ON THE PROJECTION OF A PLANE SET OF FINITE LINEAR MEASURE *

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Introduction

For any set X we denote by $\Lambda(X)$ the linear measure (1) of X and by $\Lambda(X_\theta)$ the linear measure of the projection of X onto a line perpendicular to the direction θ. We write $\mu(X)$ for the greatest lower bound of $\Lambda(X_\theta)$ taken over all directions θ. We shall consider three classes of planar sets, namely measurable sets, connected sets, and arcs. For each class we shall find the upper bound of the ratio $\mu(X)/\Lambda(X)$.

For the class of measurable sets the result is connected with the properties of regular and irregular sets and is a consequence of the properties of these sets established by Besicovitch. For the class of connected sets and for the class of arcs $\mu(X)$ is the minimal width of the convex cover of X or its convex hull as it is sometimes called. The problem of the relationship between this function and $\Lambda(X)$ is one between a set and its convex cover. There are of course a large number of such properties and a further result of this type is given in Section 5.

An interesting feature of this problem is the difficulty of determining completely the class of extremal figures. For the class of measurable sets the upper bound of $\mu(X)/\Lambda(X)$ is never attained, but we give examples to show that the upper bound which we establish is in fact the least upper bound. On the other hand both the upper bounds for the class of connected sets and for the class of arcs are attained; in the first class by a set composed of three equal segments equally inclined to one another and in the second case by an arc

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* *Editor's note.*—This paper was received on January 4, 1957. Without our knowledge it has appeared during 1957 as part of the book *Problems in Euclidean Space*, London 1957, by the same author.

(1) Hausdorff one-dimensional measure. See [1], where it is referred to as Carathéodory measure.

composed of four linear segments and two circular arcs (which will be specified more exactly later). To simplify the proofs we shall consider in both cases the subclasses of connected sets or of arcs whose convex covers are polygons with at most n vertices. Since, in fact, one extremal figure for the class of arcs is not of this nature we have no hope, by this means, of specifying all the extremal figures. But even for the case of connected sets when the only known extremal figure is of this kind I have not been able to specify completely all the extremal figures. Some further remarks about this point will be given later (see Section 6).

The actual results proved in the following paragraphs are

(i) for any measurable set E with $\Lambda(E) > 0$,

$$\mu(E) < \frac{2}{\pi}\Lambda(E),$$

(ii) for any connected set E

$$\mu(E) \leqslant \tfrac{1}{2}\Lambda(E),$$

(iii) for any simple arc E

$$\mu(E) \leqslant \Lambda(E)/(\sec\alpha + 2\tan\alpha + \pi - 4\beta - 2\alpha)$$

where α and β are defined by

$$\tfrac{1}{2} + \sin\alpha = 4\cos^2\alpha/(1 + 4\cos^2\alpha)$$

and

$$\tan\beta = \tfrac{1}{2}\sec\alpha.$$

The results proved in Section 5 are stated in that paragraph. I am indebted to the referee for suggesting simplifications of some of the properties established in Section 3.

§ 1. E any measurable plane set of finite positive linear measure

We can write $E = E_1 \cup E_2$ where E_1 is a regular and E_2 an irregular set (see [1], p. 304). Further $E_1 = E_1' \cup E_1''$ where $\Lambda(E_1'') = 0$ and E_1' is a measurable subset of the union of an enumerable infinity of rectifiable arcs (see [1], pp. 324 and 304). Another property that we require is that the projection of an irregular set is of zero measure in almost all directions (see [2], p. 357). Since we do not require many other properties of regular and irregular sets I shall not give their definitions nor the derivation of the properties stated above. They can be found in the papers [1] and [2].

Write $P(X, \theta)$ for the set which is the projection of X in the direction θ. The following lemmas are needed.

LEMMA 1. $[P(E_1', \theta)]$ *depends continuously on* θ.

Let A_i be a sequence of arcs each of finite linear measure such that $\bigcup_{i=1}^{\infty} A_i \supset E_1'$ and let ε_i be a sequence of positive numbers decreasing to zero. For each integer i there exists a closed subset F_i of E_1' and a positive integer N_i such that

$$\Lambda(F_i) > \Lambda(E_1') - \varepsilon_i \quad \Lambda(F_i \cap \bigcup_{j=1}^{N_i} A_j) > \Lambda(F_i) - \varepsilon_i .$$

The set $A_j \cap F_i$ is a closed subset of the arc A_j and its complement in A_j is an at most enumerable infinity of open subintervals of A_j say $B_{j,1}, B_{j,2}, \ldots$ These subintervals of A_j are open relative to A_j and there may of course be only a finite number of them. We can choose an integer $M_{i,j}$ such that

$$\Lambda(\bigcup_{k \geq M_{ij}} B_{jk}) < \frac{1}{N_i} \varepsilon_i .$$

The complement of $\bigcup_{1 \leqslant k < M_{ij}} B_{jk}$ in A_j consists of a finite number of arcs or points, say $A_{j,1}, \ldots, A_{j,h}$, where h depends on both i and j. We omit the points in this set and rename the set of all these arcs for all j from 1 to N_i as $C_1, C_2, \ldots, C_{L_i}$. Write C for $\bigcup_{j=1}^{L_i} C_j$, then if H denotes the set of points omitted in renaming the $A_{j,k}$ as C_l, we have

$$C \cup H \supset F_i \cap \bigcup_{1 \leq j \leq N_i} A_j , \tag{1}$$

$$C - F_i \subset \bigcup_{1 \leq j \leq N_i} \bigcup_{k \geq M_{ij}} B_{jk} . \tag{2}$$

From (2) it follows that

$$\Lambda(C - F_i) < \varepsilon_i . \tag{3}$$

Hence
$$\Lambda(C - E_1') + \Lambda(E_1' - C) < 3\varepsilon_i .$$

Thus
$$|\Lambda[P(E_1', \theta)] - \Lambda[P(C, \theta)]| < 3\varepsilon_i . \tag{4}$$

But $\Lambda[P(C, \theta)]$ depends continuously on θ, and (4) shows that $\Lambda[P(E_1', \theta)]$ is the uniform limit of a sequence of continuous functions. Thus $\Lambda[P(E_1', \theta)]$ is continuous and the lemma is proved.

LEMMA 2. $\int_0^{2\pi} \Lambda[P(E_1', \theta)] d\theta \leqslant 4\Lambda(E_1')$.

As in lemma 1 there is a sequence of sets $\{C_i\}$ each of which is a union of a finite number of rectifiable arcs and such that $\Lambda[P(C_i, \theta)]$ tends to $\Lambda[P(E_1', \theta)]$ uniformly in θ and

$\Lambda(C_i)$ tends to $\Lambda(E_1')$. Thus we need only prove Lemma 2 when E_1' is a union of a finite number of rectifiable arcs. Clearly this case will follow if we can establish the inequality for one arc. But we can approximate to an arc A by a polygonal line R such that, given $\varepsilon > 0$ every point of A is within a distance $\frac{1}{2}\varepsilon$ of some point of R and $\Lambda(R) \leqslant \Lambda(A)$. Then $\Lambda[P(R, \theta)] \geqslant \Lambda[P(A, \theta)] - \varepsilon$ and it is sufficient to prove the inequality for a polygonal line. Finally this case will follow if the inequality is true for a single segment. But the truth of the inequality in this last case is easily verified. The lemma is proved.

In the next Lemma we need to consider the relationship between the set E_1' and the union of an enumerable infinity of rectifiable arcs of which E_1' is a measurable subset. There are of course many such sets of arcs. We select one A and call the arcs of which it is the union $A_1, A_2, \ldots$ Let p be a point of E_1' lying on arc A_i of A. The densities of A_i and of $E_1' \cap A_i$ at p are defined to be

$$\lim_{r\to 0} \frac{\Lambda(A_i \cap \overline{C(p, r)})}{2r}, \quad \lim_{r\to 0} \frac{\Lambda(E_1' \cap A_i \cap \overline{C(p, r)})}{2r}$$

respectively, when these limits exist where $\overline{C(p, r)}$ is the closed set of points whose distance from p is less than or equal to r. It is known that at almost all(1) points p of A_i the first density exists and is equal to unity and at almost all points p of $E_1' \cap A_i$ the second density exists and is equal to unity (see [1], p. 303–304). Further, since A_i is a rectifiable arc it is known that at almost all points of it there is a tangent to it. Thus finally at almost all points p of E_1', the densities of A_i and $E_1' \cap A_i$ are unity and the tangent to A_i exists. There is of course a certain ambiguity in this since p may belong to more than one arc A_i. But in this case we simply select one A_i corresponding to each p and consider this arc A_i associated with p throughout what follows. The tangent to p will be denoted by $t(p)$ and any point p of E_1' with the above properties will be called an R-point.

LEMMA 3. *Either*

(a) *almost all points of E_1' lie on one straight line or*

(b) *there are two R-points of E_1', say p_1, p_2, such that p_2 does not lie on $t(p_1)$ and p_1 does not lie on $t(p_2)$.*

If (a) is false we can select an R-point of E_1', q_1 and a second R-point q_2 that does not lie on $t(q_1)$. If q_1 does not lie on $t(q_2)$ then q_1, q_2 have the properties required. If q_1 lies on $t(q_2)$ we select, if possible, a third R-point q_3 not on $t(q_1)$ nor $t(q_2)$. Now $t(q_3)$ cannot contain both q_1 and q_2 since if it did q_3 would lie on $q_1 q_2$, i.e. $t(q_2)$. Thus one of the pairs $q_1 q_3$ or

(1) "almost all" means "all but a set of zero linear measure".

$q_2 q_3$ has the required properties. If we cannot select a point such as q_3 then almost all of E_1' lies on $t(q_1) \cup t(q_2)$, and there are points of E_1' other than q_1 or q_2 on each of these lines. Let q_4 be an R-point of E on $t(q_1)$ distinct from q_1. Since $t(q_4)$ must coincide with $t(q_1)$ we can take the pair q_2, q_4 as the pair p_1, p_2.

The lemma is proved.

We are now in a position to prove the main result. If $\Lambda(E_2) = \delta > 0$, then for almost all θ

$$\Lambda[P(E_2, \theta)] = 0. \tag{5}$$

By Lemma 2 we can choose an angle θ such that

$$\Lambda[P(E_1', \theta)] < \frac{2}{\pi}(\Lambda(E_1') + \delta), \tag{6}$$

and since by Lemma 1 $\Lambda[P(E_1', \theta)]$ is a continuous function of θ we may suppose that both (5) and (6) hold for the same value of θ. Since $\Lambda(E_1'') = 0$ we have $\Lambda[P(E_1'', \theta)] = 0$ for all θ. Thus finally

$$\Lambda[P(E, \theta)] < \frac{2}{\pi}\Lambda(E).$$

If $\Lambda(E_2) = 0$, and (a) of Lemma 3 holds for E_1', then almost all points of E lie on one straight line and projecting parallel to this line we see that $\mu(E) = 0$. This implies the required result.

If $\Lambda(E_2) = 0$ and (a) of Lemma 3 is false for E_1' let p_1 and p_2 be two R-points of E_1' for which (b) holds. We now require the property that if p, an R-point of E_1' projects onto the point q of the set $P(E_1', \theta)$ and the direction of projection is not parallel to $t(p)$, then the set $P(E_1', \theta)$ has unit density at q. We suppose A_i is the arc associated with p, $\overline{C(p, \delta)}$ is the closed disc centre p and radius δ (as above), and write $I(q, \delta)$ for the linear closed interval perpendicular to the direction of projection with q as mid-point and of length 2δ. Given a positive number ε we can find a positive number δ_0 such that

$$\Lambda(E_1' \cap A_i \cap \overline{C(p, \delta)}) > (1 - \varepsilon) 2\delta \tag{7}$$

$$\Lambda(A_i \cap \overline{C(p, \delta)}) < (1 + \varepsilon) 2\delta$$

for all $\delta < \delta_0$. Now write A_i^* for $A_i \cap \overline{C(p, \delta)}$, then

$$\Lambda[P(E_1' \cap A_i^*, \theta) \cap I(q, \delta)] \geqslant \Lambda[P(A_i^*, \theta) \cap I(q, \delta)] - \Lambda[P(A_i^* - E_1', \theta) \cap I(q, \delta)], \tag{8}$$

and

$$\Lambda[P(A_i^* - E_1', \theta) \cap I(q, \delta)] \leqslant \Lambda[(A_i^* - E_1') \cap \overline{C(p, \delta)}] < 4\varepsilon\delta \tag{9}$$

if $\delta < \delta_0$. But if δ is sufficiently small, say $\delta < \delta_1$, then

$$P(A_i^*, \theta) \supset I(q, \delta).$$

Thus from (8) and (9)

$$\Lambda[P(E_1' \cap A_i^*, \theta) \cap I(q, \delta)] \geqslant 2\delta(1 - 2\varepsilon)$$

for $\delta < \min(\delta_0, \delta_1)$. Since obviously

$$\Lambda[P(E_1' \cap A_i^*, \theta) \cap I(q, \delta)] \leqslant 2\delta,$$

it follows that q is a point of unit density of $P(E_1', \theta)$.

Now suppose that the direction of the line joining the two R-points $p_1 p_2$ of E_1' is θ_0. We divide E_1' into two sets, E_1^* formed from those points of E_1' whose distance from p_1 is less than one half the distance of p_1 from p_2 and E_1^{**} defined by $E_1^{**} = E_1' - E_1^*$. Then since $P(E_1^*, \theta_0)$ and $P(E_1^{**}, \theta_0)$ have a common density point,

$$\Lambda[P(E_1', \theta_0)] < \Lambda[P(E_1^*, \theta_0)] + \Lambda[P(E_1^{**}, \theta_0)].$$

By continuity established in lemma 1 and by lemma 2 applied to E_1^* and E_1^{**} it follows that

$$\int_0^{2\pi} \Lambda[P(E_1', \theta)]\,d\theta < 4\Lambda(E_1^*) + 4\Lambda(E_1^{**}) = 4\Lambda(E_1').$$

Since we have $\Lambda(E) = \Lambda(E_1')$ we conclude that for some θ

$$\Lambda[P(E, \theta)] < \frac{2}{\pi}\Lambda(E).$$

Thus in all cases we have

$$\mu(E) < \frac{2}{\pi}\Lambda(E).$$

Example. We next construct an example to show that this result is the best possible.

Let ε be a given positive number and n a large positive integer, the actual lower bound of which will be specified later. Let $M_1, M_2, \ldots, M_{4n}$ be $4n$ points such that all the lines $M_i M_j$ have different directions. Let L_i be a segment of length $\delta/4n$ in a direction making an angle $2\pi i/4n$ with a fixed direction, and with mid-point at M_i. Choose δ so small that if we project the segments in any direction at most two of the segements overlap. Denote $\bigcup_{i=1}^{4n} L_i$ by E. Then

$$\Lambda[P(E, \theta)] \geqslant \sum_{j=1}^{4n} \frac{\delta}{4n}\left|\cos\left(\frac{2\pi j}{4n} + \theta\right)\right| - \frac{\delta}{4n}, \tag{10}$$

and since the expression on the right hand side of (10) is periodic in θ with period $\pi/2n$ we may assume that $0 \leqslant \theta \leqslant \pi/2n$. Substitute those values of θ which lie in this range and reduce the right hand side terms to their least values, i.e. for $0 < j < n$ and $2n \leqslant j < 3n$, $j = 4n$, put $\theta = \pi/2n$; for $n \leqslant j < 2n$ and $3n \leqslant j < 4n$ put $\theta = 0$. Then

$$\Lambda[P(E, \theta)] \geqslant \frac{\delta}{4n} \sum_{j=0}^{4n-1} \left| \cos \frac{\pi j}{2n} \right| - \frac{3\delta}{4n} = \delta \left(\int_0^{2\pi} |\cos 2\pi x| \, dx + o(1) \right), \tag{11}$$

as $n \to \infty$. Thus choosing first n sufficiently large, and then points $M_1 \ldots M_{4n}$, and δ we have for all θ

$$\Lambda[P(E, \theta)] > \delta \left(\frac{2}{\pi} - \varepsilon \right). \tag{12}$$

Thus

$$\frac{\mu(E)}{\Lambda(E)} \geqslant \frac{2}{\pi} - \varepsilon,$$

and this shows that the result obtained is the best possible.

§ 2. Some preliminary results

In the following two theorems the containing space is R^2.

THEOREM 3.1. *Let T be a closed connected set of finite linear measure and let $H(T)$ be its convex cover. Then either there is a tree T_1 contained in T such that the convex cover of T_1 coincides with that of T, or there is a simple closed convex curve K contained in T such that its convex cover coincides with that of T.*

We use $H(X)$ to denote the convex cover of the set X.

There is a subset K of T which is irreducible with respect to the three properties,

(i) $K \subset T$,

(ii) $H(K) = H(T)$,

(iii) K is closed and connected.

There certainly exist sets with these three properties since T is one such set. If possible form a sequence of sets K_i such that each K_i has properties (i), (ii), (iii) and K_j is a proper subset of K_i if $j > i$. If it is only possible to define a finite sequence of such sets then the last member of the sequence is irreducible. If the sequence has infinitely many members it can contain at most an enumerable infinity of members(1) (since the sequence of sets

(1) The sequence K_i may of course be transfinite but since the cardinal is less than $\aleph_1$ we can always find an enumerable sequence of ordinals as stated.

complementary to K_i in T form a strictly increasing sequence of sets open in T). But then $\bigcap_i K_i = K^*$ has properties (i) and (iii). We shall show that it also has property (ii). If $p \in H(T)$ then $p \in H(K_i)$ and therefore, since K_i is connected by Bunt's refinement of Carathéodory's theorem (see [5]) there exist two points k_i, k_i' of K_i such that p is a point of the segment $k_i k_i'$. We can select a subsequence of ordinals n_i such that for every j of the sequence there exists an i with $n_i > j$ and such that $k_{n_i} \to k$, and $k'_{n_i} \to k'$. Then since $k_i \in K_j$ if $i > j$ and K_j is closed $k \cup k' \in K_j$, all j. Thus $k \cup k' \in K^*$. Also p is a point of the segment kk'. Hence $p \in H(K^*)$ and this means that $H(T) \subset H(K^*)$: since the reverse inclusion is trivial (ii) is proved. Clearly K^* is irreducible and the statement is proved.

If K^* is a tree we have the desired result. If K^* is not a tree, then there are two points p_1, p_2 of K such that two arcs exist α_1, α_2 both contained in K^* and having in common only their end points p_1 and p_2. (K^* is of finite linear measure and therefore both locally connected and arc-wise connected.) If these two arcs lie in Fr $(H(T)) =$ Fr $(H(K^*))$ then they comprise the whole of that frontier and form a closed convex curve with the properties stated in the theorem. Otherwise there is a point say p on them which is an interior point of $H(T)$. Let the distance of p from Fr $H(T)$ be δ.

Now every component of $\overline{K^* - (\alpha_1 \cup \alpha_2)}$ meets $\alpha_1 \cup \alpha_2$ in a single point, for if this were not the case we could join two distinct points of $\alpha_1 \cup \alpha_2$ say p and q by an arc that lies in $\overline{K^* - (\alpha_1 \cup \alpha_2)}$. This arc cannot lie in $\alpha_1 \cup \alpha_2$ since K^* is locally connected, and thus this arc contains a subarc meeting $\alpha_1 \cup \alpha_2$ only at its end points p_1 and q_1. But this means that in K^* there are three distinct arcs joining p_1 to q_1 and intersecting only in their end points. Then one of these arcs lies in the bounded domain of which the other two form the frontier. Denote this open domain by D. $K^* - D$ has the same convex cover as K^*, is closed connected and is a proper subset of K^*. This is impossible by the irreducibility property of K^*.

Let β be a subset of $\alpha_1 \cup \alpha_2$ contained in $\overline{C(p, \frac{1}{2}\delta)}$. Since K^* is irreducible every component of $K^* - (\alpha_1 \cup \alpha_2)$ meets Fr $H(K^*)$. If it also meets β such a component must have linear measure of at least $\frac{1}{2}\delta$. Since K^* is of finite linear measure there are at most a finite number of such components and hence a subarc β_1 of β which is disjoint from $\overline{K^* - (\alpha_1 \cup \alpha_2)}$. But then $\overline{K^* - \beta_1}$ is a closed connected set with the same convex cover as K^* (since β_1 is interior to this convex cover) and is a proper subset of K^*. This is impossible since K^* is irreducible.

Thus arcs such as α_1, α_2 do not exist and Theorem 3.1 is proved.

DEFINITION: A polygonal tree is a tree formed from a finite number of linear segments. We always consider such a tree to have a simplicial decomposition into linear segments. So that if two segments meet they do so only in a common end point, and every end point of a segment is either an end point of the tree or an end point of at least one other segment of the tree. A point which belongs to more than one segment of the tree is called a singular point of the tree.

THEOREM 3.2. *Let $f(X)$ be an increasing continuous function of the convex set X, i.e. $X_1 \supset X_2$ implies $f(X_1) \geqslant f(X_2)$. Let $\mathcal{T}$ be the class of connected closed sets of finite positive linear measure. Let $\mathcal{P}(n)$ be the subclass of $\mathcal{T}$ of those polygonal trees whose convex covers are polygons with at most n sides. Then*

$$\sup_{T \in \mathcal{T}} \frac{f(H(T))}{\Lambda(T)} = \sup_n \sup_{P \in \mathcal{P}(n)} \frac{f(H(P))}{\Lambda(P)}.$$

By the previous result there is a tree K contained in T such that the convex cover of K coincides with that of T, or a simple closed curve K contained in T for which the convex covers of T and K coincide.

Let $k_1, k_2, \ldots, k_n$, be a sequence of points dense in K and consider the class of polygonal trees which contain $k_1, \ldots, k_n$. Amongst these we select one with least length and denote it by K_n. Then, whether K is a tree or a simple closed curve,

$$\Lambda(K_n) \leqslant \Lambda(K) \qquad H(K) \supset \bigcup_n H(K_n) \supset (H(K))^0.$$

Thus given $\varepsilon > 0$ there exists an integer n such that

$$\frac{f(H(K_n))}{\Lambda(K_n)} \geqslant \frac{f(H(K))}{\Lambda(K)} - \varepsilon.$$

But $K_n \in \mathcal{P}(m)$ for some m, thus

$$\sup_n \sup_{P \in \mathcal{P}(n)} \frac{f(H(P))}{\Lambda(P)} \geqslant \sup_{T \in \mathcal{T}} \frac{f(H(T))}{\Lambda(T)}.$$

The inequality in the reverse direction is trivial. Thus the theorem is proved.

There is a similar result for the class of arcs.

THEOREM 3.3. *Let $f(X)$ be an increasing continuous function of the convex set X. Let $\mathcal{A}$ be the class of arcs of finite positive linear measure and $\mathcal{A}(n)$ be the subclass of those members of $\mathcal{A}$ formed from at most n segments. Then*

$$\sup_{A \in \mathcal{A}} \frac{f(H(A))}{\Lambda(A)} = \sup_n \sup_{A^* \in \mathcal{A}(n)} \frac{f(H(A^*))}{\Lambda(A^*)}.$$

The proof is omitted.

§ 3. *E* a closed connected plane set of finite positive linear measure

Denote by $\mathfrak{C}(n)$ the class of closed connected plane sets which are of finite positive linear measure and such that their convex covers are polygons with at most n vertices.

Since the subclass of $\mathfrak{C}(n)$ contained in a bounded part of the plane forms a compact space under the closed-set metric (see [1], p. 316, and [3]) it follows that there is a member T of $\mathfrak{C}(n)$ such that

$$\frac{\mu(T)}{\Lambda(T)} = \sup_{E\epsilon\mathfrak{C}(n)} \frac{\mu(E)}{\Lambda(E)}. \tag{13}$$

We shall show that $\mu(T) = \frac{1}{2}\Lambda(T)$. This will imply that for any connected set E $\mu(E) \leqslant \frac{1}{2}\Lambda(E)$, for the general case when the convex cover of E is not a polygon can be dealt with by Theorem 2, Section 2.

Our argument will be such that we can specify the extremal figures T exactly, in so far as T is a member of some $\mathfrak{C}(n)$ but not when T is not a member of some $\mathfrak{C}(n)$. When E is composed of three equal segments equally inclined to one another, $\mu(E) = \frac{1}{2}\Lambda(E)$.

Thus we have
$$\mu(T) \geqslant \tfrac{1}{2}\Lambda(T), \tag{*}$$

and our aim in the rest of this paragraph is to show that $\mu(T) \leqslant \frac{1}{2}\Lambda(T)$. One method is to assume the contrary,(1) namely that $\mu(T) > \frac{1}{2}\Lambda(T)$ and show that this leads to a contradiction. I have not followed that method here because it is not then possible to particularize the extremal figures. The method is to use (13) and (*) to establish by variational arguments a number of properties of T which will specify it more and more exactly until finally we can assert that $\mu(T) \leqslant \frac{1}{2}\Lambda(T)$.

Denote the polygon which is the convex cover of T by P. $\mu(T)$ is the minimal width of P. A support line of P which is at a distance $\mu(T)$ from the parallel support line will be referred to as a minimal support line. A vertex of P which lies on a minimal support line of P will be referred to as a minimal vertex. There are two properties of minimal support lines of which we shall make frequent use.

(A) A pair of minimal support lines is such that at least one of the lines meets P in a segment. Otherwise we could give each of the lines an equal rotation about the vertices of P through which they passed and reduce the distance apart of the two lines. This would contradict the fact that they are a pair of minimal support lines of P.

(B) If the lines l_1 and l_1 are a pair of minimal support lines and meet P in X_1 and X_2 respectively, then the projection of X_1 onto l_2 by means of lines perpendicular to both l_1 and l_2 is a set Y_1 which intersects X_2.

(1) I feel no aversion to this type of argument but I find it repugnant to have to illustrate a hypothetical argument by drawing a diagram which cannot exist! (See Fig. 3 later.)

For if this were not the case there would be a line m perpendicular to l_1 and l_2 separating X_1 from X_2. Suppose m meets l_1 in L_1 and l_2 in L_2. If we give to l_1 a rotation about L_1 and to l_2 an equal rotation about L_2 we should reduce the distance between the two parallel lines. But since X_1 and X_2 lie on opposite sides of m we can choose this rotation to be in such a sense and of such a magnitude that the rotated strip still contains P. But this contradicts the fact that the distance apart of l_1 and l_2 is the minimal width of P.

We shall later require the following lemma: it is inserted here for convenience of reference.

LEMMA. *Let ABC be a triangle every angle of which is less than $\frac{2}{3}\pi$. Let K be the unique point such that $\angle AKB = \angle BKC = \angle CKA = \frac{2}{3}\pi$. On AB erect the triangle ADB which is equilateral and such that D lies on the side of AB opposite to C, then*

(i) *of all connected sets containing A, B and C the tree formed from the three segments AK, BK, CK, has the least length,*

(ii) *the sum of the lengths $AK + BK + CK$ is equal to the length CD.*

Let $\mathfrak{V}$ be a connected set joining A, B and C. If $\mathfrak{V}$ has infinite linear measure we need not consider it further. If $\mathfrak{V}$ has finite linear measure then it contains an arc γ_1 joining A to B and an arc γ_2 joining A to C. Let K_1 be the last point of $\gamma_1 \cap \gamma_2$ on γ_2 in the order A to C. Then arc AK_1 of γ_1 has length greater than or equal to segment AK_1 : arc K_1B of γ_1 has length greater than or equal to that of segment K_1B : arc K_1C of γ_2 has length greater than or equal to that of K_1C. Thus

$$\Lambda(\mathfrak{V}) \geqslant AK_1 + K_1B + K_1C.$$

We next consider a variable point X and the function $XA + XB + XC = F(X)$.(1) There is a position of X for which $F(X)$ attains its least value. Let this position be X_0. It is easy to see that X_0 does not coincide with any of A or B or C since each angle of triangle ABC is less than $\frac{2}{3}\pi$. If we move X from X_0 in the direction perpendicular to AX_0 then $AX = AX_0 + O(XX_0)^2$ and therefore $BX + CX = BX_0 + CX_0 + O(XX_0)^2$, i.e. XX_0 is perpendicular to the internal bisector of $\angle BX_0C$. Thus $\angle AX_0C = \angle AX_0B$ and similarly both these angles are equal to $\angle BX_0C$., i.e. X_0 coincides with the point K. Thus

$$K_1A + K_1B + K_1C \geqslant KA + KB + KC$$

and (i) is proved.

To prove (ii) we have $\angle AKB + \angle ADB = \pi$ so that $ADBK$ is a cyclic quadrilateral (see Fig. 1). Also $\angle AKD = \angle ABD = \frac{1}{3}\pi$ so the points C, K, D are collinear, and we

(1) Here XA denotes the length of the segment joining X to A.

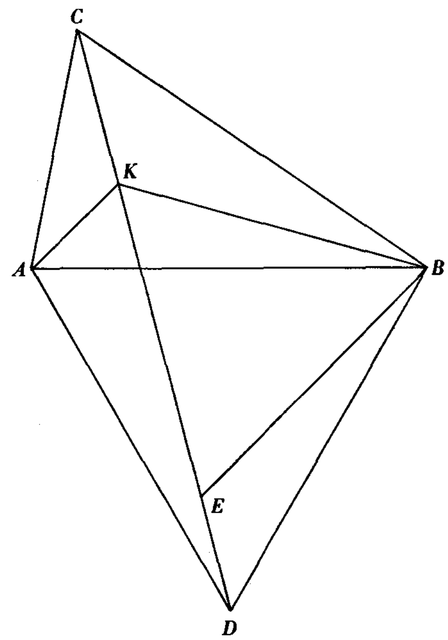

Fig. 1.

need only show that $KD = AK + KB$. Take E on KD so that $\angle KEB = \angle KBE$. Then, since $\angle EKB = \frac{1}{3}\pi$, KEB is an equilateral triangle and $KE = KB$. Since $\angle KBE = \frac{1}{3}\pi$, $\angle KBA = \angle EBD$. Hence in triangles AKB, DEB, $\angle KAB = \angle EDB$ since $ADBK$ is a cyclic quadrilateral, $AB = DB$ since ABD is an equilateral triangle, $\angle KBA = \angle EBD$ proved above.

Thus triangle AKB is congruent to triangle DEB and

$$ED = AK.$$

Hence $AK + KB = ED + KE = KD$ and the proof of (ii) is complete.

Properties of the extremal figure T

1. *Every vertex of P belongs to T.*
2. *Of all connected sets containing the vertices of P, T has the least length.*
3. *T is a polygonal tree formed from a finite number of linear segments.*
4. *Every end-point of T and every singular point of T is a vertex of P.*

If an end-point t of T was not a vertex of P we could remove from T a small segment with one end point at t and obtain $T_1 \in \mathfrak{L}(n)$. Since $\Lambda(T_i) < \Lambda(T)$, and (if the segment removed is sufficiently small) T_1 contains the vertices of P, we have a contradiction with 2. Thus every end point of T is a vertex of P.

Similarly if q is the end point of the two segments of T, pq, qr and q is not a vertex of P, then we can select a point q_1 on pq near to q and replace pq, qr by pq_1, q_1r to obtain T_1. Again $T_1 \in \mathfrak{C}(n)$, $\Lambda(T_1) < \Lambda(T)$, and we have a contradiction with 2.

5. *The angle between two adjacent segments of T is not less than $\frac{2}{3}\pi$.*

For if t_1t_2 and t_2t_3 are two adjacent segments of T and $\angle t_1t_2t_3 < \frac{2}{3}\pi$ we can replace these segments by a connected set containing t_1, t_2, t_3 and of less length. By 2 this is impossible.

6. *Every node of T is of order 3 and is an interior point of P. The three segments of T which abut at a node of T are inclined to one another at an angle of $\frac{2}{3}\pi$.*

This follows immediately from 5.

7. *T has either 3 or 4 end-points.*

Suppose that T has r end-points and that δ is a positive number less than the least length of a segment of T. Let T_1 be the subtree of T obtained from T by removing r segments each of length δ and such that each of these segments has one end point at an end point of T and each end point of T is an end point of one of these r segments. Then

$$\Lambda(T_1) = \Lambda(T) - r\delta,$$

and since every point of T is distant at most δ from some point of T_1,

$$\mu(T_1) \geqslant \mu(T) - 2\delta$$

(T_1 is not void because every node of T is a point of T_1, and if T has no nodes it is an arc and must contain at least two segments for otherwise $\mu(T) = 0$). If δ is small the convex cover of T_1 has the same number of vertices as P. Now if $r \geqslant 5$,

$$\frac{\mu(T_1)}{\Lambda(T_1)} \geqslant \frac{\mu(T) - 2\delta}{\Lambda(T) - 5\delta} > \frac{\mu(T)}{\Lambda(T)}, \tag{14}$$

since we know that $\Lambda(T) \leqslant 2\mu(T)$. But (14) is in contradiction with (13). Thus $r = 2$, 3 or 4.

If $r = 2$, projection in the direction of the line joining the end points of T shows that

$$\mu(T) < \tfrac{1}{2}\Lambda(T)$$

in contradiction with (*). Thus $r \neq 2$, and property 7 is proved.

8. *If T has four end points then P is a quadrilateral with these four points as vertices.*

If P has more than four vertices then one of them, say p, is not an end-point of T. Let the two segments pq_1, pq_2 of T meet at p and let p' be a point on pq_1, distant δ from

p. In T replace pq_1, pq_2 by $p'q_1$, $p'q_2$ and remove segments of length δ from each end-point of T as in 7. We obtain a connected set T_1 with

$$\Lambda(T_1) < \Lambda(T) - 4\delta$$
$$\mu(T_1) \geqslant \mu(T) - 2\delta,$$

and since

$$\frac{\mu(T_1)}{\Lambda(T_1)} > \frac{\mu(T)}{\Lambda(T)},$$

we again have a contradiction with 13. Thus P has at most four vertices. But by 4, P has at least four vertices and these vertices are end-points of T. Property 8 is established.

9. *T has exactly three end-points.*

Otherwise by 7, 8 and 2, P is a quadrilateral and T is the connected set of least length joining the vertices of P. In this case T is a polygonal tree with two third-order nodes and is formed from five segments. Let the vertices of P be a, b, c, d (in order round Fr P) and the nodes of T be $k_1 k_2$ with the notation chosen so that the segments of T are ak_1, bk_2, ck_2, dk_1 and k_1k_2.

The line through a perpendicular to ak_1 is a support line of P. For otherwise $\angle bak_1 > \frac{1}{2}\pi$ (since $\angle dak_1 \leqslant \frac{1}{3}\pi$). Suppose $\angle bak_1 > \frac{1}{2}\pi$. Let a_1 be the foot of the perpendicular from k_1 to the line ab. In T replace segment ak_1 by the segment a_1k_1 to obtain the tree T_1. The convex cover of T_1 contains P and $\Lambda(T_1) < \Lambda(T)$. Thus we have

$$\frac{\mu(T_1)}{\Lambda(T_1)} > \frac{\mu(T)}{\Lambda(T)},$$

in contradiction with (13). Thus the line through a perpendicular to a k_1 is a support line of P.

Since ck_2 is parallel to ak_1 we have a pair of parallel support lines, one each through a and c. Thus, projecting the polygonal line ak_1k_2c perpendicular to ak_1 we have

$$ak_1 + \tfrac{1}{2}k_1k_2 + k_2c \geqslant \mu(T). \tag{15}$$

Similarly, by projecting bk_2k_1d perpendicular to bk_2,

$$dk_1 + \tfrac{1}{2}k_1k_2 + k_2b \geqslant \mu(T). \tag{16}$$

Adding, we obtain

$$\Lambda(T) \geqslant 2\mu(T). \tag{17}$$

Now strict inequality in (17) is impossible (by (*)). Thus equality must hold in (17) and therefore in each of (15) and (16). Hence the lines through a and c perpendicular to ak_1 and those through d and b perpendicular to dk_1 are all minimal support lines.

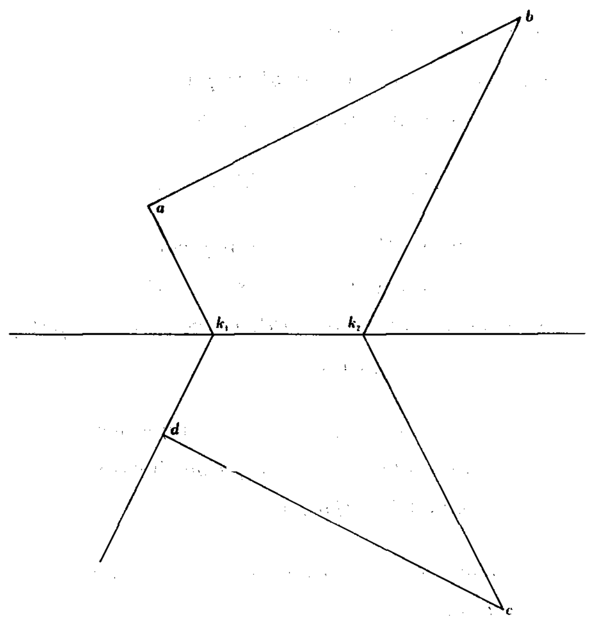

Fig. 2.

By property (A) of minimal support lines applied to the pair of minimal support lines perpendicular to $a k_1$, one of the segments ab, ad, bc, cd is perpendicular to $a k_1$. Clearly this is not true of ad and bc since $\angle d a k_1$ and $\angle b c k_2$ are both less than $\frac{1}{2}\pi$. Thus either ab or cd is perpendicular to $a k_1$. But in an exactly similar way we see from the pair of minimal support lines perpendicular to $b k_2$ that either ab or cd is perpendicular to $b k_2$. Since $a k_1$ and $b k_2$ are not parallel we conclude that either ab is perpendicular to $a k_1$ and cd is perpendicular to $b k_2$ or ab is perpendicular to $b k_2$ and cd is perpendicular to $c k_2$. The arguments in the two cases are the same and we shall consider the first case only. Remove from $a k_1$ a segment of length δ with end point at a and similarly from $d k_1$ a segment of length δ with end point at d. Denote the resulting tree by T_1. Then

$$\Lambda(T_1) = \Lambda(T) - 2\delta,$$
$$\mu(T_1) > \mu(T) - \delta.$$

But this is impossible since it implies a contradiction with (13).

Thus T has not got four end-points and by 7 must have exactly three end-points.

REMARK. T has one node and it is of order three. We shall denote it by k and the three arcs of T which terminate at k by α, β and γ. Denote the vertices of P on α, β, γ by $a_1, a_2, \ldots, a_h$; $b_1, b_2, \ldots, b_i$ and $c_1, c_2, \ldots, c_j$ where α is $a_1, \ldots, a_h, k$ and this order is the order in which these points lie on α. Similarly for β and γ.

10. *Every vertex of P is a minimal vertex.*

Suppose that the vertex p of P is not minimal. If p is an end-point of T we can remove a small segment one of whose end-points is p from T to obtain a subtree T_1, for which

$$\mu(T_1) = \mu(T),\ \Lambda(T_1) < \Lambda(T),$$

which leads to a contradiction with (13). Similarly if p is a point common to two segments pq, pq_2 of T we could move it into a new position p' on the internal bisector of the angle of these two segments in such a way that $\Lambda(T)$ is reduced but $\mu(T)$ remains unaltered. This again leads to a contradiction with (13).

DEFINITION. Two vertices of P joined by a single segment lying in the frontier of P (belonging to T or not) are said to be *P-adjacent*. Two singular points of T joined by a single segment of T are called *T-adjacent*.

11. *To each pair of end-points of T say a_1, b_1 there corresponds a pair of parallel minimal support lines l_1 and l_2 such that l_1 contains a_1 and l_2 contains b_1.*

Suppose that this is not the case. Remove length δ from the segment of T terminating at a_1 and another equal length from the segment of T terminating at b_1 to obtain the tree T_1. Let the new end-points be a_1' in place of a_1 and b_1' in place of b_1, and let the convex cover of T_1 be P_1. We shall assume that δ is a small number. Then by (13),

$$\frac{\mu(T_1)}{\Lambda(T_1)} \leqslant \frac{\mu(T)}{\Lambda(T)}, \tag{18}$$

and by construction,

$$\Lambda(T_1) = \Lambda(T) - 2\delta. \tag{19}$$

Since $\mu(T) \geqslant \frac{1}{2}\Lambda(T)$, (18) and (19) imply

$$\mu(T_1) \leqslant \frac{\mu(T)}{\Lambda(T)}(\Lambda(T) - 2\delta) \leqslant \mu(T) - \delta.$$

Further, if $\mu(T) > \frac{1}{2}\Lambda(T)$, then

$$\mu(T_1) < \mu(T) - \delta.$$

Now by the method of construction of T_1 from T,

$$\mu(T_1) \geqslant \mu(T) - \delta.$$

For of the two lines which form a pair of minimal support lines of P, one is a support line of P_1 and the other is distant at most δ from a parallel support line of P_1. By the inequality for $\mu(T_1)$ proved above it follows that

$$\mu(T_1) = \mu(T) - \delta.$$

Thus $\mu(T) = \frac{1}{2}\Lambda(T)$ and therefore

$$\frac{\mu(T_1)}{\Lambda(T_1)} = \frac{\mu(T) - \delta}{\Lambda(T) - 2\delta} = \frac{\mu(T)}{\Lambda(T)}.$$

Thus T_1 is also an extremal connected set for which $\dfrac{\mu(T_1)}{\Lambda(T_1)}$ assumes its least upper bound.

The results proved about T apply equally well to T_1.

By 10 every vertex of P_1 is a minimal vertex and since $\mu(T_1) = \mu(T) - \delta$ any pair of minimal support lines of P_1 are obtained from a pair of minimal support lines of P by keeping one line of the pair fixed and moving the other line a distance δ into a parallel position. There are at most two support lines of P for which the parallel corresponding support line of P_1 is distant δ, and these are the two lines perpendicular respectively to $a_1 a_1'$ and to $b_1 b_1'$; further this is so only if these lines contain no points of P apart from a_1 and b_1 respectively. Now P_1 must have at least two pairs of parallel minimal support lines. For otherwise, a small affine contraction orthogonal to the single pair of parallel minimal support lines would reduce $\Lambda(T_1)$ without altering $\mu(T_1)$.

Thus the lines through a_1 and b_1 perpendicular respectively to $a_1 a_1'$ and $b_1 b_1'$ are minimal support lines of P and contain no points of P apart from a_1 and b_1. Let the line through a_1 perpendicular to $a_1 a_1'$ be m_1 and the parallel support line of P be m_1'. Let the line through b_1 perpendicular to $b_1 b_1'$ be m_2 and the parallel support line be m_2'. Since every vertex of P is a vertex of P_1 apart from a_1 and b_1 (assuming that δ is sufficiently small) it follows from 10 that every vertex of P apart from a_1 and b_1 lies on m_1' or m_2'.

Denote the rhombus bounded by $m_1 m_2' m_2 m_1'$ by R, let its vertices be $ABCD$ in order where a_1 lies on AB and b_1 on BC. Let $a_1 a_1'$ produced meet $b_1 b_1'$ produced in s. Let $a_1 s$ produced meet DC in a_1^* and $b_1 s$ produced meet AD in b_1^*. Property (B) of minimal support lines implies that a_1^* is a point of the segment DC and b_1^* a point of the segment AD. Then $a_1 a_1^*$ and $b_1 b_1^*$ lie inside R. Thus $a_1 s$ and $b_1 s$ contain no vertices of T. (Every vertex of T is a vertex of P, see 4.) If a singular point of T lay on $a_1 s$ apart from a_1 and s, it would have to be a node k. Of the segments of T terminating at k, one has points interior to the quadrilateral $a_1 B b_1 s$. This segment cannot meet $a_1 B$ or $B b_1$ since no points of P lie on these segments apart from a_1 and b_1. Nor can it terminate in the interior of this quadrilateral for such a termination would be a singular point of T, therefore a vertex

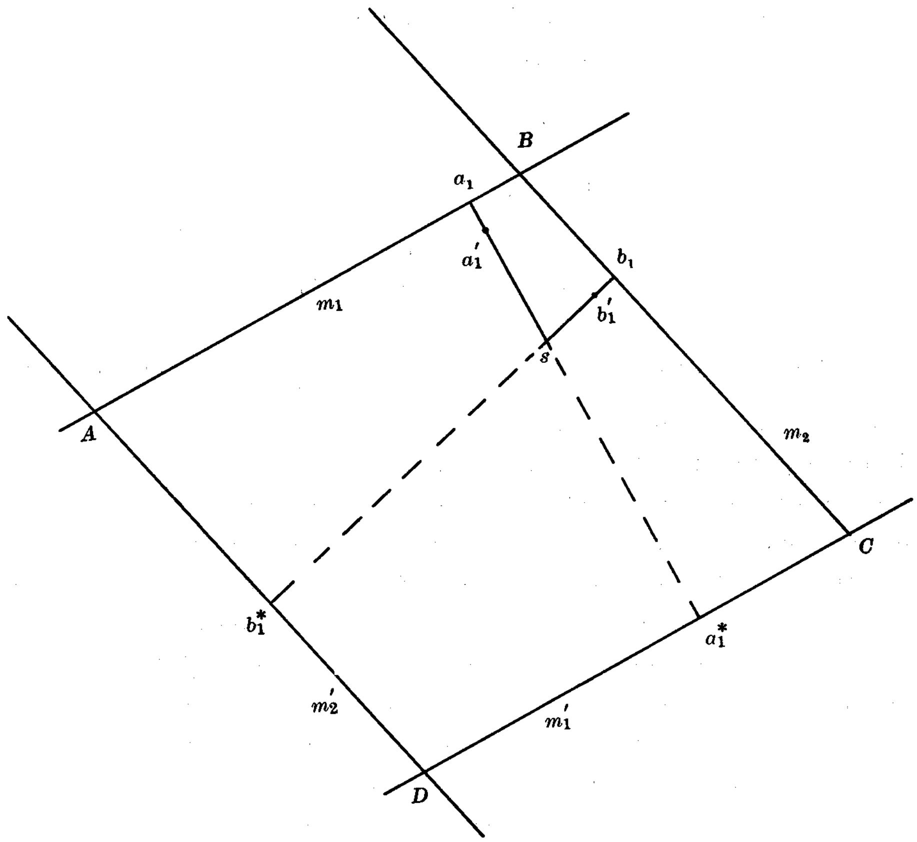

Fig. 3.

of P. But there are no such vertices of P. Thus this segment must meet segment $b_1 s$ in say k_1. But then segment $b_1 s$ contains a singular point of T and this singular point must be a node of T. Since T has only one node it follows that it lies at s. In the notation which we have adopted s is k. Let the third segment of T at $k = s$ meet the frontier of R in d. By a similar argument to that used above kd is a segment of T. Now the three segments ka_1, $kb_1 kd$ divide R into three domains one of which denoted by D_1 contains a_1^* and another, denoted by D_2 contains b_1^*. By property (B) a_1^* and b_1^* are points of P and neither a_1 nor b_1 are vertices of R (since they do not lie one each on a pair of parallel minimal support lines of P). Thus both D_1 and D_2 contain points of T on Fr R other than d. Since T is a tree with one node of order three and since a_1 and b_1 are end-points of T we have a contradiction. If for example the third end point of T lay in D_1 so would the whole of the arc of T joining this point to d and would thus have no points in D_2.

Thus we are led to a contradiction. The original assumption is false and 11 is proved.

REMARK. Of any two parallel support lines of P at least one passes through an end point of T. For if two parallel support lines exist neither of which contains an end-point of T we can take two points of the frontier of P one on each of these lines. But then these two points divide the frontier of P into two arcs one of which must contain two end-points of T. Through these two end-points there is then no pair of parallel support lines. This contradiction with 11 establishes the above statement. Then the three end-points of T, a_1, b_1 and c_1, divide the frontier of P into three non-overlapping arcs which are denoted by $A(a_1, b_1)$, $A(b_1, c_1)$, and $A(c_1, a_1)$ where arc $A(a_1, b_1)$ does not contain c_1 etc. Then any support line to P at a point of $A(a_1, b_1)$ is parallel to a support line of P at c_1 and there are similar relations for $A(b_1, c_1)$ and $A(c_1, a_1)$.

12. *The angle between two end-segments of T that lie in the frontier of P must be greater than $\frac{1}{3}\pi$.*

For if it were less than or equal to $\frac{1}{3}\pi$ then the removal of segments of length δ from the two end-segments concerned to produce a new tree T_1 would imply $\mu(T_1) \geqslant \mu(T) - \delta$ and the only directions in which T_1 can have minimal support lines are those orthogonal to the end-segments and (when the angle is equal to $\frac{1}{3}\pi$) that parallel to the bisector of the angle between the end-segments. This last case is impossible by (A) and the argument of 11 can then be used to establish property 12.

13. *Each of the three arcs α, β, γ has length less than $\mu(T)$.*

If, for example, $\Lambda(\alpha) \geqslant \mu(T)$ then from 11 there are a pair of parallel support lines to P through the end points of $\beta \cup \gamma$. Thus

$$\Lambda(\beta \cup \gamma) > \mu(T)$$

and

$$\Lambda(T) > 2\mu(T)$$

in contradiction with (*).

14. *If a_2 exists then a_1, a_2 lies in the frontier of P. i.e. if a_1 is not T-adjacent to k then $a_1 a_2$ lies in the frontier of P.*

The points a_1, a_2 belong to the frontier of P and thus if $a_1 a_2$ does not lie in the frontier of P it divides P into two non-empty domains. Thus there is a vertex of P on each side of the line containing $a_1 a_2$. By 1 there are points of T on each side of the line containing $a_1 a_2$. These points are joined by an arc of T inside P. Since these points lie on opposite sides of $a_1 a_2$ this arc meets $a_1 a_2$. But this is not so since there is no node of T on $a_1 a_2$, a contradiction which establishes 14.

15. *If three vertices of α, say a_s, a_{s+1}, a_{s+2}, are such that a_s, a_{s+1} and a_{s+1}, a_{s+2} are P-adjacent (as well as T adjacent) then $a_s s_{s+1}$ and $a_{s+1}a_{s+2}$ are tangent to a circle whose radius is $\mu(T)$ and whose centre is b_1 or c_1.*

The three points a_1, b_1, c_1 divide the frontier of P into three non-overlapping arcs which we shall denote, as before, by $A(a_1, b_1)$, $A(b_1, c_1)$ and $A(c_1, a_1)$. The three vertices $a_s a_{s+1} a_{s+2}$ cannot belong to $A(b_1 c_1)$, since if they did, the support line parallel to $a_s a_{s+1}$ would pass through a_1 (by the remark after 11) and this implies $\Lambda(\alpha) \geqslant \mu(T)$ in contradiction with 13. Thus $a_s a_{s+1}, a_{s+2}$ belong entirely either to $A(a_1, b_1)$ or to $A(c_1, a_1)$. Suppose that they belong to $A(a_1 b_1)$. The argument in the alternative case is similar. The support line of P parallel to the line $a_s a_{s+1}$ passes through c_1. Thus $a_s a_{s+1}$ is either tangent to the circle whose centre is c_1 and radius $\mu(T)$, $c(c_1, \mu(T))$, or the line containing $a_s a_{s+1}$ lies outside this circle. In the second case select a point a'_{s+1} on $a_{s+1}a_{s+2}$ near to a_{s+1} such that $a_s a'_{s+1}$ lies outside the circle $c(c_1, \mu(T))$. In T replace segments $a_s a_{s+1}, a_{s+1}a_{s+2}$ by $a_s a'_{s+1}$, $a'_{s+1}a_{s+2}$. If a'_{s+1} is not coincident with a_{s+1} the effect is to reduce $\Lambda(T)$ without altering $\mu(T)$. This is impossible by (13).

Property 15 is proved.

16. *If the vertex a_2 exists and if p is the other vertex of P which is P-adjacent to a_1, then either the line through a_1 perpendicular to $a_1 a_2$ is a minimal support line of P or the line containing $a_1 p$ is a minimal support line of P.*

We assume, without any real loss of generality that the points $a_2 a_1 p$ are in the clockwise sense round the frontier of P. Let the class of minimal support lines through a_1 be denoted by $\mathcal{J}$. Any member l of $\mathcal{J}$ together with the line $a_1 a_2$ divides the plane into four sectors of which one contains k. The angle of this sector is denoted by $\phi(l)$.

The set of values $\phi(l)$ is closed. If the line containing $a_1 p$ is not a minimal support line of P and if there is an l of $\mathcal{J}$ with $\phi(l) < \frac{1}{2}\pi$, this line l meets P in the single point a_1. For it cannot coincide with $a_1 a_2$ since $\phi(l) < \frac{1}{2}\pi$, nor with $a_1 p$ since by assumption this is not a minimal support line. By (B) the line through a_1 perpendicular to l must meet P in a segment of length $\mu(T)$. But in fact this line meets P in the single point a_1. Thus if $a_1 p$ is not a minimal support line then for every l of $\mathcal{J}$, $\phi(l) \geqslant \frac{1}{2}\pi$.

If $\phi(l) > \frac{1}{2}\pi$ for all l of $\mathcal{J}$ then there exists a small positive number ε such that $\phi(l) > \frac{1}{2}\pi + \varepsilon$ for all l of $\mathcal{J}$. Thus we can rotate $a_1 a_2$ about a_2 in the anti-clockwise sense so that a_1 becomes a'_1. Replace $a_1 a_2$ by $a'_1 a_2$ to obtain the tree T'. Now if $a_1 p$ is not a minimal support line and if the rotation is sufficiently small $\mu(T') = \mu(T)$. But $\Lambda(T') = \Lambda(T)$ so that T' is an extremal figure. By 10 a'_1 is an extremal vertex of T': but by the construction a'_1 is not an extremal vertex of T'.

This contradiction shows that either the line containing $a_1 p$ is a minimal support line or the line l with $\phi(l) = \frac{1}{2}\pi$ is a minimal support line of P. Thus property 16 is established.

17. *Any two vertices of P which are T-adjacent are also P-adjacent, i.e. the points $a_1 a_2 \dots a_h$ are in order round the frontier of P and so are $b_1 \dots b_i$ and $c_1, c_2 \dots c_j$.*

By 14 a_1 and a_2 are P-adjacent. We shall show first that a_2 and a_3 are also P-adjacent.(1) The vertex a_3 is P-adjacent either to a_2 or to a_1. For otherwise the segment $a_3 a_2$ divides P into two domains each of which contains vertices of P, say p, q such that neither p nor q is any one of a_1, a_2 or a_3. There is an arc in T joining p to q. This arc must cut $a_2 a_3$ which therefore contains a node of T. But this is not so.

We shall assume that a_3 is P-adjacent to a_1 and show that this leads to a contradiction.

We assume for definiteness that the order $a_2 a_1 a_3$ round the frontier of P is clockwise. Consider the minimal support lines that pass through a_1. We shall show that $a_1 a_3$ is not a minimal support line. Let q be the vertex of P that is P-adjacent and not T-adjacent to a_2. Let a_2' be a point on the line $q a_2$ such that a_2 lies between a_2' and q, and let T' be the tree obtained from T by replacing segments $a_1 a_2$ and $a_3 a_2$ by $a_1 a_2'$ and $a_3 a_2'$. Since the convex cover of T' includes P it follows from (13) that $\Lambda(T') \geqslant \Lambda(T)$. This in turn is true for any choice of a_2' as described above only if

$$\angle q a_2 a_3 \leqslant \angle a_2' a_2 a_1 .$$

But by 5 $\angle a_1 a_2 a_3 \geqslant \frac{2}{3}\pi$ and therefore

$$\angle q a_2 a_3 \leqslant \tfrac{1}{6}\pi.$$

Now if $a_1 a_3$ is a minimal support line of P there is a point of P on the parallel support line inside the strip which is bounded by the lines through a_1 and a_3 perpendicular to $a_1 a_3$. By 5 again $\angle a_2 a_1 a_3 \leqslant \frac{1}{3}\pi$; thus, if we produce $a_2 q$ to meet the line through a_3 perpendicular to $a_1 a_3$, it will do so in a point r on the same side of $a_1 a_3$ as a_2 (see Fig. 4). Thus it follows that the lines through points of segment $a_1 a_3$ perpendicular to $a_1 a_3$ intersect the quadrilateral $a_1 a_2 r a_3$ in segments of which the largest has length greater than or equal to $\mu(T)$. The largest segment (or one of them) is either the perpendicular from a_2 to $a_1 a_3$ or it is the segment $a_3 r$. In the first case $\Lambda(\alpha)$ is greater than the length of the segment $a_1 a_2$ and is therefore greater than $\mu(T)$. This is impossible by 13. In the second case we consider triangle $a_2 r a_3$. We have $\angle a_2 a_3 r < \frac{1}{2}\pi$ and thus $\angle a_3 a_2 r + \angle a_3 r a_2 > \frac{1}{2}\pi$. But we have

(1) It is assumed that such vertices as a_2, a_3 etc. exist. Otherwise there is nothing to prove.

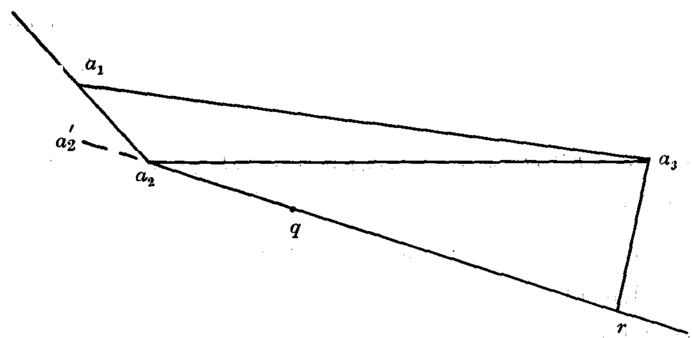

Fig. 4.

already seen that $\angle q a_2 a_3 = \angle r a_2 a_3 < \frac{1}{6}\pi$. Thus $\angle a_3 r a_2 > \frac{1}{3}\pi$ and hence $\angle a_3 r a_2 > \angle r a_2 a_3$. This implies that $a_3 a_2 > r a_3 \geq \mu(T)$. Finally we again obtain $\Lambda(\alpha) > a_3 a_2 > \mu(T)$. By 13 this is impossible. Thus $a_1 a_3$ is not a minimal support line of P.

But by 16 this implies that the line perpendicular to $a_1 a_2$ through a_1 is a minimal support line of P. This line meets P only in the point a_1 (since $\angle a_2 a_1 a_3 \leq \frac{1}{3}\pi$, see above) and thus the line through $a_1 a_2$ meets P in a segment of length $\mu(T)$, i.e. $a_1 a_2$ is of length at least $\mu(T)$. By 14 this is not so since it implies that $\Lambda(\alpha) > \mu(T)$. Thus finally a_3 is not P-adjacent to a_1 and a_3 must therefore be P-adjacent to a_2.

Next we suppose that there is a first integer m such that a_m and a_{m+1} are not P-adjacent. Then $m \geq 3$ and, by an argument similar to that used for a_3 above, it can be seen that a_1 and a_{m+1} are P-adjacent. The points $a_2, \ldots, a_m$ all belong to $A(a_1, b_1)$ or to $A(a_1, c_1)$. Suppose that they belong to $A(a_1, b_1)$ then by 15 each segment $a_1 a_2, a_2 a_3, \ldots, a_{m-1} a_m$ is part of a tangent to the circle centre c_1 and radius $\mu(T)$ and by (B) the segment $a_{m-1} a_m$ actually touches this circle. Thus $\angle a_{m-1} a_m c_1 < \frac{1}{2}\pi$. Since by 5 $\angle a_{m-1} a_m a_{m+1} \geq \frac{2}{3}\pi$ and since a_{m+1} and c_1 lie on the same side of $a_{m-1} a_m$ (they are points of P and $a_m a_{m-1}$ is part of a support line of P), it follows that a_{m-1} and a_{m+1} lie on opposite sides of the line $a_m c_1$. Hence a_1 and a_{m+1} lie on opposite sides of the line $a_m c_1$. But a_m and c_1 are both vertices of P. Thus a_1 and a_{m+1} are not P-adjacent.

This contradiction establishes the required result.

18. *If the vertex a_2 exists and if the vertices $a_2 a_1 b_1 p$ are in order round the frontier of P (i.e. $a_2 a_1$, $a_1 b_1$, $b_1 p$ are P-adjacent), then the line $a_1 a_2$ is not parallel to the line $b_1 p$.*

Remove a small segment of length δ from $a_1 a_2$ at a_1 and from the end b_1 of the segment of T that terminates at b_1. Denote the new tree by T' with end points a_1' in place of a_1 and b_1' in place of b_1. Now if $a_1 a_2$ is parallel to $b_1 p$ then any pair of parallel support lines of the convex cover of T' are such that at most one goes through a_1' or b_1' (except when

b_1 and p are T-adjacent, in which case the pair of parallel lines $a_1 a_2$ and $b_1 p$ are support lines of the convex cover of T' and go through a_1' and b_1' respectively). But in any case

$$\Lambda(T') = \Lambda(T) - 2\delta, \quad \mu(T') \geqslant \mu(T) - \delta.$$

As in 11 it follows that T' is an extremal figure, that $\mu(T') = \mu(T) - \delta$ and that the line through a_1 perpendicular to $a_1 a_2$ is a minimal support line of P meeting P in the single point a_1. By Property (B) of minimal support lines the line $a_1 a_2$ meets P in a segment of length $\mu(T)$. Thus the length of $a_1 a_2$ is $\mu(T)$ and

$$\Lambda(\alpha) > \mu(T)$$

in contradiction with 13.

Thus the assumption that $a_1 a_2$ is parallel to $b_1 p$ is false and 18 is proved.

We next consider the various cases that might arise according to the different orders of $a_1, \ldots, a_h$; $b_1, \ldots, b_i$ and $c_1, \ldots, c_j$ on the frontier of P, and according as α, β, γ are formed from one segment or more than one segment.

Case I. Each arc α, β, γ is made up of more than one segment and the orders $a_1, \ldots, a_h$; $b_1, \ldots b_i$; $c_1, \ldots, c_j$ on frontier P are all the same.

There is no real loss of generality in supposing that the vertices of P in clockwise order are $a_1, \ldots, a_h, b_1, \ldots, b_i, c_1, \ldots, c_j$. The other cases are obtained either by a change of notation or by an argument similar to the following.

Produce $b_1 a_h$ to d (see Fig. 5). Then

$$\angle k a_h b_1 \leqslant \angle d a_h a_{h-1}$$

for if this was not the case we could replace a_h by a_h' on $b_1 a_h$ such that a_h lies between a_h' and b_1 and such that the new tree obtained from T by replacing $k a_h$ and $a_{h-1} a_h$ by $k a_h'$ and $a_{h-1} a_h'$ has less length than T (see the argument in 17). Thus we have

$$\angle k a_h b_1 \leqslant \tfrac{1}{6}\pi$$

and further $\angle k a_h b_i < \frac{1}{6}\pi$. Since $\angle b_i k a_h = \frac{2}{3}\pi$ it follows that

$$k b_i < k a_h.$$

Similarly $k a_h < k c_j$, $k c_j < k b_i$. Thus we are led to the contradiction

$$k a_h < k a_h$$

and this case cannot occur.

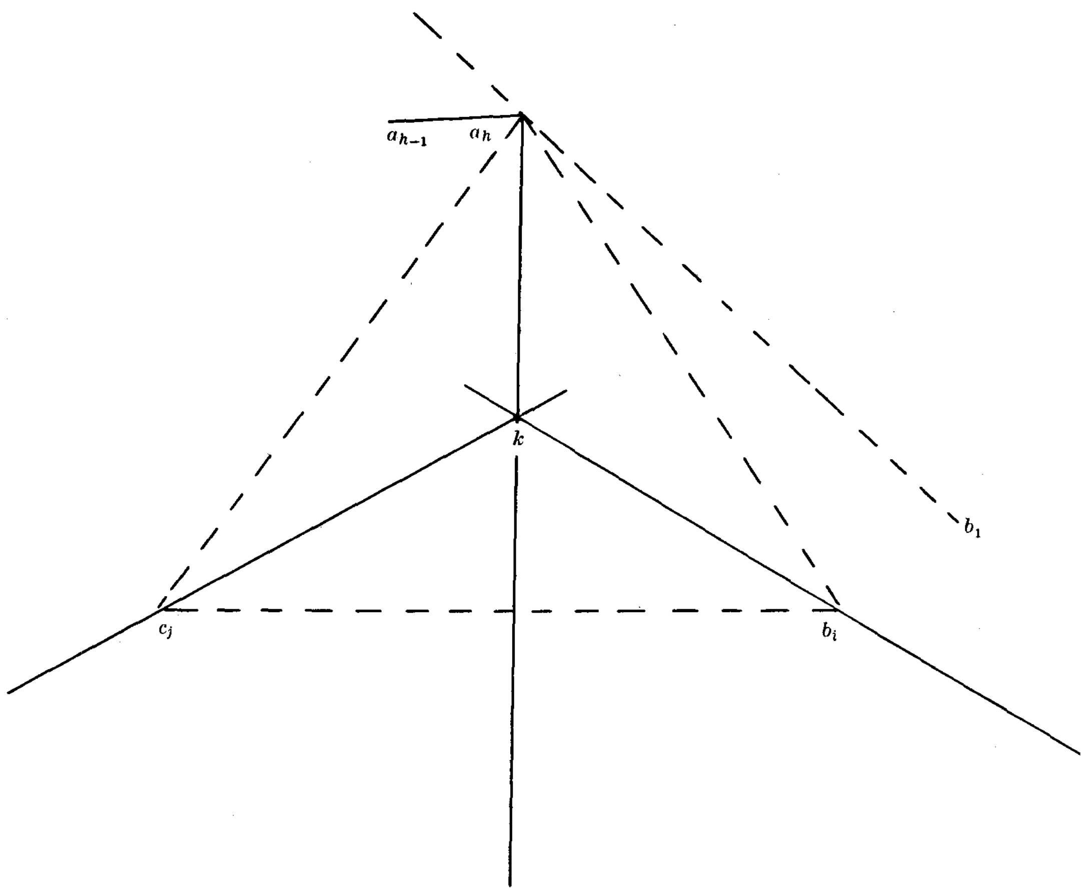

Fig. 5.

Case II. Two arcs α, β, γ, say α and β, are each formed from more than one segment and the orders $a_1, \ldots, a_h$; $b_1, \ldots, b_i$ are such that the sector of angle $\frac{2}{3}\pi$ bounded by the half lines containing $k a_h$, $k b_i$ respectively and terminating at k, is void of the points $a_1, \ldots, a_{h-1} b_1, \ldots, b_{i-1}$.

By an argument of the same type as that used in Case I we have $\angle k a_h b_i = \angle k b_i a_h = \frac{1}{6}\pi$ and $\angle k a_h a_{h-1} = \angle k b_i b_{i-1} = \frac{2}{3}\pi$. Take k' on $c_j k$ distant δ from k and a'_h, on $a_h a_{h-1}$, b'_i on $b_i b_{i-1}$ so that $k' a'_h$ is parallel to $k a_h$ and $k' b'_i$ to $k b_i$. In T replace $c_j k$, $k a_h$, $k b_i$, $a_h a_{h-1}$ $b_i b_{i-1}$ by $c_j k'$, $k' a'_h$, $k' b'_i$, $a'_h a_{h-1}$, $b'_i b_{i-1}$ to obtain T'. Then

$$\Lambda(T') = \Lambda(T) - \delta, \quad \mu(T') \geqslant \mu(T) - \tfrac{1}{2}\delta$$

and by an argument similar to that in 11 T' is an extremal figure. But this is not so since, for example, a_1 is not a minimal vertex of T'. This case cannot occur.

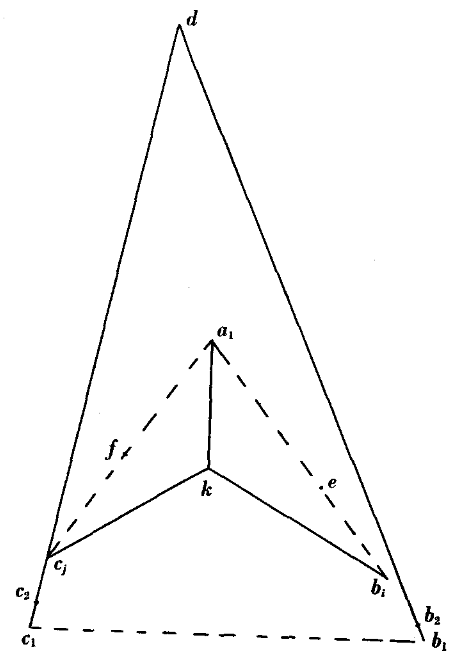

Fig. 6.

Case III. One arc, say α, is a single segment and the other two arcs each contain more than one segment. The orders of $b_1, \ldots, b_i$ and $c_1, \ldots, c_j$ are such that all these points belong to the sector whose angle is $\frac{2}{3}\pi$ and which is bounded by half-rays containing kb_i and kc_j respectively and terminating at k.

In this case the vertices b_1 and c_1 are P-adjacent.

We remark first that the line $b_1 b_2$ meets the line $c_1 c_2$ at a point d which lies on the same side of $b_1 c_1$ as k and that, of the four sectors into which the lines $b_1 b_2$ and $c_1 c_2$ divide the plane, the sector containing k has an angle greater than or equal to $\frac{1}{3}\pi$ (from 12). By the argument in 15 the perpendicular distance from c_1 to $b_1 b_2$ is equal to $\mu(T)$ and so is that from b_1 to $c_1 c_2$. Thus in triangle $d b_1 c_1$, $\angle b_1 d c_1 > \frac{1}{3}\pi$ and $\angle d b_1 c_1 = \angle d c_1 b_1 < \frac{1}{3}\pi$. But this implies that the distance from d to $b_1 c_1$ is less than $\mu(T)$. Since this is impossible this case cannot occur.

Case IV. One arc, say α, is a single segment and the other two arcs are not single segments. The vertices $b_1, \ldots, b_i, c_1, \ldots, c_j$ are in order on the frontier of P.

Either the three pairs a_1, c_j; c_1, b_i; b_1, a_1 are P-adjacent or the three pairs a_1, c_1; c_j, b_1; b_i, a_1 are P-adjacent. We suppose that the first alternative holds: the argument when the second alternative holds is the same with b's and c's interchanged.

Any line through c_j that supports P apart from $a_1 c_j$ and $c_{j-1} c_j$ is parallel to another support line of P that meets P in the single point b_1. By (A) it follows that no such line can be a minimal support line of P. If now $a_1 c_j$ were not a minimal support line of P we could replace c_j on $c_j c_{j-1}$ by c_j' lying between c_j and c_{j-1}. In T replace segments kc_j, $c_j c_{j-1}$, by kc_j', $c_j' c_{j-1}$. The effect is to obtain a tree T, with $\Lambda(T_1) < \Lambda(T)$, $\mu(T_1) = \mu(T)$. This is impossible and thus $a_1 c_j$ is a minimal support line of P. Similarly $c_1 b_i$ is a minimal support line of P.

Produce $c_1 b_i$ in both directions to meet $a_1 c_j$ produced in e and $a_1 b_1$ produced in f.

Now each of the angles $e a_1 f$, $a_1 f e$, $a_1 e f$ is not greater than $\frac{1}{2}\pi$. For, since $c_1 b_i$ is a minimal support line and the parallel support line through a_1 meets P in the single point a_1 (otherwise we should have $\Lambda(\beta) > \mu(T)$ in contradiction with (13)) it follows from (B) that the perpendicular from a_1 to ef intersects the segment $b_i c_1$ and therefore

$$\angle a_1 e f < \tfrac{1}{2}\pi, \quad \angle a_1 f e < \tfrac{1}{2}\pi.$$

Also by 18, $b_1 b_2$ is not parallel to $a_1 e$ and thus, by a similar argument, the perpendicular from b_1 to $a_1 e$ meets segment $a_1 c_j$, thus

$$\angle f a_1 e \leqslant \tfrac{1}{2}\pi.$$

By 5 $\angle k c_j c_{j-1} \geqslant \frac{2}{3}\pi$ and by the argument used in Case I $\angle a_1 c_j k \leqslant \angle e c_j c_{j-1}$. Thus $\angle a_1 c_j k \leqslant \frac{1}{6}\pi$, and this implies from triangle $a_1 k c_j$ that $\angle e a_1 k \geqslant \frac{1}{6}\pi$. Also $\angle k b_i e \leqslant \frac{1}{6}\pi$.

Project the polygonal line a_1, k, b_i in the direction of $c_1 b_i$. We have

$$a_1 k + \tfrac{1}{2} k b_i \geqslant \mu(T).$$

Project the polygonal line $c_1, c_2, \ldots, c_j, k, b_i, b_{i-1}, \ldots, b_1$ in the direction $a_1 b_i$. We have

$$k c_j + \ldots + c_2 c_1 + k b_i \sin \angle k b_i a_1 + b_i b_{i-1} + \ldots + b_1 b_2 \geqslant \mu(T).$$

Now if $kb_i > ka_1$, then $\angle k b_i a_1 < \frac{1}{6}\pi$, and on adding the above inequalities we obtain

$$\Lambda(T) > 2\mu(T)$$

in contradiction with (*). Thus $kb_i \leqslant ka_1$.

But in triangle $k c_j b_i$

$$\angle k b_i c_j < \angle k b_i e \leqslant \tfrac{1}{6}\pi, \quad \angle k c_j b_i + \angle k b_i c_j = \tfrac{1}{3}\pi.$$

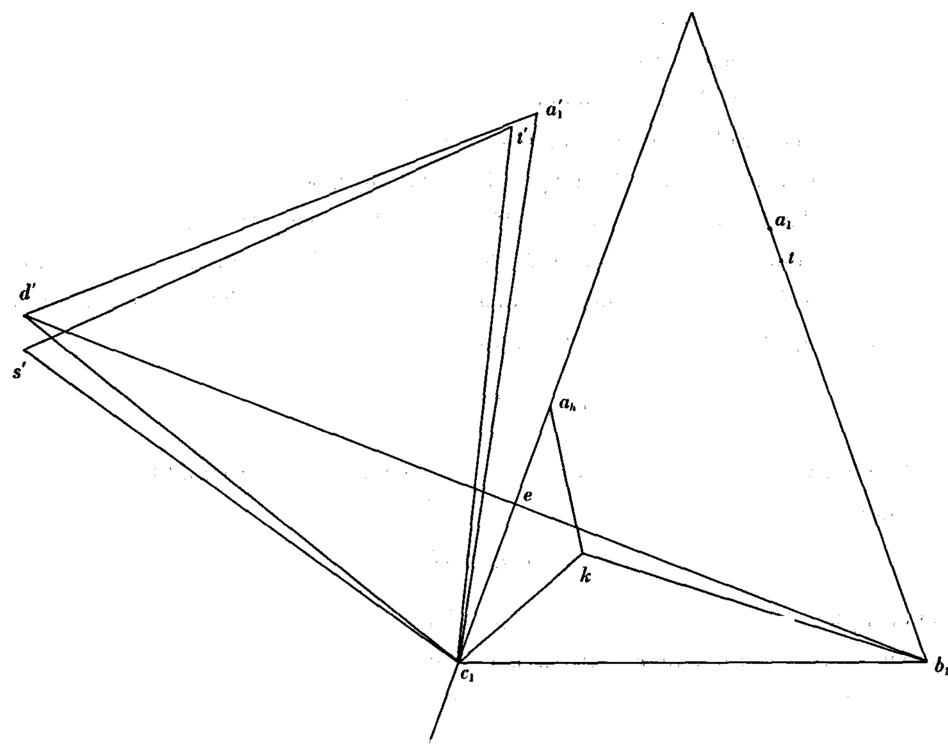

Fig. 7.

Thus $$\angle k b_i c_j < \angle k c_j b_i,$$

and this implies that $$k c_j < k b_i.$$

Similarly $$k a_1 \leqslant k c_j.$$

Thus $$k a_1 < k b_i$$

and we have a contradiction.

This case cannot occur.

Case V. Two of the arcs are single segments and one is composed of more than one segment.

We assume without any loss of generality that the arc α is the only arc with more than one segment and that the points $a_h a_{h-1}, \ldots, a_1, b_1, c_1$ are in the clockwise order round the frontier of P (see Fig. 7).

As in the previous case, $c_1 a_h$ is a minimal support line and the perpendicular distance from b_1 to $c_1 a_h$ is $\mu(T)$. We show first that $a_1 b_1$ is a minimal support line of P. By 16 if this is not the case the line through a_1 perpendicular to a_1a_2 is a minimal support line. It cannot therefore coincide with a_1b_1 and must meet P in the single point a_1. By property (B) the line a_1a_2 meets P in a segment of length $\mu(T)$, i.e. the length of segment a_1a_2 is $\mu(T)$. This is in contradiction with 13. Thus a_1b_1 is a minimal support line of P.

The perpendicular distances of b_1 and c_1 from a_hc_1 and a_1b_1 respectively are both equal to $\mu(T)$. Thus $\angle a_hc_1b_1 = \angle a_1b_1c_1$. If these angles are less than $\frac{1}{3}\pi$ the perpendicular distance of a_1 from b_1c_1 is less than $\mu(T)$. This is not so. If these angles are equal to $\frac{1}{3}\pi$, then a_1 must be the third angle of an equilateral triangle $a_1b_1c_1$. There are then no other vertices $a_2, \ldots, a_h$. This case is considered later (see Case VI). Thus in fact

$$\angle a_hc_1b_1 = \angle a_1b_1c_1 > \tfrac{1}{3}\pi.$$

Since the perpendicular distance from a_1 to b_1c_1 is greater than or equal to that of c_1 from a_1b_1 we have

$$a_1b_1 \geqslant c_1b_1.$$

On a_1b_1 let t be such that $tb_1 = c_1b_1$. Let a_1', t' be the reflections of a_1, t in a_hc_1 respectively. On $a_1'c_1$ erect the equilateral triangle whose third vertex d' lies on the side of $a_1'c_1$ opposite to b_1, and on c_1t' erect the equilateral triangle whose third vertex s' lies on the side of $t'c_1$ opposite to b_1. Let s and d be the reflections of s' and d' respectively in a_hc_1. Then by the lemma

$$\Lambda(T) \geqslant d'b_1.$$

Let $d'b_1$ meet line a_hc_1 in e. Now, since $\angle tb_1c_1 > \frac{1}{3}\pi$ and $b_1t = b_1c_1$ we have

$$\angle c_1tb_1 = \angle tc_1b_1 < \tfrac{1}{3}\pi.$$

Therefore b_1 is a point of the equilateral triangle c_1ts. The vector $\vec{sd}$ is equal to the vector $\vec{ta_1}$ rotated in the clockwise sense through an angle of $\frac{1}{3}\pi$. Since $\angle a_hc_1b_1 = \angle a_1b_1c_1 > \frac{1}{3}\pi$ it follows that the perpendicular distance of d from a_hc_1 is greater than that of s from a_hc_1, and since b_1 is a point of triangle c_1st, this last distance is greater than $\mu(T)$. Thus

$$de > \mu(T).$$

Since $b_1e \geqslant \mu(T)$ we have

$$\Lambda(T) \geqslant b_1d' = b_1e + ed > 2\mu(T).$$

This is in contradiction with (*).

This case cannot occur.

Case VI. Each arc α, β, γ is a single segment.

On the largest side of a_1, b_1, c_1, say on b_1c_1, erect the equilateral triangle whose third vertex d lies on the side of b_1c_1 opposite to a_1. Triangle b_1c_1d has area greater than or equal to that of triangle $a_1b_1c_1$. Thus the perpendicular distance of d from b_1c_1 is greater than or equal to that of a_1 from b_1c_1. If $a_1b_1c_1$ is not equilateral we have

$$\Lambda(T) = a_1d > 2\mu(T).$$

This is in contradiction with (*). Thus $a_1b_1c_1$ is equilateral.

This concludes the proof that $\mu(T) \leqslant \frac{1}{2}\Lambda(T)$ and that the only extremal figure whose convex cover is a polygon is formed from three equal equally inclined segments.

§ 4. E is a simple arc

Let $A_{(n)}$ be the class of all simple polygonal arcs of unit length composed of at most n segments. Define K by

$$K = (\sec\alpha + 2\tan\alpha + \pi - 4\beta - 2\alpha),$$

where

$$\tfrac{1}{2} + \sin\alpha = 4\cos^2\alpha/(1 + 4\cos^2\alpha)$$

and

$$\tan\beta = \tfrac{1}{2}\sec\alpha.$$

By Theorem 3 of Section 2 it is sufficient to show that for any member E of $A_{(n)}$

$$\frac{1}{\mu(E)} \geqslant K.$$

Write

$$\inf_{E\in A_{(n)}} \frac{1}{\mu(E)} = \tau.$$

By the arguments used by P. A. P. Moran [6] there is a member T of $A_{(n)}$ for which $\mu(T) = \tau^{-1}$.

We shall assume that
$$K > \tau \tag{31}$$

and show that this assumption leads to a contradiction. The method is similar to that used in Section 3 in that it depends upon appropriately chosen variations of T.

Denote the polygon which is the convex cover of T by P, and let the end points of the segments of T be $t_1, t_2, \ldots, t_n$ in order.

1. *The points common to two segments of T and the two end-points of T are vertices of P.*

   Obvious: Cf. Section 3.4.

2. *Every vertex of P is either a point common to two segments of T or is an end-point of T.*

   Obvious: Cf. Section 3.1.

3. *The polygon P subtends an angle of not more than $\frac{1}{2}\pi$ at each end point of T.*

By the same argument as that used in Section 3.14, t_1 and t_2 are P-adjacent. Suppose that t_h is the other vertex of P P-adjacent to t_1. If $\angle t_2 t_1 t_h > \frac{1}{2}\pi$ let t_1' be a point on the line $t_h t_1$ such that t_1 lies between t_1' and t_h and $\angle t_2 t_1' t_h \geqslant \frac{1}{2}\pi$. In T replace segment $t_2 t_1$ by segment $t_2 t_1'$. We suppose that t_1' is so close to t_1 that the new connected set T' is an arc. Then

$$\Lambda(T') < \Lambda(T), \quad \mu(T') \geqslant \mu(T).$$

Since $T' \in A_{(n)}$ we have a contradiction with the minimal property of T. Hence $\angle t_2 t_1 t_h \leqslant \frac{1}{2}\pi$.

4. *There are parallel support lines of P, one through each of the end points t_1, t_n of T.*

   This is an immediate consequence of 3.

5. *Let t_i be a vertex of P which is not an end point of T, such that of the vertices t_{i-1}, t_{i+1} at most one, say t_{i-1}, is P-adjacent to t_i. Let t_j be the other vertex of P P-adjacent to t_i then*

$$\angle t_{i+1} t_i t_j + \angle t_{i-1} t_i t_j \leqslant \pi.$$

On the line $t_j t_i$ let p be a point such that t_i lies between p and t_j. Then if $\angle t_{i+1} t_i t_j + \angle t_{i-1} t_i t_j > \pi$, it follows that

$$\angle t_{i+1} t_i t_j > \angle t_{i-1} t_i p.$$

But if we move t_i along $t_i p$ towards p through a small distance to the position t_i', and in T replace segments $t_{i-1} t_i$, $t_i t_{i+1}$ by $t_{i-1} t_i'$, $t_i' t_{i+1}$ respectively, we obtain a new member T' of $A_{(n)}$ for which

$$\Lambda(T') < \Lambda(T), \quad \mu(T') \geqslant \mu(T).$$

This is impossible because of the extremal property of T. Thus 5 is established.

6. *It is possible to find two vertices of T say t_i, t_j, $i < j$, with the following properties.*

   (a) *t_i and t_j are P-adjacent.*

   (b) *The support line of P parallel to $t_i t_j$, other than the line $t_i t_j$ itself, meets P in a vertex t_h with $i < h < j$.*

Consider two vertices for which (a) is true (there are such vertices).

Denote by $T(i, j)$ the subarc $t_i t_j$ of T and by $K(i, j)$ the set of points of intersection of the line joining t_i to t_j with support lines of P that pass through some vertex t_k of $T(i, j)$ other than t_i or t_j. If t_i, t_j do not satisfy (a) we define $K(i, j)$ to be the void set.

If $K(i, j)$ is non-void and unbounded then t_i, t_j satisfy (b). We shall assume that each non-void $K(i, j)$ is bounded and show that this assumption leads to a contradiction.

If t_i, t_j are P-adjacent $i < j$, then distinct consecutive members of $t_{i+1}, \ldots, t_{j-1}$ are also P-adjacent (if there are any). For if for example t_k was not P-adjacent to t_{k+1} then the segment $t_k t_{k+1}$ would divide P into two domains. Of these domains one must contain both t_i and t_j since they are P-adjacent. The other domain contains a vertex t_r with either $r < k$ or $r > k + 1$. If $r < k$, $T(r, i)$ which joins t_r to t_i, cuts $t_k t_{k+1}$. This is not so since T is an arc. Similarly we cannot have $r > k + 1$ and in fact t_k and t_{k+1} are P-adjacent.

It follows that two members of $t_{i+1}, \ldots, t_{j-1}$ which are not T-adjacent are also not P-adjacent. For if there were two such members say t_h and t_g, $h < g$, then, in the sequence $t_h, t_{h+1}, \ldots, t_{g-1}, t_g, t_h$, each consecutive pair is P-adjacent and thus the segments $t_h t_{h+1}, \ldots$ $t_{g-1} t_g$, $t_g t_h$ would comprise the whole of the frontier of P. This is not so since t_i belongs to the frontier of P and to none of these segments. Thus since $K(g, g+1)$ is void for all g we see that $K(g, h)$ is void for all g, h satisfying $i \leqslant g < h \leqslant j$ except $g = i$, $h = j$.

If $K(i, j)$ is non-void and bounded it is a closed segment. For it is the union of segments one corresponding to each t_k with $i < k < j$ and $t_k t_{k+1}$ is a support line of P and thus intersects $t_i t_j$ in a point belonging to the segment corresponding to t_k and to the segment corresponding to t_{k+1}. Thus these segments abut to form one segment.

The end points t_1, t_n of T are each end points of exactly one segment say $K(1, i_1)$ and $K(j_1, n)$ respectively since $t_1 t_2$ and $t_{n-1} t_n$ are P-adjacent pairs. Now an end point e of $K(i, j)$ other than t_1 or t_n lies on $t_i t_j$ and on a support line through t_k, $i < k < j$. This support line must pass through a second vertex t_l of P or e would not be an end point of $K(i, j)$. If $i < l < j$ then t_k and t_e are T-adjacent, i.e. $l = k - 1$ or $k + 1$ but then this again contradicts the fact that e is an end point of $K(i, j)$. Thus either $l < i < k$ or $k < j < l$. Suppose the former. Then $K(l, k)$ is not void; it contains e. Now no three of the segments $K(i, j)$ can meet, for if they did it would imply that three support lines of P would be concurrent. Also no two segments $K(i, j)$, $K(g, h)$ can meet except possibly at end points of each. For if they did each of the segments $K(i, j)$, $K(g, h)$ would be on support lines of P and since there are at most two support lines through any one point the line containing $K(g, h)$ would be a line used in the definition of $K(i, j)$, i.e. it would meet P in a vertex t_k with $i < k < j$. But any non-end point of $K(i, j)$ lies on a support line of P that meets P exclusively in points of $T(i, j)$. Thus $i \leqslant g < h \leqslant j$ and as remarked above this implies that $K(g, h)$ is void. Thus $K(g, h)$ meets $K(i, j)$ in an end point of $K(i, j)$. Similarly this end point is an end point of $K(g, h)$.

It follows that the union of all the non-void sets $K(i, j)$ contains a simple arc joining t_1 to t_n.

By 4 there are two parallel support lines of P, one each through t_1 and t_n. Denote the open strip bounded by these support lines by U. We may assume that t_1 and t_n are not P-adjacent for if they are then (b) obviously holds with $i = 1, j = n$.

The line $t_1 t_n$ divides the frontier of P into two arcs which are disjoint except for the fact that they both have t_1 and t_n as end points. Denote these two arcs by X_1 and X_2. Of the two P-adjacent vertices $t_i t, i < j$, either both belong to X_1 or both to X_2 or one is t_1 or t_n and in any case the segment $t_i t_j$ of the frontier of P is contained in X_1 or X_2. If $t_i t_j$ is contained in X_1 and $K(i, j)$ is non-void then all vertices t_k, $i < k < j$, belong to X_2 and vice-versa. In any case the part of the line $t_i t_j$ contained in U is separated from t_k by $t_1 t_n$. Thus no part of the line $t_i t_j$ in U can belong to $K(i, j)$, for such a point is joined to t_k by a segment which on the one hand is contained in U and on the other cannot meet the part of $t_1 t_n$ contained in U.

Hence $$K(i, j) \cap U = \phi.$$

But U separates t_1 from t_n and $K(i, j)$ joins t_1 to t_n thus for some pair i, j $K(i, j) \cap U \neq \phi$. This contradiction shows that for some i, j $K(i, j)$ is unbounded and (b) holds.

We can now complete the proof of the inequality $\tau \geqslant K$ by considering two possible cases and by showing that in each case the assumption (31) leads to a contradiction.

Case I. There is a pair of integers i, j such that $t_i t_j$ satisfy 6 *and one of t_i, t_j is not an end point of T.*

Suppose for definiteness that $1 \leqslant i < j < n$. Let t_k, $i < k < j$, be the vertex of P at which a support line is parallel to $t_i t_j$.

If $$\angle t_k t_j t_i \leqslant \tfrac{1}{4}\pi$$

the length of the segment $t_k t_j$ is at least $\sqrt{2}\,\mu(T)$, and since that of $t_i t_k$ is at least $\mu(T)$ we see that

$$\Lambda(T) \geqslant (1 + \sqrt{2})\,\mu(T);$$

since calculation shows that

$$\mu(T) = \frac{1}{\tau} > \frac{1}{K} > \frac{1}{2 \cdot 28},$$

we have $$\Lambda(T) > 1.$$

By our original assumption this is not so. Thus $\angle t_k t_j t_i > \frac{1}{4}\pi$.

Since, by 5

$$\angle t_i t_j t_{j+1} + \angle t_{j-1} t_j t_{j+1} \leqslant \pi$$

and

$$\angle t_{j-1} t_j t_i \geqslant \angle t_k t_j t_i > \tfrac{1}{4}\pi,$$

we have

$$\angle t_{j-1} t_j t_{j+1} \leqslant \tfrac{3}{8}\pi.$$

Now construct a new arc from T by removing a segment of length δ from the end of $t_{n-1}t_n$ at t_n and moving t_j along the internal bisector of $\angle t_{j-1} t_j t_{j+1}$ a distance δ to t_j' and replacing segments $t_{j-1}t_j$, $t_j t_{j+1}$ by $t_{j-1}t_j'$ and $t_j' t_{j+1}$. If δ is small we do in fact obtain a new arc. We denote it by T_1. Then since there are not two parallel support lines of P through t_j and t_n(1) we have

$$\mu(T_1) \geqslant \mu(T) - \delta.$$

Also

$$\Lambda(T_1) \leqslant \Lambda(T) - \delta - 2\delta \cos\frac{3\pi}{16} + O(\delta^2).$$

But these inequalities imply, if δ is small,

$$\frac{\mu(T_1)}{\Lambda(T_1)} > \frac{\mu(T)}{\Lambda(T)},$$

and this is impossible by the extremal property of T.

This case cannot occur.

Case II. The only pair of integers i, j for which t_i, t_j satisfy 6 are $i = 1$ and $j = n$.

In this case t_1, t_n are P-adjacent and this implies that the whole arc T lies in the frontier of P. Let t_k be the vertex of T, $1 < k < n$, at which there is a support line parallel to $t_1 t_n$.

Denote the common part of the two circular discs whose centres are t_1 and t_n and whose radii are $\mu(T)$ by D. The part of D on the same side of $t_1 t_n$ as t_k is contained in P. Denote it by D_1. Denote the convex cover of t_1, t_k, t_n and D_1 by P_1 and the length of the frontier of P_1 excluding the segment $t_1 t_n$ by $X(P_1)$.

---

(1) If there were, each subarc $T(i, k)$, $T(k, j)$, $T(j, n)$ of T would be of length greater than or equal to $\mu(T)$. Hence

$$\Lambda(T) \geqslant 3\mu(T) > 1,$$

a contradiction since $\Lambda(T) = 1$.

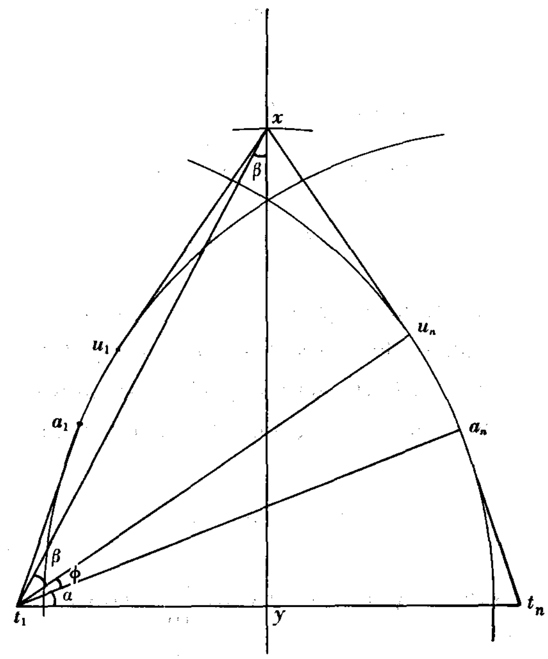

Fig. 8.

Steiner symmetrisation about the perpendicular bisector of $t_1 t_n$ shows that $X(P_1)$ is least when t_k lies on the perpendicular bisector of $t_1 t_n$. Also $X(P_1)$ is least when the distance of t_k from $t_1 t_n$ is $\mu(T)$. Denote this position of t_k by x and the corresponding convex cover of t_1, x, t_n, D_1 by P_2. Let the points of contact of the lines of support from x to D_1 be u_1 and u_n and those from t_1 and t_n to be a_1 and a_n respectively where the point u_1 is on the same side of the perpendicular bisector of t_1 as is t_1. Denote the length of the frontier of P_2 excluding $t_1\, t_n$ by L and letting y be the mid-point of $t_1 t_n$ (see Fig. 8).

Suppose the points $t_1 a_1 u_1 x u_n a_n t_n$ are in order on the frontier of P_2.

If δ is a small positive number and we move t_1 along $t_1 t_n$ a distance δ to t_1' and t_n along $t_n t_1$ a distance δ to t_n' and then form P_2' and L' from t_1', t_n', x in exactly the same way that P_2 and L were formed from t_1, t_n, x, we have

$$L' = L + 2\delta \sin \angle y x u_n - 4\delta \sin \angle a_n t_1 t_n + o(\delta),$$

since, to within a term in $o(\delta)$ the effect is to translate u_n, a_n by δ in the sense $\overrightarrow{t_1 t_n}$ parallel to $t_1 t_n$ and u_1, a_1 by an equal amount in the opposite sense. Thus L is least when either

(i) x, t_1, t_n are all distant $\frac{2}{\sqrt{3}}\mu(T)$ from one another, or

(ii) t_1, t_n are distant $\mu(T)$ from one another, or

(iii) $\sin \angle yxu_n = 2 \sin \angle a_n t_1 t_n$.

In the third case write β for $\angle xt_1u_n = \angle yxt_1$ (this equality is because $xy = t_1u_n = \mu(T)$), and α for $\angle a_n t_1 t_n$. Then calculating t_1t_n in two different ways we have,

$$t_1t_n = t_1a_n \sec \alpha = \mu(T) \sec \alpha,$$

$$t_1t_n = 2\, x\, y \tan \beta = 2\mu(T) \tan \beta.$$

Thus
$$\tan \beta = \tfrac{1}{2} \sec \alpha. \tag{32}$$

Also
$$\angle yxu_n = \angle u_n t_1 y = \phi + \alpha$$

where $\phi = \angle u_n t_1 a_n$. Thus by (iii)

$$\sin(\phi + \alpha) = 2 \sin \alpha.$$

But from triangle xyt_1 we have

$$\beta = \tfrac{1}{2}\pi - (\beta + \alpha + \phi).$$

Hence,
$$\cos 2\beta = 2 \sin \alpha.$$

Substituting for β from (32) we have

$$\tfrac{1}{2} + \sin \alpha = \frac{4 \cos^2 \alpha}{1 + 4 \cos^2 \alpha}. \tag{33}$$

Also
$$L = (2 \tan \alpha + 2\phi + 2 \tan \beta)\mu(T)$$
$$= (2 \tan \alpha + \sec \alpha + \pi - 4\beta - 2\alpha)\mu(T).$$

Calculation shows that in the third case $L = 2.273\ \mu(T)$ approximately and that in (i) $L = 2.309\ \mu(T)$, in (ii) $L = 2.28\ \mu(T)$. Thus L is least in the third case, and we have proved that

$$\Lambda(T) \geqslant L \geqslant K\mu(T) > 1$$

But this is not so by assumption. Thus (31) leads to a contradiction in all cases and must itself be false.

Thus the required inequality is established.

§ 5. Further problems

There are many other problems of the same type as those considered in Section 3 and Section 4. If T is any connected set of finite linear measure and $f(X)$ an increasing functional of the convex set X, then the number

$$\sup_T \frac{f(H(T))}{\Lambda(T)} = \mu_f$$

(where $H(T)$ is the convex cover of T) conveys certain information about the relationship between a connected set and its convex cover. Examples of the function $f(X)$ are the area of X, the inradius of X, the circumradius of X, the perimeter of X, the diameter of X, the moment of inertia of X about its centroid, etc. Of these some lead only to trivial results, either because an extremal figure is obvious or because the ratio $f(H(T))/\Lambda(T)$ is not an invariant under similarity transformation.

We consider here the case when $f(X)$ is the square root of the area of X. This problem can be replaced by another one as follows. Consider a finite set of n points in R^2, say the set E. Let A be the area of the convex cover of E. What is the least measure of any connected set which contains E, expressed in terms of A and n? We shall show that

$$\Lambda(K) \geqslant 2[A(n-1)\tan \pi/(2(n-1))]^{\frac{1}{2}} \quad n > 3, \tag{34}$$

$$\Lambda(K) \geqslant 2[A\sqrt{3}]^{\frac{1}{2}} \quad n = 3. \tag{35}$$

Since as $n \to \infty$ the right-hand side of (34) decreases to $(2\pi A)^{\frac{1}{2}}$, it follows that μ_f calculated for f equal to the square root of the area is $(2\pi)^{-\frac{1}{2}}$ (making use of Theorem 3.2). In turn the fact that $\mu_f = \dfrac{1}{(2\pi)^{\frac{1}{2}}}$ implies a result of P. A. P. Moran, who proved a conjecture of S. Ulam, namely that the convex cover of an arc of unit length has area less than or equal to $\frac{1}{2}\pi$. This result is best possible since equality is attained when the arc is a semi-circle; whether this is the only extremal curve is not known. The results given in (34) and (35) are also best possible. In (34) equality is attained when E is a set of consecutive vertices of a regular $2(n-1)$ – gon and K is the arc joining them. In 35 equality holds when E is the set of vertices of an equilateral triangle and K is formed from three equal segments inclined at an angle of $\frac{2}{3}\pi$ with one another.

The proof of (34) and (35) is quite simple. As in Section 3 let $\mathfrak{L}(n)$ be the class of closed connected plane sets which are of finite positive linear measure and whose convex covers are polygons with at most n vertices. Denote the area of the convex cover of T by $A(T)$. Write

$$K_n = \sup_{T \in \mathfrak{L}(n)} \frac{[A(T)]^{\frac{1}{2}}}{\Lambda(T)}.$$

It is not difficult to prove that there is a member T_0 of $\mathfrak{L}(n)$ for which

$$K_n = \frac{[A(T_0)]^{\frac{1}{2}}}{\Lambda(T_0)}.$$

There may be more than one such member of $\mathfrak{L}(n)$. If there is we select one whose convex cover has the least possible number of vertices. Denote this set by T^* and the convex cover of T^* by P^*.

As in the problem considered in Section 3, T^* is a polygonal tree and is a connected set of the least possible measure containing the vertices of P^*. Every end point of T^* is a vertex of P^* and every node of T^* is an interior point of P^*.

Next, the segment joining any two end-points of T^* lies in the frontier of P^*. For suppose that these were two end points p_1, p_2 of T^* such that the segment p_1p_2 met the interior of P^*. Let p_1q_1 and p_2q_2 be the segments of T^* which terminate at p_1 and p_2 respectively. Take points p_1' on line p_1q_1 distant x_1 from p_1, where x_1 is positive if p_1' lies between p_1 and q_1, and negative otherwise, and p_2' on line p_2q_2 distant x_2 from p_2. Both x_1 and x_2 are not greater than the least length of segments p_1q_1 and p_2q_2. In T^* replace p_1q_1 by $p_1'q_1$ and p_2q_2 by $p_2'q_2$. Denote the new polygonal tree by $T^*(x_1, x_2)$ and its area by $A(x_1, x_2)$. Now if $x_1 > 0$ is small,

$$\Lambda(T^*(x_1, -x_1)) = \Lambda(T^*).$$

Thus $A(x_1, -x_1) \leqslant A(T^*)$. But if $A(x_1, -x_1) < A(T^*)$, then $A(-x_1, x_1) > A(T^*)$. This is impossible. Hence

$$A(x_1, -x_1) = A(T^*).$$

We increase x_1 until either $p_1'p_2'$ lies in the frontier of P^* or p_1' coincides with q_1. This is possible. In each case we obtain an extremal figure whose convex cover has less vertices than P^*. This is impossible by the choice of P^*. Thus every segment joining two end points of T^* lies in the frontier of P^*.

Thus T^* has either three end points or two end points. If T^* has three end points the segments joining them in pairs lie in the frontier of P^*; thus P^* is a triangle and T^* is formed from three segments inclined to one another at an angle of $\frac{2}{3}\pi$. If T^* has two end points it is an arc and must lie entirely in the frontier of P^*.

Consider the first alternative. Let the lengths of the three segments be l_1, l_2, l_3. Then

$$\begin{aligned} A(T^*) &= \tfrac{1}{4}\sqrt{3}\cdot(l_1 l_2 + l_2 l_3 + l_3 l_1) \\ &= [2\,(\Lambda\,(T^*))^2 - (l_1-l_2)^2 - (l_1-l_3)^2 - (l_2-l_3)^2]/8\sqrt{3} \\ &\leqslant [\Lambda\,(T^*)]^2/4\sqrt{3}. \end{aligned}$$

Thus (35) (and a fortiori (34)) is true in this case.

Consider the second alternative. Let the arc T^* be $p_1 p_2 \ldots p_k$ where each p_1 is a vertex of P^*. Then segment $p_i p_{i+1}$ is of equal length to segment $p_{i-1} p_i$ $i = 2, \ldots, k-1$. For otherwise we can symmetrize the triangle $p_{i-1} p_i p_{i+1}$ about the perpendicular bisector of segment $p_{i-1} p_{i+1}$ to reduce $\Lambda(T^*)$ without affecting $A(T^*)$. This is impossible because of the extremal property of T^*.

Consider next the second alternative. If P'' has only three vertices, then T^* is the sum of the lengths of the two shortest sides of P^*. Since T^* is the connected set of least length that contains the vertices of P^*, this implies that one of the angles of P^* is at least $\frac{2}{3}\pi$ and T^* is the two sides adjacent to this angle. But then $A(T^*)$ can be increased without altering $\Lambda(T^*)$ by rotating one of these sides relative to the other until they form an angle equal to $\frac{1}{2}\pi$. By the extremal property of T^* this is impossible. Thus P^* has at least four vertices. We consider any four consecutive vertices of T^*, say p_1, p_2, p_3, p_4, for definiteness. We shall show that $p_2 p_3$ is parallel to $p_1 p_4$, and thus, since $p_1 p_2$ and $p_3 p_4$ are segments of equal length, that $\angle p_1 p_2 p_3 = \angle p_2 p_3 p_4$. If now $p_2 p_3$ is not parallel to $p_1 p_4$, suppose that p_3 is nearer to $p_1 p_4$ than is p_2. Let the line through p_3 parallel to the line $p_1 p_4$ cut the segment $p_1 p_2$ in p_2'. Symmetrize the trapezium $p_1 p_2' p_3 p_4$ about the perpendicular bisector of $p_1 p_4$ to obtain the trapezium $p_1 p_2^* p_3^* p_4$. On $p_2^* p_3^*$ construct a triangle $t p_2^* p_3^*$ congruent to and similarly situated to $p_2 p_2' p_3^*$. Now since p_3 is nearer to $p_1 p_4$ than is p_2, we have $\angle p_2 p_1 p_4 > \angle p_3 p_4 p_1$ and thus $\angle t p_2^* p_3^* > \angle p_2^* p_1 p_4$. It follows that p_2^* is an interior point of the convex cover of p_1, t, p_3^*, p_4. In T^* we replace $p_1 p_2$, $p_2 p_3$, $p_3 p_4$ by $p_1 t$, $t p_3^*$, $p_3^* p_4$. The effect is to reduce $\cap\ (T^*)$ and to increase $A(T^*)$; since however the new polygonal tree still is a member of $\mathfrak{L}(n)$, we have a contradiction with the extremal property of T^*. It follows that all the angles $p_{i-1} p_i p_{i+1}$ are equal, $i = 2, \ldots, k-1$, and therefore that all the points $p_1, \ldots, p_k$ lie on a circle, say C. Now $p_1 p_k$ is a diameter of C, for if $\angle p_1 p_2 p_k \neq \frac{1}{2}\pi$ we could increase the area of triangle $p_2 p_1 k$ by a suitable small rotation of $p_1 p_2$ about p_2. This is not so by the extremal property of T^*. Thus $p_1 p_k$ is a diameter of C. Direct calculation now leads to (34).

§ 6. Remarks

Although the arguments used in the three preceding paragraphs are both long and complicated, they do not completely solve the problems concerned. They fail to characterize completely the extremal figures. In each case we are able to give one extremal figure but out methods are such that we are unable to say whether or not the figure is unique. Our method is to classify some of the possible figures into classes which are not difficult to deal with and then to obtain the final result by an approximation argument. In Section 4 it is not surprising that we are unable to define all the extremal figures, since the one which we actually specify does not belong to any of the classes that we argue with, its convex cover is not a polygon. In Section 3 the extremal figure belongs to all these classes and is almost certainly unique. The methods used here are by no means exhausted. There are many other possible variations available and it may be possible to establish the uniqueness of the extremal set without using any really new ideas.

The argument in Section 3 could have been substantially simplified by the assumption $\mu(T) > \frac{1}{2}\Lambda(T)$ instead of $\mu(T) \geqslant \frac{1}{2}\Lambda(T)$. For the two key steps in the argument are to show that T has 3 end points and that every two end points lie on a pair of minimal support lines. Now (14) implies that T has at most 3 end points (if we assume $\mu(T) > \frac{1}{2}\Lambda(T)$) and the arguments given in 9 and 10 are unnecessary. Similarly (18) and (19) together imply the second key property of T without the complicated succeeding argument in 13. But of course such a procedure abandons any hope of finding all the extremal figures.

There are many other problems similar to those solved here. For example, we can consider the analogues of the problem of Section 1, 3, 4, 5 in R^3. The analogues of Section 5 in R^3 (i.e. to find the largest volume of the convex cover of a connected set of given length) are particularly interesting. The case when the connected set is restricted to be an arc, that is to say, the three dimensional analogues of Ulam's conjecture, has not been solved. It is likely that the solution is a certain equi-angular spiral (see Egerváry [4]), and, that unlike the situation in R^2, the solution of the connected set problem does not imply that of the arc problem.

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