THE INHOMOGENEOUS MINIMUM OF A TERNARY QUADRATIC FORM (II)

BY

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1. Let $Q(x, y, z)$ be an indefinite ternary quadratic form with real coefficients and determinant $D \neq 0$. For any real numbers x_0, y_0, z_0 we write

$$M(Q; x_0, y_0, z_0) = \text{g.l.b.}\ |Q(x, y, z)|, \tag{1.1}$$

where the lower bound is taken over all sets

$$x, y, z \equiv x_0, y_0, z_0 \pmod 1.$$

Then
$$M(Q) = \text{l.u.b.}\ M(Q; x_0, y_0, z_0), \tag{1.2}$$

over all sets x_0, y_0, z_0, is called the *inhomogeneous minimum* of Q.

In a recent paper [1] I showed that

$$M(Q) < (\tfrac{4}{15}|D|)^{\frac{1}{3}} \tag{1.3}$$

unless Q is equivalent to a multiple of one of

$$Q_1 = x^2 - y^2 - z^2 + xy - 7yz + zx \tag{1.4}$$

or
$$Q_2 = 2x^2 - y^2 + 15z^2; \tag{1.5}$$

while
$$M(Q_1) = (\tfrac{27}{100}|D|)^{\frac{1}{3}}, \qquad M(Q_2) = (\tfrac{4}{15}|D|)^{\frac{1}{3}}, \tag{1.6}$$

the upper bound (1.2) being attained only when $x_0, y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \pmod 1$.

These results were an extension of those given by Davenport [5], who showed that there existed a constant $\delta > 0$ such that

$$M(Q) \leq (1 - \delta)(\tfrac{27}{100}|D|)^{\frac{1}{3}}$$

unless Q is equivalent to a multiple of Q_1, while $M(Q_1)=(\frac{27}{100}|D|)^{\frac{1}{3}}$ (and is an attained upper bound).

At the end of [1] I mentioned the possibility of extending the method and finding more complete results. Such an extension has in fact proved possible and leads to:

THEOREM 1. (i) *If $Q(x, y, z)$ is not equivalent to a multiple of either of the forms Q_1, Q_2 of* (1.4), (1.5), *then*

$$M(Q)\leq(\tfrac{1}{4}|D|)^{\frac{1}{3}}. \tag{1.7}$$

(ii) *For the special forms Q_1, Q_2, we have*

$$M(Q_i;x_0,y_0,z_0)\leq(\tfrac{1}{4}|D|)^{\frac{1}{3}} \qquad (i=1,2) \tag{1.8}$$

unless $x_0, y_0, z_0=\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ (mod 1); *further*

$$M(Q_1;\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})=(\tfrac{27}{100}|D|)^{\frac{1}{3}}=M(Q_1), \tag{1.9}$$

$$M(Q_2;\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})=(\tfrac{4}{15}|D|)^{\frac{1}{3}}=M(Q_2). \tag{1.10}$$

This theorem has the same statement as [1] Theorem 1, save that the constant $\frac{4}{15}$ there has been replaced by $\frac{1}{4}$ in (1.7), (1.8). The constant $\frac{1}{4}$ was found veryr simply by Davenport [5] to correspond to the precise upper bound of $M(Q)$ fo *zero* forms Q and which, by analogy with corresponding results for binary quadratic forms and the product of three real linear forms, might be regarded as the "natural" constant for this problem. Davenport's example ([5], Theorem 2) of the form

$$Q_3=x^2-y^2+4z^2 \tag{1.11}$$

for which

$$M(Q_3;\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})=1=(\tfrac{1}{4}|D|)^{\frac{1}{3}} \tag{1.12}$$

shows explicitly that Theorem 1 becomes false if $\frac{1}{4}$ is replaced by any smaller constant.

Using Davenport's result for zero forms, and the proof of (1.9), (1.10) given in [1], § 2, we can reduce the proof of Theorem 1 to that of

THEOREM 2. *If Q is not a zero form and*

$$M(Q;x_0,y_0,z_0)>(\tfrac{1}{4}|D|)^{\frac{1}{3}}, \tag{1.13}$$

then Q is equivalent to a multiple of either Q_1 or Q_2 with $x_0, y_0, z_0\equiv\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ (mod 1).

It would be very interesting to know whether or not the sequence of "succesive minima" continues below $(\frac{1}{4}|D|)^{\frac{1}{3}}$. This problem appears to be of a much greater order of difficulty and would almost certainly need a different technique from that used in this paper [which would break down at the proof of Lemma 2.5].

2. For the proof of Theorem 2, we may assume that $D<0$ (considering $-Q$ in place of Q, if necessary). Let a be any value properly represented by Q and satisfying

$$0<a\leq(\tfrac{4}{3}|D|)^{\frac{1}{3}}; \tag{2.1}$$

that such a value exists follows from Barnes [2], Theorem 1. Then $\dfrac{1}{a}Q(x, y, z)$ is equivalent to a form

$$f(x, y, z)=(x+hy+gz)^2-\phi(y, z), \tag{2.2}$$

where $\phi(y, z)$ is indefinite, of discriminant

$$\Delta^2=\frac{4|D|}{a^3}\geq 3. \tag{2.3}$$

For any such form f we define

$$d=d(f)=(\tfrac{1}{2}\Delta^2)^{\frac{1}{3}}, \tag{2.4}$$

so that, by (2.3),

$$d\geq(\tfrac{3}{2})^{\frac{1}{3}}>1\cdot 144. \tag{2.5}$$

Now (1.13) of Theorem 2 is equivalent to the assertion that*

$$M(f;\, x_0, y_0, z_0)>(\tfrac{1}{16}\Delta^2)^{\frac{1}{3}}=\tfrac{1}{2}d. \tag{2.6}$$

Defining $\mu>0$, $\nu>0$ by

$$\mu\Delta=\tfrac{1}{2}d-\tfrac{1}{4}, \tag{2.7}$$

$$\nu\Delta=\tfrac{1}{2}d+\tfrac{1}{4}[d]^2, \tag{2.8}$$

we see as in [1] Lemma 4 that (2.6) certainly cannot hold if there exists a solution of

$$-\mu\Delta\leq\phi(y, z)\leq\nu\Delta, \qquad y, z\equiv y_0, z_0 \pmod 1. \tag{2.9}$$

We may therefore suppose henceforward that (2.9) cannot be satisfied.

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* The sets x_0, y_0, z_0 in (1.13) and (2.6.) are of course not necessarily the same; we adopt the convention, as in [1], that when we pass to (a multiple of) an equivalent form, any particular set x_0, y_0, z_0 is subjected to the same transformation as the variables x, y, z.

The form $\frac{1}{\mu\Delta}\phi(y, z)$, of discriminant $1/\mu^2$, with $y, z \equiv y_0, z_0 \pmod 1$, runs over the values of $\xi\eta$ corresponding to an inhomogeneous lattice $\mathfrak{L}$:

$$\left.\begin{aligned}\xi &= \alpha y + \beta z\\ \eta &= \gamma y + \delta z\end{aligned}\right\}, \qquad y, z \equiv y_0, z_0 \pmod 1$$

in the ξ, η-plane, of determinant $\Delta(\mathfrak{L}) = 1/\mu$. For any $m \geq 1$ we denote by $\mathcal{R}_m$ the region

$$-1 \leq \xi\eta \leq m.$$

In the usual terminology, we say that a lattice $\mathfrak{L}$ is admissible for $\mathcal{R}_m$ if $\mathfrak{L}$ has no point in the interior of $\mathcal{R}_m$; and we define D_m, the (inhomogeneous) critical determinant of $\mathcal{R}_m$, to be the lower bound of $\Delta(\mathfrak{L})$ over all lattices $\mathfrak{L}$ admissible for $\mathcal{R}_m$.

Since (2.9) has no solution, *the lattice $\mathfrak{L}$ corresponding to $\frac{1}{\mu\Delta}\phi(y, z)$ can have no point in $\mathcal{R}_m$, where*

$$m = \frac{\nu}{\mu} = \frac{2d + [d]^2}{2d - 1}. \tag{2.10}$$

This implies that certainly

$$\frac{1}{\mu} \geq D_m. \tag{2.11}$$

We use (2.11) to deduce:

Lemma 2.1. *If* (2.6) *holds for any x_0, y_0, z_0, then*

$$1\cdot 144 < d < 9. \tag{2.12}$$

Proof. If $d \geq 9$, (2.10) gives

$$m > \frac{2d + (d-1)^2}{2d - 1} = \frac{d^2 + 1}{2d - 1} \geq \frac{82}{17} > 3,$$

so that*

$$D_m \geq \sqrt{(m+1)(m+9)} \qquad (m \geq 3), \tag{2.13}$$

then it follows from (2.10), (2.11), (2.13) that

$$1 \geq \sqrt{(\nu + \mu)(\nu + 9\mu)},$$

or

$$([d]^2 + 4d - 1)([d]^2 + 20d - 9) \leq 16\Delta^2 = 32d^3. \tag{2.14}$$

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* See, for example, Barnes and Swinnerton-Dyer [4], Theorem 3.

If now $9 \le d < 10$, $[d] = 9$ and (2.14) gives

$$(d+20)(5d+13) \le 2d^3,$$

which is easily seen to be false for $d \ge 9$.

If $d \ge 10$, (2.14), with $[d] > d-1$, gives

$$(d^2+2d)(d^2+18d-8) \le 32d^3,$$

$$d^3 - 12d^2 + 28d - 16 \le 0;$$

and it is easily verified that this is false for $d \ge 10$.

Thus (2.11) cannot hold if $d \ge 9$; this, with (2.5), proves the lemma.

In order to find further restrictions on the possible values of d, we shall need more precise information than (2.13) on the value of D_m. For this, and for the later stages of the proof, we use the technique of Barnes and Swinnerton-Dyer [4], which is outlined in [1], § 4. The main results are stated here for convenience.

We denote generally by $[b_1, b_2, b_3, \dots]$ the continued fraction

$$b_1 - \frac{1}{b_2} - \frac{1}{b_3} - \cdots,$$

where b_n is integral and $b_n \ge 2$; all continued fractions we consider are infinite and satisfy the conditions $b_n \ge 4$ for infinitely many n. The value of any such continued fraction is increased if any partial quotient b_n is increased; this gives, in particular,

$$[b_1, b_2, \dots, b_{n-1}, b_n - 1] < [b_1, b_2, \dots, b_n, \dots] < [b_1, b_2, \dots, b_n]. \tag{2.15}$$

Let $\{a_n\}$ $(-\infty < n < \infty)$ be any chain of positive even integers such that $a_n \ge 4$ for infinitely many n of each sign. For each n we define

$$\left.\begin{aligned} \theta_n &= [a_n, a_{n-1}, a_{n-2}, \dots] \\ \phi_n &= [a_{n+1}, a_{n+2}, a_{n+3}, \dots] \end{aligned}\right\}; \tag{2.16}$$

then, using (2.15), $\theta_n > 1$, $\phi_n > 1$. For any real λ, μ with $\lambda\mu > 0$, the inhomogeneous lattice $\mathcal{L}$ defined, for any n, by

$$\xi = \lambda\{\theta_n(u-\tfrac{1}{2}) + (v-\tfrac{1}{2})\}$$
$$\eta = \mu\{(u-\tfrac{1}{2}) + \phi_n(v-\tfrac{1}{2})\},$$

where u, v run through all integral values, is called a symmetrical lattice corresponding to the chain $\{a_n\}$. If $\mathcal{L}$ has determinant $\Delta > 0$, we have $\Delta = \lambda\mu(\theta_n\phi_n - 1)$, so that, for points of $\mathcal{L}$,

$$\xi\eta = \frac{\Delta}{\theta_n\phi_n - 1}(\theta_n y + z)(y + \phi_n z), \qquad y, z \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1 \tag{2.17}$$

A symmetrical lattice $\mathfrak{L}$ of determinant Δ is admissible for $\mathcal{R}_m$ $(m > 1)$ if and only if the inequalities

$$\left.\begin{aligned} \frac{\Delta}{m} &\geq \frac{4(\theta_n\phi_n - 1)}{(\theta_n + 1)(\phi_n + 1)} = \Delta_n^+ \\ \Delta &\geq \frac{4(\theta_n\phi_n - 1)}{(\theta_n - 1)(\phi_n - 1)} = \Delta_n^- \end{aligned}\right\} \tag{2.18}$$

hold for all n.

For any $m > 1$, all critical lattices of $\mathcal{R}_m$ (i.e. admissible lattices of determinant D_m) are symmetrical. Moreover the inequality

$$\Delta(\mathfrak{L}) \geq 2(m+1) \text{ if } 1 < m \leq 3 \tag{2.19}$$

holds for any $\mathcal{R}_m$-admissible $\mathfrak{L}$ which is not symmetrical.

Finally, if

$$0 < G < 2(k+1) \tag{2.20}$$

and, for any n,

$$\Delta_n^- \leq G, \qquad \Delta_n^+ \leq \frac{G}{k}, \tag{2.21}$$

then the inequality

$$\left|\alpha - \frac{2(k-1)}{2(k+1) - G}\right| \leq \frac{\sqrt{G^2 - 16k}}{2(k+1) - G} \tag{2.22}$$

holds if $\alpha = \theta_n$ or if $\alpha = \phi_n$.

We now specialize some of these results to the problem in hand.

LEMMA 2.2. *Suppose that, for the form $f(x, y, z)$ of (2.2), $\phi(y, z)$, with $y, z \equiv y_0, z_0 \pmod 1$, corresponds to a symmetrical lattice $\mathfrak{L}$ with chain $\{a_n\}$. Then, for each n, $f(x, y, z)$ is equivalent to*

$$f_n(x, y, z) = (x + h_n y + h_{n-1} z)^2 - g_n(y, z), \tag{2.23}$$

where

$$g_n(y, z) = \frac{\Delta}{\theta_n\phi_n - 1}(\theta_n y + z)(y + \phi_n z), \tag{2.24}$$

$$h_{n+1} = a_{n+1} h_n - h_{n-1} \quad \textit{for all } n, \tag{2.25}$$

and

$$x_0, y_0, z_0 \equiv x_0, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1 \tag{2.26}$$

in each form $f_n(x, y, z)$.

PROOF. By (2.1), $\phi(y,z)$ is equivalent to $g_n(y,z)$ for each n, with $y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2}$ (mod 1). Hence, under a transformation on y, z which leaves x, and therefore x_0, fixed, f is equivalent to a form

$$f_n(x, y, z) = (x + h_n y + k_n z)^2 - g_n(y, z),$$

say, for each n. From the relations

$$\theta_{n+1} = [a_{n+1}, a_n, a_{n-1}, \ldots] = a_{n+1} - \frac{1}{\theta_n},$$

$$\phi_n = [a_{n+1}, a_{n+2}, \ldots] = a_{n+1} - \frac{1}{\phi_{n+1}},$$

we deduce easily that

$$g_{n+1}(y, z) \equiv g_n(a_{n+1} y + z, -y).$$

Hence if we make the transformation

$$y = a_{n+1} y' + z', \qquad z = -y'$$

we obtain

$$f_n(x, y, z) = \{x + (a_{n+1} h_n - g_n) y' + h_n z'\}^2 - g_{n+1}(y', z');$$

this is $f_{n+1}(x, y', z')$ if we take

$$h_{n+1} = a_{n+1} h_n - g_n, \quad g_{n+1} = h_n.$$

The lemma now follows immediately.

LEMMA 2.3. *Suppose that $f(x, y, z)$, x_0, y_0, z_0 satisfy* (2.6) *and the hypotheses of Lemma* 2.2. *Then, for all n,*

$$\Delta_n^- = \frac{4(\theta_n \phi_n - 1)}{(\theta_n - 1)(\phi_n - 1)} < \frac{1}{\mu} = \frac{4\Delta}{2d - 1}, \tag{2.27}$$

$$\Delta_n^+ = \frac{4(\theta_n \phi_n - 1)}{(\theta_n + 1)(\phi_n + 1)} < \frac{1}{\nu} = \frac{4\Delta}{2d + [d]^2}. \tag{2.28}$$

PROOF. Since (2.6) holds, the inequality (2.9) cannot be satisfied. By Lemma 2.2, $\phi(y, z)$, $y, z \equiv y_0, z_0$ (mod 1), is equivalent to $g_n(y, z)$, $y, z \equiv \frac{1}{2}, \frac{1}{2}$ (mod 1), for all n. Now

$$-g_n(\tfrac{1}{2}, -\tfrac{1}{2}) = \frac{\Delta}{\theta_n \phi_n - 1}(\tfrac{1}{2}\theta_n - \tfrac{1}{2})(\tfrac{1}{2}\phi_n - \tfrac{1}{2}) = \frac{\Delta}{\Delta_n^-} > 0;$$

$$g_n(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{\Delta}{\theta_n \phi_n - 1}(\tfrac{1}{2}\theta_n + \tfrac{1}{2})(\tfrac{1}{2}\phi_n + \tfrac{1}{2}) = \frac{\Delta}{\Delta_n^+} > 0.$$

The insolubility of (2.9) therefore gives

$$\frac{\Delta}{\Delta_n^-} = -g_n\left(\tfrac{1}{2},\ -\tfrac{1}{2}\right) > \mu\,\Delta,$$

$$\frac{\Delta}{\Delta_n^+} = g_n\left(\tfrac{1}{2},\ \tfrac{1}{2}\right) > \nu\,\Delta,$$

for all n; these inequalities are just (2.27) and (2.28).

LEMMA 2.4. *The inequalities*

$$\Delta_n^- \leq \frac{1}{\mu} \tag{2.27'}$$

$$\Delta_n^+ \leq \frac{1}{\nu} \tag{2.28'}$$

cannot be satisfied by any chain $\{a_n\}$ *if* $5 \leq d < 9$.

PROOF. (i) If $[d] = 8$, $8 \leq d < 9$ and (2.27′), (2.28′) give

$$\Delta_n^- \leq \frac{4\,(2\,d^3)^{\frac{1}{2}}}{2\,d-1} < 9,$$

$$\Delta_n^+ \leq \frac{4\,(2\,d^3)^{\frac{1}{2}}}{2\,d+64} < 9\left(\frac{17}{82}\right)$$

for all n. Thus (2.20), (2.21) hold with $G = 9$, $k = 82/17 > 4.82$, and so, by (2.22) with $\alpha = \theta_n$,

$$\left|\theta_n - \frac{7.64}{2.64}\right| < \frac{\sqrt{3.88}}{2.64} < \frac{1.97}{2.64},$$

$$2.14 < \theta_n < 3.65.$$

Using (2.15), we see that these inequalities imply that $a_n = 4$ for all n; but then

$$\theta_n = [4,\ 4,\ 4,\ \ldots] = 2 + \sqrt{3} > 3.7,$$

which is false.

(ii) If $[d] = 7$, $7 \leq d < 8$ and (2.27′), (2.28′) give

$$\Delta_n^- \leq \frac{4\,(2\,d^3)^{\frac{1}{2}}}{2\,d-1} < \frac{128}{15}$$

$$\Delta_n^+ \leq \frac{4\,(2\,d^3)^{\frac{1}{2}}}{2\,d+49} < \frac{128}{65} = \frac{3}{13}\left(\frac{128}{15}\right)$$

for all n. Thus (2.20), (2.21) hold with $G=128/15$, $k=13/3$, and so (2.22) gives

$$\left|\theta_n-\frac{25}{8}\right|<\frac{7}{8},$$

$$2.25<\theta_n<4$$

for all n. Hence $a_n=4$ for all n. But then, for all n,

$$\theta_n=\phi_n=2+\sqrt{3},\qquad \Delta_n^+=\frac{4}{\sqrt{3}}>9\left(\frac{17}{82}\right),$$

which is false.

(iii) If $[d]=6$, $6\leq d<7$, and (2.27′) (2.28′) give

$$\Delta_n^-\leq\frac{4\,(2\,d^3)^{\frac{1}{2}}}{2\,d-1}<\frac{28\sqrt{14}}{13}<8.059,$$

$$\Delta_n^+\leq\frac{4\,(2\,d^3)^{\frac{1}{2}}}{2\,d+36}<\frac{28\sqrt{14}}{50}<2.096$$

for all n. Thus (2.20), (2.21) hold with $G=28\sqrt{14}/13<8.059$, $k=50/13>3.846$, and so (2.22) gives

$$\left|\theta_n-\frac{5.692}{1.633}\right|<\frac{1.847}{1.633},$$

$$2.35<\theta_n<4.62$$

for all n. Hence $a_n=4$ for all n, and, as above,

$$\Delta_n^+=\frac{4}{\sqrt{3}}=2.3094\,\ldots>2.096,$$

which is false.

(iv) If $[d]=5$, $5\leq d<6$, and (2.27′), (2.28′) give

$$\Delta_n^-\leq\frac{4\,(2\,d^3)^{\frac{1}{2}}}{2\,d-1}<\frac{48\sqrt{3}}{11}<7.56,$$

$$\Delta_n^+\leq\frac{4\,(2\,d^3)^{\frac{1}{2}}}{2\,d+25}<\frac{48\sqrt{3}}{37}<2.25$$

for all n. Thus (2.20), (2.21) hold with $G=48\sqrt{3}/11<7.56$, $k=37/11>3.36$, and so (2.22) gives

$$\left|\theta_n - \frac{4.72}{1.16}\right| < \frac{1.82}{1.16},$$

$$2.5 < \theta_n < 5.64$$

for all n. Hence $4 \leq a_n \leq 6$ for all n.

If some $a_n = 6$, $\theta_n \geq [6, 4, 4, 4, \ldots] = 4 + \sqrt{3} > 5.7$, a contradiction. Hence $a_n = 4$ for all n. But then, as above

$$\Delta_n^+ = \frac{4}{\sqrt{3}} = 2.3094 \ldots > 2.25,$$

which is false.

LEMMA 2.5. *If* (2.6) *holds, then*

$$1.144 < d < 5,$$

and the hypotheses of Lemma 2.2 *are satisfied.*

PROOF. We known, by Lemma 2.1, that $1.144 < d < 9$.

(i) If $5 \leq d < 9$, Lemma 2.4 shows that, for any symmetrical lattices, at least one of the inequalities

$$\Delta_n^- > \frac{1}{\mu}, \quad \Delta_n^+ > \frac{1}{\nu}$$

holds for some n. Using (2.18), we see that any $\mathcal{R}_m$-admissible symmetrical lattice $\mathcal{L}$ (with $m = \nu/\mu$) has

$$\Delta(\mathcal{L}) > \frac{1}{\mu}\cdot$$

Since $m > 1$, all critical lattices of $\mathcal{R}_m$ are symmetrical hence

$$D_m > \frac{1}{\mu},$$

which is inconsistent with (2.11).

(ii) Suppose next that $1.144 < d < 5$. We have to show that the $\mathcal{R}_m$-admissible lattice $\mathcal{L}$, of determinant $1/\mu$, associated with the form $\dfrac{1}{\mu\Delta}\phi(y, z)$, $y, z \equiv y_0, z_0$, is necessarily symmetrical.

Suppose to the contrary that $\mathcal{L}$ is not symmetrical. Then (2.19) gives

$$\frac{1}{\mu} \geq 2(m+1) = 2\left(\frac{\nu}{\mu} + 1\right) \quad \text{if } m \leq 3. \tag{2.29}$$

This reduces to

$$[d]^2 + 4d - 1 \leq 2\Delta = 2(2d^3)^{\frac{1}{2}},$$

which is easily found to be false for $[d] = 1, 2, 3$ or 4. Also

$$m = \frac{2d + [d]^2}{2d - 1} \leq 3$$

if $[d] = 1, 2$ or 3 or if $4.75 \leq d < 5$. We therefore certainly have a contradiction from (2.29) unless

$$4 \leq d < 4.75. \tag{2.30}$$

If however (2.30) is satisfied, then $m > 3$ and the analysis of Lemma 2.1 holds (using only the fact that $\mathfrak{L}$ is admissible). Thus (2.14), with $[d] = 4$, gives

$$(4d + 15)(20d + 5) \leq 32 d^3,$$

and this is false for $d < 4.75$.

3. The results of § 2 show that the binary form $\phi(y, z)$ in (2.2) must correspond to a symmetrical lattice $\mathfrak{L}$ satisfying (2.27), (2.28) for all n, with some d with $1.144 < d < 5$.

The next step in the proof of Theorem 2 is the investigation of such lattices. In each of the intervals $4 \leq d < 5$, $3 \leq d < 4$ we find (Lemmas 3.1, 3.2) that there exists a unique chain $\{a_n\}$ satisfying (2.27), (2.28), so that $\phi(y, z)$ is determinate within an arbitrary (positive) multiple. In each of the intervals $2 \leq d < 3$, $1.144 < d < 2$ there are uncountably many distinct chains $\{a_n\}$ satisfying (2.27), (2.28) (Lemmas 3.3, 3.4); we are able, however, to specify the general structure of these chains sufficiently well for our purposes.

LEMMA 3.1. *If $4 \leq d < 5$, the only chain satisfying (2.27), (2.28) for all n is $\{\overset{\times}{4}\}$. Thus, for all n,*

$$g_n(y, z) = \frac{\Delta}{\sqrt{12}}(y^2 + 4yz + z^2), \qquad y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1. \tag{3.1}$$

PROOF. The inequalities (2.27), (2.28) give

$$\Delta_n^- < \frac{4(2d^3)^{\frac{1}{2}}}{2d - 1} < \frac{20\sqrt{10}}{9} < 7.028, \tag{3.2}$$

$$\Delta_n^+ < \frac{4(2d^3)^{\frac{1}{2}}}{2d + 16} < \frac{20\sqrt{10}}{26} < 2.433. \tag{3.3}$$

Thus (2.20), (2.21) hold with $G = 20\sqrt{10}/9 < 7.028$, $k = 26/9 > 2.888$, and so (2.22) gives

$$\left|\theta_n - \frac{3.776}{0.748}\right| < \frac{1.778}{0.748},$$

$$\theta_n > \frac{1.998}{0.748} > 2$$

for all n. Hence $a_n \geq 4$ for all n.

If now $a_n \geq 6$ for some n, we should have

$$\theta_n \geq [6, \overset{\times}{4}] = 4 + \sqrt{3}, \qquad \phi_n \geq [\overset{\times}{4}] = 2 + \sqrt{3},$$

$$\Delta_n^+ \geq \frac{4(10 + 6\sqrt{3})}{(5 + \sqrt{3})(3 + \sqrt{3})} = \frac{4(9 + 7\sqrt{3})}{33} > 2.5,$$

contradicting (3.3). It follows that $a_n = 4$ for all n, as asserted.

The rest of the Lemma follows immediately, since for the chain $\{\overset{\times}{4}\}$ we have

$$\theta_n = \phi_n = [\overset{\times}{4}] = 2 + \sqrt{3}$$

for all n.

Lemma 3.2 *If* $3 \leq d < 4$, *the only chain* $\{a_n\}$ *satisfying* (2.27), (2.28) *for all* n *is* $\{\overset{\times}{4}, \overset{\times}{6}\}$. *Thus we have (taking* $a_{2n} = 6$, $a_{2n+1} = 4$)

$$g_{2n}(y, z) = g_{2n+1}(z, y) = \frac{\Delta}{\sqrt{120}}(3y^2 + 12yz + 2z^2) \tag{3.4}$$

with $y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2} \pmod 1$.

Proof. (2.27), (2.28) give

$$\Delta_n^- < \frac{4(2d^3)^{\frac{1}{2}}}{2d - 1} < \frac{32\sqrt{2}}{7} < 6.465, \tag{3.5}$$

$$\Delta_n^+ < \frac{4(2d^3)^{\frac{1}{2}}}{2d + 9} < \frac{32\sqrt{2}}{17} < 2.663. \tag{3.6}$$

Thus (2.20), (2.21) hold with $G = 32\sqrt{2}/7 < 6.465$, $k = 17/7 > 2.428$, and so give

$$\left|\theta_n - \frac{2.856}{.391}\right| < \frac{1.715}{.391},$$

$$\theta_n > \frac{1.141}{.391} > 2,$$

so that $a_n \geq 4$ for all n.

If now $a_n \geq 8$ for some n, we find

$$\theta_n \geq [8, \overset{\times}{4}] = 6 + \sqrt{3} > 7.7, \qquad \phi_n \geq [\overset{\times}{4}] = 2 + \sqrt{3} > 3.7,$$

$$\Delta_n^+ > \frac{4\,(27.49)}{(4.7)\,(8.7)} > 2.68,$$

contrary to (3.6). Hence $a_n \leq 6$ for all n.

If $a_n = a_{n+1} = 4$, then

$$\theta_n \leq [4, \overset{\times}{6}] = 1 + \sqrt{8}, \qquad \phi_n \leq [4, \overset{\times}{6}] = 1 + \sqrt{8},$$

$$\Delta_n^- \geq \frac{4\,(2 + \sqrt{8})}{\sqrt{8}} = 4 + \sqrt{8} > 6.8,$$

contrary to (3.5); and if $a_n = a_{n+1} = 6$, then

$$\theta_n \geq [6, \overset{\times}{4}] = 4 + \sqrt{3}, \qquad \phi_n \geq [6, \overset{\times}{4}] = 4 + \sqrt{3},$$

$$\Delta_n^+ \geq \frac{4\,(3 + \sqrt{3})}{5 + \sqrt{3}} = \frac{4}{11}(3 + \sqrt{3}) > 2.8,$$

contrary to (3.6).

The only remaining possibility is that $\{a_n\}$ is the periodic chain $\{\overset{\times}{4}, \overset{\times}{6}\}$, which proves the first part of the lemma. The second part of the lemma follows at once, since (choosing the enumeration so that $a_{2n} = 6$, $a_{2n+1} = 4$)

$$\phi_{2n+1} = \theta_{2n} = [\overset{\times}{6}, \overset{\times}{4}] = \frac{6 + \sqrt{30}}{2}, \qquad \phi_{2n} = \theta_{2n+1} = [\overset{\times}{4}, \overset{\times}{6}] = \frac{6 + \sqrt{30}}{3}\,.$$

Lemma 3.3. *If* $2 \leq d < 3$ *and the chain* $\{a_n\}$ *satisfies* (2.27), (2.28) *for all* n, *then either*

(i) $\{a_n\}$ *is* $\{\overset{\times}{6}\}$, *when*

$$g_n(y, z) = \frac{\Delta}{\sqrt{32}}(y^2 + 6yz + z^2)\; y_0, \quad z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1; \tag{3.7}$$

or (ii) $\{a_n\}$ *satisfies*

$$a_{2n} = 4, \quad 10 \leq a_{2n+1} \leq 14 \quad \text{for all } n. \tag{3.8}$$

PROOF. (2.27) and (2.28) give

$$\Delta_n^- < \frac{4(2d^3)^{\frac{1}{2}}}{2d-1} < \frac{12\sqrt{6}}{5} < 5.8788, \tag{3.9}$$

$$\Delta_n^+ < \frac{4(2d^3)^{\frac{1}{2}}}{2d+4} < \frac{12\sqrt{6}}{10} < 2.9394. \tag{3.10}$$

Thus (2.20), (2.21) hold with $G = 12\sqrt{6}/5 < 5.8788$, $k = 2$, and so (2.22) gives

$$\left|\theta_n - \frac{2}{.1212}\right| < \frac{1.6}{.1212},$$

$$\theta_n > \frac{.4}{.1212} > 3.$$

Hence $a_n \geq 4$ for all n.

Case I: Suppose that $a_n = 4$ for some n, say $a_{2n} = 4$ (by suitable choice of the enumeration); then $a_{2n-1} \geq 10$, $a_{2n+1} \geq 10$. For otherwise we may suppose, by symmetry, that $a_{2n+1} \leq 8$; then $\theta_{2n} < 4$, $\phi_{2n} < 8$ and so

$$\Delta_{2n}^- > \frac{4 \times 31}{3 \times 7} = \frac{124}{21} > 5.9,$$

contradicting (3.9).

Further, if $a_{2n+1} \geq 10$ for some n, then $a_{2n} = a_{2n+2} = 4$. For otherwise we may suppose, by symmetry, that $a_{2n} \geq 6$, when

$$\theta_{2n} \geq [6, \overset{\times}{4}] = 4 + \sqrt{3}, \qquad \phi_{2n} \geq [10, \overset{\times}{4}] = 8 + \sqrt{3},$$

$$\Delta_{2n}^+ \geq \frac{4(34 + 12\sqrt{3})}{(5 + \sqrt{3})(9 + \sqrt{3})} = \frac{4(282 + 25\sqrt{3})}{429} > 3,$$

contradicting (3.10).

From these results it follows at once that, if $a_{2n} = 4$ for some n, then, for all n, $a_{2n} = 4$, $a_{2n+1} \geq 10$. Finally, if $a_{2n+1} \geq 16$ for some n, we obtain

$$\theta_{2n} > [\overset{\times}{4}] = 2 + \sqrt{3} > 3.7, \qquad \phi_{2n} > [16, \overset{\times}{4}] = 14 + \sqrt{3} > 15.7,$$

$$\Delta_{2n}^+ > \frac{4(57.09)}{(4.7)(16.7)} > 2.96,$$

contradicting (3.10). Hence $a_{2n+1} \leq 14$ for all n. This gives (ii) of the lemma.

Case II: Suppose that $a_n \geq 6$ for all n. If then $a_n \geq 8$ for some n, we obtain

$$\theta_n \geq [8, \overset{\times}{6}] = 5 + 2\sqrt{2}, \qquad \phi_n \geq [\overset{\times}{6}] = 3 + 2\sqrt{2},$$

$$\Delta_n^+ \geq \frac{8 + 9\sqrt{2}}{7} > 2.96,$$

contrary to (3.10). Hence $a_n = 6$ for all n.

Finally, for the chain $\{\overset{\times}{6}\}$ we have, for all n,

$$\theta_n = \phi_n = 3 + 2\sqrt{2},$$

$$g_n(y, z) = \frac{\Delta}{\sqrt{32}}(y^2 + 6yz + z^2), \qquad y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1.$$

This gives (i) of the lemma.

LEMMA 3.4. *If* $1.144 < d < 2$ *and the chain* $\{a_n\}$ *satisfies* (2.27), (2.28) *for all* n, *then every pair* (a_n, a_{n+1}) *and* (a_{n+1}, a_n) *of consecutive elements of* $\{a_n\}$ *is one of* (6, 10), (6, 12), (6, 14), (6, 16), (8, 8) *or* (8, 10).

PROOF. (2.27) and (2.28) give

$$\Delta_n^- < \frac{4(2d^3)^{\frac{1}{2}}}{2d-1} < \frac{16}{3} = 5.3333 \ldots \tag{3.11}$$

$$\Delta_n^+ < \frac{4(2d^3)^{\frac{1}{2}}}{2d+1} < \frac{16}{5} = 3.2. \tag{3.12}$$

Now*

$$\Delta_n^- = \frac{4(\theta_n \phi_n - 1)}{(\theta_n - 1)(\phi_n - 1)} = 4\left(1 + \frac{1}{\theta_n - 1} + \frac{1}{\phi_n - 1}\right) > 4\left(1 + \frac{1}{\theta_n - 1}\right),$$

so that (3.11) gives, for all n,

$$1 + \frac{1}{\theta_n - 1} < \frac{4}{3}, \qquad \theta_n > 4.$$

Hence $a_n \geq 6$ for all n. The lemma will now follow from the following results:

---

* The inequality (2.22) cannot be used, since with $G = 16/3$, $k = 5/3$ we have $G = 2(k+1)$.

(i) If $a_n = 6$, then $10 \leq a_{n+1} \leq 16$;

(ii) if $a_n = 8$, then $8 \leq a_{n+1} \leq 10$;

(iii) if $a_n = 10$, then $6 \leq a_{n+1} \leq 8$;

(iv) if $a_n \geq 12$, then $a_{n+1} = 6$;

and the corresponding results (which follow by symmetry) with a_{n+1} replaced by a_{n-1}. For the proof of these it is convenient to write (3.11), (3.12) in the alternative forms:

$$\phi_n > \frac{4\,\theta_n - 7}{\theta_n - 4}, \tag{3.13}$$

$$\phi_n < \frac{4\,\theta_n + 9}{\theta_n - 4}. \tag{3.14}$$

If $a_n = 6$, then, since $a_n \geq 6$ for all n,

$$3 + \sqrt{8} = [\overset{\times}{6}] \leq \theta_n < 6.$$

(3.13) gives $\phi_n > 17/2 = 8.5$, whence $a_{n+1} \geq 10$. (3.14) gives

$$\phi_n < \tfrac{1}{7}(25\sqrt{8} + 53) < 17.7;$$

since $\phi_n \geq [a_{n+1}, \overset{\times}{6}] = a_{n+1} - (3 - \sqrt{8}) > a_{n+1} - 0.172$, it follows that $a_{n+1} \leq 16$. This proves (i).

If $a_n = 8$, then

$$5 + \sqrt{8} = [8, \overset{\times}{6}] \leq \theta_n < 8.$$

(3.13) gives $\phi_n > 25/4 > 6$, whence $a_{n+1} \geq 8$. (3.14) gives

$$\phi_n < \tfrac{1}{7}(25\sqrt{8} + 3) < 10.54,$$

whence, as above, $a_{n+1} \leq 10$. This proves (ii).

If $a_n = 10$, then

$$7 + \sqrt{8} = [10, \overset{\times}{6}] \leq \theta_n < 10.$$

(3.13) gives $\phi_n > 11/2 = 5.5$, whence $a_{n+1} \geq 6$. (3.14) gives

$$\phi_n < 79 - 25\sqrt{8} < 8.29,$$

whence $a_{n+1} \leq 8$. This proves (iii).

Finally, if $a_n \geq 12$, then $\theta_n \geq [12, \overset{\times}{6}] = 9 + \sqrt{8}$; (3.14) gives

$$\phi_n < \frac{1}{17}(193 - 25\sqrt{8}) < 7.2,$$

whence $a_{n+1} \leq 6$ and so $a_{n+1} = 6$. This proves (iv).

4. The final stage in the proof of Theorem 2 is the direct examination of $f(x, y, z)$ by means of the chain $f_n(x, y, z)$ of equivalent forms given by Lemma 2.2, together with the results on the chain $g_n(y, z)$ found in § 3.

LEMMA 4.1. *It is impossible that* $4 \leq d < 5$.

PROOF. By Lemma 3.1,

$$g_n(y, z) = k(y^2 + 4yz + z^2), \qquad y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1 \tag{4.1}$$

where

$$12k^2 = \Delta^2 = 2d^3. \tag{4.2}$$

Now (2.27), (2.28) become

$$4\sqrt{3} < \frac{4\Delta}{2d-1}, \qquad 3(2d-1)^2 < \Delta^2 = 2d^3,$$

$$\frac{4}{\sqrt{3}} < \frac{4\Delta}{2d+16}, \qquad 2(d+8)^2 < \tfrac{3}{2}\Delta^2 = 3d^3,$$

of which the former gives

$$d > 4.8. \tag{4.3}$$

Since $4.8 < d < 5$, (4.2) gives

$$4.294 < k < 4.565. \tag{4.4}$$

Now, for all n,

$$f_n(x, -\tfrac{1}{2}, \tfrac{1}{2}) = (x - \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1})^2 + \tfrac{1}{2}k,$$

$$f_n(x, \tfrac{1}{2}, \tfrac{1}{2}) = (x + \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1})^2 - \tfrac{3}{2}k.$$

We can choose $x \equiv x_0 \pmod 1$ to satisfy either of

$$0 \leq \alpha_n = |x - \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1}| \leq \tfrac{1}{2},$$

$$\tfrac{5}{2} \leq \beta_n = |x + \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1}| \leq 3.$$

Since, by (2.6), $|f_n(x, y, z)| > \tfrac{1}{2}d$ whenever $x, y, z \equiv x_0, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$, it follows that

$$\alpha_n^2 + \tfrac{1}{2}k > \tfrac{1}{2}d, \tag{4.5}$$

$$\beta_n^2 - \tfrac{3}{2}k > \tfrac{1}{2}d \tag{4.6}$$

(since clearly

$$\beta_n^2 - \tfrac{3}{2}k > \tfrac{25}{4} - \tfrac{3}{2}(4.294) = -0.191 > -\tfrac{1}{2}d).$$

(4.5) gives, with $d < 5$,

$$\alpha_n^2 > \tfrac{1}{2}d - \tfrac{1}{2}k = \tfrac{1}{2}d - \left(\frac{d^3}{24}\right)^{\frac{1}{2}}$$

$$> \tfrac{5}{2} - (\tfrac{125}{24})^{\frac{1}{2}} > .217,$$

$$\alpha_n > .4658.$$

(4.6) gives, with $d > 4.8$, $k > 4.294$,

$$\beta_n^2 > \tfrac{1}{2}d + \tfrac{3}{2}k > 8.841,$$

$$\beta_n > 2.9735.$$

These results, together with the definitions of α_n, β_n, show that for all n

$$.4658 < x_0 - \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1} < .5342 \pmod 1, \tag{4.7}$$

$$-.0265 < x_0 + \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1} < .0265 \pmod 1. \tag{4.8}$$

Subtracting, we have

$$-.5607 < h_n < -.4393 \pmod 1,$$

so that

$$|h_n - \tfrac{1}{2}| < .0607 \pmod 1. \tag{4.9}$$

By making a parallel transformation on x in $f_1(x, y, z)$ (and the corresponding transformation on x_0) we may suppose without loss of generality that

$$0 \le h_0 < 1, \qquad 0 \le h_1 < 1.$$

Then (4.9) gives

$$|h_0 - \tfrac{1}{2}| < .0607, \qquad |h_1 - \tfrac{1}{2}| < .0607, \tag{4.10}$$

and (4.8), with $n = 1$, then gives

$$|x_0 - \tfrac{1}{2}| < .0872 \pmod 1. \tag{4.11}$$

Since $a_n = 4$ for all n, the recurrence relation (2.25) gives

$$h_2 = 4h_1 - h_0,$$

whence, by (4.10)

$$|h_2 - \tfrac{3}{2}| = |4(h_1 - \tfrac{1}{2}) - (h_0 - \tfrac{1}{2})| < .3035.$$

(4.9) now shows that

$$|h_2 - \tfrac{3}{2}| < .0607.$$

But now (4.8), with $n = 2$, gives

$$|x_0| < .0872 \pmod 1,$$

which is incompatible with (4.11). This proves the lemma.

LEMMA 4.2. *If* $3 \leq d < 4$, *then* $f(x, y, z)$ *is equivalent to* $Q_2(x, y, z) = 2x^2 - y^2 + 15z^2$ *with* $x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$.

PROOF. By Lemma 3.2, we have

$$\left.\begin{aligned} f_{2n}(x, y, z) &= (x + h_{2n}y + h_{2n-1}z)^2 - k(3y^2 + 12yz + 2z^2) \\ f_{2n+1}(x, y, z) &= (x + h_{2n+1}y + h_{2n}z)^2 - k(2y^2 + 12yz + 3z^2) \end{aligned}\right\} \tag{4.12}$$

for all n, where $k > 0$, $y_0, z_0 = \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$ and

$$120k^2 = \Delta^2 = 2d^3. \tag{4.13}$$

Now (2.27), (2.28) become

$$\frac{8\sqrt{30}}{7} = \Delta_n^- < \frac{4\Delta}{2d-1}, \qquad 60(2d-1)^2 < \frac{49}{2}\Delta^2 = 49d^3,$$

$$\frac{8\sqrt{30}}{17} = \Delta_n^+ < \frac{4\Delta}{2d+9}, \qquad 60(2d+9)^2 < \frac{289}{2}\Delta^2 = 289d^3,$$

of which the second gives $d > 3.87$; hence

$$3.87 < d < 4. \tag{4.14}$$

Now (4.13) gives

$$.933 < k < 1.033. \tag{4.15}$$

We can ensure, by suitable transformation on x in $f_0(x, y, z)$, that

$$|h_0| \leq \tfrac{1}{2}, \qquad |h_{-1}| \leq \tfrac{1}{2}. \tag{4.16}$$

After these preliminaries, our first step is to show that $h_n = 0$ for all n. We first obtain bounds for h_n (mod 1) by the method used in Lemma 4.1.

For all n,

$$g_n(\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{17}{4}k, \qquad g_n(\tfrac{1}{2}, -\tfrac{1}{2}) = -\tfrac{7}{4}k.$$

We can choose $x \equiv x_0 \pmod 1$ to satisfy either of

$$0 \le \alpha_n = |x - \tfrac{1}{2} h_n + \tfrac{1}{2} h_{n-1}| \le \tfrac{1}{2},$$

$$\tfrac{3}{2} \le \beta_n = |x + \tfrac{1}{2} h_n + \tfrac{1}{2} h_{n-1}| \le 2.$$

Since, by (2.6), $|f_n(x, y, z)| > \tfrac{1}{2} d$ whenever $x, y, z \equiv x_0, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$, it follows that

$$\alpha_n^2 + \tfrac{7}{4} k > \tfrac{1}{2} d, \tag{4.17}$$

$$\tfrac{17}{4} k - \beta_n^2 > \tfrac{1}{2} d \tag{4.18}$$

(since clearly

$$\frac{17}{4} k - \beta_n^2 > \frac{17}{4}(.933) - 4 = -0.03 \ldots > -\tfrac{1}{2} d).$$

(4.17) gives, with $d < 4$,

$$\alpha_n^2 > \tfrac{1}{2} d - \tfrac{7}{4} k > 2 - \tfrac{7}{4}(1.033) = .19225,$$

$$\alpha_n > .4384.$$

(4.18) gives, with $d < 4$,

$$\beta_n^2 < \frac{17}{4} k - \tfrac{1}{2} d < \frac{17}{4}(1.033) - 2 = 2.39025,$$

$$\beta_n < 1.5461.$$

These results, together with the definitions of α_n, β_n, show that for all n

$$.4384 < x_0 - \tfrac{1}{2} h_n + \tfrac{1}{2} h_{n-1} < .5616 \pmod 1 \tag{4.19}$$

$$.4539 < x_0 + \tfrac{1}{2} h_n + \tfrac{1}{2} h_{n-1} < .5461 \pmod 1. \tag{4.20}$$

Subtracting, we find

$$|h_n| < .1077 \pmod 1. \tag{4.21}$$

Since $a_{2n} = 6$, $a_{2n+1} = 4$ for all n, the recurrence relation (2.5) gives

$$\left.\begin{aligned} h_{2n+1} &= 4 h_{2n} - h_{2n-1} \\ h_{2n+2} &= 6 h_{2n+1} - h_{2n} = 23 h_{2n} - 6 h_{2n-1} \end{aligned}\right\}. \tag{4.22}$$

Consider now a Euclidean plane with rectangular coordinates ξ, η, and let $\mathcal{R}$ be the region defined by

$$|\xi| < .1077, \qquad |\eta| < .1077.$$

Denote by P_n $(-\infty < n < \infty)$ the point (h_{2n}, h_{2n-1}). By (4.21), P_n is congruent to a point of $\mathcal{R}$ for all n; and, by (4.16), $P_0 \in \mathcal{R}$.

Let T be the unimodular matrix

$$T = \begin{pmatrix} 23 & -6 \\ 4 & -1 \end{pmatrix};$$

then by (4.22) $$P_{n+1} = T(P_n).$$

Let now $P = (\xi, \eta)$ be any point of $\mathcal{R}$, and let

$$Q = (\xi', \eta') = T(P).$$

Then $$|\eta'| = |4\xi - \eta| < .5385,$$

$$\xi' = 23\xi - 6\eta = 6\eta' - \xi.$$

If now Q is congruent (mod 1) to a point of $\mathcal{R}$, we clearly require $|\eta'| < .1077$; then, since

$$|\xi'| = |6\eta' - \xi| < 6(.1077) + (.1077) = .7539 < 1 - .1077,$$

we require also $|\xi'| < .1077$.

Thus Q is congruent to a point of $\mathcal{R}$ only if $Q \in \mathcal{R}$.

Since $P_0 \in \mathcal{R}$ and $P_n = T^n(P_0)$ is congruent to a point of $\mathcal{R}$ for all n, it follows* that P_0 is the (unique) point F of $\mathcal{R}$ satisfying.

$$T(F) = F,$$

i.e. that $P_0 = (0, 0)$. Thus $h_0 = 0$, $h_{-1} = 0$, and so $h_n = 0$ for all n, as asserted.

This result, with (4.20), shows also that

$$|x_0 - \tfrac{1}{2}| < .0461 \pmod 1. \tag{4.23}$$

The final step of the argument is to show that $x_0 \equiv \frac{1}{2} \pmod 1$, $k = 1$.

Since $f(x, y, z)$ is equivalent to $f_0(x, y, z)$, we may take

$$f(x, y, z) = f_0(x, y, z) = x^2 - k(3y^2 + 12yz + 2z^2)$$

with $x, y, z \equiv x_0, \frac{1}{2}, \frac{1}{2} \pmod 1$.

Let

$$a = f(2, 1, 0) = 4 - 3k.$$

Then, by (4.15), $$.901 < a < 1.201. \tag{4.24}$$

If we make the equivalence transformation

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* We are appealing here to the general Lemma 3 of [1]. As remarked there, this lemma is due to CASSELS, and a proof is given in BARNES and SWINNERTON-DYER [3], Theorem D.

$$x = 2x' + y', \qquad y = x' + y', \qquad z = z',$$

we find that

$$f(x, y, z) = a x'^2 - (6k - 4) x' y' - 12 k x' y' + \cdots$$
$$= a f'(x', y', z'),$$

say, where

$$f'(x', y', z') = \left(x - \frac{3k-2}{a} y' - \frac{6k}{a} z'\right)^2 - \phi'(y', z'), \tag{4.25}$$

$$x', y', z' \equiv x_0 - \tfrac{1}{2}, \ -x_0, \ \tfrac{1}{2} \pmod 1. \tag{4.26}$$

Writing $d' = d(f')$, it is easy to see that $d' = d/a$, so that, by (4.14) and (4.24),

$$3.22 < d' < 4.44.$$

It follows from Lemma 4.1 that in fact

$$3.22 < d' < 4.$$

Also, since f' is equivalent to a positive multiple of f,

$$M(f'; x_0 - \tfrac{1}{2}, \ -x_0, \ \tfrac{1}{2}) > \tfrac{1}{2} d'.$$

By what we have already proved, we see that there exists an equivalence transformation of the type

$$x' = X + \lambda Y + \mu Z, \qquad y' = \alpha Y + \beta Z, \qquad z' = \gamma Y + \delta Z$$

such that
$$f'(x', y', z') = X^2 - k'(3Y^2 + 12YZ + Z^2)$$

is of the same form as $f(x, y, z)$ (with possibly a different value k' of k) with

$$X_0, Y_0, Z_0 \equiv X_0, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1. \tag{4.27}$$

Since $\lambda, \mu, \alpha, \beta, \gamma$ and δ are integers, a comparison of (4.26), (4.27) shows at once that $x_0 \equiv 0$ or $\tfrac{1}{2} \pmod 1$, and then (4.23) gives $x_0 \equiv \tfrac{1}{2} \pmod 1$, as required.

Finally, it follows by the same argument from (4.25) that $(3k-2)/a$ and $6k/a$ must be integral; hence $4/a = 6k/a - 2(3k-2)/a$ is integral and, by (4.24),

$$\frac{4}{5} < a < \frac{4}{3}, \qquad 3 < \frac{4}{a} < 5.$$

Thus $a = 1$, and so $k = \tfrac{1}{3}(4 - a) = 1$, as required.

We have therefore shown that $f(x, y, z)$ is equivalent to $x^2 - (3y^2 + 12yz + 2z^2)$, with $x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$; since

$$x^2-3y^2-12yz-2z^2=2(x-3y-z)^2-(x-6y-2z)^2+15y^2,$$

f is therefore equivalent to Q_2, with $x_0, y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ (mod 1).

LEMMA 4.3. *If* $2 \leq d < 3$, *then Lemma* 3.3 (ii) *cannot hold.*

PROOF. The proof follows the same lines as that of Lemma 4.1, with some numerical complication arising from the fact that we have no explicit expression for $g_n(y, z)$.

Suppose, contrary to the assertion of the lemma, that $\{a_n\}$ satisfies $a_{2n}=4$, $10 \leq a_{2n+1} \leq 14$ for all n. Then

$$\left.\begin{aligned}
\phi_{2n+1},\ \theta_{2n} &\leq [\overset{\times}{4},\ \overset{\times}{14}]=\tfrac{1}{7}(14+\sqrt{182}),\\
\theta_{2n+1},\ \phi_{2n} &\leq [\overset{\times}{14},\ \overset{\times}{4}]=\tfrac{1}{2}(14+\sqrt{182}),\\
\phi_{2n+1},\ \theta_{2n} &\geq [\overset{\times}{4},\ \overset{\times}{10}]=\tfrac{1}{5}(10+3\sqrt{10}),\\
\theta_{2n+1},\ \phi_{2n} &\geq [\overset{\times}{10},\ \overset{\times}{4}]=\tfrac{1}{2}(10+3\sqrt{10}),
\end{aligned}\right\} \tag{4.28}$$

so that, for all n,

$$\left.\begin{aligned}
\frac{8\sqrt{182}}{19} &< \Delta_n^- = \frac{4(\theta_n\phi_n-1)}{(\theta_n-1)(\phi_n-1)} < \frac{24\sqrt{10}}{13},\\
\frac{8\sqrt{10}}{9} &< \Delta_n^+ = \frac{4(\theta_n\phi_n-1)}{(\theta_n+1)(\phi_n+1)} < \frac{8\sqrt{182}}{37}.
\end{aligned}\right\} \tag{4.29}$$

Now (2.27), (2.28) give

$$\frac{8\sqrt{182}}{19} < \frac{4(2d^3)^{\frac{1}{2}}}{2d-1}, \qquad \frac{8\sqrt{10}}{9} < \frac{4(2d^3)^{\frac{1}{2}}}{2d+4},$$

the latter of which yields the inequality

$$d > 2.85.$$

Also (4.29) shows that, for all n,

$$\frac{37\Delta}{8\sqrt{182}} < g_n(\tfrac{1}{2}, \tfrac{1}{2}) < \frac{9\Delta}{8\sqrt{10}}, \tag{4.30}$$

$$0 < -g_n(\tfrac{1}{2}, -\tfrac{1}{2}) < \frac{19\Delta}{8\sqrt{182}}. \tag{4.31}$$

Choosing $x \equiv x_0 \pmod 1$ to satisfy either of

$$0 \leq \alpha_n = |x - \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1}| \leq \tfrac{1}{2},$$

$$\tfrac{3}{2} \leq \beta_n = |x + \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1}| \leq 2,$$

we require

$$\alpha_n^2 - g_n(\tfrac{1}{2}, -\tfrac{1}{2}) > \tfrac{1}{2}d, \tag{4.32}$$

$$|\beta_n^2 - g_n(\tfrac{1}{2}, \tfrac{1}{2})| > \tfrac{1}{2}d. \tag{4.33}$$

From (4.32) and (4.31), with $d < 3$, we obtain

$$\alpha_n^2 > \tfrac{1}{2}d - \frac{19\,\Delta}{8\sqrt{182}} > \tfrac{3}{2} - \frac{19\sqrt{54}}{8\sqrt{182}} > 0.207,$$

$$\alpha_n > 0.4549.$$

From (4.33) and (4.30), with $d > 2.85$, we obtain

$$\beta_n^2 > \tfrac{1}{2}d + \frac{37\,\Delta}{8\sqrt{182}} > 1.425 + \frac{37\sqrt{46.298}}{8\sqrt{182}} > 3.757,$$

$$\beta_n > 1.938.$$

It follows that, for all n,

$$0.4549 < x_0 - \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1} < 0.5451 \pmod 1, \tag{4.34}$$

$$-0.062 < x_0 + \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1} < 0.062 \pmod 1; \tag{4.35}$$

subtracting, we obtain

$$0.3929 < h_n < 0.6071 \pmod 1. \tag{4.36}$$

Supposing, as we may without loss of generality, that

$$0 \leq h_0 < 1, \qquad 0 \leq h_1 < 1,$$

we have, by (4.36),

$$|h_0 - \tfrac{1}{2}| < 0.1071, \qquad |h_1 - \tfrac{1}{2}| < 0.1071; \tag{4.37}$$

then (4.34) with $n = 1$ gives

$$|x_0 - \tfrac{1}{2}| < 0.1522 \pmod 1. \tag{4.38}$$

Since $a_{2n} = 4$, the recurrence relation (2.25) gives

$$h_2 = 4h_1 - h_0,$$

whence, by (4.37),

$$|h_2 - \tfrac{3}{2}| = |4(h_1 - \tfrac{1}{2}) - (h_0 - \tfrac{1}{2})| < 0.5355.$$

By (4.36), this implies that

$$|h_2 - \tfrac{3}{2}| < 0.1071.$$

But now (4.34), with $n = 2$, gives

$$|x_0| < 0.1522 \pmod 1,$$

which is incompatible with (4.38).

LEMMA 4.4. *Suppose that* $2 \leq d < 3$. *Then* $f(x, y, z)$ *is equivalent to* $Q_1(x, y, z)$ *with* $x_0, y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \pmod 1$.

PROOF. By Lemma 4.3, the conclusion (i) of Lemma 3.3 must hold whenever $2 \leq d < 3$. Thus

$$g_n(y, z) = k(y^2 + 6yz + z^2), \qquad y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1,$$

where

$$k > 0, \qquad 32k^2 = \Delta^2 = 2d^3. \tag{4.39}$$

(2.27), (2.28) become

$$4\sqrt{2} = \Delta_n^- < \frac{4\Delta}{2d-1}, \qquad \frac{4}{\sqrt{2}} = \Delta_n^+ < \frac{4\Delta}{2d+4},$$

the latter of which yields

$$(d+2)^2 < \tfrac{1}{2}\Delta^2 = d^3, \qquad d > 2.875.$$

Thus

$$2.875 < d < 3, \tag{4.40}$$

and so, by (4.39),

$$1.218 < k < 1.300. \tag{4.41}$$

Choosing $x \equiv x_0 \pmod 1$ to satisfy either of

$$0 \leq \alpha_n = |x - \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1}| \leq \tfrac{1}{2},$$

$$\tfrac{3}{2} \leq \beta_n = |x + \tfrac{1}{2}h_n + \tfrac{1}{2}h_{n-1}| \leq 2,$$

we then require

$$\alpha_n^2 + k > \tfrac{1}{2}d,$$

$$|\beta_n^2 - 2k| > \tfrac{1}{2}d.$$

Using (4.40), (4.41), we deduce that

$$\alpha_n^2 > \tfrac{3}{2} - 1.3 = 0.2, \qquad \alpha_n > 0.447,$$

$$\beta_n^2 > 1.437 + 2.436 = 3.873, \qquad \beta_n > 1.967,$$

so that, for all n,

$$0.447 < x_0 - \tfrac{1}{2} h_n + \tfrac{1}{2} h_{n-1} < 0.553 \pmod 1, \tag{4.42}$$

$$0.033 < x_0 + \tfrac{1}{2} h_n + \tfrac{1}{2} h_{n-1} < 0.033 \pmod 1; \tag{4.43}$$

subtraction yields

$$|h_n - \tfrac{1}{2}| < 0.086 \pmod 1. \tag{4.44}$$

We may suppose without loss of generality that

$$0 \le h_0 < 1, \qquad 0 \le h_1 < 1. \tag{4.45}$$

Let $\mathcal{R}$ be the region of the ξ, η-plane defined by

$$|\xi - \tfrac{1}{2}| < 0.086, \qquad |\eta - \tfrac{1}{2}| < 0.086;$$

P_n the point (h_n, h_{n-1}); and T the unimodular matrix

$$T = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}.$$

The recurrence relation (2.25), with $a_{n+1} = 6$, gives

$$h_{n+1} = 6 h_n - h_{n-1},$$

so that

$$P_{n+1} = (h_{n+1}, h_n) = T(P_n).$$

By (4.44), $P_{n+1} = T^n(P_1)$ is congruent (mod 1) to a point of $\mathcal{R}$ for all n; and by (4.45), $P_1 \in \mathcal{R}$. Further, if $P = (\xi, \eta)$ is any point of $\mathcal{R}$ and $Q = T(P) = (\xi', \eta')$, then

$$|\xi' - \tfrac{5}{2}| = |6(\xi - \tfrac{1}{2}) - (\eta - \tfrac{1}{2})| < 0.602,$$

$$|\eta' - \tfrac{1}{2}| = |\xi - \tfrac{1}{2}| < 0.086,$$

hence Q can be congruent to a point of $\mathcal{R}$ only if $Q - (2, 0)$ belongs to $\mathcal{R}$.

From these results, and [1] Lemma 3, it follows that P_1 is the unique point F of $\mathcal{R}$ satisfying

$$T(F) - (2, 0) = F;$$

i.e.

$$P_1 = F = (\tfrac{1}{2}, \tfrac{1}{2}), \qquad h_1 = \tfrac{1}{2}, \qquad h_0 = \tfrac{1}{2}.$$

Thus we may take

$$f(x, y, z) = f_1(x, y, z) = (x + \tfrac{1}{2} y + \tfrac{1}{2} z)^2 - k(y^2 + 6yz + z^2) \tag{4.46}$$

with $x_0, y_0, z_0 \equiv x_0, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$; and, by (4.43) with $n = 1$,

$$|x_0 - \tfrac{1}{2}| < 0.033 \pmod 1. \tag{4.47}$$

Now let $$a = f(1, 1, 0) = \tfrac{9}{4} - k, \tag{4.48}$$

so that, by (4.41), $$0.95 < a < 1.032. \tag{4.49}$$

Making the equivalence transformation

$$x = x' + y', \qquad y = x', \qquad z = z',$$

we find that

$$f(x, y, z) = a\,x'^2 + 3\,x'y' - (6\,k - \tfrac{3}{2})\,x'y' + \cdots$$
$$= a\,f'(x', y', z'),$$

say, where therefore

$$f'(x', y', z') = \left(x' + \frac{3}{2\,a}\,y' - \frac{12\,k - 3}{4\,a}\,z'\right) - \phi'(y', z')$$

with $$x_0', y_0', z_0' \equiv \tfrac{1}{2}, x_0 - \tfrac{1}{2}, \tfrac{1}{2} \pmod{1}. \tag{4.50}$$

Writing $d' = d(f')$, we have $d' = d/a$, so that, by (4.40) and (4.49),

$$2.78 < d' < 3.16.$$

Using Lemma 4.2, (4.14), we see that in fact

$$2.78 < d' < 3.$$

Also, since f' is equivalent to a multiple of f,

$$M(f'; \tfrac{1}{2}, x_0 - \tfrac{1}{2}, \tfrac{1}{2}) > \tfrac{1}{2}\,d'.$$

By Lemma 4.3 and what we have already proved in this lemma, $\phi'(y', z')$ is equivalent to a positive multiple of

$$y'^2 + 6\,y'z' + z'^2, \; y_0', \quad z_0' \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod{1},$$

and each of the coefficients $3/2\,a$ and $(12\,k - 3)/4\,a$ must be congruent to 0 or $\tfrac{1}{2}$ (mod 1) [arguing precisely as in Lemma 4.2].

(4.50) and (4.47) show at once that

$$x_0 \equiv \tfrac{1}{2} \pmod{1}.$$

Also, by (4.49),

$$1.45 < \frac{3}{2\,a} < 1.58,$$

so that we must have

$$\frac{3}{2\,a} = \frac{3}{2}, \qquad a = 1;$$

and now (4.48) gives

$$k = \tfrac{5}{4}.$$

Thus

$$f(x, y, z) = (x + \tfrac{1}{2} y + \tfrac{1}{2} z)^2 - \tfrac{5}{4}(y^2 + 6yz + z^2)$$
$$= Q_1(x, y, z),$$

with $x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1$, as required.

Lemma 4.5. *Suppose that* $1.144 < d < 2$. *Then* $f(x, y, z)$ *is equivalent to a positive multiple of a form* $f'(x, y, z)$, *satisfying the same conditions* (2.2), (2.6), *for which*

$$d' = d(f') > 3.4.$$

Proof. By Lemma 3.4, every pair (a_n, a_{n+1}) and (a_{n+1}, a_n) of consecutive elements of the chain $\{a_n\}$ is one of (6, 10), (6, 12), (6, 14), (6, 16), (8, 8) or (8, 10).

If some a_n is 6, say $a_1 = 6$, we have

$$\theta_1 \le [\overset{\times}{6}, \overset{\times}{16}] = \frac{12 + \sqrt{138}}{4}, \qquad \phi_1 \le [\overset{\times}{16}, \overset{\times}{6}] = \frac{12 + \sqrt{138}}{3}, \tag{4.51}$$

$$\theta_1 \ge [\overset{\times}{6}, \overset{\times}{10}] = \frac{15 + \sqrt{210}}{5}, \qquad \phi_1 \ge [\overset{\times}{10}, \overset{\times}{6}] = \frac{15 + \sqrt{210}}{3}, \tag{4.52}$$

whence

$$\frac{(\theta_1 - 1)(\phi_1 - 1)}{\theta_1 \phi_1 - 1} \le \frac{37}{4\sqrt{138}} < 0.7875, \tag{4.53}$$

$$1.2557 < \frac{59}{4\sqrt{138}} \le \frac{(\theta_1 + 1)(\phi_1 + 1)}{\theta_1 \phi_1 - 1} \le \frac{19}{\sqrt{210}} < 1.3112. \tag{4.54}$$

If however no a_n is 6, then $a_n = 8$ or 10 for all n, and $a_n = 8$ for some n. Taking $a_1 = 8$ without loss of generality, we therefore have

$$\theta_1 \le [\overset{\times}{8}, \overset{\times}{10}] = \frac{20 + \sqrt{380}}{5}, \qquad \phi_1 \le [10, 8] = \frac{20 + \sqrt{380}}{4}, \tag{4.55}$$

$$\theta_1 \ge [\overset{\times}{8}] = 4 + \sqrt{15}, \qquad \phi_1 \ge [\overset{\times}{8}] = 4 + \sqrt{15}, \tag{4.56}$$

whence

$$\frac{(\theta_1 - 1)(\phi_1 - 1)}{\theta_1 \phi_1 - 1} \le \frac{31}{2\sqrt{380}} < 0.7952, \tag{4.57}$$

$$1.2568 < \frac{49}{2\sqrt{380}} \le \frac{(\theta_1 + 1)(\phi_1 + 1)}{\theta_1 \phi_1 - 1} \le \frac{5}{\sqrt{15}} < 1.2910. \tag{4.58}$$

Thus in either case the basic inequalities (2.27), (2.28) yield

$$\frac{4\,\Delta}{2\,d-1} > \frac{4\,(\theta_1\,\phi_1-1)}{(\theta_1-1)\,(\phi_1-1)} \geq \frac{8\sqrt{380}}{31},$$

$$\frac{4\,\Delta}{2\,d+1} > \frac{4\,(\theta_1\,\phi_1-1)}{(\theta_1+1)\,(\phi_1+1)} \geq \frac{4\sqrt{210}}{19};$$

from the second of these we obtain by a straightforward calculation

$$361\,d^3 > 105\,(2\,d+1)^2,$$

$$d > 1.85.$$

To prove the lemma, it now suffices to show that $f(x, y, z)$ assumes primitively a value a satisfying

$$0 < a < 0.5437. \tag{4.59}$$

For then $(1/a)\,f(x, y, z)$ represents 1 and is equivalent to a form $f'(x, y, z)$ of the type (2.2) with

$$d' = d\,(f') = \frac{d}{a} > \frac{1.85}{0.5437} > 3.4,$$

as required.

By applying a parallel transformation on x in $f_1(x, y, z)$, and changing the signs of y and z, if necessary, we may take without loss of generality

$$0 \leq h_0 \leq \tfrac{1}{2}. \tag{4.60}$$

We shall now show that

$$a = f_1(1,\,0,\,-1) = (1-h_0)^2 - \frac{\phi_1\,\Delta}{\theta_1\,\phi_1-1} \tag{4.61}$$

satisfies (4.56) (where $a_1 = 6$ or 8 is chosen as above).

We first prove that

$$0.4563 < \frac{\phi_1\,\Delta}{\theta_1\,\phi_1-1} < 0.6901. \tag{4.62}$$

For if $a_1 = 6$, (4.51) and (4.52) give

$$5.7965 < \tfrac{2}{5}\sqrt{210} \leq \theta_1 - \frac{1}{\phi_1} \leq \tfrac{1}{2}\sqrt{138} < 5.8737,$$

while if $a_1 = 8$, (4.53) and (4.54) give

$$7.7459 < 2\sqrt{15} \leq \theta_1 - \frac{1}{\phi_1} \leq \tfrac{2}{5}\sqrt{380} < 7.7975,$$

thus in either case we have

$$5.7965 < \theta_1 - \frac{1}{\phi_1} < 7.7975.$$

Also, since $\Delta^2 = 2d^3$, $1.85 < d < 2$, we have

$$3.5580 < \Delta < 4. \tag{4.63}$$

Division of these two inequalities gives (4.59).

Next we show that

$$0 \le h_0 < 0.1186. \tag{4.64}$$

Choosing $x \equiv x_0 \pmod 1$ to satisfy either of

$$0 \le \alpha = |x + \tfrac{1}{2}h_1 - \tfrac{1}{2}h_0| \le \tfrac{1}{2},$$

$$1 \le \beta = |x + \tfrac{1}{2}h_1 + \tfrac{1}{2}h_0| \le \tfrac{3}{2},$$

we require, since $|f_1(x, \tfrac{1}{2}, \pm\tfrac{1}{2})| > \tfrac{1}{2}d$,

$$\alpha^2 - g_1(\tfrac{1}{2}, -\tfrac{1}{2}) > \tfrac{1}{2}d, \qquad |\beta^2 - g_1(\tfrac{1}{2}, \tfrac{1}{2})| > \tfrac{1}{2}d.$$

Now by (4.53) or (4.57)

$$0 < -4g_1(\tfrac{1}{2}, -\tfrac{1}{2}) = \frac{(\theta_1 - 1)(\phi_1 - 1)}{\theta_1\phi_1 - 1}\Delta < (0.7952)\Delta,$$

and by (4.54) or (4.58)

$$4g_1(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{(\theta_1 + 1)(\phi_1 + 1)}{\theta_1\phi_1 - 1}\Delta > (1.2557)\Delta.$$

Hence, using $1.85 < d < 2$, $3.5580 < \Delta < 4$, we obtain

$$4\alpha^2 > 2d + 4g_1(\tfrac{1}{2}, -\tfrac{1}{2}) > 4 - 4(0.7952) = 0.8192,$$

$$\alpha > \sqrt{0.2048} > 0.4525;$$

$$4\beta^2 > 2d + 4g_1(\tfrac{1}{2}, \tfrac{1}{2}) > 3.7 + (1.2557)(3.5580) > 8.1677,$$

$$\beta > \sqrt{2.0419} > 1.4289.$$

It follows that

$$0.4525 < x_0 + \tfrac{1}{2}h_1 - \tfrac{1}{2}h_0 < 0.5475 \pmod 1,$$

$$0.4289 < x_0 + \tfrac{1}{2}h_1 + \tfrac{1}{2}h_0 < 0.5711 \pmod 1,$$

and so, by subtraction

$$-0.1186 < h_0 < 0.1186 \pmod 1.$$

This inequality, with (4.60), gives (4.64) at once.

Inserting (4.64) and (4.62) in (4.61), we find

$$a > (0.8814)^2 - 0.6901 > 0,$$

$$a < 1 - 0.4563 = 0.5437,$$

so that (4.59) is satisfied. This completes the proof of the lemma.

The proof of Theorem 2 follows immediately from the lemmas of this section, together with the fact (Lemma 2.5) that $1.144 < d < 5$. For Lemmas 4.1–4.4 give the required result if $2 \leq d < 5$; while Lemma 4.5 shows that, if $1.144 < d < 2$, an appropriate multiple of $f(x, y, z)$ satisfies $3.4 < d < 5$. [It is easily verified *a posteriori* that if $1.144 < d < 2$, then $\{a_n\}$ must be $\{\overset{\times}{8}\}$ and then $f(x, y, z)$ is equivalent to a multiple of $Q_2(x, y, z)$; this corresponds to the form $x^2 - \frac{1}{2}(y^2 + 8yz + z^2)$ found explicitly in [1] Lemma 12.]

The calculations of this paper were carried out on a Brunsviga, supplied to me by the University of Sydney.

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