair and any point P_0 of $\mathfrak{L}$, we call the parallelogram with vertices P_0, $P_0 + A$, $P_0 + B$, $P_0 + A + B$ a *cell* of $\mathfrak{L}$: a parallelogram with vertices at points of $\mathfrak{L}$ is a cell of $\mathfrak{L}$ if and only if it has area Δ.

A cell is said to be *divided* if it has one vertex in each of the four quadrants. Delauney [5] has proved that if $\mathfrak{L}$ has no point on either of the coordinate axes $\xi = 0$, $\eta = 0$, then $\mathfrak{L}$ has at least one divided cell[1]; we outline his proof in § 2. We then develop an algorithm for finding a new divided cell from a given one, thus obtaining in general[2] a chain of divided cells $A_n B_n C_n D_n$ $(-\infty < n < \infty)$. The analytical

[1] This result fills the gap, noted by CASSELS [3], in the very simple proof of Minkowski's theorem on the product of two inhomogeneous linear forms given by SAWYER [6].

[2] The condition that the chain does not break off is simply that $\mathfrak{L}$ shall have no lattice-vector parallel to a coordinate axis.

formulation of the algorithm leads uniquely to a specification of $\mathfrak{L}$ in terms of a chain of integer pairs h_n, k_n.

In §§ 3, 4 and 5 we apply these results to the problem of evaluating the critical determinant D_m of the asymmetric hyperbolic region

$$R_m: \qquad -1 \leq \xi \eta \leq m \quad (m \geq 1),$$

i.e. the lower bound of the determinants of lattices $\mathfrak{L}$ having no point in the interior of R_m.

The value of D_m has been established for infinitely many values of m by Blaney [1], who also gives estimates valid for general m[1]. His main results are:

$$D_m \geq 4\sqrt{m} \text{ if } m \geq 1, \tag{1.2}$$

with equality if and only if m is of the form $1 + \frac{2}{r}$ $(r = 1, 2, 3, \ldots)$; and

$$D_m \geq \sqrt{(m+1)(m+9)} \text{ if } m \geq 3, \tag{1.3}$$

with equality if and only if m is of the form $4r-1$ $(r = 1, 2, 3, \ldots)$. An alternative proof of these inequalities will be given in §§ 3 and 4.

The complete evaluation of D_m appears to be extremely difficult. Defining a lattice $\mathfrak{L}$ as *admissible* for R_m if it has no point in the interior of R_m, and *critical* for R_m if it is admissible and has $\Delta(\mathfrak{L}) = D_m$, we prove in § 4 that *all critical lattices of R_m are of the form*[2]

$$\left.\begin{aligned} \xi &= \alpha(x-\tfrac{1}{2}) + \beta(y-\tfrac{1}{2}) \\ \eta &= \gamma(x-\tfrac{1}{2}) + \delta(y-\tfrac{1}{2}) \end{aligned}\right\}. \tag{1.4}$$

This reduces the problem to that of the admissibility of lattices of the type (1.4), which are discussed in § 4. Using the results obtained there, it would be possible to evaluate D_m for any particular value of m, though the arithmetical labour involved

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[1] Blaney's results are formulated in terms of inhomogeneous binary quadratic forms. Thus (1.2) is equivalent to saying that if $f(x, y)$ is a quadratic form of discriminant $D > 0$, x_0, y_0 are any real numbers, and $m \geq 1$, then there exist integers x, y satisfying

$$-\frac{1}{4\sqrt{m}}\sqrt{D} \leq f(x+x_0, y+y_0) \leq \frac{\sqrt{m}}{4}\sqrt{D}.$$

It is sufficient, by homogeneity, to consider R_m in place of the more general region:

$$-m_1 \leq \xi\eta \leq m_2 \quad (m_1 > 0,\ m_2 > 0).$$

[2] The existence of a critical lattice of R_m follows from a general theorem of Swinnerton-Dyer [7].

might well be excessive. We illustrate the method in § 5, where we find all critical lattices of R_m for a small range of values of m.

The last two sections, 6 and 7, contain some results on the isolation of D_m and an example (quoted by Swinnerton-Dyer [7]) of an automorphic star-body none of whose (inhomogeneous) critical lattices has more than one point on the boundary.

The methods of this paper may be applied to other problems involving inhomogeneous lattices, e.g. to those considered in parts I and II of this series. One of us hopes to publish further applications in the near future.

2. The divided cells of a lattice

We first sketch the proof given by Delauney [5] of

Theorem 1. *If $\mathfrak{L}$ is a two-dimensional inhomogeneous lattice having no point on either of the coordinate axes $\xi=0$, $\eta=0$, then $\mathfrak{L}$ has at least one divided cell.*

Since the origin O is not a point of $\mathfrak{L}$, we can draw a square, with diagonals lying along the axes, containing no point of $\mathfrak{L}$. We now expand the square homothetically until a point P of $\mathfrak{L}$ first appears on some side. By symmetry, we may suppose that P has positive coordinates ξ_0, η_0.

Suppose first that there is no point (ξ, η) of $\mathfrak{L}$ with $0<\eta<\eta_0$; then it is easy to see that there is another point of $\mathfrak{L}$ with $\eta=\eta_0$. For, by Minkowski's fundamental theorem, $\mathfrak{L}$ has a point $Q(\xi, \eta)$ other than P satisfying

$$|\xi-\xi_0|<K, \quad |\eta-\eta_0|<\eta_0,$$

if K is sufficiently large. Since neither Q nor $2Q-P$ (its image in P) satisfies $0<\eta<\eta_0$, it follows that $\eta=\eta_0$. Thus there is a lattice-step PQ parallel to the ξ-axis. If now we take the least such steps AB, CD which intersect the η-axis and lie nearest the origin on either side, it is clear that $ABCD$ is a divided cell.

Suppose next that $\mathfrak{L}$ contains some point with $0<\eta<\eta_0$. We deform the square into a rhombus by continuously moving the corners on the ξ-axis away from the origin and those on the η-axis towards the origin, keeping the point P on one side of the rhombus throughout. We continue this deformation until for the first time a lattice-point $Q(\xi_1, \eta_1)$, other than P, appears on a side. There are now three cases to distinguish:

(a) Q and P lie on the same side, so that $\xi_1>0$, $\eta_1>0$. The next line of $\mathfrak{L}$ parallel to PQ and on the same side of it as O must be beyond O and at least as far from it as PQ; for otherwise the rhombus would cut off from it a segment of length greater than $|PQ|$ and so would have a lattice-point in its interior, contrary to the construction. Hence the segment of this lattice-line in the quadrant $\xi < 0$, $\eta < 0$ has length greater than $|PQ|$ and so contains a lattice-point. Take on this lattice-line and on PQ the unique lattice step intersecting the η-axis; the four lattice points so determined form a divided cell.

(b) Q and P lie on adjacent sides. By symmetry it is sufficient to suppose that $\xi_1 < 0$, $\eta_1 > 0$, so that PQ intersects the η-axis. Let AB be the lattice-step which intersects the η-axis, is parallel to PQ, and lies nearest to PQ on the same side of PQ as O is. By an argument similar to that in (a), the points P, Q, A, B form a divided cell.

(c) Q and P lie on opposite sides, so that $\xi_1 < 0$, $\eta_1 < 0$. Take the next parallels to PQ in the lattice on each side of PQ. The segments of these intercepted by the sides of the rhombus on which P and Q lie (produced if necessary) have length $|PQ|$, and so each contains a lattice-point. Since these points lie outside the rhombus, they must lie one in each of the second and fourth quadrants; and it is then easy to see that, with P and Q, they form a divided cell.

Let now $\mathfrak{L}$ have no point on $\xi = 0$ or $\eta = 0$, so that by Theorem 1 it has a divided cell $A_0 B_0 C_0 D_0$. It is convenient to choose the notation so that the points A_0, B_0, C_0, D_0 are respectively *either* in the first, fourth, third and second quadrants, *or* in the third, second, first and fourth quadrants.

We now define non-zero integers h_0, k_0 as follows:

(i) If $A_0 D_0$ and $B_0 C_0$ are parallel to the ξ-axis, we write conventionally

$$h_0 = k_0 = -\infty.$$

(ii) If $A_0 D_0$ and $B_0 C_0$, produced either way, intersect the ξ-axis, we define h_0 as the unique integer for which the η-coordinates of the lattice points

$$A_1 = A_0 + (h_0 + 1)(D_0 - A_0)$$
$$B_1 = A_0 + h_0 (D_0 - A_0)$$

have opposite sign. (Thus $A_1 B_1$ is the unique lattice step of the line $A_0 D_0$ which intersects the ξ-axis.) Similarly k_0 is the unique integer for which the η-coordinates of the lattice-points

$$C_1 = C_0 + (k_0 + 1)(B_0 - C_0)$$
$$D_1 = C_0 + k_0 (B_0 - C_0)$$

have opposite sign.

Since $\eta(A_0)$ and $\eta(D_0)$ have the same sign, it is clear that $h_0 \neq 0$; similarly $k_0 \neq 0$. Also h_0 and k_0 have the same sign, since $\eta(D_0 - A_0)$ and $\eta(C_0 - B_0)$ do. Finally, it is easy to see from the construction that $A_1 B_1 C_1 D_1$ is again a divided cell, where A_1, C_1 lie one in each of the first and third quadrants, and B_1, D_1 lie one in each of the second and fourth quadrants.

In a precisely similar manner we may define non-zero integers h_{-1}, k_{-1} by considering the intersections of the lattice-lines $C_0 D_0$ and $A_0 B_0$ with the η-axis, and obtain a divided cell $A_{-1} B_{-1} C_{-1} D_{-1}$. Clearly these may be defined so as to coincide with the integers h_{-1}, k_{-1} obtained from $A_{-1} B_{-1} C_{-1} D_{-1}$ by the above process.

We may continue this process indefinitely, obtaining an infinity of divided cells $A_n B_n C_n D_n$ and of integer pairs h_n, k_n $(-\infty < n < \infty)$, unless some pair h_n, k_n is infinite (when the process terminates in one direction). The relation between successive cells is, as above,

$$\left.\begin{aligned} A_{n+1} &= A_n + (h_n + 1)(D_n - A_n) \\ B_{n+1} &= A_n + h_n (D_n - A_n) \\ C_{n+1} &= C_n + (k_n + 1)(B_n - C_n) \\ D_{n+1} &= C_n + k_n (B_n - C_n) \end{aligned}\right\}. \tag{2.1}$$

Writing $\underset{\sim}{V}_n$ for the lattice vector $A_n - D_n$, we have

$$\underset{\sim}{V}_n = A_n - D_n = B_n - C_n = B_{n+1} - A_{n+1} = C_{n+1} - D_{n+1}, \tag{2.2}$$

and (2.1) may be written as

$$\left.\begin{aligned} A_{n+1} &= A_n - (h_n + 1)\underset{\sim}{V}_n \\ B_{n+1} &= A_n - h_n \underset{\sim}{V}_n \\ C_{n+1} &= C_n + (k_n + 1)\underset{\sim}{V}_n \\ D_{n+1} &= C_n + k_n \underset{\sim}{V}_n \end{aligned}\right\}. \tag{2.3}$$

Thus, taking $n = 0$ as a reference, (2.3) gives the expressions

$$\left.\begin{aligned} A_n &= A_0 - (h_0 + 1)\underset{\sim}{V}_0 - (h_1 + 1)\underset{\sim}{V}_1 - \cdots - (h_{n-1} + 1)\underset{\sim}{V}_{n-1} \\ C_n &= C_0 + (k_0 + 1)\underset{\sim}{V}_0 + (k_1 + 1)\underset{\sim}{V}_1 + \cdots + (k_{n-1} + 1)\underset{\sim}{V}_{n-1} \end{aligned}\right\} \quad (n \geq 1) \tag{2.4}$$

$$\left.\begin{aligned} A_{-n} &= A_0 + (h_{-1} + 1)\underset{\sim}{V}_{-1} + (h_{-2} + 1)\underset{\sim}{V}_{-2} + \cdots + (h_{-n} + 1)\underset{\sim}{V}_{-n} \\ C_{-n} &= C_0 - (k_{-1} + 1)\underset{\sim}{V}_{-1} - (k_{-2} + 1)\underset{\sim}{V}_{-2} + \cdots - (k_{-n} + 1)\underset{\sim}{V}_{-n} \end{aligned}\right\} \quad (n \geq 1) \tag{2.5}$$

We have also from (2.1) and (2.2)

$$\underset{\sim}{V}_{n+1} = A_{n+1} - D_{n+1} = A_n + (h_n + 1)(D_n - A_n) - C_n - k_n(B_n - C_n)$$
$$= D_n - C_n - h_n(A_n - D_n) - k_n(B_n - C_n),$$

i.e.

$$\underset{\sim}{V}_{n+1} = -(h_n + k_n)\underset{\sim}{V}_n - \underset{\sim}{V}_{n-1}. \tag{2.6}$$

Now suppose the coordinates A_n, B_n, C_n, D_n are given by

$$C_n = (\xi_n, \eta_n),\ B_n = (\xi_n + \alpha_n, \eta_n + \gamma_n),\ D_n = (\xi_n + \beta_n, \eta_n + \delta_n),$$
$$A_n = (\xi_n + \alpha_n + \beta_n, \eta_n + \gamma_n + \delta_n). \tag{2.7}$$

Since these points, for any n, form a cell of $\mathfrak{L}$, $\mathfrak{L}$ is given by

$$\xi = \xi_n + \alpha_n x + \beta_n y,$$
$$\eta = \eta_n + \gamma_n x + \delta_n y, \tag{2.8}$$

where x, y run through all integral values; and $\Delta = \Delta(\mathfrak{L}) = |\alpha_n \delta_n - \beta_n \gamma_n|$. Also, by (2.2), $\underset{\sim}{V}_n$ has components

$$V_n = \{\alpha_n, \gamma_n\} = \{-\beta_{n+1}, -\delta_{n+1}\}. \tag{2.9}$$

Writing now for all n

$$a_{n+1} = h_n + k_n, \tag{2.10}$$

so that a_{n+1} is integral (possibly infinite) and $|a_{n+1}| \geq 2$, we have by (2.6), (2.9) and (2.10)

$$\left.\begin{aligned} \underset{\sim}{V}_{n+1} &= -a_{n+1}\underset{\sim}{V}_n - \underset{\sim}{V}_{n-1}, \\ \alpha_{n+1} &= -a_{n+1}\alpha_n - \alpha_{n-1}, \\ \gamma_{n+1} &= -a_{n+1}\gamma_n - \gamma_{n-1} \end{aligned}\right\}. \tag{2.11}$$

Setting

$$V_n = \{\alpha_n, \gamma_n\} = (-1)^n \{\alpha_0 p_n - \beta_0 q_n, \gamma_0 p_n - \delta_0 q_n\}, \tag{2.12}$$

we therefore have

$$p_{-1} = 0,\ q_{-1} = -1;\ \ p_0 = 1,\ q_0 = 0;\ \ p_1 = a_1,\ q_1 = 1; \tag{2.13}$$

$$\left.\begin{aligned} p_{n+1} &= a_{n+1}p_n - p_{n-1} \\ q_{n+1} &= a_{n+1}q_n - q_{n-1} \end{aligned}\right\}. \tag{2.14}$$

It follows that, for $n \geq 1$, p_n/q_n is the continued fraction

$$a_1 - \frac{1}{a_2} - \frac{1}{a_3} - \cdots \frac{1}{a_n},$$

which we shall write as

$$\frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n] \quad (n \geq 1). \tag{2.15}$$

Similarly, since

$$p_{n-1}=a_{n+1}p_n-p_{n+1},$$

$$q_{n-1}=a_{n+1}q_n-q_{n+1},$$

we have

$$\frac{(-q_{-n})}{(-p_{-n})}=[a_0, a_{-1}, \dots, a_{-n+2}] \quad (n\geq 2). \tag{2.16}$$

In order to justify a passage to the limit in the formulae so far established, we need two lemmas.

Lemma 1. *It is impossible, either for all* $n\geq n_0$ *or for all* $n\leq -n_0$, *that either* (i) $h_n=-1$ *or* (ii) $k_n=-1$. *It is also impossible that* $h_{n_0+2r}=k_{n_0+2r+1}=1$ *either for all* $r\geq 0$ *or for all* $r\leq 0$.

Proof. If for example $h_n=-1$ for all $n\geq n_0$, (2.1) shows that $A_n=A_{n_0}=A$, say, for all $n\geq n_0$. Since the triangle $A_n B_n C_n$ has the constant area $\frac{1}{2}\Delta$ and $\eta(A_n)=$ $=\eta(A)\neq 0$, it is easy to see that B_n and C_n must lie in a bounded part of the plane (since $\eta(B_n)$ and $\eta(C_n)$ are of the opposite sign to $\eta(A_n)$, and B_n, C_n lie in different quadrants). Hence there can be only a finite number of distinct points B_n, C_n for $n\geq n_0$. We show that this is impossible.

By (2.2),

$$\eta(B_{n+1})-\eta(A_{n+1})=\eta(C_{n+1})-\eta(D_{n+1})=\eta(\underline{V}_n).$$

Since $\eta(B_{n+1})$ and $\eta(A_{n+1})$ have opposite signs, as do also $\eta(C_{n+1})$ and $\eta(D_{n+1})$, this gives

$$|\eta(A_{n+1})|<|\eta(\underline{V}_n)|, \quad |\eta(D_{n+1})|<|\eta(\underline{V}_n)|,$$

whence

$$|\eta(\underline{V}_{n+1})|=|\eta(A_{n+1})-\eta(D_{n+1})|<|\eta(\underline{V}_n)|,$$

since $\eta(A_{n+1})$ and $\eta(D_{n+1})$ have the same sign. It follows that $|\eta(\underline{V}_n)|=|\eta(B_n)-\eta(C_n)|$ is strictly decreasing, so that there cannot be only a finite number of distinct points B_n, C_n.

Next, if $h_{n_0+2r}=k_{n_0+2r+1}$ for all $r\geq 0$, two applications of (2.1) give

$$D_{n_0+2r+2}=B_{n_0+2r+1}=D_{n_0+2r},$$

so that $D_{n_0+2r}=D$, say, for all $r\geq 0$. We may now deduce a contradiction precisely as above.

The other cases of the lemma are provided similarly; for $n\leq -n_0$, we use the fact that $|\xi(\underline{V}_{-n})|$ is strictly decreasing.

Lemma 2. *As $n \to +\infty$, each of*

$$\eta(\underset{\sim}{V}_n),\ \eta(A_n),\ \eta(B_n),\ \eta(C_n),\ \eta(D_n)$$

and

$$\xi(\underset{\sim}{V}_{-n}),\ \xi(A_{-n}),\ \xi(B_{-n}),\ \xi(C_{-n}),\ \xi(D_{-n})$$

tends to zero (or is undefined for large n)[1].

Proof. Supposing that h_n, k_n are defined for all large n, we have as in Lemma 1

$$|\eta(\underset{\sim}{V}_{n+1})| < |\eta(\underset{\sim}{V}_n)|; \tag{2.17}$$

and by (2.11)

$$|\eta(\underset{\sim}{V}_{n+1})| \geq |a_{n+1}|\,|\eta(\underset{\sim}{V}_n)| - |\eta(\underset{\sim}{V}_{n-1})|.$$

Combining these,

$$|\eta(\underset{\sim}{V}_n)| < \frac{1}{|a_{n+1}|-1}\,|\eta(\underset{\sim}{V}_{n-1})|. \tag{2.18}$$

If now $|a_{n+1}| \geq 3$ for arbitrarily large values of n, (2.18) shows that $\eta(\underset{\sim}{V}_n) \to 0$ as $n \to +\infty$. Otherwise, $|a_{n+1}| = 2$ for all large n; since $a_{n+1} = \pm 2$ if and only if $h_n = k_n = \pm 1$, Lemma 1 shows that a_{n+1} must change sign for arbitrarily large values of n. Now by two applications of (2.11) we have

$$\underset{\sim}{V}_{n+2} = (a_{n+1}a_{n+2} - 1)\,\underset{\sim}{V}_n + a_{n+2}\,\underset{\sim}{V}_{n-1},$$

whence

$$|a_{n+1}a_{n+2} - 1|\,|\eta(\underset{\sim}{V}_n)| \leq |\eta\,\underset{\sim}{V}_{n+2})| + |a_{n+2}|\,|\eta(\underset{\sim}{V}_{n-1})|;$$

using (2.17) this gives

$$|a_{n+1}a_{n+2} - 1|\,|\eta\,\underset{\sim}{V}_n)| < (|a_{n+2}| + 1)\,|\eta(\underset{\sim}{V}_{n-1})|.$$

Thus, for any n for which $|a_{n+1}| = |a_{n+2}| = 2$, $a_{n+1}a_{n+2} < 0$, we have

$$|\eta(\underset{\sim}{V}_n)| < \tfrac{3}{5}\,|\eta(\underset{\sim}{V}_{n-1})|.$$

Since this inequality holds for arbitrarily large n, it follows again that $\eta(\underset{\sim}{V}_n) \to 0$.

From the relations

$$\eta(\underset{\sim}{V}_{n-1}) = \eta(B_n) - \eta(A_n) = \eta(C_n) - \eta(D_n),$$

where $\eta(A_n)\,\eta(B_n) < 0$, $\eta(C_n)\,\eta(D_n) < 0$, we see that $\eta(\underset{\sim}{V}_n) \to 0$ implies that each of $\eta(A_n)$, $\eta(B_n)$, $\eta(C_n)$ and $\eta(D_n)$ tends to zero as $n \to +\infty$.

The fact that $\xi(\underset{\sim}{V}_{-n}) \to 0$ as $n \to +\infty$ may be proved in a precisely similar way.

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[1] This happens when some h_n and k_n become infinite, i.e. when there is a lattice-vector parallel to a coordinate axis.

Theorem 2. *Suppose that no a_n is infinite, and set*

$$\varphi_0 = [a_1, a_2, a_3, \dots],$$

$$\theta_0 = [a_0, a_{-1}, a_{-2}, \dots].$$

Then

$$\frac{\delta_0}{\gamma_0} = \varphi_0, \quad \frac{\alpha_0}{\beta_0} = \theta_0, \tag{2.19}$$

$$\left.\begin{aligned} \xi_0 &= \sum_{n=1}^{\infty} (k_{-n}+1)\,\xi(V_{-n}) = \sum_{n=1}^{\infty} (-1)^n (k_{-n}+1)(\alpha_0 p_{-n} - \beta_0 q_{-n}) \\ \eta_0 &= \sum_{n=0}^{\infty} -(k_n+1)\,\eta(V_n) = \sum_{n=0}^{\infty} (-1)^{n-1} (k_n+1)(\gamma_0 p_n - \delta_0 q_n) \end{aligned}\right\}, \tag{2.20}$$

where p_n, q_n are defined by (2.13) *and* (2.14).

If further we define for each n

$$\left.\begin{aligned} \varphi_n &= [a_{n+1}, a_{n+2}, \dots], \\ \theta_n &= [a_n, a_{n-1}, a_{n-2}, \dots], \end{aligned}\right. \tag{2.21}$$

$$\varepsilon_n = h_n - k_n, \tag{2.22}$$

then

$$\left.\begin{aligned} 2\xi_0 + \alpha_0 + \beta_0 &= \beta_0 \sum_{n=0}^{\infty} (-1)^n \frac{\varepsilon_{-n-1}}{\theta_{-1}\theta_{-2}\cdots\theta_{-n}} \\ 2\eta_0 + \gamma_0 + \delta_0 &= \gamma_0 \sum_{n=0}^{\infty} (-1)^n \frac{\varepsilon_n}{\varphi_1\varphi_2\cdots\varphi_n} \end{aligned}\right\}. \tag{2.23}$$

Proof. By (2.12) and Lemma 2,

$$|\gamma_0 p_n - \delta_0 q_n| = |\eta(V_n)| \to 0 \quad \text{as} \quad n \to +\infty.$$

Also it follows easily, by induction, from (2.14) that $|q_{n+1}| \geq |q_n| + 1$, so that $|q_n| \to \infty$. Hence

$$\varphi_0 = \lim_{n\to\infty} \frac{p_n}{q_n} = \frac{\delta_0}{\gamma_0};$$

and the second relation of (2.19) follows similarly.

The formulae (2.20) are now immediate consequences of (2.12) and (2.4), (2.5), since by Lemma 2

$$\eta(C_n) \to 0, \quad \xi(C_{-n}) \to 0 \quad \text{as} \quad n \to +\infty,$$

and

$$\xi_0 = \xi(C_0), \quad \eta_0 = \eta(C_0).$$

Next, by adding the two relations of (2.4), we obtain

$$A_0 + C_0 = A_n + C_n + \sum_{r=0}^{n-1} (h_r - k_r)\, \underset{\sim}{V}_r,$$

whence

$$2\eta_0 + \gamma_0 + \delta_0 = \eta(A_n) + \eta(C_n) + \sum_{r=0}^{n-1} (-1)^r \varepsilon_r (\gamma_0 p_r - \delta_0 q_r),$$

using the definitions (2.12) and (2.22). By Lemma 2,

$$\eta(A_n) \to 0, \quad \eta(C_n) \to 0 \quad \text{as} \quad n \to +\infty.$$

and so

$$2\eta_0 + \gamma_0 + \delta_0 = \sum_{n=0}^{\infty} (-1)^n \varepsilon_n (\gamma_0 p_n - \delta_0 q_n).$$

Now

$$\varphi_0 = [a_1, a_2, \ldots, a_n, a_{n+1}, \ldots] = [a_1, \ldots, a_n, \varphi_n] = \frac{\varphi_n p_n - p_{n-1}}{\varphi_n q_n - q_{n-1}},$$

whence

$$p_{n-1} - \varphi_0 q_{n-1} = \varphi_n (p_n - \varphi_0 q_n);$$

since $p_0 - \varphi_0 q_0 = 1$, it follows that

$$p_n - \varphi_0 q_n = \frac{1}{\varphi_1 \varphi_2 \cdots \varphi_n}, \quad \gamma_0 p_n - \delta_0 q_n = \frac{\gamma_0}{\varphi_1 \varphi_2 \cdots \varphi_n}.$$

This establishes the second formula of (2.23); and the first formula may be proved similarly.

3. Asymmetric Hyperbolic Regions

Suppose now that an inhomogeneous lattice $\mathfrak{L}$ of determinant Δ is admissible for the region

$$R_m: \qquad -1 \le \xi\eta \le m \qquad (m \ge 1).$$

This implies, in particular, that $\mathfrak{L}$ has no point on either of the coordinate axes $\xi = 0$, $\eta = 0$, so that the theory of § 2 applies. With the notation of § 2, we now establish the following inequalities:

(i) If any pair h_n, k_n is negative or infinite, then

$$\Delta \ge 2(m+1), \tag{3.1}$$

where equality is possible only if $h_n = k_n = -\infty$.

(ii) If, for any n, $h_n > 0$, $k_n > 0$, $h_n \neq k_n$, then

$$\Delta \geq \sqrt{m+1}\,(2+\sqrt{m+5}) \quad \text{if} \quad h_n = 1 \quad \text{or} \quad k_n = 1; \tag{3.2}$$

$$\Delta \geq (1+\sqrt{2})\,(m+1) \quad \text{if} \quad m \leq 3; \tag{3.3}$$

$$\Delta \geq (\tfrac{1}{2}+\sqrt{2})\,(m+1) \quad \text{if} \quad m \geq 3,\ h_n \geq 2,\ k_n \geq 2. \tag{3.4}$$

(iii) We have always

$$\Delta \geq 4\sqrt{m}, \tag{3.5}$$

where equality is possible only if $h_n = k_n = \dfrac{m+1}{m-1}$ for all n.

(iv) If $m \geq 3$ and $h_n = k_n \geq 2$ for any n, then

$$\Delta \geq \sqrt{(m+1)\,(m+9)}. \tag{3.6}$$

For the proof it is convenient, in order to avoid enumeration of cases, to suppose that $\mathfrak{L}$ has no point in the interior of the region

$$R: \qquad -m_1 \leq \xi\eta \leq m_2 \qquad (m_1 > 0,\ m_2 > 0). \tag{3.7}$$

Suppose that P_1 and P_2 are two vertices of any divided cell of $\mathfrak{L}$, where P_1 lies in the second quadrant and P_2 in the first; suppose also that $P_1 P_2$ produced intersects the ξ-axis. Then there exists a unique lattice-step $P_3 P_4$ cutting the ξ-axis, where P_3 lies in the first quadrant and P_4 in the fourth. For some integer $h \geq 1$ we now have

$$P_4 = P_1 + (h+1)\,(P_2 - P_1)$$

$$P_3 = P_1 + h\,(P_2 - P_1),$$

so that

$$|P_1 P_2| = \frac{|P_2 P_3|}{h-1} = |P_3 P_4|.$$

The lattice-line $P_1 P_2 P_3 P_4$ has an equation of the form

$$\xi \cos\theta + \eta \sin\theta = \lambda \qquad \left(0 < \theta < \frac{\pi}{2},\ \lambda > 0\right).$$

For the intersections of this lattice line with any hyperbola $\xi\eta = \mu$, we find the equation

$$(\xi \cos\theta - \eta \sin\theta)^2 = \lambda^2 - 4\mu \sin\theta \cos\theta,$$

so that the intersections have coordinates

$$\xi = \frac{\lambda \pm \sqrt{\lambda^2 - 4\mu \sin\theta\cos\theta}}{2\cos\theta}, \quad \eta = \frac{\lambda \mp \sqrt{\lambda^2 - 4\mu \sin\theta\cos\theta}}{2\sin\theta}.$$

The length of the intercept made on the line by the hyperbola is therefore

$$\frac{\sqrt{\lambda^2 - 4\mu \sin\theta\cos\theta}}{\sin\theta\cos\theta}.$$

Now since $\mathfrak{L}$ is admissible for R, P_1 and P_4 satisfy $\xi\eta \leq -m_1$, and P_2 and P_3 satisfy $\xi\eta \geq m_2$. We deduce that

$$|P_1 P_4| \geq \frac{\sqrt{\lambda^2 + 4m_1 \sin\theta\cos\theta}}{\sin\theta\cos\theta},$$

$$|P_2 P_3| \leq \frac{\sqrt{\lambda^2 - 4m_2 \sin\theta\cos\theta}}{\sin\theta\cos\theta}$$

(the radicals being necessarily real). Since

$$\frac{|P_1 P_4|}{|P_2 P_3|} = \frac{h+1}{h-1},$$

we have

$$\left(\frac{h+1}{h-1}\right)^2 \geq \frac{\lambda^2 + 4m_1 \sin\theta\cos\theta}{\lambda^2 - 4m_2 \sin\theta\cos\theta},$$

whence

$$\lambda^2 \geq \frac{m_1(h-1)^2 + m_2(h+1)^2}{h} \sin\theta\cos\theta,$$

$$|P_1 P_4| \geq \frac{(h+1)\sqrt{m_1+m_2}}{\sqrt{h}\sin\theta\cos\theta},$$

$$|P_1 P_2| = \frac{|P_1 P_4|}{h+1} \geq \sqrt{\frac{m_1+m_2}{h\sin\theta\cos\theta}}. \tag{3.8}$$

Now suppose that the opposite side of the divided cell has equation

$$\xi\cos\theta + \eta\sin\theta = -\lambda' \qquad (\lambda' > 0),$$

and that an integer $k \geq 1$ is similarly defined for it. It is convenient to suppose, as we may by symmetry, that $k \geq h$.

As above, the intercept made on this lattice line by the hyperbola $\xi\eta = m_2$ has length

$$\frac{\sqrt{\lambda'^2 - 4m_2 \sin\theta\cos\theta}}{\sin\theta\cos\theta}.$$

This intercept contains $(k-1)$ lattice-steps and so has length at least $(k-1)|P_1P_2|$. Inserting the bound (3.8) for $|P_1P_2|$ we deduce that

$$\lambda'^2 \geq \left\{4\,m_2 + \frac{(k-1)^2}{h}(m_1+m_2)\right\}\sin\theta\cos\theta.$$

Finally, since $\Delta = (\lambda+\lambda')|P_1P_2|$ (the area of a cell), we derive the inequality

$$\Delta \geq \sqrt{\frac{m_1+m_2}{h}}\left\{\sqrt{\frac{m_1(h-1)^2+m_2(h+1)^2}{h}} + \sqrt{4\,m_2 + \frac{(k-1)^2}{h}(m_1+m_2)}\right\}, \tag{3.9}$$

from which we shall deduce the inequalities (3.1)–(3.7) above.

(a) Suppose that $\mathfrak{L}$ is admissible for R_m and that, for some n, h_n and k_n are negative. The above analysis now applies with $h=-h_n$, $k=-k_n$, $m_1=1$, $m_2=m$, and so by (3.9)

$$\Delta \geq \sqrt{\frac{1+m}{h}}\left\{\sqrt{\frac{(h-1)^2+m\,(h+1)^2}{h}} + \sqrt{4\,m + \frac{(k-1)^2}{h}(1+m)}\right\}.$$

Using $k \geq h$, $m \geq 1$, we easily obtain the estimate

$$\Delta > \sqrt{\frac{1+m}{h}}\,\{\sqrt{h\,(m+1)} + \sqrt{h\,(m+1)}\} = 2\,(m+1).$$

If any pair h_n, k_n is infinite, we may proceed to the limit $h\to\infty$, $k\to\infty$ in the above and obtain $\Delta \geq 2\,(m+1)$.

This proves (3.1) under the given hypotheses.

(b) Suppose now that $\mathfrak{L}$ is admissible for R_m and for some n, $h_n>0$, $k_n>0$. The above analysis applies (supposing $k_n \geq h_n$ without loss of generality) with $h=h_n$, $k=k_n$, $m_1=m$, $m_2=1$: as is most easily seen by considering the lattice $\mathfrak{L}'$ derived from $\mathfrak{L}$ by changing the sign of either ξ or η.

If now $h \neq k$, we have $k \geq h+1$, and so (3.9) gives

$$\Delta \geq \sqrt{\frac{m+1}{h}}\left\{\sqrt{\frac{m\,(h-1)^2+(h+1)^2}{h}} + \sqrt{4+h\,(m+1)}\right\}. \tag{3.10}$$

For $h=1$, this gives

$$\Delta \geq \sqrt{m+1}\,(2+\sqrt{m+5}),$$

which is (3.2)

Next, writing (3.10) in the form

$$\frac{\Delta}{m+1} \geq \sqrt{1-\frac{2}{h}\frac{m-1}{m+1}+\frac{1}{h^2}} + \sqrt{h+\frac{4}{m+1}}, \tag{3.11}$$

we observe that the right hand side is a decreasing function of m. For $m \leq 3$ we therefore have

$$\frac{\Delta}{m+1} \geq \sqrt{1-\frac{1}{h}+\frac{1}{h^2}}+\sqrt{h+1}$$

which is at least $1+\sqrt{2}$, its value at $h=1$, for any positive integral h: this gives (3.3).

For $m \geq 3$, we let $m \to \infty$ on the right of (3.11) and obtain

$$\frac{\Delta}{m+1} \geq 1-\frac{1}{h}+\sqrt{h}.$$

Thus, for $h \geq 2$,

$$\Delta \geq (\tfrac{1}{2}+\sqrt{2})(m+1),$$

which gives (3.4).

(c) It is easily verified that (3.5) holds if any of (3.1)–(3.4) are true. Hence for the proof of (3.5) it suffices to consider the case $h_n = k_n > 0$ for all n. Taking $m_1 = m$, $m_2 = 1$, $h_n = k_n = h$ in (3.9) gives

$$\begin{aligned}\Delta &\geq \sqrt{\frac{m+1}{h}}\left\{\sqrt{\frac{m(h-1)^2+(h+1)^2}{h}}+\sqrt{4+\frac{(h-1)^2}{h}(m+1)}\right\} \\ &= 2\sqrt{4m+\left(m-1-\frac{m+1}{h}\right)^2} \qquad (3.12) \\ &\geq 4\sqrt{m},\end{aligned}$$

with equality only if $h=\dfrac{m+1}{m-1}$.

(d) If $m \geq 3$, $h \geq 2$, we have

$$m-1-\frac{m+1}{h} \geq m-1-\frac{m+1}{2}=\frac{m-3}{2} \geq 0.$$

(3.12) now gives

$$\Delta \geq 2\sqrt{4m+\left(\frac{m-3}{2}\right)^2}=\sqrt{(m+1)(m+9)},$$

which establishes (3.6).

Defining the critical determinant D_m of R_m, as in § 1, as the lower bound of the determinants of R_m-admissible lattices, we deduce

Theorem 3. (i) *For all* m,

$$D_m \geq 4\sqrt{m}, \qquad (3.13)$$

where equality is possible only if $\dfrac{m+1}{m-1}$ *is integral (or infinite).*

(ii) *If* $m \geq 3$, *then*

$$D_m \geq \sqrt{(m+1)(m+9)}. \tag{3.14}$$

Proof. (i) (3.13) follows from (3.5), with the equality clause.

(ii) To prove (ii), we first note that the estimates (3.1), (3.2) and (3.4) imply that $\Delta > \sqrt{(m+1)(m+9)}$ for $m \geq 3$. Since it is impossible that $h_n = k_n = 1$ for all n it follows that, for any admissible $\mathfrak{L}$, *either* one of (3.1), (3.2) or (3.4) is true, *or* $h_n = k_n \geq 2$ for some n; in this latter case, (3.6) gives the result.

We note that Theorem 3 gives the bounds (1.2), (1.3) quoted in § 1 from Blaney [1], which are known to be precise for infinitely many m. We see from the proofs of (3.1)–(3.7) that the inequality is strict unless $h_n = k_n > 0$ for each divided cell of $\mathfrak{L}$, which suggests that, in order to evaluate D_m, it will suffice to consider only such lattices. We shall examine these lattices in detail in the following section.

4. Symmetrical Lattices

We suppose now that $\mathfrak{L}$ has determinant Δ and that $h_n = k_n > 0$ for all n. For such lattices, the sequence $\{a_n\}$ therefore consists of *positive even* integers; and by Lemma 1, the inequality $a_n > 2$ must hold for arbitrarily large n of each sign.

We first establish some fundamental properties of the continued fractions[1] $[a_1, a_2, a_3, \ldots]$, and obtain inequalities, which will be useful in what follows, for fractions with positive partial quotients.

The successive convergents p_n/q_n are defined by (2.14), i.e.

$$\left.\begin{aligned} p_{n+1} &= a_{n+1}\, p_n - p_{n-1} \\ q_{n+1} &= a_{n+1}\, q_n - q_{n-1} \end{aligned}\right\} \quad (n \geq 1) \tag{4.1}$$

with

$$p_0 = 1,\ q_0 = 0; \quad p_1 = a_1,\ q_1 = 1. \tag{4.2}$$

The identity

$$p_{n-1}\, q_n - p_n\, q_{n-1} = 1 \qquad (n \geq 1) \tag{4.3}$$

follows immediately by induction on n.

[1] The continued fraction $[a_1, a_2, \ldots]$ is easily transformed into a semi-regular continued fraction

$$a_1 + \frac{\mu_2}{|a_2|} + \frac{\mu_3}{|a_3|} + \cdots \qquad (\mu_i = \pm 1),$$

whose convergents have the same value (though the signs of p_n and q_n may be different). Hence some of our results below follow from the classical theory of semi-regular fractions; see for example Perron [8].

Lemma 3. *For all* $n \geq 1$,

$$|p_n| \geq n+1, \ |q_n| \geq n, \ \left|\frac{p_n}{q_n}\right| \geq 1 + \frac{1}{n}. \tag{4.4}$$

If further $a_i > 0$ *for* $i = 1, 2, \ldots, n$, *then* $p_n > 0, q_n > 0$.

Proof. Since $|a_i| \geq 2$, (4.1) gives

$$|p_{n+1}| - |p_n| \geq |p_n| - |p_{n-1}|, \ |q_{n+1}| - |q_n| \geq |q_n| - |q_{n-1}|.$$

The first two inequalities of (4.4) follow at once by induction. Now (4.3) gives

$$\left|\frac{p_n}{q_n}\right| = \left|\frac{p_{n-1}}{q_{n-1}} - \frac{1}{q_n q_{n-1}}\right| \geq \left|\frac{p_{n-1}}{q_{n-1}}\right| - \frac{1}{n(n-1)} \quad (n \geq 2);$$

since

$$\left|\frac{p_1}{q_1}\right| = |a_1| \geq 2,$$

the last inequality of (4.4) follows at once by induction.

If all $a_i > 0$, so that $a_i \geq 2$, (4.1) gives

$$p_{n+1} - p_n \geq p_n - p_{n-1}, \ q_{n+1} - q_n \geq q_n - q_{n-1} \qquad (n \geq 1)$$

provided that $p_n \geq 0, q_n \geq 0$; since in fact $p_0 = 1, q_0 = 0, p_1 = a_1 \geq 2, q_1 = 1$, it follows by induction that p_n and q_n are positive.

Lemma 4. *The infinite continued fraction* $[a_1, a_2, a_3, \ldots]$ *converges to a real number* α *satisfying* $|\alpha| \geq 1$; *and if* a_n *is not constantly equal to* 2 *or to* -2 *for large* n, *we have* $|\alpha| > 1$. *If further* $a_n > 0$ *for all* n, *then* α *is positive and the convergents* p_n/q_n *form a strictly decreasing sequence.*

Proof. (i) By (4.3) and (4.4),

$$\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{|q_n q_{n-1}|} \leq \frac{1}{n(n-1)} \quad (n \geq 2),$$

so that the series $\sum \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right)$ is convergent. It follows that $\alpha = \lim \frac{p_n}{q_n}$ exists; and by (4.4), $|\alpha| \geq 1$.

(ii) If $a_n > 0$ for all n, Lemma 3 shows that p_n and q_n are positive, so that α is positive. Also, by (4.3),

$$\frac{p_n}{q_n} = \frac{p_{n-1}}{q_{n-1}} - \frac{1}{q_n q_{n-1}} < \frac{p_{n-1}}{q_{n-1}},$$

since q_n is positive.

(iii) If $a_1 \geq 3$, we have

$$\alpha = [a_1, a_2, \ldots] = [a_1, \alpha_2],$$

where, by (i), $|\alpha_2| \geq 1$; hence

$$\alpha = a_1 - \frac{1}{\alpha_2} \geq a_1 - 1 \geq 2.$$

Next, if $a_1 = 2$, $a_2 \leq -2$, we have

$$\alpha = [2, a_2, \alpha_3], \ |\alpha_3| \geq 1;$$

so that

$$\alpha = 2 - \cfrac{1}{a_2 - \cfrac{1}{\alpha_3}} = 2 + \cfrac{1}{|a_2| + \cfrac{1}{\alpha_3}} \geq 2 + \frac{1}{|a_2| + 1} > 2.$$

In the same way we may show that $\alpha \leq -2$ either if $a_1 \leq -3$ or if $a_1 = -2$, $a_2 \geq 2$.

Thus finally if $a_1 = a_2 = \cdots = a_r = \pm 2 \neq a_{r+1}$, it follows that

$$\alpha = [a_1, \ldots, a_{r-1}, \alpha_r], \ |\alpha_r| \geq 2,$$

so that by Lemma 3

$$|\alpha| \geq 1 + \frac{1}{r} > 1.$$

This completes the proof of the lemma.

Lemma 5. *The continued fraction*

$$\alpha = [a_1, a_2, a_3, \ldots]$$

with positive partial quotients is increased if any a_n *is increased and* $a_{n+1}, a_{n+2}, \ldots$ *are replaced by any integers exceeding* 1.

Proof. Let

$$\beta = [a_1, a_2, \ldots, a_{n-1}, b_n, b_{n+1}, \ldots],$$

where $b_n > a_n$, $b_i \geq 2$ for $i \geq n+1$. We may write

$$\alpha = [a_1, a_2, \ldots, a_{n-1}, x_n], \quad \beta = [a_1, a_2, \ldots, a_{n-1}, y_n],$$

where

$$x_n = [a_n, a_{n+1}, \ldots]$$

$$y_n = [b_n, b_{n+1}, \ldots] = b_n - a_n + [a_n, b_{n+1}, \ldots].$$

By Lemma 4, each of $[a_n, a_{n+1}, \ldots]$, $[a_n, b_{n+1}, \ldots]$ lies between $a_n - 1$ and a_n and is strictly less than a_n; and by hypothesis $b_n - a_n \geq 1$; hence

$$x_n \geq 1,$$

$$y_n - x_n > 1 + (a_n - 1) - a_n = 0.$$

From the identity

$$\beta - \alpha = \frac{y_n p_{n-1} - p_{n-2}}{y_n q_{n-1} - q_{n-2}} - \frac{x_n p_{n-1} - p_{n-2}}{x_n q_{n-1} - q_{n-2}}$$

$$= \frac{y_n - x_n}{(y_n q_{n-1} - q_{n-2})(x_n q_{n-1} - q_{n-2})},$$

where both numerator and denominator are positive, it follows that $\beta > \alpha$, as required.

Corollary. *For any* $n \geq 1$,

$$[a_1, a_2, \ldots, a_{n-1}, a_n - 1] < \alpha < [a_1, a_2, \ldots, a_{n-1}, a_n]. \tag{4.5}$$

The second inequality follows from the last sentence of Lemma 4. The first follows from Lemma 5, by comparison of α with $[a_1, a_2, \ldots, a_n, 2, 2, 2, \ldots]$, on noting that $[2, 2, 2, \ldots] = 1$.

We note finally that any irrational α has a unique expansion as

$$\alpha = [a_1, a_2, a_3, \ldots]$$

where $a_n \geq 2$ for $n \geq 2$ and $a_n \geq 3$ for some arbitrarily large n. This may be described as its continued fraction expansion 'by the nearest integer above'. On the other hand, if the a_n are restricted only by the conditions that $|a_n| \geq 2$ and that a_n shall not be constantly 2 or -2 for all large n, any irrational α has infinitely many expansions.

Returning now to the lattices $\mathfrak{L}$ with $h_n = k_n > 0$ for each n, we see that $\varepsilon_n = h_n - k_n = 0$ and so, by (2.23) of Theorem 2,

$$2\xi_0 + \alpha_0 + \beta_0 = 0, \quad 2\eta_0 + \gamma_0 + \delta_0 = 0.$$

Since the results of Theorem 2 are clearly independent of the enumeration of the sequence $\{a_n\}$, we have for all n

$$\xi_n = -\tfrac{1}{2}(\alpha_n + \beta_n), \quad \eta_n = -\tfrac{1}{2}(\gamma_n + \delta_n), \tag{4.6}$$

where

$$\frac{\alpha_n}{\beta_n} = \theta_n = [a_n, a_{n-1}, a_{n-2}, \ldots], \quad \frac{\delta_n}{\gamma_n} = \varphi_n = [a_{n+1}, a_{n+2}, a_{n+3}, \ldots]. \tag{4.7}$$

Thus $\mathfrak{L}$ is defined for any n by

$$\left.\begin{aligned}\xi &= \alpha_n (x-\tfrac{1}{2}) + \beta_n (y-\tfrac{1}{2}) \\ \eta &= \gamma_n (x-\tfrac{1}{2}) + \delta_n (y-\tfrac{1}{2})\end{aligned}\right\}; \tag{4.8}$$

we therefore describe it as *symmetrical*. Since $\theta_n > 1$, $\varphi_n > 1$ for all n (by Lemma 4) and $\mathfrak{L}$ has determinant Δ, (4.8) gives

$$\xi\eta = \frac{\Delta}{\theta_n \varphi_n - 1} \{\theta_n (x-\tfrac{1}{2}) + (y-\tfrac{1}{2}\} \{(x-\tfrac{1}{2}) + \varphi_n (y-\tfrac{1}{2})\}. \tag{4.9}$$

Theorem 4. *A symmetrical lattice $\mathfrak{L}$ is admissible for R_m if and only if the inequalities*

$$\frac{\Delta}{m} \geq \frac{4(\theta_n \varphi_n - 1)}{(\theta_n + 1)(\varphi_n + 1)} = \Delta_n^+, \quad \text{say,} \tag{4.10}$$

$$\Delta \geq \frac{4(\theta_n \varphi_n - 1)}{(\theta_n - 1)(\varphi_n - 1)} = \Delta_n^-, \quad \text{say,} \tag{4.11}$$

hold for all n.

This theorem is a corollary of the following general result:

Theorem 5. *Let $\mathfrak{L}$ be an inhomogeneous lattice with no point on the coordinate axes $\xi=0$, $\eta=0$, and let the chain of divided cells $A_n B_n C_n D_n$ be defined for $\mathfrak{L}$ as in § 2. Suppose there exists a point of $\mathfrak{L}$ in the region*

$$-m_1 < \xi\eta < m_2 \qquad (m_1 > 0,\ m_2 > 0).$$

Then there exists some cell of the chain which has a vertex in this region.

Proof. For each n, let P_n denote that one of A_n, C_n which is in the first quadrant: $\xi > 0, \eta > 0$. Then it is clear by the construction of one cell from the next that no point of $\mathfrak{L}$ lies in the interior of the triangle formed by the positive axes and the line $P_n P_{n+1}$ produced. It now follows from the strict convexity of the region $\xi\eta \geq m_2$ that all first quadrant points of $\mathfrak{L}$ satisfy $\xi\eta \geq m_2$ if and only if this equality holds for all P_n. This argument may clearly be applied to each of the four quadrants.

For the proof of Theorem 4, we note that the points A_n, B_n, C_n, D_n correspond to the values $x=0$ or 1, $y=0$ or 1 in (4.8) and (4.9). The conditions that $\xi\eta \geq m$ for A_n, C_n and that $\xi\eta \leq -1$ for B_n, D_n are thefore just (4.10) and (4.11).

We now consider some general classes of symmetrical lattices and thus obtain upper bounds for D_m.

(i) Suppose that

$$a_n = 2a \geq 4 \qquad \text{for all } n.$$

By (4.7) we have

$$\theta_n = \varphi_n = [2\overset{\times}{a}]$$

(the cross denoting infinite repetition). Thus $\theta_n = \varphi_n$ is a root of the equation

$$x = 2a - \frac{1}{x},$$

and so

$$\theta_n = \varphi_n = a + \sqrt{a^2 - 1},$$

$$\Delta_n^+ = \frac{4(\theta_n \varphi_n - 1)}{(\theta_n + 1)(\varphi_n + 1)} = 4\sqrt{\frac{a-1}{a+1}},$$

$$\Delta_n^- = \frac{4(\theta_n \varphi_n - 1)}{(\theta_n - 1)(\varphi_n - 1)} = 4\sqrt{\frac{a+1}{a-1}}.$$

By theorem 4, the corresponding symmetrical lattice, $\mathfrak{L}_a$ say, is admissible for R_m if

$$\Delta(\mathfrak{L}_a) = \max\left\{4m\sqrt{\frac{a-1}{a+1}},\ 4\sqrt{\frac{a+1}{a-1}}\right\}. \tag{4.12}$$

(ii) Suppose that for all n

$$a_{2n} = 2a, \quad a_{2n+1} = 2b, \quad a > b \geq 1.$$

Then, by (4.7),

$$\theta_{2n} = [2\overset{\times}{a}, 2\overset{\times}{b}] = \varphi_{2n+1}, \quad \varphi_{2n} = [2\overset{\times}{b}, 2\overset{\times}{a}] = \theta_{2n+1},$$

whence

$$\theta_{2n} = \frac{ab + \sqrt{ab(ab-1)}}{b}, \quad \varphi_{2n} = \frac{ab + \sqrt{ab(ab-1)}}{a}.$$

A simple calculation gives, for all n,

$$\Delta_n^+ = \frac{8\sqrt{ab(ab-1)}}{2ab + a + b}, \quad \Delta_n^- = \frac{8\sqrt{ab(ab-1)}}{2ab - a - b}.$$

By Theorem 4, the corresponding symmetrical lattice, $\mathfrak{L}_{a,b}$ say, is admissible for R_m if

$$\Delta(\mathfrak{L}_{a,b}) = \max\left\{\frac{8m\sqrt{ab(ab-1)}}{2ab + a + b},\ \frac{8\sqrt{ab(ab-1)}}{2ab - a - b}\right\}. \tag{4.13}$$

(iii) Suppose that $p \geq 1$ and

$$a_n = 4 \quad \text{if } p \text{ divides } n,$$

$$a_n = 2 \quad \text{otherwise},$$

so that $\{a_n\}$ is the periodic sequence $\{\overset{\times}{4}, 2, 2, \dots \overset{\times}{2}\}$ (with $p-1$ elements 2). Then, with an obvious notation,

$$\theta_0 = [\overset{\times}{4}, (\overset{\times}{2})_{p-1}] = \frac{2p + \sqrt{p^2+2p}}{p},$$

$$\varphi_0 = [(\overset{\times}{2})_{p-1}, \overset{\times}{4}] = \frac{2p + \sqrt{p^2+2p}}{3p-2},$$

whence

$$\Delta_0^+ = \frac{4\sqrt{p^2+2p}}{4p-1}, \quad \Delta_0^- = 4\sqrt{p^2+2p}. \tag{4.14}$$

We now show that

$$\max \Delta_n^+ = \Delta_0^+, \quad \max \Delta_n^- = \Delta_0^-. \tag{4.15}$$

Clearly Δ_n^+ and Δ_n^- are, like a_n, periodic with period p, and so it suffices for the proof of (4.15) to consider only $n = 0, 1, \dots, p-1$. Since $\varphi_{p-1} = \theta_0$, $\theta_{p-1} = \varphi_0$, we have $\Delta_{p-1}^{\pm} = \Delta_0^{\pm}$; in particular (4.15) is trivially true if $p \leq 2$. Supposing then that $p \geq 3$ and $0 < n < p-1$ we see that the continued fraction for at least one of θ_n, φ_n begins $[2, 2, \dots]$, while the other begins $[2, 2, \dots]$ or $[2, 4, \dots]$. Hence

$$\min(\theta_n, \varphi_n) < [2, 2] = \tfrac{3}{2}, \quad \max(\theta_n, \varphi_n) < [2, 4] = \tfrac{7}{4},$$

$$\Delta_n^+ < \frac{4\left(\frac{21}{8} - 1\right)}{\left(\frac{3}{2}+1\right)\left(\frac{7}{4}+1\right)} = \tfrac{52}{55} < 1 < \Delta_0^+,$$

which proves the first equation of (4.15). The second is an immediate consequence of the following simple lemma, which shows that Δ_n^- is in fact constant for our special sequence:

Lemma 6. *If* $a_n = 2$ *and* Δ_n^- *is defined by* (4.11), *then* $\Delta_{n-1}^- = \Delta_n^-$.

Proof. We have

$$\varphi_{n-1} = [2, a_{n+1}, \dots] = [2, \varphi_n] = 2 - \frac{1}{\varphi_n},$$

$$\theta_n = [2, a_{n-1}, \dots] = [2, \theta_{n-1}] = 2 - \frac{1}{\theta_{n-1}},$$

whence

$$\tfrac{1}{4}\Delta_{n-1}^- = \frac{\theta_{n-1}\varphi_{n-1} - 1}{(\theta_{n-1}-1)(\varphi_{n-1}-1)} = \frac{\varphi_{n-1} - \dfrac{1}{\theta_{n-1}}}{\left(1 - \dfrac{1}{\theta_{n-1}}\right)(\varphi_{n-1}-1)}$$

$$= \frac{\theta_n - \dfrac{1}{\varphi_n}}{(\theta_n - 1)\left(1 - \dfrac{1}{\varphi_n}\right)} = \tfrac{1}{4}\Delta_n^-.$$

From (4.14), (4.15) and Theorem 4 we see that the symmetric lattice, $\mathcal{L}_p'$ say, corresponding to the given sequence $\{a_n\}$ is admissible for R_m if

$$\Delta(\mathcal{L}_p') = \max\left\{\frac{4m\sqrt{p^2+2p}}{4p-1},\ 4\sqrt{p^2+2p}\right\}. \tag{4.16}$$

These results now enable us to prove:

Theorem 6. *Suppose that $\mathcal{L}$ is admissible for R_m $(m>1)$ and is not a symmetrical lattice. Then there exists a symmetrical lattice $\mathcal{L}'$ which is R_m-admissible and has*

$$\Delta(\mathcal{L}') < \Delta(\mathcal{L}).$$

We shall take $\mathcal{L}'$ to be a suitable chosen one of the special lattices $\mathcal{L}_a$, $\mathcal{L}_{a,b}$, $\mathcal{L}_p'$ discussed above.

(i) Suppose first that $1 < m \leq 3$.

By (3.1) and (3.3), any R_m-admissible $\mathcal{L}$ which is not symmetrical has $\Delta(\mathcal{L}) \geq$ $\geq 2(m+1)$, and so for the proof of Theorem 6 it suffices to establish the inequality

$$\Delta(\mathcal{L}') < 2(m+1) \tag{4.17}$$

for a suitable symmetrical $\mathcal{L}'$. We define an integer $a \geq 2$ by

$$1+\frac{2}{a} < m \leq 1+\frac{2}{a-1}. \tag{4.18}$$

By (4.13), $\mathcal{L}_{a,a+1}$ is admissible if

$$\Delta(\mathcal{L}_{a,a+1}) = \max\left\{\frac{8m\sqrt{a(a+1)(a^2+a-1)}}{2(a+1)^2-1},\ \frac{8\sqrt{a(a+1)(a^2+a-1)}}{2a^2-1}\right\},$$

and so if

$$\Delta(\mathcal{L}_{a,a+1}) = \frac{8\sqrt{a(a+1)(a^2+a-1)}}{2a^2-1} \quad \text{for } m \leq \frac{2(a+1)^2-1}{2a^2-1},$$

$$\Delta(\mathcal{L}_{a,a+1}) = \frac{8m\sqrt{a(a+1)(a^2+a-1)}}{2(a+1)^2-1} \quad \text{for } m \geq \frac{2(a+1)^2-1}{2a^2-1}.$$

Also, by (4.12), $\mathcal{L}_a$ and $\mathcal{L}_{a+1}$ are admissible if

$$\Delta(\mathcal{L}_a) = 4\sqrt{\frac{a+1}{a-1}},\quad \Delta(\mathcal{L}_{a+1}) = 4m\sqrt{\frac{a}{a+2}}$$

for m in the range (4.18). Hence to establish (4.17) we need only prove that

$$\min\left\{4m\sqrt{\frac{a}{a+2}},\ \frac{8\sqrt{a(a+1)(a^2+a-1)}}{2a^2-1}\right\}<2(m+1)$$

$$\text{if}\ \ 1+\frac{2}{a}<m\le\frac{2(a+1)^2-1}{2a^2-1}; \tag{4.19}$$

$$\min\left\{\frac{8m\sqrt{a(a+1)(a^2+a-1)}}{2(a+1)^2-1},\ 4\sqrt{\frac{a+1}{a-1}}\right\}<2(m+1)$$

$$\text{if}\ \ \frac{2(a+1)^2-1}{2a^2-1}\le m\le 1+\frac{2}{a-1}. \tag{4.20}$$

Since $\frac{m}{m+1}$ increases with m, we need only verify that (4.19) and (4.20) hold at the particular values of m for which the two expressions to which the 'min' refers are equal. Thus, for (4.19) we have to show that

$$4m\sqrt{\frac{a}{a+2}}<2(m+1)\ \ \text{at}\ \ m=\frac{2\sqrt{(a+2)(a+1)(a^2+a-1)}}{2a^2-1}=m_1,\ \ \text{say}; \tag{4.21}$$

$$4\sqrt{\frac{a+1}{a-1}}<2(m+1)\ \ \text{at}\ \ m=\frac{2(a+1)^2-1}{2\sqrt{(a-1)a(a^2+a-1)}}=m_2,\ \ \text{say}. \tag{4.22}$$

Now

$$\begin{aligned}(a+2)(a+1)(a^2+a-1)&=a^4+4a^3+4a^2-a-2\\&<a^4+4a^3+4a^2=(a^2+2a)^2,\end{aligned}$$

so that

$$m_1<\frac{2a(a+2)}{2a^2-1}.$$

Hence

$$\begin{aligned}\frac{1}{2}\left(1+\frac{1}{m_1}\right)&>\frac{4a^2+4a-1}{4a(a+2)}=\frac{a^2+a-\frac{1}{4}}{\sqrt{a^3(a+2)}}\sqrt{\frac{a}{a+2}}\\&>\sqrt{\frac{a}{a+2}},\end{aligned}$$

this last inequality being equivalent to

$$\begin{aligned}(a^2+a-\tfrac{1}{4})^2&=a^4+2a^3+\tfrac{1}{2}a^2-\tfrac{1}{2}a+\tfrac{1}{16}\\&>a^4+2a^3=a^3(a+2),\end{aligned}$$

which is trivially true. This proves (4.21).

Next,

$$(a-1)\,a\,(a^2+a-1)=a^4-2a^2+a$$
$$<a^4-2a^2+a+1-\frac{1}{a}+\frac{1}{4a^2}$$
$$=\left(a^2-1+\frac{1}{2a}\right)^2,$$

whence

$$m_2>\frac{2(a+1)^2-1}{2\left(a^2-1+\dfrac{1}{2a}\right)},$$

$$\tfrac{1}{2}(m_2+1)>\frac{4a^2+4a-1+\dfrac{1}{a}}{4a^2-4+\dfrac{2}{a}}.$$

We suppose $a\geq 3$, since (4.22) can be simply verified for $a=2$; then we can replace this last inequality by

$$\tfrac{1}{2}(m_2+1)>\frac{4a^2+4a-2}{4a^2-4}=\frac{a^2+a-\frac{1}{2}}{\sqrt{(a-1)(a+1)^3}}\sqrt{\frac{a+1}{a-1}}.$$

This gives

$$\tfrac{1}{2}(m_2+1)>\sqrt{\frac{a+1}{a-1}},$$

as required, since

$$(a^2+a-\tfrac{1}{2})^2=a^4+2a^3-a+\tfrac{1}{4}>a^4+2a^3-2a-1=(a-1)(a+1)^3.$$

This completes the proof of Theorem 6 for $1<m\leq 3$.

(ii) We next consider the range $m\geq 3$. From the estimates (3.1), (3.2) and (3.4), we see that it suffices to give a symmetrical $\mathfrak{L}'$ with

$$\Delta(\mathfrak{L}')<\min\{(\tfrac{1}{2}+\sqrt{2})(m+1),\ \sqrt{m+1}\,(2+\sqrt{m+5})\}\qquad (m\geq 3). \tag{4.23}$$

We note that

$$\frac{m+1}{m},\quad \frac{\sqrt{m+1}\,(2+\sqrt{m+5})}{m+1}\quad\text{and}\quad\frac{\sqrt{m+1}\,(2+\sqrt{m+5})}{m}$$

are all decreasing functions of m.

By (4.12), $\mathfrak{L}_2$ is R_m-admissible if

$$\Delta(\mathfrak{L}_2)=\max\left\{\frac{4m}{\sqrt{3}},\ 4\sqrt{3}\right\}=\frac{4m}{\sqrt{3}}\quad\text{for } m\geq 3;$$

and by (4.13), $\mathfrak{L}_{1,3}$ and $\mathfrak{L}_{1,2}$ are R_m-admissible if

$$\Delta(\mathfrak{L}_{1,3}) = \max\left\{\frac{4m\sqrt{6}}{5},\ 4\sqrt{6}\right\} = \frac{4m\sqrt{6}}{5} \quad \text{for } m \geq 5,$$

$$\Delta(\mathfrak{L}_{1,2}) = \max\left\{\frac{8m\sqrt{2}}{2},\ 8\sqrt{2}\right\} = \frac{8m\sqrt{2}}{7} \quad \text{for } m \geq 7.$$

It is easily verified that (4.23) holds in $3 \leq m \leq 8$ by taking

$$\mathfrak{L}' = \mathfrak{L}_2 \quad \text{if } 3 \leq m \leq 5,$$

$$\mathfrak{L}' = \mathfrak{L}_{1,3} \quad \text{if } 5 \leq m \leq 7,$$

$$\mathfrak{L}' = \mathfrak{L}_{1,2} \quad \text{if } 7 \leq m \leq 8$$

(since the inequalities need be tested only at $m=5$, $m=7$ and $m=8$ respectively).

For $m > 8$, (4.23) becomes

$$\Delta(\mathfrak{L}') < \sqrt{m+1}\,(2+\sqrt{m+5}) \quad (m > 8). \tag{4.24}$$

Define an integer $p \geq 2$ by

$$4p - 1 \leq m < 4p + 3. \tag{4.25}$$

By (4.16), $\mathfrak{L}'_{p+1}$ is admissible for R_m if

$$\Delta(\mathfrak{L}'_{p+1}) = \max\left\{\frac{4m\sqrt{p^2+4p+3}}{4p+3},\ 4\sqrt{p^2+4p+3}\right\} = 4\sqrt{p^2+4p+3}. \quad \text{for } m < 4p+3.$$

Taking $\mathfrak{L}' = \mathfrak{L}'_{p+1}$ for m satisfying (4.25), we shall have established (4.24) when we show that

$$4\sqrt{p^2+4p+3} < \sqrt{m+1}\,(2+\sqrt{m+5}) \quad \text{for } m \geq 4p-1, \tag{4.26}$$

i.e. that

$$4\sqrt{p^2+4p+3} < \sqrt{4p}\,(2+\sqrt{4p+4}).$$

On squaring twice, this last inequality reduces to

$$4p(p^2-3) > 9,$$

which is obviously true if $p \geq 3$.

If $p = 2$, we need only verify that (4.26) holds for $m > 8$, i.e. that

$$4\sqrt{15} < 3(2+\sqrt{13}).$$

Thus (4.24) is established, and the proof of Theorem 6 complete.

As an immediate deduction from our inequalities (3.1)–(3.7) and our results for the special lattices $\mathfrak{L}_a$, $\mathfrak{L}'_p$ we have

Theorem 7. (Blaney [1]) (i) *If* m *is of the form* $1+\frac{2}{r}$ $(r=1, 2, 3, \ldots)$, *then*

$$D_m = 4\sqrt{m}.$$

(ii) *If* m *is of the form* $4s-1$ $(s=1, 2, 3, \ldots)$, *then*

$$D_m = \sqrt{(m+1)(m+9)}.$$

Proof. (i) If $m=1+\frac{2}{r}$ $(r\geq 1)$, the symmetrical lattice $\mathfrak{L}_{r+1}$ is, by (4.12), R_m-admissible for

$$\Delta(\mathfrak{L}_{r+1}) = \max\left\{4m\sqrt{\frac{r}{r+2}},\ 4\sqrt{\frac{r+2}{r}}\right\} = 4\sqrt{m},$$

so that $D_m \leq 4\sqrt{m}$. By Theorem 3 (i), $D_m \geq \sqrt{m}$.

(ii) If $m=4s-1$ $(s\geq 1)$, the symmetrical lattice $\mathfrak{L}'_s$ is, by (4.16), R_m-admissible for

$$\Delta(\mathfrak{L}'_s) = \max\left\{\frac{4m\sqrt{s^2+2s}}{4s-1},\ 4\sqrt{s^2+2s}\right\} = 4\sqrt{s^2+2s},$$

and so

$$D_m \leq 4\sqrt{s^2+2s} = \sqrt{(m+1)(m+9)}.$$

By Theorem 3 (ii),

$$D_m \geq \sqrt{(m+1)(m+9)}.$$

5. The evaluation of D_m

The foregoing analysis provides a 'local' method for the evaluation of D_m analogous to that provided by ordinary simple continued fractions for corresponding problems connected with homogeneous binary quadratic forms. Here we apply the method to determine D_m for the range

$$\frac{21}{11} = 1\cdot 9090\cdots \leq m \leq \frac{1098\sqrt{10}+6750}{4810} = 2\cdot 1251\cdots. \tag{5.1}$$

We first establish some simple inequalities:

Lemma 7. *Suppose that* $\theta_n > 1$, $\varphi_n > 1$, *and*

$$\Delta_n^- = \frac{4(\theta_n\varphi_n - 1)}{(\theta_n - 1)(\varphi_n - 1)} \leq D, \tag{5.2}$$

$$\Delta_n^+ = \frac{4(\theta_n\varphi_n - 1)}{(\theta_n + 1)(\varphi_n + 1)} \leq \frac{D}{k}, \tag{5.3}$$

where $D < 2(k+1)$. *Then*

$$\frac{D(\theta_n - 1) - 4}{D(\theta_n - 1) - 4\theta_n} \le \varphi_n \le \frac{4 + \dfrac{D}{k}(\theta_n + 1)}{4\theta_n - \dfrac{D}{k}(\theta_n + 1)} \tag{5.4}$$

and

$$\left|\theta_n - \frac{2(k-1)}{2(k+1) - D}\right| \le \frac{\sqrt{D^2 - 16k}}{2(k+1) - D}. \tag{5.5}$$

These inequalities also hold if θ_n, φ_n *are replaced by* φ_n, θ_n.

Proof. We note that Δ_n^- is a decreasing function of θ_n and φ_n, while Δ_n^+ increases with θ_n and φ_n. In particular, from (5.2),

$$D \ge \Delta_n^- > \frac{4\theta_n}{\theta_n - 1}. \tag{5.6}$$

Now (5.2) and (5.3) give (5.4), noting that $D(\theta_n - 1) - 4\theta_n > 0$ by (5.6). Also (5.4) implies that

$$\frac{D(\theta_n - 1) - 4}{D(\theta_n - 1) - 4\theta_n} \le \frac{4 + \dfrac{D}{k}(\theta_n + 1)}{4\theta_n - \dfrac{D}{k}(\theta_n + 1)},$$

i.e.

$$\theta_n^2(2k + 2 - D) - 4(k-1)\theta_n + D + 2k + 2 \le 0.$$

Since $D < 2k + 2$, by hypothesis, this holds if and only if (5.5) is satisfied.

Finally it is clear by symmetry that we may interchange θ_n and φ_n in the above argument.

Theorem 8. *We have*

$$D_m = \frac{16m\sqrt{15}}{21} \quad \text{if} \quad \frac{21}{11} \le m \le \frac{7\sqrt{30}}{20}, \tag{5.7}$$

$$D_m = 4\sqrt{2} \quad \text{if} \quad \frac{7\sqrt{30}}{20} \le m \le 2, \tag{5.8}$$

$$D_m = 2m\sqrt{2} \quad \text{if} \quad 2 \le m \le \frac{12\sqrt{5}}{13}, \tag{5.9}$$

$$D_m = \frac{24\sqrt{10}}{13} \quad \text{if} \quad \frac{12\sqrt{5}}{13} \le m \le \frac{27}{13}, \tag{5.10}$$

$$D_m = \frac{8m\sqrt{10}}{9} \quad \text{if} \quad \frac{27}{13} \le m \le \frac{1098\sqrt{10} + 6750}{4810}. \tag{5.11}$$

All critical lattices are given by the symmetrical lattices corresponding to the sequences

$$(\overset{\times}{4}, \overset{\times}{16}) \quad \textit{for} \quad \frac{21}{11} \leq m \leq \frac{7\sqrt{30}}{20}, \tag{5.12}$$

$$(\overset{\times}{6}) \quad \textit{for} \quad \frac{7\sqrt{30}}{20} \leq m \leq \frac{12\sqrt{5}}{13}, \tag{5.13}$$

$$(\overset{\times}{4}, \overset{\times}{10}) \quad \textit{for} \quad \frac{12\sqrt{5}}{13} \leq m \leq \frac{1098\sqrt{10}+6750}{4810}, \tag{5.14}$$

$$(\,_\infty(4, 10), 4, 8, (4, 10)_\infty) \quad \textit{for}^1 \quad m = \frac{1098\sqrt{10}+6750}{4810}. \tag{5.15}$$

Proof. By Theorems 6 and 4, we have only to show that the sequences in (5.12)–(5.15) are the only sequences of positive even integers for which the inequality

$$\max\,(\Delta_n^-, m\,\Delta_n^+) \leq D_m \tag{5.16}$$

holds for each n, where D_m is defined by (5.7)–(5.11); and that, for some n, equality holds in (5.16) for each of the given sequences in the stated range of values of m.

(i) Thus, we begin by considering sequences $\{a_n\}$ satisfying

$$\Delta_n^- \leq 4\sqrt{2}, \tag{5.17}$$

$$m\,\Delta_n^+ \leq \frac{16\,m\sqrt{15}}{21} \tag{5.18}$$

for each n, and prove that the only such sequences are $(\overset{\times}{4}, \overset{\times}{16})$ and $(\overset{\times}{6})$.

The hypotheses of Lemma 7 are satisfied, for each n, with

$$D = 4\sqrt{2} = 5\cdot 65685\ldots,$$

$$\frac{D}{k} = \frac{16\sqrt{15}}{21} = 2\cdot 95084\ldots,$$

$$k = \frac{7\sqrt{30}}{20} = 1\cdot 91702\ldots.$$

Working with sufficient accuracy to four places of decimals, (5.5) gives, for any n,

$$\left|\theta_n - \frac{1\cdot 8340}{0\cdot 1772}\right| < \frac{1\cdot 1530}{0\cdot 1772},$$

[1] The suffixes ∞ imply infinite repetition to the left and right respectively. Thus this is the sequence $(\ldots 4, 10, 4, 10, 4, 8, 4, 10, 4, 10, \ldots)$.

whence

$$3.8 < \theta_n < 16.9.$$

By Lemma 5, Corollary, $\theta_n = [a_n, a_{n-1}, \ldots]$ lies between $a_n - 1$ and a_n; since a_n is even, it follows that a_n can take only the values 4, 6, 8, 10, 12, 14, 16.

Suppose that some $a_r = 4$. Since all $a_n \leq 16$, Lemma 5 gives

$$\theta_r \leq [4, 16, \ldots] < [4, 16] = \tfrac{63}{16} = 3 \cdot 9375,$$

and so, by (5.4),

$$\varphi_r > \frac{(5 \cdot 6569)(2 \cdot 9375) - 4}{(5 \cdot 6569)(2 \cdot 9375) - 4(3 \cdot 9375)} > \frac{12 \cdot 616}{0 \cdot 867} > 14 \cdot 5.$$

Since $\varphi_r = [a_{r+1}, \ldots] < a_{r+1}$, it follows that $a_{r+1} = 16$. By symmetry, $a_{r-1} = 16$ (since we may replace θ_r, φ_r by φ_{r-1}, θ_{r-1} in the above argument).

Suppose next that some $a_r = 16$. Then $\theta_r > 15$ and (5.4) gives

$$\varphi_r < \frac{4 + 16(2 \cdot 9509)}{60 - 16(2 \cdot 9509)} = \frac{51 \cdot 2144}{12 \cdot 7856} < 5,$$

whence $a_{r+1} = 4$; by symmetry, $a_{r-1} = 4$.

It follows at once from these last two results that *either* the sequence $\{a_n\}$ is $(\overset{\times}{4}, \overset{\times}{16})$ *or* $6 \leq a_n \leq 14$ for all n. We now show that the second alternative can hold only for the sequence $(\overset{\times}{6})$. For suppose that some $a_n \geq 8$. Then, using Lemma 5, we have

$$\theta_n \geq [8, 6_\infty] = 5 + 2\sqrt{2}, \quad \varphi_n \geq [6_\infty] = 3 + 2\sqrt{2},$$

whence

$$\Delta_n^+ \geq \frac{4[(5 + 2\sqrt{2})(3 + 2\sqrt{2}) - 1]}{(6 + 2\sqrt{2})(4 + 2\sqrt{2})} = \frac{8 + 9\sqrt{2}}{7} = 2 \cdot 961 \cdots > \frac{16\sqrt{15}}{21},$$

contradicting (5.18).

Thus the only sequences which can satisfy (5.17), (5.18) are $(\overset{\times}{4}, \overset{\times}{16})$ and $(\overset{\times}{6})$. Also (using the results of § 4, (4.12), (4.13)) we have

$$\Delta_n^+ = 2\sqrt{2}, \quad \Delta_n^- = 4\sqrt{2} \quad \text{for } (\overset{\times}{6}),$$

$$\Delta_n^+ = \frac{16\sqrt{15}}{21}, \quad \Delta_n^- = \frac{16\sqrt{15}}{11} \quad \text{for } (\overset{\times}{4}, \overset{\times}{16}).$$

This proves (5.7) and (5.8), and establishes the assertions on critical lattices for the range $\frac{21}{11} \leq m \leq 2$.

(ii) To prove (5.9) and (5.10). we begin by proving that the only sequences satisfying

$$\Delta_n^+ \leq 2\sqrt{2} = 2\cdot 82842\ldots \tag{5.19}$$

$$\Delta_n^- \leq \frac{24\sqrt{10}}{13} = 5\cdot 83805\ldots \tag{5.20}$$

for each n are $(\overset{\times}{6})$ and $(\overset{\times}{4}, \overset{\times}{10})$.

The inequalities (5.19), (5.20) imply that the hypotheses of Lemma 7 are satisfied, for each n, with

$$D = \frac{24\sqrt{10}}{13} = 5\cdot 83805\ldots,$$

$$\frac{D}{k} = 2\sqrt{2} = 2\cdot 82842\ldots,$$

$$k = \frac{12\sqrt{5}}{13} = 2\cdot 06406\ldots$$

Hence (5.5) gives, with sufficient accuracy,

$$\left|\theta_n - \frac{2\cdot 1281}{0\cdot 2900}\right| < \frac{1\cdot 0290}{0\cdot 2900},$$

i.e.

$$3\cdot 79 < \theta_n < 10\cdot 89.$$

Thus a_n can take only the values 4, 6, 8, 10.

Now if $a_n = 4$ and $a_{n+1} \leq 8$, we have $\theta_n < 4$, $\varphi_n < 8$ and so

$$\Delta_n^- > \frac{4\times 31}{3\times 7} = \frac{124}{21} > 5\cdot 9,$$

contradicting (5.20). Hence if $a_n = 4$, we require $a_{n+1} = 10$; and, by symmetry, also $a_{n-1} = 10$.

Next, if $a_n = 10$ and $a_{n+1} \geq 6$, we have $\theta_n > 9$, $\varphi_n > 5$ and so

$$\Delta_n^+ > \frac{4\times 44}{10\times 6} = \frac{44}{15} > 2.9,$$

contradicting (5.19). Hence if $a_n = 10$ we require $a_{n+1} = 4$; and, by symmetry, also $a_{n-1} = 4$.

The results of the last two paragraphs show that *either* $\{a_n\}$ is $(\overset{\times}{4}, \overset{\times}{10})$ *or* $6 \leq a_n \leq 8$ for each n. The second alternative can hold only if $\{a_n\}$ is $(\overset{\times}{6})$. For otherwise we obtain $\Delta_n^+ \geq \frac{8+9\sqrt{2}}{7} = 2\cdot 961\ldots$, precisely as in (i), and this contradicts (5.19).

Thus the only sequences satisfying (5.19) and (5.20) are $(\overset{\times}{6})$, for which

$$\Delta_n^+ = 2\sqrt{2}, \quad \Delta_n^- = 4\sqrt{2} < \frac{24\sqrt{10}}{13} \quad \text{for all } n,$$

and $(\overset{\times}{4}, \overset{\times}{10})$, for which

$$\Delta_n^- = \frac{24\sqrt{10}}{13}, \quad \Delta_n^+ = \frac{8\sqrt{10}}{9} < 2\sqrt{2} \quad \text{for all } n,$$

This proves (5.9) and (5.10) and establishes the assertions on critical lattices for the range $2 \le m \le \frac{27}{13}$.

(iii) For convenience, we set

$$m_0 = \frac{1098\sqrt{10} + 6750}{4810} = 2 \cdot 12519 \ldots.$$

For the proof of (5.11) we have first to show that the only sequences satisfying

$$\Delta_n^+ \le \frac{8\sqrt{10}}{9} = 2 \cdot 81091 \ldots \tag{5.21}$$

$$\Delta_n^- \le \frac{8 m_0 \sqrt{10}}{9} = \frac{976 + 600\sqrt{10}}{481} = 5 \cdot 97373 \ldots \tag{5.22}$$

for all n are $(\overset{\times}{4}, \overset{\times}{10})$ and $({}_\infty(4, 10), 4, 8, (4, 10)_\infty)$.

Using Lemma 7 with

$$D = 5 \cdot 97373 \ldots, \quad \frac{D}{k} = 2 \cdot 81091 \ldots, \quad k = m_0 = 2 \cdot 12519 \ldots$$

we have to sufficient accuracy

$$\left| \theta_n - \frac{2 \cdot 2504}{\cdot 2767} \right| < \frac{1 \cdot 6827}{\cdot 2767},$$

i.e.

$$2 \cdot 05 < \theta_n < 14 \cdot 22.$$

Hence $4 \le a_n \le 14$ for all n.

Since $a_n \ge 4$ for all n, $\theta_n \ge [\overset{\times}{4}] = 2 + \sqrt{3} > 3 \cdot 732$. If now $\varphi_n \ge 11$ we have

$$\Delta_n^+ > \frac{4\,(41 \cdot 052 - 1)}{4 \cdot 732 \times 12} = \frac{40 \cdot 052}{14 \cdot 196} > 2 \cdot 82,$$

contradicting (5.21). It follows that $\varphi_n < 11$ for all n and so that $a_n \le 10$ for all n.

Now if $a_n = 6$ for some n, we have

$$\theta_n \le [6, \overset{\times}{10}] = 1 + \sqrt{24}, \quad \theta_n \ge [6, \overset{\times}{4}] = 4 + \sqrt{3}.$$

Now (5.4) gives, to sufficient accuracy,

$$\frac{(5\cdot 9738)(4\cdot 8990)-4}{(5\cdot 9738)(4\cdot 8990)-4(5\cdot 8990)}<\varphi_n<\frac{4+(2\cdot 8110)(6\cdot 8990)}{4(5\cdot 8990)-(2\cdot 8110)(6\cdot 8990)},$$

whence

$$4\cdot 4<\varphi_n<5\cdot 6,$$

and so $a_{n+1}=6$. By symmetry, also $a_{n-1}=6$. By repeated application, it follows that $a_n=6$ for all n. But this involves $\Delta_n^+=2\sqrt{2}=2\cdot 828\ldots$, contradicting (5.21). Thus $a_n\neq 6$ for all n.

No two consecutive elements of the sequence $\{a_n\}$ can be 4. For if $a_n=a_{n+1}=4$ we have $\theta_n<4$, $\varphi_n<4$,

$$\Delta_n^->\frac{4\times 15}{3\times 3}=\frac{20}{3},$$

contradicting (5.22). Also no two consecutive elements can be 8 or 10. For if $a_n\geq 8$, $a_{n+1}\geq 8$ we have $\theta_n>7$, $\varphi_n>7$,

$$\Delta_n^+>\frac{4\times 48}{8\times 8}=3,$$

contradicting (5.21). Thus the sequence must be of the form

$$\ldots,4,a_{-2},4,a_0,4,a_2,\ldots,$$

where each a_{2n} is 8 or 10.

If $a_{2n}=10$ for all n, the sequence is $(\overset{\times}{4},\overset{\times}{10})$. Otherwise some $a_{2n}=8$; then

$$\theta_{2n}=[8,4,a_{2n-2},\ldots]\leq[8,\overset{\times}{4},\overset{\times}{10}]=\frac{6+3\sqrt{10}}{2},$$

$$\varphi_{2n}=[4,a_{2n+2},\ldots]\leq[\overset{\times}{4},\overset{\times}{10}]=\frac{10+3\sqrt{10}}{5},$$

whence

$$\Delta_{2n}^-\geq\frac{4\left[\dfrac{(6+3\sqrt{10})(10+3\sqrt{10})}{10}-1\right]}{\left(\dfrac{4+3\sqrt{10}}{2}\right)\left(\dfrac{5+3\sqrt{10}}{5}\right)}=\frac{976+600\sqrt{10}}{481};$$

this is consistent with (5.22) only if the equality sign holds throughout, i.e. if the sequence is that given in (5.15).

We must now consider the values of Δ_n^+, Δ_n^- for the sequences of (5,14), (5,15). If $\{a_n\}$ is $(\overset{\times}{4},\overset{\times}{10})$ we have, as in (ii),

$$\Delta_n^+=\frac{8\sqrt{10}}{9},\quad\Delta_n^-=\frac{24\sqrt{10}}{3}=5\cdot 83\ldots<\frac{8m_0\sqrt{10}}{9}=5\cdot 97\ldots\quad\text{for all } n.$$

For the sequence (5.15), choose the enumeration so that $a_0 = 8$. As in the above calculation, we have $\theta_0 = \varphi_{-1} = [8, \overset{\times}{4}, \overset{\times}{10}]$, $\varphi_0 = \theta_{-1} = [\overset{\times}{4}, \overset{\times}{10}]$,

$$\Delta_0^- = \Delta_{-1}^- = \frac{976 + 600\sqrt{10}}{481} = \frac{8 m_0 \sqrt{10}}{9}.$$

For $n > 0$,

$$\varphi_{2n} = [\overset{\times}{4}, \overset{\times}{10}] = \varphi_0,$$

$$\theta_{2n} = [10, 4, \ldots] > \theta_0,$$

whence

$$\Delta_{2n}^- < \Delta_0^-$$

also

$$\varphi_{2n+1} = [\overset{\times}{10}, \overset{\times}{4}] = \frac{10 + 3\sqrt{10}}{2} > 9 \cdot 74$$

$$\theta_{2n+1} = [4, 10, \ldots, 4, 10, 4, 8, 4, 10, \ldots]$$

$$\geq [4, 8, \overset{\times}{4}, \overset{\times}{10}] = \frac{38 - \sqrt{10}}{9} > 3 \cdot 87,$$

whence

$$\Delta_{2n+1}^- < \frac{4\{(9 \cdot 74)(3 \cdot 87) - 1\}}{(8 \cdot 74)(2 \cdot 87)} = \frac{146 \cdot 7752}{25 \cdot 0838} = 5 \cdot 85 \ldots < \frac{8 m_0 \sqrt{10}}{9}.$$

Thus

$$\max_n \Delta_n^- = \frac{8 m_0 \sqrt{10}}{9},$$

for $n \geq 0$ and so, by symmetry, over all n.

Finally, the values of θ_n and φ_n at any point of the sequence (5.15) are less than or equal to the corresponding values of θ_n and φ_n for the sequence $(\overset{\times}{4}, \overset{\times}{10})$, with strict inequality for one of θ_n, φ_n, so that

$$\Delta_n^+ < \frac{8\sqrt{10}}{9} \quad \text{for all } n.$$

Since θ_n, φ_n tend to $[\overset{\times}{4}, \overset{\times}{10}]$, $[\overset{\times}{10}, \overset{\times}{4}]$, in some order, as $n \to \pm\infty$, it follows that

$$\lim_{n \to \pm\infty} \Delta_n^+ = \frac{8\sqrt{10}}{9}.$$

The above results establish (5.11) and the assertions on the critical lattices for $\frac{27}{13} \leq m \leq m_0$.

6. It is clear from the proof of Theorem 8 that, for the range (5.1) of values of m with the exception of the end-point

$$m = m_0 = \frac{1098\sqrt{10} + 6750}{4810}, \tag{6.1}$$

any admissible lattice $\mathfrak{L}$ *either* is a multiple of a critical lattice (essentially unique) *or* satisfies an inequality

$$\Delta(\mathfrak{L}) \geq D_m (1+\delta_m), \quad \delta_m > 0. \tag{6.2}$$

Thus we may say that D_m is *isolated.*[1] For any particular value of m it is not difficult to obtain an explicit value for δ_m. As an example we prove

Theorem 9. *If* $m=2$ *and* $\mathfrak{L}$ *is* R_m*-admissible, then either* $\Delta(\mathfrak{L}) \geq \sqrt{33}$ *or* $\mathfrak{L}$ *is a symmetrical lattice corresponding to the sequence* $(\overset{\times}{6})$.

Proof. The result will follow if we show that the only sequence $\{a_n\}$ satisfying

$$\left.\begin{aligned} \Delta_n^- &< \sqrt{33} \\ 2\Delta_n^+ &< \sqrt{33} \end{aligned}\right\} \tag{6.3}$$

for all n is $(\overset{\times}{6})$.

The hypotheses of Lemma 7 are satisfied, for each n, with

$$D = \sqrt{33}, \quad k=2,$$

and so we find from (5.5)

$$\left| \theta_n - \frac{2}{6-\sqrt{33}} \right| \leq \frac{1}{6-\sqrt{33}},$$

i.e.

$$[\overset{\times}{4}, \overset{\times}{12}] = \frac{6+\sqrt{33}}{3} \leq \theta_n \leq 6+\sqrt{33} = [\overset{\times}{12}, \overset{\times}{4}]. \tag{6.4}$$

It follows easily from (6.4) that $4 \leq a_n \leq 12$ for each n, and that if any element of the sequence is 4 or 12 then the sequence is $(\overset{\times}{4}, \overset{\times}{12})$.

For this sequence we have

$$\Delta_n^- = \sqrt{33}, \quad \Delta_n^+ = \tfrac{1}{2}\sqrt{33}$$

for each n, and (6.3) is not satisfied.

Thus (6.3) implies that $6 \leq a_n \leq 10$ for all n. If now some $a_n \geq 8$ we have, precisely as in the proof of Theorem 8 (i),

$$\Delta_n^+ \geq \frac{8+9\sqrt{2}}{7} > \tfrac{1}{2}\sqrt{33},$$

contradicting (6.3). It follows that the sequence must be $(\overset{\times}{6})$, as required.

On the other hand, it is easy to see that D_{m_0} (where m_0 is defined by (6.1)) is *not* isolated. We can approximate to the sequence (5.15) by sequences of the type

$$\ldots, 4, 8, (4, 10)_{n_0}, 4, 8, (4, 10)_{n_1}, \ldots \tag{6.5}$$

[1] The existence of such a result for $m=2$ was first established by Davenport [4], Lemma 14.

where the n_i are arbitrarily large positive integers. For such sequences the values of $\max \Delta_n^-$ and $\max \Delta_n^+$ are arbitrarily close to those for the sequence (5.15). Hence these sequences yield admissible symmetrical lattices of determinant arbitrarily close to D_{m_0}.

We have been unable to evaluate D_m for general m near (but greater than) m_0. It is not difficult to show that, for m sufficiently close to m_0, all critical lattices are derived from sequences of the type (6.5); and there is good reason to believe that the subscripts n_i behave in a similar way to those occurring in Markoff's classical papers on the (homogeneous) minimum of a binary quadratic form.[1] This type of difficulty arises for infinitely many values of m in the range $1 < m < 3$.

7. We conclude by giving the region K referred to in Swinnerton-Dyer [7], which satisfies the conditions of Theorem 4 of that paper and whose critical lattices have either no or one point on the boundary of K. We define K by the inequalities[2]

$$xy > -1,$$
$$xy < 2 \quad \text{if } x, y \text{ are both negative,}$$
$$xy < 2 + 2\varepsilon \cos^2 \pi \{\log (x/y)/\log 2\} \quad \text{if } x, y \text{ are both positive,}$$

where ε is a fixed constant satisfying

$$0 < \varepsilon < \sqrt{\frac{33}{32}} - 1.$$

K is automorphic, since it admits the automorph

$$x' = 2^n x, \quad y' = 2^{-n} y$$

for all integral n; and it is easily seen to satisfy the other conditions of [7] Theorem 4. Since K lies inside $-1-\varepsilon < xy < 2 + 2\varepsilon$ and contains $-1 < xy < 2$, we deduce from Theorem 8 that

$$4\sqrt{2} \leq \Delta(K) \leq 4(1+\varepsilon)\sqrt{2} < \sqrt{33}$$

(where $\Delta(K)$ is the inhomogeneous critical determinant of K). It now follows from [7] Theorem 4 that there are K-admisssible lattices Λ with $d(\Lambda) \leq 4(1+\varepsilon)\sqrt{2}$, and from Theorem 9 that all such lattices are of the form

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[1] See also L. E. Dickson, Studies in the Theory of Numbers (Chicago, 1930), 79–107.

[2] This body is an obvious analogue of that given by Cassels [2] in order to prove a corresponding result for the homogeneous critical lattices of an automorphic star body.

$$x=\lambda\{(\sqrt{2}+1)(u-\tfrac{1}{2})+(\sqrt{2}-1)(v-\tfrac{1}{2})\}$$
$$y=\mu\{(\sqrt{2}-1)(u-\tfrac{1}{2})+(\sqrt{2}+1)(v-\tfrac{1}{2})\},$$

where u, v run through all integral values and λ, μ satisfy

$$1\leq\lambda\mu=\frac{d(\Lambda)}{4\sqrt{2}}\leq 1+\varepsilon.$$

We now examine what further conditions on λ, μ are needed to make Λ K-admissible. The only doubtful points of Λ are those which lie in the first quadrant and have $xy=2\lambda\mu$; for $xy/\lambda\mu$ is an integer for every point of Λ. These points are given by $x=\lambda(\sqrt{2}+1)^{2n}\sqrt{2}$, $y=\mu(\sqrt{2}-1)^{2n}\sqrt{2}$ for any integer n. For these points to lie outside K we require

$$\lambda\mu\geq 1+\varepsilon\cos^2\pi\left\{\frac{\log(\lambda/\mu)+4n\log(\sqrt{2}+1)}{\log 2}\right\}$$

for all n; and since $\dfrac{\log(\sqrt{2}+1)}{\log 2}$ is irrational we can choose n so that the expression in curly brackets is arbitrarily near an integer. Thus Λ is K-admissible if and only if $\lambda\mu\geq 1+\varepsilon$, and critical for K if and only if $\lambda\mu=1+\varepsilon$; while $\Delta(K)=4\sqrt{2}(1+\varepsilon)$.

It is easy to see that a critical lattice Λ cannot have more than one point on the boundary of K, and has one if and only if

$$\frac{\lambda}{\mu}=2^m(\sqrt{2}-1)^{4n}$$

for some integers, m, n.

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References

[1] H. Blaney, Some asymmetric inequalities, *Proc. Cam. Phil. Soc.* 46 (1950), 359–376.
[2] J. W. S. Cassels, On two problems of Mahler, *Proc. K. Ned. Akad. v. Wetensch.* 51 (1948), 854–857.
[3] ——, Yet another proof of Minkowski's theorem on the product of two inhomogeneous linear forms, *Proc. Cam. Phil. Soc.* 49 (1953), 365–366.
[4] H. Davenport, Non-homogeneous ternary quadratic forms, *Acta Math.* 80 (1948), 65–95.
[5] B. N. Delauney, An algorithm for divided cells, *Izvestia Acad. Nauk. SSSR*, 11 (1947), 505–538 (in Russian).
[6] D. B. Sawyer, The product of two non-homogeneous linear forms, *J. Lond. Math. Soc.* 23 (1948), 250–251.
[7] H. P. F. Swinnerton-Dyer, Inhomogeneous Lattices, *Proc. Cam. Phil. Soc.* 50 (1954), 20–25.
[8] O. Perron, *Kettenbrüche* (Leipzig, 1929).