ON THE REGULARITY OF THE SOLUTIONS OF BOUNDARY PROBLEMS

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0. Introduction

As is well known, the application of operator theory or calculus of variations to the study of differential equations leads to the question whether the solutions thus obtained are sufficiently smooth to be solutions in a classical sense. Rather complete results are known concerning the regularity of the solutions in the interior of their domain of existence (cf. Hörmander [6], Malgrange [9] and the references given there). The regularity at the boundary of solutions of boundary problems has been far less studied, although quite recently very important progress has been made (cf. Browder [1], Gusev [4], Lopatinski [8], Morrey-Nirenberg [10], Nirenberg [11]). The purpose of this paper is to give a complete description of the boundary conditions which give rise to regularity at the boundary, in the special case where the coefficients of the differential operators considered (in the interior and in the boundary conditions) are constant and the boundary is plane. This case can be studied essentially in the same way as the corresponding problem of interior regularity was studied by Hörmander [5], but considerable technical difficulties are added.

In the case of boundary conditions satisfying condition (a) of Theorem 3.3, it is easy to extend the study to the case of variable coefficients and curved boundaries. (In this case one can also admit overdetermined systems of differential operators and boundary operators.) Indeed, one can argue as in Morrey-Nirenberg [1] using a priori estimates for the case of constant coefficients obtained by Fourier transformations from estimates for ordinary differential operators. However, these results will not be developed here since the author has been informed by Professor Nirenberg that he, Agmon, Douglis and Schecter have also independently obtained the same theorems.

A classical prototype for the results in this article is contained in Schwarz' reflection principle. According to this principle, a function u satisfying the differential equation

$$\Delta u = 0 \tag{0.1}$$

in an open set Ω and vanishing on a plane piece ω of the boundary of Ω, can be extended as a solution of (0.1) across ω. Thus it is *analytic* in $\Omega \cup \omega$ since every solution of (0.1) is analytic, that is, it has a power series expansion in a neighbourhood of each point in this set.

In extending this result we shall consider solutions of a partial differential equation with constant coefficients

$$P(D)\, u = 0 \tag{0.2}$$

(for notations see section 2 or Hörmander [5]), which are defined in an open set Ω and on a plane piece ω of the boundary satisfy a number of boundary conditions with constant coefficients

$$Q_\nu(D)\, u = 0, \quad \nu = 1, \cdots \quad \text{on } \omega. \tag{0.3}$$

In order that the interpretation of these equations shall be elementary we assume that the derivatives of u of order $\leqslant k$ are continuous in $\Omega \cup \omega$, where k is the maximum order of $P(D)$ and $Q_\nu(D)$. Since the coefficients are constants there is in fact no real difficulty in extending the study to weak solutions, but for the sake of simplicity we shall not do so. Our purpose is to investigate the following two questions:

A) *What are the conditions on P and Q_ν in order that every solution of* (0.2), (0.3) *shall be infinitely differentiable in $\Omega \cup \omega$?*

B) *What are the conditions for these solutions to be analytic in $\Omega \cup \omega$?*

In studying A (B) it is natural to assume that the operator $P(D)$ is hypoelliptic (elliptic), that is, such that every solution of the equation $P(D)u=0$ in an open set is infinitely differentiable (analytic) there. Indeed, if $P(D)$ is not elliptic and ω is a bounded subset of a hyperplane, we can take Ω such that a solution of $P(D)u=0$ vanishes in a neighbourhood of ω but not in the whole of Ω (Hörmander [5], Theorem 3.2). Then u satisfies all boundary conditions (0.3) but is not analytic in Ω. Thus one must assume that $P(D)$ is elliptic in studying question B, at least if one wants a result independent of Ω as will be the case here. Similarly, it is natural to assume that $P(D)$ is hypoelliptic when studying A.

Algebraic characterizations of hypoelliptic and elliptic operators have been given by Hörmander [5] and Petrowsky [13]. $P(D)$ is *hypoelliptic* if and only if

$$\operatorname{Im}\zeta\to\infty \quad \text{when } \zeta\to\infty \text{ on the surface } P(\zeta)=0, \tag{0.4}$$

$P(D)$ is *elliptic* if and only if

$$P^0(\xi)\neq 0 \text{ for real } \xi\neq 0, \tag{0.5}$$

where $P^0(\xi)$ denotes the principal part of $P(\xi)$, that is, the homogeneous part of highest degree.

Thus in all that follows *we assume that* (0.4) *is fulfilled.* We also suppose that Ω is contained in one of the half spaces bounded by the plane through ω and denote by N an interior normal of this half space. This means that $\langle x-x_0, N\rangle>0$ if $x\in\Omega$ and $x_0\in\omega$. We shall consider the roots of the equation in τ

$$P(\xi+\tau N)=0, \tag{0.6}$$

where ξ is real. If τ is a *real* root, it follows from (0.4) that $\xi+\tau N$ belongs to the compact set in R^n defined by $P=0$. Let $\dot\xi$ be the residue class of ξ modulo $\{\tau N\}$. Then, if $\dot\xi$ is outside a compact set K in $R^n/\{\tau N\}$, equation (0.6) has no real roots. Now the roots are continuous functions of ξ, for the coefficient of the highest power of τ in (0.6) is independent of ξ (cf. Hörmander [5, p. 239]). Hence in each component of $\complement K$, the number of roots with positive imaginary part is constant. Assuming as we may that K is a sphere, $\complement K$ has 1 component if $n-1>1$ and 2 components if $n-1=1$. When $n=2$ we therefore add to our hypothesis (0.4) that the number of zeros with positive imaginary part is the same for $\dot\xi$ in the two components. We shall say that P is of (*determined*) *type* μ, if the number of zeros with positive imaginary part is μ for all $\dot\xi\in\complement K$; when $n>2$ all hypoelliptic operators are thus of determined type.

Example. When $n=2$ and $P(\xi)=\xi_1+i\xi_2$, $N=(0,1)$, we have one root with positive imaginary part if $\xi_1>0$ but none if $\xi_1<0$. Hence, although elliptic, P is not of determined type.

Remark. The above argument shows in particular that every elliptic operator in $n>2$ variables is of even order. This was also observed by Lopatinski [8].

We can now formulate our final hypothesis: *The number of boundary conditions* (0.3) *shall be* $\leqslant \mu$. Since we can add a finite number of identically satisfied boundary conditions ($Q_\nu=0$), we may and will assume that there are precisely μ boundary conditions. Our reason for this restriction has been that the problems A) and B) would otherwise be analogous to the problem of characterizing the overdetermined differential systems which only have infinitely differentiable solutions. The solution of this problem has been given quite recently by C. Lech but was not known when the results of this paper were obtained.

Summing up: Given a hypoelliptic (elliptic) differential operator $P(D)$ of determined type μ, we are going to characterize the systems (0.3) of μ boundary conditions such that the solutions of (0.2), (0.3) are infinitely differentiable (analytic) in $\Omega \cup \omega$.

DEFINITION 0.1. *The boundary conditions* (0.3) *are called hypoelliptic* (*elliptic*), *with respect to the hypoelliptic* (*elliptic*) *operator* $P(D)$, Ω *and* ω, *if all k times continuously differentiable solutions in* $\Omega \cup \omega$ *of* (0.2), (0.3) *are infinitely differentiable* (*analytic*) *in* $\Omega \cup \omega$. (1)

We recall that k denotes the maximum of the orders of $P(D)$ and $Q_\nu(D)$. The assumption that $u \in C^k$ can easily be relaxed, as mentioned previously.

The plan of the paper is as follows. In section 1 we present some basic facts concerning ordinary differential equations, and in section 2 some algebraic preliminaries. In section 3 we then state our main result, the algebraic characterization of hypoelliptic and elliptic boundary conditions and discuss some special cases. The necessity and sufficiency of these conditions is proved in sections 4 and 5–6, respectively. In doing so we also study inhomogeneous boundary problems.

1. Preliminaries concerning ordinary differential equations

Let
$$k(\delta)=\delta^\mu+a_{\mu-1}\delta^{\mu-1}+\cdots+a_0, \tag{1.1}$$
where $\delta=-i\,d/dt$, be an ordinary differential operator with constant coefficients and

(1) This terminology differs from that of BROWDER [1].

order μ. As is well known, the solutions of the equation $k(\delta)u=0$ are the linear combinations of the exponential solutions $G(t)e^{it\tau}$, where τ is a zero of the polynomial

$$k(\tau)=\tau^{\mu}+a_{\mu-1}\tau^{\mu-1}+\cdots+a_0$$

of higher order than the degree of the polynomial G.

Let $q_\nu(\delta)$, $\nu=1, \ldots, \mu$, be some other ordinary differential operators. We are going to determine a condition in order that

$$k(\delta)u=0; \qquad (q_\nu(\delta)u)(0)=0, \quad \nu=1, \ldots, \mu, \tag{1.2}$$

shall imply that $u=0$. The equations

$$k(\delta)u=0; \qquad (q_\nu(\delta)u)(0)=\psi_\nu, \quad \nu=1, \ldots, \mu, \tag{1.3}$$

then have one and only one solution for arbitrary complex ψ_ν, and we are going to give a formula for this solution.

Assume for a moment that the zeros $\tau_1, \ldots, \tau_\mu$ are all *different*. Then the solutions of $k(\delta)u=0$ are the functions

$$u(t)=\sum_1^{\mu} C_j e^{i\tau_j t}$$

with constant C_j. In order that u shall satisfy (1.2) we must have

$$\sum_1^{\mu} C_j q_\nu(\tau_j)=0, \ \ \nu=1, \ldots, \mu.$$

Thus one can find a non trivial solution of (1.2) if and only if $\det q_\nu(\tau_j)=0$.

In order to study the case of coinciding zeros also, we introduce a rational function of the indeterminates $\tau_1, \ldots, \tau_\mu$ defined by

$$R(k; q_1, \ldots, q_\mu)=\det q_\nu(\tau_j)/\prod_{l<j}(\tau_j-\tau_l).$$

Since it is obvious that each factor in the denominator divides the numerator, the right hand side is a polynomial in the variables τ_j and is thus defined also in the case of coinciding zeros. Since it is a symmetric function of these variables, it can be expressed as a polynomial in the coefficients of k. Hence:

$R(k; q_1, \ldots, q_\mu)$ is a polynomial in the coefficients of $k, q_1, \ldots, q_\mu$, which is linear and antisymmetric in $q_1, \ldots, q_\mu$ and vanishes when k is a factor of some q_ν.

It is easy to see that these properties characterize R up to a function which is independent of q_ν. Furthermore, one can verify that R vanishes if and only if some linear combination of the polynomials q_ν,

$$\sum_1^\mu a_\nu q_\nu$$

where all a_ν do not vanish, has k as a factor. This fact connects our results with those of Agmon, Douglis, Nirenberg and Schecter.

For future reference we also observe that R does not change if we simultaneously make a translation of k and all q_ν, and only multiplies by a constant factor if we make some other linear transformation of the independent variable τ.

The important role of this function here is due to the following theorem.

THEOREM 1.1. *A necessary and sufficient condition in order that* (1.2) *shall possess a non trivial solution is that*

$$R(k;\, q_1, \ldots, q_\mu) = 0. \tag{1.4}$$

Proof. In the case where the zeros of k are simple, the result is already proved. Now suppose that $\tau_1, \ldots, \tau_r$ are different zeros with multiplicities $\mu_1, \ldots, \mu_r$ and that

$$\mu_1 + \cdots + \mu_r = \mu$$

so that there are no other zeros. The solutions of $k(\delta)u = 0$ are then

$$u = \Sigma\, C_{js}(it)^s e^{i\tau_j t},$$

where C_{js} are constants and the summation is extended over

$$0 \leqslant s < \mu_j,\ 1 \leqslant j \leqslant r.$$

Application of Leibniz' formula gives (cf. Hörmander [5, p. 177])

$$(q_\nu(\delta)u)(0) = \Sigma\, C_{js} q_\nu^{(s)}(\tau_j).$$

Hence (1.2) has a non trivial solution if and only if

$$\begin{vmatrix} q_1(\tau_1) & q_1'(\tau_1) & \ldots & q_1^{(\mu_1-1)}(\tau_1) & q_1(\tau_2) & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_\mu(\tau_1) & q_\mu'(\tau_1) & \ldots & q_\mu^{(\mu_1-1)}(\tau_1) & q_\mu(\tau_2) & \ldots \end{vmatrix} = 0. \tag{1.5}$$

On the other hand, noting that R is a continuous function of $\tau_1, \ldots, \tau_\mu$ and passing to the limit from the case where all μ zeros are different, using Taylor series expansions, it is easy to see that in the present case the function R is precisely the determinant in (1.5) divided by

$$\prod_{j=1}^{r} \prod_{s<\mu_j} s! \prod_{1\leq k<j\leq r} (\tau_j - \tau_k)^{\mu_j \mu_k}.$$

This shows that (1.4) is equivalent to (1.5), which completes the proof.

When solving the inhomogeneous equations (1.3) we shall again need to consider expressions defined by

$$R(k; f_1, \dots, f_\mu) = \det f_\nu(\tau_j)/\prod_{k<j}(\tau_j - \tau_k) \tag{1.6}$$

when the zeros τ_j are different and by continuity otherwise. However, all f_j will not be polynomials, which requires some preliminary study of expressions of this form.

We first recall some notions from difference calculus. If f is an analytic function of a complex variable τ, its divided differences are defined as follows (cf. Nörlund [12]), when all τ_i are different:

$$f(\tau_1, \tau_2) = (f(\tau_1) - f(\tau_2))/(\tau_1 - \tau_2), \dots$$
$$f(\tau_1, \dots, \tau_n) = (f(\tau_1, \dots, \tau_{n-1}) - f(\tau_2, \dots, \tau_n))/(\tau_1 - \tau_n).$$

It is easy to show that $f(\tau_1, \dots, \tau_n)$ is a symmetric function of $\tau_1, \dots, \tau_n$. Assume for simplicity that all τ_i are situated within a Jordan curve C and that f is analytic there. It is then immediately established that

$$f(\tau_1, \dots, \tau_n) = (2\pi i)^{-1} \int_C f(z)\, dz/(z-\tau_1) \dots (z-\tau_n).$$

This formula has a sense even for coinciding zeros and we take it as a definition in that case; $f(\tau_1, \dots, \tau_n)$ is then an analytic function of all its variables.

From another formula for the divided differences one obtains the following useful estimate (cf. Nörlund [12, p. 16])

$$|f(\tau_1, \dots, \tau_n)| \leq \frac{1}{(n-1)!} \sup_{z\in K} |f^{(n-1)}(z)|, \tag{1.7}$$

where K is the convex hull of the points τ_i, provided that f is analytic in a neighbourhood of K.

By subtracting columns in the determinant defining $R(k; f_1, \dots, f_\mu)$ and using the definition of the divided differences, one immediately obtains

$$R(k; f_1, \dots, f_\mu) = \det f_\nu(\tau_1, \dots, \tau_j), \quad \nu, j = 1, \dots, \mu, \tag{1.8}$$

when all τ_i are different. If all f_ν are analytic, we thus obtain a definition of R also in the case of multiple zeros, and R becomes an analytic function of all τ_i.

Applying the estimate (1.7) to (1.8) we can get estimates of R. Thus if K is the convex hull of the zeros of k we have

$$|R(k; f_1, \ldots, f_\mu)| \leq \prod_{\nu=1}^{\mu} \left(\sum_{j=0}^{\mu-1} \sup_{z \in K} |f_\nu^{(j)}(z)|/j! \right). \tag{1.9}$$

We shall now give the solution of (1.3).

Theorem 1.2. *If* $R(k; q_1, \ldots, q_\mu) \neq 0$, *equations* (1.3) *have one and only one solution and this is given by*

$$u(t) = \sum_{1}^{\mu} \psi_\nu R(k; q_1, \ldots, q_{\nu-1}, e^{it\tau}, q_{\nu+1}, \ldots, q_\mu)/R(k; q_1, \ldots, q_\mu). \tag{1.10}$$

Proof. The existence and uniqueness of the solution follows from Theorem 1.1. Since the right hand side of (1.10) is a continuous function of $\tau_1, \ldots, \tau_\mu$ as long as $R(k; q_1, \ldots, q_\mu) \neq 0$, it is sufficient to prove (1.10) when all τ_i are different. In that case we shall determine u by means of the equations

$$u = \sum_{1}^{\mu} a_j e^{i\tau_j t}, \quad \psi_\nu = \sum_{1}^{\mu} a_j q_\nu(\tau_j), \quad \nu = 1, \ldots, \mu.$$

Considering this as a homogeneous system in the variables a_j and 1 and using the definition of R, we immediately obtain (1.10).

Next assume that $k(\tau)$ is a factor of a polynomial $p(\tau)$,

$$p(\tau) = \tau^\sigma + \text{lower order terms},$$

and that *the zeros of* $k(\tau)$ *and* $p(\tau)/k(\tau)$ *have positive and negative imaginary parts, respectively*. Denoting by u an infinitely differentiable function vanishing for large t, we set

$$p(\delta) u = f; \qquad (q_\nu(\delta) u)(0) = \psi_\nu, \quad \nu = 1, \ldots, \mu, \tag{1.11}$$

and are going to give a formula for u in terms of f and ψ_ν, assuming that *the degree of* p *is greater than the degree of* q_ν for all $\nu = 1, \ldots, \mu$.

First write

$$g_0(t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{it\tau}/p(\tau)\, d\tau. \tag{1.12}$$

This integral is absolutely convergent if $\sigma \geq 2$ and for $\sigma = 1$ it converges when $t \neq 0$. The function $g_0(t-s)$ is a fundamental solution of the differential operator $p(\delta)$ with pole at s. We shall modify it in order to obtain a fundamental solution $g(t, s)$ satisfying

$$q_\nu(\delta)\, g(t, s)_{t=0} = 0, \quad \nu = 1, \ldots, \mu.$$

According to Theorem 1.2 the function

$$g(t, s) = g_0(t-s) - \sum_1^\mu (q_\nu(\delta)\, g_0)(-s)\, h_\nu(t), \tag{1.13}$$

where for the sake of brevity we have written

$$h_\nu(t) = R(k; q_1, \ldots, q_{\nu-1}, e^{it\tau}, q_{\nu+1}, \ldots, q_\mu)/R(k; q_1, \ldots, q_\mu), \tag{1.14}$$

satisfies the desired boundary conditions, and the compensating term w satisfies as a function of t the equation $k(\delta)\, w = 0$, hence $p(\delta)\, w = 0$. Thus writing

$$u_1(t) = \int_0^\infty g(t, s)\, f(s)\, ds$$

we get

$$p(\delta)\, u_1(t) = p(\delta) \int_0^\infty g_0(t-s)\, f(s)\, ds = f(t), \quad t > 0.$$

We also obtain $(q_\nu(\delta)\, u_1)(0) = 0$ for $\nu = 1, \ldots, \mu$. With $u_2 = u - u_1$ it therefore follows that

$$p(\delta)\, u_2 = 0, \qquad (q_\nu(\delta)\, u_2)(0) = \psi_\nu, \quad \nu = 1, \ldots, \mu.$$

The first equation can be replaced by $k(\delta)\, u_2 = 0$, for u_2 is bounded when $t > 0$, since u and u_1 are bounded, and $p(\tau)/k(\tau)$ has all zeros in the lower half plane. Hence u_2 is given by Theorem 1.2, and we obtain the formula

$$u(t) = \int_0^\infty g(t, s)\, f(s)\, ds + \sum_1^\mu \psi_\nu\, h_\nu(t), \tag{1.15}$$

where g and h_ν are given by (1.13) and (1.14).

When using this formula in section 5 we shall need some rough estimates of the kernels g and h_ν. They will be consequences of the following theorem.

THEOREM 1.3. *If the zeros of $p(\tau)$ satisfy the inequalities*

$$|\tau| \leqslant \Lambda_1, \ |\mathrm{Im}\, \tau| \geqslant 1 + \Lambda_0, \ \Lambda_0 > 0,$$

then for $t \neq 0$ we have

$$|g_0^{(j)}(t)| \leqslant 2^{\sigma+j} \Lambda_1^j e^{-\Lambda_0 |t|}, \tag{1.16}$$

where σ is the degree of p and g_0 is defined by (1.12).

Proof. If $\sigma = 1$ we have $p(\tau) = (\tau - \lambda)$, and assuming for example that $\mathrm{Im}\, \lambda < 0$, we get

$$g_0(t)=0 \text{ if } t>0, \qquad g_0(t)=-i\,e^{it\lambda} \text{ if } t<0.$$

Thus the result is true when $\sigma=1$.

Next assume that $j=0$, $\sigma>1$. Moving the line of integration in (1.12) we obtain

$$g_0(t)=(2\pi)^{-1}\int e^{it(\tau\pm i\Lambda_0)}/p(\tau\pm i\Lambda_0)\,d\tau.$$

Now the integrals $\int d\tau/|p(\tau\pm i\Lambda_0)|$ are $\leqslant\pi$. In fact, if λ_j are the zeros of $p(\tau\pm i\Lambda_0)$, we have $|\operatorname{Im}\lambda_j|\geqslant 1$, hence the integrand can be estimated by $|\tau-\lambda_1|^{-1}|\tau-\lambda_2|^{-1}\leqslant(|\tau-\lambda_1|^{-2}+|\tau-\lambda_2|^{-2})/2$, and if we calculate the integral of the right hand side, the assertion follows. Thus (1.16) holds when $j=0$.

Now we prove the result in the general case, assuming that it has already been proved for derivatives of order $<j$ when p is of order $<\sigma$. Let λ be a zero of $p(\tau)$. Then

$$-i\,g_0'-\lambda g_0=(\delta-\lambda)\,g_0$$

is the fundamental solution (1.12) belonging to the operator $p(\delta)/(\delta-\lambda)$, and hence by hypothetis

$$|\delta^{s+1}g_0-\lambda\,\delta^s g_0|\leqslant 2^{\sigma-1+s}\Lambda_1^s\,e^{-\Lambda_0|t|},\ s<j.$$

Multiplying this inequality by λ^{j-1-s} and adding for $s=1,\ldots,j-1$, we obtain

$$|\delta^j g_0-\lambda^j g_0|\leqslant\Lambda_1^{j-1}e^{-\Lambda_0|t|}\sum_0^{j-1}2^{\sigma-1+s}\leqslant\Lambda_1^{j-1}e^{-\Lambda_0|t|}\,2^{\sigma-1+j}.$$

Hence, using the estimate of g_0 already proved and the fact that $\Lambda_1>1$,

$$|\delta^j g_0|\leqslant\Lambda_1^j\,e^{-\Lambda_0|t|}\,(1+2^{\sigma-1+j}),$$

which implies (1.16). The proof is complete.

We shall finally prove an inequality for the solutions of (1.1), which in a somewhat weaker form will be useful in section 4.

THEOREM 1.4. *There is a constant γ depending only on μ such that, if u is a solution of an equation $k(\delta)\,u=0$ where all zeros of $k(\tau)$ have non negative imaginary parts, we have*

$$|u(a)|\leqslant\gamma\,a^{-1}\int_0^a|u(t)|\,dt,\quad a>0, \tag{1.17}$$

$$\int_0^b|u(t)|\,dt\leqslant(b/a)^\gamma\int_0^a|u(t)|\,dt,\quad 0<a\leqslant b. \tag{1.18}$$

Proof. First note that (1.18) follows from (1.17). Indeed, if we write

$$I(b)=\int_0^b |u(t)|\,dt,$$

(1.17) gives $I'(b)\leqslant \gamma b^{-1} I(b)$. Dividing by $I(b)$ and integrating from a to b, we obtain (1.18).

Next note that to prove (1.17) it is sufficient to assume that $a=1$, since the general case reduces to this by means of the substitution $t=as$. One may furthermore assume in the proof that $k(0)=0$. For let τ_0 be one of the zeros of $k(\tau)$ with the smallest imaginary part and write $k_1(\tau)=k(\tau+\tau_0)$, $u_1(t)=u(t)\,e^{i(a-t)\tau_0}$. Then the zeros of k_1 have non negative imaginary parts, too, $k_1(0)=0$, and from $k(\delta)u=0$ it follows that $k_1(\delta)u_1=0$. Since $|u_1(a)|=|u(a)|$ but $\int_0^a |u_1(t)|\,dt\leqslant \int_0^a |u(t)|\,dt$, the inequality (1.17) follows from the corresponding one where u is replaced by u_1.

In the proof we may also assume that the theorem has already been proved for solutions of differential equations of order lower than μ, for it is trivial when $\mu=1$.

Denote the zeros of $k(\tau)$ by $\tau_1, \dots, \tau_\mu$ and set

$$M(\tau_1, \dots, \tau_\mu)=\sup |u(1)|/\int_0^1 |u(t)|\,dt,$$

where u varies over the solutions of the (fixed) equation $k(\delta)u=0$. We have to prove that M is bounded when $\operatorname{Im}\tau_j\geqslant 0$, $j=1, \dots, \mu$.

LEMMA 1.1. *$M(\tau_1, \dots, \tau_\mu)$ is a continuous function of $\tau_1, \dots, \tau_\mu$.*

Proof. Denote by $\varphi_j(t;\tau_1, \dots, \tau_\mu)$ the divided differences of the function $e^{it\tau}$ at the arguments $\tau_1, \dots, \tau_j$; $1\leqslant j\leqslant\mu$. Considering for example Cauchy's problem for the equation (1.1) and using (1.10) and (1.8), we find that the functions φ_j constitute a basis for the solutions of (1.1) regardless whether the zeros are multiple or not. Furthermore the functions φ_j are continuous functions of all variables. Now

$$M(\tau_1, \dots, \tau_\mu)=\sup \Big|\sum_1^\mu a_j\varphi_j(1;\tau_1, \dots, \tau_\mu)\Big|/\int_0^1\Big|\sum_1^\mu a_j\varphi_j(t;\tau_1, \dots, \tau_\mu)\Big|\,dt$$

and the supremum obviously does not change if we add the restriction $\sum_1^\mu |a_j|=1$. Since the supremum is then taken over a fixed compact set and the denominator does not vanish, the continuity is obvious.

From Lemma 1.1 it follows that M is bounded when the absolute values of all τ_j are $\leqslant \mu$. It therefore only remains to consider the case where some zero has a larger absolute value. According to Lemma 1.1 it is enough to consider the case of simple zeros τ_j and as we have remarked above we may assume that some of them equals 0. Then there exists an integer $\nu < \mu$ such that there is no zero of k in the annulus $\nu < |\tau| < \nu + 1$. For if each of these contained a zero of k we would get $\mu + 1$ zeros altogether, counting the one at 0 and the one with absolute value $> \mu$, which is impossible. We now decompose a solution of $k(\delta)u = 0$ in the form

$$u = u_1 + u_2,$$

where $u_1 (u_2)$ is the sum of those exponential components of u with exponent $\leqslant \nu\ (\geqslant \nu + 1)$ in absolute value. We shall prove that with a constant C only depending on μ we have

$$\int_0^{\frac{1}{2}} |u_i(t)|\, dt \leqslant C \int_0^1 |u(t)|\, dt, \quad i = 1, 2. \tag{1.19}$$

Since u_i is a solution of a differential equation of order $< \mu$ of the type described in Theorem 1.4, and we have assumed that the theorem is already proved in that case, it follows from (1.17) and (1.18) that, with another constant C_1 depending only on μ,

$$|u_i(1)| \leqslant C_1 \int_0^1 |u(t)|\, dt,$$

and the theorem will be proved.

In order to prove (1.19) we argue in the following way. Let $\varphi(x)$ be a continuous function with support in $(-\frac{1}{2}, 0)$ and set

$$U(t) = u * \varphi(t) = \int u(t-s)\, \varphi(s)\, ds.$$

Then
$$\int_0^{\frac{1}{2}} |U(t)|\, dt \leqslant \int_0^1 |u(t)|\, dt \max |\varphi(s)|. \tag{1.20}$$

Let $\hat{\varphi}$ be the Fourier-Laplace transform of φ. If

$$\begin{aligned} &\hat{\varphi}(\tau) = 0 \text{ when } k(\tau) = 0 \text{ and } |\tau| \geqslant \nu + 1, \\ &\hat{\varphi}(\tau) = 1 \text{ when } k(\tau) = 0 \text{ and } |\tau| \leqslant \nu, \end{aligned} \tag{1.21}$$

we have $U = u_1$. To prove (1.19) with $i = 1$ (and hence with $i = 2$) we thus only have to prove that one can find a function φ having these properties and which is bounded by a constant depending on μ only.

Take a *fixed* function $\psi \in C^\infty$ with compact support in $(-\frac{1}{2}, 0)$ such that $\hat{\psi}(\tau) \neq 0$ when $|\tau| \leqslant \mu$. Write

$$k_1(\tau) = \prod_{|\tau_j| \leqslant \nu} (\tau - \tau_j), \qquad k_2(\tau) = \prod_{|\tau_j| \geqslant \nu+1} (1 - \tau/\tau_j)$$

and, with the operator $h(\delta)$, of order lower than that of k_1, still to be determined,

$$\varphi = h(\delta)\, k_2(\delta)\, \psi.$$

Then

$$\hat{\varphi}(\tau) = h(\tau)\, k_2(\tau)\, \hat{\psi}(\tau),$$

so that (1.21) will be fulfilled if and only if

$$h(\tau) = 1/k_2(\tau)\,\hat{\psi}(\tau) \quad \text{when } k_1(\tau) = 0. \tag{1.22}$$

This condition determines the polynomial $h(\tau)$. Moreover, if $\tau_1, \ldots, \tau_r$ are the zeros of k_1 and we write $F(\tau) = 1/k_2(\tau)\,\hat{\psi}(\tau)$, Newton's interpolation formula (Nörlund [12, p. 11]) gives

$$h(\tau) = \sum_{j=1}^{r} F(\tau_1, \ldots, \tau_j)(\tau - \tau_1) \ldots (\tau - \tau_{j-1}).$$

Now all τ_j with $1 \leqslant j \leqslant r$ have absolute values $\leqslant \mu$, and to estimate the divided differences of F we just have to use (1.7), noting that if τ_j is a zero of $k_2(\tau)$ we have $|\tau_j| \geqslant 1$ and

$$|(1 - \tau/\tau_j)| \geqslant (1 - (\mu - 1)/\mu) = 1/\mu \text{ if } |\tau| \leqslant \nu.$$

Since the estimates of the coefficients of h thus obtained depend on μ only, and the inequality $|\tau_j| \geqslant 1$ is valid for every zero of $k_2(\delta)$, the inequality (1.19) and hence the theorem follows.

2. Notations and algebraic preliminaries

In order to give our results an invariant form we shall in this and the next section use a formalism which avoids the use of coordinate systems. Thus let G be a real vector space of dimension n and G^* its complex dual space, i.e. the space of all complex linear forms on G. We shall denote the elements of G by $x, y, \ldots$ and those of G^* by Greek letters $\xi, \eta, \zeta \ldots$. Usually ξ and η will denote real elements in G^*, i.e. real linear forms. If $P(\zeta)$ is a polynomial in G^* we denote by $P(D)$ the differential operator acting on the functions in G such that

$$P(D)\, e^{i\langle x, \zeta\rangle} = P(\zeta)\, e^{i\langle x, \zeta\rangle};$$

if x^k and ζ_k are coordinates in G and G^* with respect to dual bases, we obtain $P(D)$ by replacing ζ_k by $-i\,\partial/\partial x^k$ in $P(\zeta)$.

In studying the problem sketched in the introduction we shall let Ω be an open set in G and ω a part of its boundary which is an open set in a hyperplane F. The dual space F^* of F is a quotient space of G^*,

$$F^* \cong G^*/F^0,$$

where $F^0 \subset G^*$ is the orthogonal space of F. In fact, if $\zeta \in G^*$, the restriction of the linear form $\langle x, \zeta \rangle$ to F defines an element $\dot{\zeta}$ of F^*, and this element is 0 if and only if ζ is orthogonal to F. Since F is a hyperplane, F^0 is 1-dimensional. By the normal N of F we shall mean one of the real elements in F^0; we assume that the half space

$$G_+ = \{x \in G;\ \langle x, N \rangle \geqslant 0\}$$

contains Ω.

We choose once for all a Euclidean norm in G. In F, G^* and F^* we use the norms obtained by restriction and duality respectively. The norm in F^* is then the quotient norm of that in G^*. For future reference we also note that obviously $|\mathrm{Re}\,\dot{\zeta}| \leqslant |\dot{\zeta}|$.

Now let $P(D)$ be hypoelliptic and of determined type μ (cf. section 0). We shall denote by A the set of all $\zeta \in G^*$ such that the equation

$$P(\zeta + \tau N) = 0 \tag{2.1}$$

has precisely μ roots with positive imaginary part and none that is real. The coefficient of the highest power of τ is independent of ζ (Hörmander [1, p. 239]) so we may assume that it equals 1. Obviously A is open and by hypothesis a real $\xi \in G^*$ is in A if $\dot{\xi}$ belongs to a suitable neighbourhood of infinity in F^*. We shall now estimate the size of A more precisely.

THEOREM 2.1. *Suppose that $P(D)$ is elliptic and of determined type μ. Then there is a constant M such that A contains all ζ satisfying*

$$|\mathrm{Re}\,\dot{\zeta}| \geqslant M(1 + |\mathrm{Im}\,\zeta|). \tag{2.2}$$

Proof. The theorem is a consequence of the following lemma.

LEMMA 2.1. *Suppose that $P(D)$ is elliptic. Then there is a constant M such that*

$$P(\zeta) \neq 0 \quad \text{if} \quad |\mathrm{Re}\,\zeta| \geqslant M(1 + |\mathrm{Im}\,\zeta|). \tag{2.3}$$

To prove that Theorem 2.1 follows from Lemma 2.1 we first note that if τ is real and (2.2) is fulfilled, we have

$$|\operatorname{Re}(\zeta+\tau N)| \geqslant |\operatorname{Re}\zeta| \geqslant M(1+|\operatorname{Im}\zeta|) = M(1+|\operatorname{Im}(\zeta+\tau N)|),$$

hence $P(\zeta+\tau N) \neq 0$ in virtue of (2.3). Thus (2.1) has no real root if (2.2) is valid and hence the number of roots of (2.1) with positive imaginary part is constant in each component of the set defined by (2.2). Now each component of this set contains real points with arbitrarily large absolute values which proves Theorem 2.1.

Proof of Lemma 2.1. Let $P = P_m + P_{m-1} + \cdots + P_0$ be the decomposition of P in homogeneous parts, the degrees being indicated by the subscripts. By hypothesis we have

$$P_m(\zeta) \neq 0 \text{ when } \zeta \neq 0 \text{ is real.}$$

Therefore $P_m(\zeta)$ has a positive lower bound in some complex neighbourhood of the real unit sphere, thus for some positive ε and c we have

$$|P_m(\zeta)| \geqslant c \quad \text{if } |\zeta| = 1 \quad \text{and } |\operatorname{Im}\zeta| \leqslant \varepsilon |\operatorname{Re}\zeta|.$$

Since P_m is homogeneous this yields

$$|P_m(\zeta)| \geqslant c|\zeta|^m \text{ if } |\operatorname{Im}\zeta| \leqslant \varepsilon|\operatorname{Re}\zeta|.$$

Estimating the lower order terms in $P(\zeta)$ in an obvious fashion we now get with another constant c_1

$$|P(\zeta)| \geqslant c|\zeta|^m - c_1(|\zeta|^{m-1} + \cdots + 1) \text{ if } |\operatorname{Im}\zeta| \leqslant \varepsilon|\operatorname{Re}\zeta|.$$

Hence $P(\zeta) \neq 0$ if $|\zeta| \geqslant c_2$ and $|\operatorname{Im}\zeta| \leqslant \varepsilon|\operatorname{Re}\zeta|$, and the lemma follows with $M = \max(\varepsilon^{-1}, c_2)$.

Remark. Conversely, it is easy to see that $P(D)$ is elliptic if (2.3) is valid for some constant M (cf. Hörmander [5, p. 217]). Thus (2.3) gives an alternative definition of an elliptic operator, which is closely related to the characterization of hypoelliptic operators given by (0.4).

THEOREM 2.2. *Suppose that $P(D)$ is hypoelliptic and of determined type μ. Then, given any number B, there is a number B' such that A contains all ζ satisfying*

$$|\operatorname{Im}\zeta| \leqslant B, \qquad |\operatorname{Re}\zeta| \geqslant B'. \tag{2.4}$$

Proof. This is merely a rephrasing of the characterization (0.4) of hypoelliptic operators. Indeed, (0.4) means that there is a number B' such that

$$P(\zeta) \neq 0 \text{ if } |\operatorname{Im} \zeta| \leqslant B \text{ and } |\operatorname{Re} \zeta| \geqslant B'.$$

(2.4) follows from this fact in the same way as (2.2) followed from (2.3). We do not repeat the argument.

We shall also consider the projection $\dot{A}$ of A into F^*. This is an open set. In the case of an elliptic operator $P(D)$, Theorem 2.1 means that $\dot{A}$ contains all $\dot{\zeta}$ with

$$|\operatorname{Re} \dot{\zeta}| \geqslant M(1 + |\operatorname{Im} \dot{\zeta}|). \tag{2.5}$$

In the case of a hypoelliptic operator, Theorem 2.2 means that $\dot{A}$ contains all $\dot{\zeta}$ with

$$|\operatorname{Im} \dot{\zeta}| \leqslant B, \quad |\operatorname{Re} \dot{\zeta}| \geqslant B'. \tag{2.6}$$

When $\zeta \in A$ we denote by $\tau_1, \ldots, \tau_\mu$ the zeros of $P(\zeta + \tau N)$ with positive imaginary part and set

$$k_\zeta(\tau) = \prod_1^\mu (\tau - \tau_j). \tag{2.7}$$

LEMMA 2.2. *The coefficients of k_ζ are analytic functions of ζ* when $\zeta \in A$.

Proof. This lemma is classical (cf. Goursat [1, pp. 289–290]).

Let us write

$$q_\zeta^\nu(\tau) = Q_\nu(\zeta + \tau N), \tag{2.8}$$

and consider the function

$$\zeta \to R(k_\zeta; q_\zeta^1, \ldots, q_\zeta^\mu), \ \zeta \in A. \tag{2.9}$$

If ζ and ζ' are in A and $\zeta - \zeta'$ is proportional to N, this function has the same value at ζ and at ζ', since k_ζ, q_ζ^ν will differ from $k_{\zeta'}$, $q_{\zeta'}^\nu$ by the same translation. Hence (2.9) is a function of $\dot{\zeta}$ only. We shall denote it by $C(\dot{\zeta})$, thus

$$C(\dot{\zeta}) = R(k_\zeta; q_\zeta^1, \ldots, q_\zeta^\mu). \tag{2.10}$$

We shall call $C(\dot{\zeta})$ the *characteristic function of the boundary problem*. In virtue of Lemma 2.2 it is an analytic branch of an algebraic function.

If $P(D)$ is of type 0, we shall have no boundary conditions at all and define $C(\dot{\zeta}) = 1$ everywhere.

In section 5 we shall need a rough estimate of the zeros of $P(\zeta + \tau N)$.

LEMMA 2.3. *There are constants C and M such that all zeros of $P(\zeta + \tau N)$ satisfy the inequality*

$$|\tau| < C(|\zeta|^M + 1). \tag{2:11}$$

Proof. Since the coefficient of the highest power of τ is independent of ζ, this follows immediately from any estimate of the zeros of a polynomial. For instance, if we write

$$P(\zeta+\tau N)=\tau^{\sigma}+a_{\sigma-1}\tau^{\sigma-1}+\cdots+a_0,$$

the coefficients a_j are polynomials in ζ, and the zeros satisfy

$$|\tau|\leqslant 1+\sum_{0}^{\sigma-1}|a_j|.$$

3. Characterization of elliptic and hypoelliptic boundary problems

The main results of this paper are the following two theorems.

THEOREM 3.1. *A necessary and sufficient condition for the boundary conditions* (0.3) *to be hypoelliptic with respect to the hypoelliptic operator $P(D)$ of determined type μ, Ω and ω, is that*

$$\text{Im}\ \dot{\zeta}\to\infty \quad \textit{if} \quad \dot{\zeta}\to\infty \quad \textit{in}\ \dot{A} \quad \textit{satisfying}\quad C(\dot{\zeta})=0. \tag{3.1}$$

The analogy between this condition and the condition (0.4) for the hypoellipticity of an operator is obvious. Note that the only geometric property of Ω and ω which is involved in (3.1) is the direction of the interior normal of Ω on ω.

THEOREM 3.2. *A necessary and sufficient condition for the boundary conditions* (0.3) *to be elliptic with respect to the elliptic operator $P(D)$ of determined type μ, Ω and ω, is that with some constant M*

$$C(\dot{\zeta})\neq 0 \quad \textit{when} \quad |\text{Re}\ \dot{\zeta}|\geqslant M(1+|\text{Im}\ \dot{\zeta}|); \tag{3.2}$$

M may be assumed so large that all $\dot{\zeta}$ satisfying the latter condition are in $\dot{A}$.

This is clearly analogous to the characterization of elliptic operators given by Lemma 2.1. The following theorem connects it with the condition (0.5) (cf. also the results of Petrowsky [13] concerning elliptic systems).

THEOREM 3.3. *Let Q_ν^0 and P^0 be the principal parts of Q_ν and P, which we assume elliptic. Let C^0 be the characteristic function of the boundary problem defined by P^0 and Q_ν^0. Then*

(a) *If $C^0(\dot{\xi})\neq 0$ for real $\dot{\xi}\neq 0$, the boundary conditions Q_ν are elliptic with respect to P.*

(b) *If the boundary conditions Q_ν are elliptic with respect to P and $n>2$, we have either $C^0=0$ identically or $C^0(\dot{\xi})\neq 0$ for real $\dot{\xi}\neq 0$.*

Proof. (a) Denote the degree of Q_ν by m_ν. Then, as is immediately verified, $C^0(\dot{\xi})$ is a homogeneous function of $\dot{\xi}$ of order

$$\mathrm{M} = m_1 + \cdots + m_\mu - 1 - 2 - \cdots - (\mu - 1).$$

Clearly it will be bounded from below in a complex neighbourhood of the real unit sphere. From this the result follows easily if we argue in the same way as in the proof of Lemma 2.1.

(b) Let $\dot{\xi} \neq 0$ be real and such that $C^0(\dot{\xi}) = 0$. We have to prove that C^0 must then vanish identically. Let $\dot{\eta}$ be a real vector and note that

$$s^{-\mathrm{M}}\, C(s(\dot{\xi} + w\,\dot{\eta})) \to C^0(\dot{\xi} + w\,\dot{\eta}) \quad \text{when } s \to \infty \text{ and is real,} \tag{3.3}$$

provided that $|w|$ is sufficiently small (cf. (a) above). Now by assumption the function on the left hand side of (3.3) is analytic in w and is $\neq 0$ if s is real and positive and

$$s(|\dot{\xi}| - |\dot{\eta}|\,|\mathrm{Re}\, w|) \geqslant M(1 + s\,|\dot{\eta}|\,|\mathrm{Im}\, w|),$$

hence if

$$|w| \leqslant (|\dot{\xi}| - M s^{-1})/(1 + M)\,|\dot{\eta}|.$$

It thus follows that the limit $C^0(\dot{\xi} + w\,\dot{\eta})$ when $s \to \infty$ is either identically zero or never zero when $|w| < |\dot{\xi}|/(1 + M)\,|\dot{\eta}|$. But by assumption this function vanishes when $w = 0$, so that it must vanish in the whole circle. If $|\dot{\eta}| < |\dot{\xi}|/(1 + M)$, the circle contains $w = 1$ and we obtain $C^0(\dot{\xi} + \dot{\eta}) = 0$. But since C^0 is analytic and vanishes in a neighbourhood of $\dot{\xi}$, it must vanish identically, which completes the proof.

The proof of Theorems 3.1 and 3.2 will be given in sections 4–6. In this section we shall only illustrate the results with a few examples.

Example 1. Let the boundary conditions be the Dirichlet conditions

$$\partial^\nu u / \partial T^\nu = 0 \text{ in } \omega, \quad \nu = 0, 1, \ldots, \mu - 1, \tag{3.4}$$

where T is a direction transversal to ω, i.e. $\langle T, N \rangle \neq 0$. A simple computation shows that $C(\dot{\zeta})$ is a constant $\neq 0$. Thus the Dirichlet boundary conditions are (hypo-)elliptic if $P(D)$ is (hypo-)elliptic. Note that it may occur that $\mu = 0$ so that no boundary conditions are present.

A remarkable feature of this example is that the Dirichlet boundary conditions are (hypo-)elliptic with respect to all (hypo-)elliptic operators of type μ, in spite of the fact that conditions (3.1) and (3.2) involve P also. We are going to study the boundary conditions which have this property.

DEFINITION 3.1. *The μ boundary conditions* (0.3) *are called completely elliptic if they are elliptic with respect to all elliptic operators $P(D)$ of determined type μ.*

Let $D(\tau_1, \dots, \tau_\mu, \xi)$ be the principal part of the polynomial

$$R\left(\prod_1^\mu (\tau - \tau_j);\ Q_1(\xi + \tau N), \dots, Q_\mu(\xi + \tau N)\right),$$

where $\tau_1, \dots, \tau_\mu$ and ξ are considered as independent variables. We shall call D *the characteristic function of the boundary conditions* (0.3).

THEOREM 3.4. *A sufficient and, if $n > 2$, also necessary condition for the boundary condition* (0.3) *to be completely elliptic is that*

$$D(\tau_1, \dots, \tau_\mu, \xi) \neq 0 \text{ if } \operatorname{Im} \tau_j > 0,\ j = 1, \dots, \mu, \text{ and } \xi \text{ is real, } \dot{\xi} \neq 0. \tag{3.5}$$

An equivalent condition is that the polynomial in λ

$$D(\tau_1 + \lambda \tau_1^0, \dots, \tau_\mu + \lambda \tau_\mu^0, \xi) \tag{3.6}$$

has only real zeros if $\tau_1^0 > 0, \dots, \tau_\mu^0 > 0$ and $\tau_1, \dots, \tau_\mu, \xi$ are real, $\dot{\xi} \neq 0$. (When $\dot{\xi} = 0$ the polynomial either vanishes identically or else it has only real zeros.)

Note that if $D(\tau_1, \dots, \tau_\mu, 0)$ does not vanish identically, this means precisely that D is hyperbolic with respect to all vectors $(\tau_1, \dots, \tau_\mu, 0)$ with all $\tau_j > 0$. (For the definition of hyperbolic polynomials cf. Gårding [2].)

Proof of Theorem 3.4. The sufficiency of (3.5) is quite obvious. Indeed, if $P(D)$ is an elliptic operator of determined type μ and P^0 its principal part, we have

$$C(\dot{\xi}) = D(\tau_1, \dots, \tau_\mu, \xi) + O(|\dot{\xi}|^{M-1}),$$

where M is the degree of D and $\tau_1, \dots, \tau_\mu$ are the zeros of the polynomial $P^0(\xi + \tau N)$ with positive imaginary part. As in Theorem 3.3 (a) this implies that (3.2) is fulfilled.

Next assume that $n > 2$ and that the boundary conditions (0.3) are completely elliptic. Take ξ' real, $\dot{\xi}' \neq 0$, and $\tau_1', \dots, \tau_\mu'$ with positive imaginary parts. We have to prove that

$$D(\tau_1', \dots, \tau_\mu', \xi') \neq 0.$$

Since D does not vanish identically and $n > 2$, we can find $\tau_1'', \dots, \tau_\mu''$ with positive imaginary parts and a real ξ'' such that $\dot{\xi}'$ and $\dot{\xi}''$ are linearly independent and $D(\tau_1'', \dots, \tau_\mu'', \xi'') \neq 0$. Let $P(\xi)$ be a homogeneous positive definite polynomial of degree 2μ such that $P(\xi' + \tau N)$ is divisible by $\prod_1^\mu (\tau - \tau_j')$ and $P(\xi'' + \tau N)$ is divisible by $\prod_1^{\mu}(\tau-\tau_j'')$. The existence of such a polynomial will be proved when $\mu=1$ in Lemma 3.1; in the general case one only has to multiply μ such factors. Now we obviously have

$$\lim_{s\to\infty} s^{-M} C(s\,\dot{\xi}) = D(\tau_1, \ldots, \tau_\mu, \xi),$$

if $\tau_1, \ldots, \tau_\mu$ now denote the zeros of $P(\xi+\tau N)$ with positive imaginary parts, and arguing as in the proof of Theorem 3.3 (b), we can thus conclude that the right hand side either vanishes identically or is $\neq 0$ for all real ξ with $\dot{\xi}\neq 0$. Now by assumption it does not vanish when $\xi=\xi''$, and hence not when $\xi=\xi'$ either. This proves (3.5). Since the equivalence between the two conditions in the theorem is obvious, it only remains to prove the following lemma.

LEMMA 3.1. *Let ξ' and ξ'' be real, $\dot{\xi}'$ and $\dot{\xi}''$ linearly independent, and let λ' and λ'' be two non real numbers. Then there is a positive definite quadratic form $S(\xi)$ such that $S(\xi'+\lambda' N)=S(\xi''+\lambda'' N)=0$.*

Proof. Since $\xi'+\lambda' N$, $\xi''+\lambda'' N$ and N are linearly independent, we can find a complex vector $y\in G+iG$ so that

$$\langle y, \xi'+\lambda' N\rangle = \langle y, \xi''+\lambda'' N\rangle = 0, \ \langle y, N\rangle = 1.$$

Write

$$S(\xi) = (\langle y, \xi\rangle + \langle \bar{y}, \xi\rangle)^2 + s(\dot{\xi});$$

$S(\xi)$ will have the desired properties if $s(\dot{\xi})$ is positive definite in $\dot{\xi}$ and

$$s(\dot{\xi}') = -(\langle y, \xi'+\lambda' N\rangle + \overline{\langle y, \xi'+\bar{\lambda}' N\rangle})^2 = -(\langle \bar{y}, (\lambda'-\bar{\lambda}')N\rangle)^2 = 4(\operatorname{Im}\lambda')^2,$$
$$s(\dot{\xi}'') = 4(\operatorname{Im}\lambda'')^2.$$

Since $4(\operatorname{Im}\lambda')^2>0$, $4(\operatorname{Im}\lambda'')^2>0$ and $\dot{\xi}'$ and $\dot{\xi}''$ are linearly independent, one can find a form $s(\xi)$ with the desired properties.

Example 2. Let $P(D)$ be the Laplace operator, that is, if we introduce coordinates so that $N=(0, \ldots, 0, 1)$,

$$P(\xi) = (\xi', \xi') + \xi_n^2,$$

where $\xi'=(\xi_1, \ldots, \xi_{n-1})$ and $(\xi', \xi')=\xi_1^2+\cdots+\xi_{n-1}^2$. We have $\mu=1$ and, changing if necessary the boundary condition with a multiple of $P(D)$, we may assume that

$$Q(D) = q_0(D')\,u - i\,q_1(D')\,\partial u/\partial x^n, \tag{3.7}$$

where q_0 and q_1 are polynomials in ξ'. If $\zeta = (\zeta', \zeta_n)$ we can obviously identify $\dot{\zeta}$ with ζ' and then have

$$C(\dot{\zeta}) = q_0(\zeta') + i(\zeta', \zeta')^{\frac{1}{2}} q_1(\zeta'), \tag{3.8}$$

where $(\zeta', \zeta')^{\frac{1}{2}}$ denotes the square root with positive real part. *The condition* (3.2) *for ellipticity is equivalent to the ellipticity of the polynomial*

$$F(\zeta') = q_0(\zeta')^2 + (\zeta', \zeta') q_1(\zeta')^2. \tag{3.9}$$

Indeed, if this polynomial is elliptic, (3.2) follows from Lemma 2.1 because C is a factor of F. On the other hand, assume that (3.2) is fulfilled. If $n = 2$ it follows if we apply (3.2) to positive and negative ξ_1 respectively, using (3.8), that

$$q_0(\xi_1) \pm i\,\xi_1 q_1(\xi_1) \not\equiv 0,$$

and hence the product F of these two polynomials does not vanish identically which proves the assertion since all polynomials $\not\equiv 0$ in one variable are elliptic. If $n > 2$ we decompose the principal part Q^0 in a form similar to (3.7) and obtain

$$C^0(\dot{\xi}) = q_0^0(\xi') + i(\xi', \xi')^{\frac{1}{2}} q_1^0(\xi').$$

This cannot vanish identically since $(\xi', \xi')^{\frac{1}{2}}$ is not a rational function when $n > 2$. Hence, according to Theorem 3.3, the boundary condition being elliptic, we have $C^0(\dot{\xi}) \neq 0$ for real $\dot{\xi} \neq 0$. Multiplying $C^0(\dot{\xi})$ and $C^0(-\dot{\xi})$ together, noting that q_0 and q_1 are homogeneous, q_0 of one degree higher than q_1, and that $(-\xi', -\xi')^{\frac{1}{2}} = (\xi', \xi')^{\frac{1}{2}}$, we find that

$$q_0^0(\xi')^2 + (\xi', \xi') q_1^0(\xi')^2 \neq 0 \text{ for real } \xi' \neq 0.$$

But this means precisely that the principal part of $F(\xi')$ satisfies the definition (0.5) of an elliptic polynomial, so that the assertion is proved.

In particular, the result shows that *the condition for ellipticity with respect to the Laplace equation does not depend on whether Ω is situated in the half space $x^n > 0$ or $x^n < 0$.*

The latter conclusion does not hold in the hypoelliptic case. Indeed, let $n = 3$ and

$$Q(\zeta) = i\,\zeta_1^2 + \zeta_3. \tag{3.10}$$

Then

$$C(\zeta') = i\,\zeta_1^2 + i\sqrt{\zeta_1^2 + \zeta_2^2} \tag{3.11}$$

if Ω is situated in the half space $x^n > 0$ and

$$C(\zeta') = i\,\zeta_1^2 - i\sqrt{\zeta_1^2 + \zeta_2^2} \tag{3.12}$$

if Ω is situated in the half space $x^n < 0$. (Note that the square root is defined so that it has a positive real part.) Now (3.12) vanishes if ζ_1 and ζ_2 are real and satisfy the equation $\zeta_1^4 = \zeta_1^2 + \zeta_2^2$. This curve has a real infinite branch, so that the boundary condition (3.10) is not hypoelliptic with respect to the Laplacean if Ω is situated in the lower half space. On the other hand, if (3.11) vanishes, we must have $\operatorname{Re} \zeta_1^2 \leqslant 0$ in view of the definition of the square root. Hence

$$|\operatorname{Re} \zeta_1| \leqslant |\operatorname{Im} \zeta_1|.$$

If a bound for $|\operatorname{Im} \zeta'|$ is prescribed, this gives an estimate of $|\operatorname{Re} \zeta_1|$ when $C(\zeta') = 0$, thus according to (3.11) we get an estimate of $|\zeta_2|$. Thus (3.10) defines a hypoelliptic boundary condition with respect to the Laplacean if Ω is situated in the upper half space.

4. Necessity of the conditions for (hypo-)ellipticity

Using the theorem on the closed graph and the category theorem we shall prove in this section that the algebraic conditions in Theorems 3.1 and 3.2 are necessary for the boundary conditions (0.3) to be (hypo-)elliptic. The results to be proved were formulated in an invariant way in these theorems, but when we prove them in this and the following sections we shall use non invariant methods. Thus we use in what follows a coordinate system such that the hyperplane F is defined by $x^n = 0$ and Ω is situated in the half space $x^n > 0$. In order to avoid unnecessary complications we assume that Ω is bounded; the modifications that are otherwise required will be indicated at the end of the section. By Ω' we shall denote a domain whose closure is contained in $\Omega \cup \omega$ but not in Ω.

LEMMA 4.1. *Suppose that the boundary conditions* (0.3) *are hypo-elliptic with respect to* $P(D)$. *Then, if k is the integer occurring in Definition* 0.1, *there is a constant C such that*

$$\sum_{|\alpha| \leqslant k+1} \sup_{x \in \Omega'} |D_\alpha u(x)| \leqslant C \sum_{|\alpha| \leqslant k} \sup_{x \in \Omega} |D_\alpha u(x)| \tag{4.1}$$

for all $u \in C^k(\Omega \cup \omega)$ *satisfying* (0.2) *and* (0.3).

Proof. Inequality (4.1) is void if the right hand side is not finite. Now let U be the set of all $u \in C^k(\Omega \cup \omega)$, which satisfy (0.2) and (0.3), such that the norm

$$N(u) = \sum_{|\alpha| \leqslant k} \sup_{x \in \Omega} |D_\alpha u(x)|,$$

which is the right hand side of (4.1), is finite. It is obvious that U is complete, and thus a Banach space, with this norm. By V we denote the space of functions

$v \in C^{k+1}(\Omega')$ with bounded derivatives up to order $k+1$ and the norm defined by

$$\sum_{|\alpha| \leqslant k+1} \sup_{x \in \Omega'} |D_\alpha v(x)|.$$

V is also a Banach space. Now by hypothesis, the restriction of a function $u \in U$ to Ω' is in V, for $\overline{\Omega'}$ is a compact subset of $\Omega \cup \omega$ so that the $(k+1)$st derivatives of u are bounded there. Hence mapping the functions in U on their restrictions to Ω' gives a linear mapping of U into V, which is defined in the whole of U, and it is clear that this mapping is closed. Hence it is bounded in virtue of the theorem on the closed graph, and this proves the lemma.

LEMMA 4.2. *Suppose that the boundary conditions* (0.3) *are elliptic with respect to* $P(D)$. *Then there is a constant* C *such that*

$$\sum_{|\alpha| \leqslant k+j} \sup_{x \in \Omega'} |D_\alpha u(x)| \leqslant C^j j! \sum_{|\alpha| \leqslant k} \sup_{x \in \Omega} |D_\alpha u(x)|, \quad j=1, 2, \ldots \tag{4.2}$$

for all $u \in C^k(\Omega \cup \omega)$ *satisfying* (0.2) *and* (0.3).

Proof. Let U be defined as in the proof of Lemma 4.1, and let F_r be the subset of those $u \in U$ such that

$$\sum_{|\alpha| \leqslant k+j} \sup_{x \in \Omega'} |D_\alpha u(x)| \leqslant r^j j!, \quad j=1, 2, \ldots$$

The sets F_r are obviously closed and increasing with r. Since by assumption and Definition 0.1 every $u \in U$ is analytic in $\Omega \cup \omega \supset \overline{\Omega'}$, every $u \in U$ belongs to F_r for some r. Hence in virtue of the category theorem we can find R so large that F_R has an interior point, and since F_R is convex and symmetric, it then contains a sphere $\{u; N(u) \leqslant \varepsilon\}$ with positive radius ε. We then have

$$\sum_{|\alpha| \leqslant k+j} \sup_{x \in \Omega'} |D_\alpha u(x)| \leqslant R^j j! \, N(u)/\varepsilon, \quad j=1, 2, \ldots,$$

in view of the homogeneity of this inequality and the fact that it is true when $N(u)=\varepsilon$. This proves (4.2) with $C=R(1+\varepsilon^{-1})$.

We shall now prove that (3.1) and (3.2) follow from (4.1) and (4.2) by applying the latter inequalities to "exponential solutions" of the boundary problem (0.2), (0.3), that is, solutions of the form

$$u = e^{i\langle x, \zeta \rangle} v(\langle x, N \rangle), \tag{4.3}$$

where v is a function of a real variable. By straightforward computations using Leibniz' formula we obtain

$$P(D)\,u = e^{i\langle x, \zeta \rangle} P(\zeta + \delta N)\, v(\langle x, N \rangle),$$

where δ means $-i$ times differentiation with respect to the argument of v. Thus u satisfies (0.2) and (0.3) if and only if

$$P(\zeta+\delta N)\,v(t)=0, \tag{4.4}$$

$$(Q_\nu(\zeta+\delta N)\,v)(0)=0, \quad \nu=1, \ldots, \mu \tag{4.5}$$

If $\zeta\in A$ and $C(\dot{\zeta})=R(k_\zeta;\,q_\zeta^1, \ldots, q_\zeta^\mu)=0$ (and only then), we can find $v\neq 0$ satisfying (4.5) and

$$k_\zeta(\delta)v=0, \tag{4.6}$$

which implies (4.4), since $k_\zeta(\tau)$ is a factor of $P(\zeta+\tau N)$.

Differentiation of an exponential solution with respect to a boundary variable x^j, $1\leqslant j\leqslant n-1$, is equivalent to multiplying by ζ_j. Thus it follows from (4.1) that

$$\left(\sum_1^{n-1}|\zeta_j|\right)\sum_{|\alpha|\leqslant k}\sup_{\Omega'}|D_\alpha u(x)|\leqslant C\sum_{|\alpha|\leqslant k}\sup_{\Omega}|D_\alpha u(x)| \tag{4.7}$$

for the exponential solution (4.3). Now $D_\alpha u(x)=e^{i\langle x,\zeta\rangle}v_\alpha(\langle x, N\rangle)$, where v_α is also a solution of (4.6). Denoting by H the supremum of $|x|$ when $x\in\Omega$ we have

$$e^{-H|\mathrm{Im}\,\zeta|}\leqslant|e^{i\langle x,\zeta\rangle}|\leqslant e^{H|\mathrm{Im}\,\zeta|}, \quad x\in\Omega.$$

Hence (4.7) gives

$$\left(\sum_1^{n-1}|\zeta_j|\right)\sum_{|\alpha|\leqslant k}\sup_{\Omega'}|v_\alpha(\langle x, N\rangle)|\leqslant C\,e^{2H|\mathrm{Im}\,\zeta|}\sum_{|\alpha|\leqslant k}\sup_{\Omega}|v_\alpha(\langle x, N\rangle)|. \tag{4.8}$$

Now let a be a positive number such that $\langle x, N\rangle$ attains all values between 0 and a when $v\in\Omega'$, and let b be an upper bound of $\langle x, N\rangle$ when $x\in\Omega$. Then it follows from (4.8) that

$$\left(\sum_1^{n-1}|\zeta_j|\right)\sum_{|\alpha|\leqslant k}\sup_{0<t<a}|v_\alpha(t)|\leqslant C\,e^{2H|\mathrm{Im}\,\zeta|}\sum_{|\alpha|\leqslant k}\sup_{0<t<b}|v_\alpha(t)|. \tag{4.9}$$

We now use the fact that all v_α are solutions of the equation (4.6) and not only of (4.4), and that the zeros of $k_\zeta(\tau)$ have non-negative imaginary parts. It follows from Theorem 1.4 that

$$\sup_{0<t<b}|v_\alpha(t)|\leqslant\gamma\,(b/a)^{\nu-1}\sup_{0<t<a}|v_\alpha(t)|. \tag{4.10}$$

Combining (4.9) and (4.10) and noting that $v_0\neq 0$ we get, replacing $|\zeta_j|$ by $|\mathrm{Re}\,\zeta_j|$, with another constant C_1

$$\sum_1^{n-1}|\mathrm{Re}\,\zeta_j|\leqslant C_1\,e^{2H|\mathrm{Im}\,\zeta|}. \tag{4.11}$$

The semi-norms $\sum_1^{n-1} |\text{Re}\,\zeta_j|$ and $|\text{Re}\,\dot{\zeta}|$ are equivalent since they vanish for the same values of ζ. Hence we get with constants C_2 and C_3

$$|\text{Re}\,\dot{\zeta}| \leqslant C_2 e^{C_3 |\text{Im}\,\zeta|} \text{ if } \zeta \in A \text{ and } C(\dot{\zeta}) = 0. \tag{4.11'}$$

In virtue of this inequality and Theorem 2.2 we can, given any number B, find a number B' so large that

$$|\text{Im}\,\zeta| \leqslant B, \quad |\text{Re}\,\dot{\zeta}| \geqslant B' \tag{4.12}$$

implies that
$$\zeta \in A \text{ and } C(\dot{\zeta}) \neq 0. \tag{4.13}$$

Hence we have $\dot{\zeta} \in \dot{A}$ and $C(\dot{\zeta}) \neq 0$ if $|\text{Im}\,\dot{\zeta}| \leqslant \text{B}$ and $|\text{Re}\,\dot{\zeta}| \geqslant B'$. For by definition this means that there is a real τ such that

$$|\text{Im}\,(\zeta + i\tau N)| \leqslant B, \quad |\text{Re}\,(\zeta + i\tau N)^{\cdot}| = |\text{Re}\,\dot{\zeta}| \geqslant B'.$$

Hence $\zeta + i\tau N \in A$ and $C((\zeta + i\tau N)^{\cdot}) = C(\dot{\zeta}) \neq 0$ in virtue of (4.13). This proves that (3.1) is a necessary condition for hypoellipticity.

We next prove that (3.2) is a necessary condition for ellipticity. Choose M_1 according to Theorem 2.1 so that A contains all ζ with

$$|\text{Re}\,\dot{\zeta}| \geqslant M_1 (1 + |\text{Im}\,\zeta|). \tag{4.14}$$

With ζ satisfying this inequality and $C(\dot{\zeta}) = 0$, we apply (4.2) to the exponential solutions as before. This now gives

$$\left(\sum_1^{n-1} |\zeta_j|\right)^j \leqslant \gamma\,(b/a)^{\gamma-1} C^j j!\, e^{2H|\text{Im}\,\zeta|}, \quad j = 1, 2, \ldots,$$

or with some other constants C_1 and C_2

$$|\text{Re}\,\dot{\zeta}|^j \leqslant C_1^j j!\, e^{C_2 |\text{Im}\,\zeta|}, \quad j = 1, 2, \ldots \tag{4.15}$$

Now let $|\text{Re}\,\dot{\zeta}| \geqslant C_1$ and let j be the largest integer not exceeding $|\text{Re}\,\dot{\zeta}|/C_1$. Then

$$(|\text{Re}\,\dot{\zeta}|/C_1)^j / j! \geqslant j^j/j! \geqslant e^{(j-1)/2} \geqslant e^{|\text{Re}\,\dot{\zeta}|/2C_1 - 1}.$$

Hence (4.15) gives

$$|\text{Re}\,\dot{\zeta}| \leqslant 2\,C_1 (1 + C_2 |\text{Im}\,\zeta|) \text{ if } C(\dot{\zeta}) = 0 \text{ and (4.14) holds, } |\text{Re}\,\dot{\zeta}| \geqslant C_1.$$

Thus there is a constant M such that

$$|\operatorname{Re} \dot{\zeta}| \geqslant M(1 + |\operatorname{Im} \zeta|)$$

implies $\zeta \in A$ and $C(\dot{\zeta}) \neq 0$. But arguing as before we can immediately conclude that this implies (3.2).

Thus the proof is complete under the assumption that Ω is bounded. If Ω is not bounded, it is necessary to modify the definition of the Banach space U so that it contains the exponential solutions. This can be done simply by adding in the definition of the norm $N(u)$ a factor $e^{-|x|^c}$ with $c > 1$. The proof of (3.1) then proceeds as before. However we get instead of (3.2) only that $C(\dot{\zeta}) \neq 0$ when

$$|\operatorname{Re} \dot{\zeta}| \geqslant C(1 + |\operatorname{Im} \dot{\zeta}|)^{c'}; \quad 1/c + 1/c' = 1.$$

Now algebraic arguments (cf. the proof of Lemma 5.3) show that there is a smallest value of c' for which such an inequality can hold. But this value has to be $\leqslant 1$ which proves that (3.2) must be valid.

5. Sufficiency of the condition for hypoellipticity

In this section we prove that condition (3.1) of Theorem 3.1 is sufficient for hypoellipticity, and moreover we shall also study the inhomogeneous case. Thus let $u \in C^k(\Omega \cup \omega)$ be a solution of the equations

$$P(D)\,u = f \text{ in } \Omega, \; Q_\nu(D)\,u = \varphi_\nu \text{ in } \omega, \; \nu = 1, \dots, \mu, \tag{5.1}$$

where f and φ_ν are infinitely differentiable in $\Omega \cup \omega$ and ω, respectively. Since $P(D)$ is hypoelliptic, u is infinitely differentiable in Ω. (The proof will be arranged so that we do not really use this fact.) Hence we only have to prove that the derivatives have limits on ω. To do so it is enough to show that every point in ω has a neighbourhood O such that all derivatives of u are continuous in the closure of $\Omega \cup O$. If σ is the transversal order of $P(D)$, that is, the degree of $P(\xi + \tau N)$ with respect to τ, it is indeed sufficient to prove the continuity of the derivatives of transversal order $< \sigma$. For if we introduce a coordinate system such that $\langle x, N\rangle = x^n$, as was done in the preceding section also, it follows from the remarks of Hörmander [5, p. 239] that we can write

$$P(D) = c\,(\partial/\partial x^n)^\sigma + \cdots$$

where c is a constant $\neq 0$ and the terms indicated by dots have transversal order $< \sigma$. Differentiating the equation $P(D)\,u = f$ repeatedly with respect to the boundary variables $x^1, \dots, x^{n-1}$, we can conclude the continuity of the derivatives of transversal order σ from the continuity of those of transversal order $< \sigma$. Differentiating again with respect to x^n it follows that the derivatives of transversal order $\sigma + 1$ are continuous, and since the process may be repeated indefinitely, all derivatives are continuous.

It is also important to note that *we may assume that the transversal order of the operator Q_ν is $< \sigma$.* For we can add to Q_ν an operator having P as a factor so that this is true, and this does not change neither the assumption that the right hand sides in (5.1) are infinitely differentiable functions, nor the definition of the characteristic function for the boundary problem.

Let Ω' be an open half sphere $\subset \Omega$ whose flat boundary $\subset \omega$ such that

$$\bar{\Omega}' \subset \Omega \cup \omega.$$

We shall estimate the derivatives of a solution u of (5.1) in Ω'.

Let χ be a function in $C_0^\infty(G_+)$, where

$$G_+ = \{x;\ \langle x, N\rangle \geqslant 0\},$$

which vanishes outside of a compact subset of $\Omega \cup \omega$ and equals 1 in a neighbourhood of $\bar{\Omega}'$. Write

$$U = \chi u, \tag{5.2}$$

and interpret this product as 0 outside of $\Omega \cup \omega$. We have obviously $U \in C_0^k(G_+)$ and

$$P(D)\,U = F^1 + F^2; \quad Q_\nu(D)\,U = \phi_\nu^1 + \phi_\nu^2, \quad \nu = 1, \dots, \mu, \tag{5.3}$$

where

$$F^1 = \chi f \in C_0^\infty(G_+), \qquad \phi_\nu^1 = \chi \varphi_\nu \in C_0^\infty(F), \tag{5.4}$$

and

$$F^2 = 0,\ \phi_\nu^2 = 0 \text{ in a neighbourhood of } \bar{\Omega}'. \tag{5.5}$$

Let R be a positive number such that, if $x \in \Omega'$, F^2 and ϕ_ν^2 vanish in the set

$$\{y = (y', y^n);\ |y' - x'| \leqslant R,\ |y^n - x^n| \leqslant R\}.$$

(Here and below we use the notation y' alternatively for $(y^1, \dots, y^{n-1})$ or $(y^1, \dots, y^{n-1}, 0)$.) We shall estimate the derivatives of $U = u$ in Ω' in terms of those of order $\leqslant k$ in G_+, the maxima of F^2 and ϕ_ν^2, the derivatives of F^1 and ϕ_ν^1, which all exist by hypothesis, and the number R. In doing so, *we will first assume that all derivatives of U with transversal order $\leqslant \sigma$ are continuous in G_+.* This regularity assumption will make convergence difficulties disappear, but since we never use it in a quantitative way it can be removed afterwards.

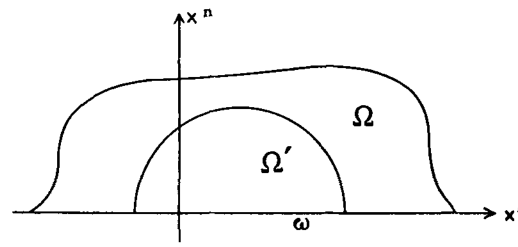

Fig. 1.

It is clearly sufficient to place the origin at a point in $\omega \cup \bar{\Omega}'$ and estimate the derivatives on the x_n-axis. Let $\hat{U}(t, \xi')$ be the Fourier transform of $U(t, x')$ with respect to the boundary variables x'

$$\hat{U}(t, \xi') = \int U(t, x')\, e^{-i\langle x', \xi'\rangle}\, dx';$$

all other Fourier transforms are defined similarly. The equations (5.3) give, if we write $P(\delta, \xi') = P(\xi' + \delta N)$ and similarly for Q_ν,

$$\left.\begin{aligned} P(\delta, \xi')\, \hat{U}(t, \xi') &= \hat{F}^1(t, \xi') + \hat{F}^2(t, \xi'), \\ Q_\nu(\delta, \xi')\, \hat{U}(t, \xi')_{t=0} &= \hat{\phi}^1_\nu(\xi') + \hat{\phi}^2_\nu(\xi'), \quad \delta = -i\, d/dt. \end{aligned}\right\} \tag{5.6}$$

Since $\hat{U}$ vanishes for large positive t, this boundary problem can be solved with the methods of section 1 provided that $C(\xi') \neq 0$, which is true by the assumption (3.1) if $|\xi'| \geqslant K$, where K is a constant. The solution is then given by (1.15). Let $G(t, s, \xi')$ be the Green's function and $H_\nu(t, \xi')$ the Poisson kernel for the problem (5.6) as given by (1.13) and (1.14). Then we have

$$\hat{U}(t, \xi') = \int_0^\infty G(t, s, \xi')\, (\hat{F}^1(s, \xi') + \hat{F}^2(s, \xi'))\, ds + \\ + \sum_1^\mu H_\nu(t, \xi')\, (\hat{\phi}^1_\nu(\xi') + \hat{\phi}^2_\nu(\xi')), \; |\xi'| \geqslant K. \tag{5.7}$$

Let ϑ be a function in $C_0^\infty(F')$ which equals 1 when $|\xi'| \leqslant K$ and write $\vartheta_1(\xi') = 1 - \vartheta(\xi')$. Then we have, for all real ξ',

$$\hat{U}(t, \xi') = \vartheta(\xi')\, \hat{U}(t, \xi') + \vartheta_1(\xi') \left\{ \int_0^\infty G(t, s, \xi')\, (\hat{F}^1(s, \xi') + \hat{F}^2(s, \xi'))\, ds + \right. \\ \left. + \sum_1^\mu H_\nu(t, \xi')\, (\hat{\phi}^1_\nu(\xi') + \hat{\phi}^2_\nu(\xi')) \right\}. \tag{5.8}$$

We shall also use the equations obtained by differentiating (5.8) with respect to t,

$$\hat{U}^{(j)}(t,\xi')=\vartheta(\xi')\,\hat{U}^{(j)}(t,\xi')+\vartheta_1(\xi')\left\{\int_0^\infty G^{(j)}(t,s,\xi')(\hat{F}^1(s,\xi')+\hat{F}^2(s,\xi'))\,ds+\right.$$
$$\left.+\sum_1^\mu H_\nu^{(j)}(t,\xi')\,(\hat{\phi}_\nu^1(\xi')+\hat{\phi}_\nu^2(\xi'))\right\}, \qquad (5.9)$$

where $^{(j)}$ denotes differentiation of order j with respect to t. This will only be used when $j\leqslant\sigma$.

Fourier's inversion formula gives, if D'_α denotes partial differentiations with respect to the boundary variables only,

$$D'_\alpha\,U^{(j)}(t,x')=(2\pi)^{-(n-1)}\int\xi'_\alpha\,\hat{U}^{(j)}(t,\xi')\,e^{i\langle x',\xi'\rangle}\,d\xi'.$$

As already observed, we are only interested in the value of this derivative on the x^n-axis, so we set $x'=0$ and obtain, if $(0,t)\in\Omega'$

$$D'_\alpha\,u^{(j)}(t,0)=(2\pi)^{1-n}\int\xi'_\alpha\,\hat{U}^{(j)}(t,\xi')\,d\xi'. \qquad (5.10)$$

We now pass to estimating each of the integrals obtained by replacing $\hat{U}^{(j)}(t,\xi')$ in (5.10) by its value given by (5.9). The various terms are denoted by a, b, ...

It is very easy to estimate the term

$$a=(2\pi)^{1-n}\int\xi'_\alpha\,\vartheta(\xi')\,\hat{U}^{(j)}(t,\xi')\,d\xi'.$$

For since ϑ has compact support, we can estimate ξ'_α by $C^{|\alpha|}$ where C is a constant. (By C we always denote constants, but not always the same every time.) Furthermore, we have

$$|\hat{U}^{(j)}(t,\xi')|\leqslant\int|U^{(j)}(t,x')|\,dx'\leqslant C\sup|U^{(j)}|.$$

Writing
$$|U|_k=\sum_{|\alpha|\leqslant k}\sup|D_\alpha U(x)|,$$

we thus obtain, since $j\leqslant\sigma\leqslant k$,

$$|a|\leqslant C^{|\alpha|+1}|U|_k. \qquad (a)$$

The study of the other integrals depends on the following lemma, the proof of which will be postponed to the end of this section.

LEMMA 5.1. *There are positive constants M, M', c, γ, γ', with $\gamma\geqslant 1$, such that the functions $G^{(j)}(t,s,\zeta')$ and $H_\nu^{(j)}(t,\zeta')$ of ζ' are analytic in the set*

$$D=\{\zeta';\ |\mathrm{Re}\,\zeta'|\geqslant M(1+|\mathrm{Im}\,\zeta'|^\gamma)\} \qquad (5.11)$$

and satisfy the inequalities

$$|G^{(j)}(t, s, \zeta')| \leqslant M' |\zeta'|^{\gamma'} e^{-c|t-s||\zeta'|^{1/\gamma}}; \quad \zeta' \in D;\ 0 \leqslant j \leqslant \sigma;\ s, t \geqslant 0, \tag{5.12}$$

$$|H_\nu^{(j)}(t, \zeta')| \leqslant M' |\zeta'|^{\gamma'} e^{-ct|\zeta'|^{1/\gamma}}; \quad \zeta' \in D;\ 0 \leqslant j \leqslant \sigma;\ 1 \leqslant \nu \leqslant \mu;\ t \geqslant 0. \tag{5.13}$$

Using only a minor part of this lemma we can study the term

$$b = (2\pi)^{1-n} \int \xi'_\alpha \vartheta_1(\xi')\, d\xi' \int_0^\infty G^{(j)}(t, s, \xi')\, \hat{F}^1(s, \xi')\, ds.$$

For using (5.12) for real ξ' only we obtain

$$|b| \leqslant C \int_{|\xi'| \geqslant K} |\xi'|^{|\alpha|+\gamma'} |\hat{F}^1(s, \xi')|\, d\xi' \leqslant C_1 \sup\left(|\xi'|^{|\alpha|+\gamma'+n} |\hat{F}^1(s, \xi')|\right) \int_{|\xi'| \geqslant K} d\xi'/|\xi'|^n.$$

Assuming as we may that γ' is an integer and using a notation introduced above, we thus obtain since the integral is convergent

$$|b| \leqslant C_\alpha |F^1|_{|\alpha|+\gamma'+n}. \tag{b}$$

In the same way it follows that for the term

$$c = (2\pi)^{1-n} \sum_1^\mu \int \xi'_\alpha \vartheta_1(\xi')\, H_\nu^{(j)}(t, \xi')\, \hat{\phi}_\nu^1(\xi')\, d\xi'$$

we have the estimate

$$|c| \leqslant C_\alpha \sum_1^\mu |\phi_\nu^1|_{|\alpha|+\gamma'+n}. \tag{c}$$

We next consider the terms

$$d = (2\pi)^{1-n} \int \xi'_\alpha \vartheta_1(\xi')\, d\xi' \int_0^\infty G^{(j)}(t, s, \xi')\, \hat{F}^2(s, \xi')\, ds$$

and

$$e_\nu = (2\pi)^{1-n} \int \xi'_\alpha \vartheta_1(\xi')\, H_\nu^{(j)}(t, \xi')\, \hat{\phi}_\nu^2(\xi')\, d\xi', \quad \nu = 1, \ldots, \mu.$$

To do so we have to use the information in Lemma 5.1 about G and H_ν in the complex domain too.

LEMMA 5.2. *Let $K(\zeta')$ be a function, analytic in the set D defined by* (5.11), *where $\gamma \geqslant 1$, and satisfying*

$$|K(\zeta')| \leqslant M' |\zeta'|^{\gamma'},\ \zeta' \in D. \tag{5.14}$$

Then there is a constant C, depending on M and γ but independent of M', γ' and $K(\zeta')$ such that

$$|D'_\beta(\xi'_\alpha K(\xi'))| \leqslant M' C^{|\alpha|+|\beta|+\gamma'} |\beta|!\ |\xi'|^{\gamma'+|\alpha|-|\beta|/\gamma}, \tag{5.15}$$

when ξ' is real and $|\xi'| \geqslant M+1$.

Proof. Since $$|\zeta'_\alpha K(\zeta')| \leqslant M' |\zeta'|^{\gamma'+|\alpha|} \text{ in } D,$$

it is sufficient to prove (5.15) when $\alpha = 0$. For the general result follows if we apply this special case to the function $\zeta'_\alpha K(\zeta')$. It follows from the definition (5.11) of D that there is a constant M_1 such that the sphere

$$\{\zeta'; |\zeta' - \xi'| \leqslant \varrho\}, \; \varrho = M_1 |\xi'|^{1/\gamma},$$

is contained in D if $|\xi'| \geqslant M + 1$, and a constant C_1 such that

$$|\zeta'| \leqslant C_1 |\xi'|$$

for all ζ' in the sphere. Hence it follows from (5.14) that

$$|K(\zeta')| \leqslant M' C_1^{\gamma'} |\xi'|^{\gamma'}$$

in the sphere. But then it follows from Cauchy's inequality for the derivatives of an analytic function in a circle (sphere) that

$$|D'_\beta K(\xi')| \leqslant M' C_1^{\gamma'} |\xi'|^{\gamma'} |\beta|!/\varrho^{|\beta|},$$

which proves (5.15) with $C = \max(C_1, 1/M_1)$.

In virtue of (5.12) and (5.13), this lemma applies to $G^{(j)}$ and $H_\nu^{(j)}$ which makes it possible to estimate the terms d and e_ν. We start with e_ν which is slightly simpler to handle than d.

The choice of R following formula (5.5) means that

$$\phi_\nu^2(x') = 0 \text{ when } |x'| \leqslant R.$$

Thus if we set, with a positive integer r to be chosen later,

$$\phi_\nu^2(x') = |x'|^{2r} \psi_r(x'),$$

it follows that $$\sup |\psi_r(x')| \leqslant R^{-2r} \sup |\phi_\nu^2(x')|.$$

Now pass to Fourier transforms. If Δ' denotes the Laplace operator along the boundary, $D_1^2 + \cdots + D_{n-1}^2$, we obtain

$$\hat{\phi}_\nu^2(\xi') = \Delta'^r \hat{\psi}_r(\xi').$$

Eliminating $\hat{\phi}_\nu^2$ in the definition of e_ν by means of this equation and integrating by parts, we obtain

$$e_\nu = (2\pi)^{1-n} \int \hat{\psi}_r(\xi') \Delta'^r \{\xi'_\alpha H_\nu^{(j)}(t, \xi') \vartheta_1(\xi')\} d\xi'.$$

Now we have $\vartheta_1(\xi')=1$ outside of a compact set, and from Lemmas 5.1 and 5.2 it follows that for $|\xi'|\geqslant M+1$

$$|\Delta'^r(\xi'_\alpha H^{(j)}(t,\xi'))|\leqslant M' C^{|\alpha|+2r+\gamma'}(2r)!(n-1)^r|\xi'|^{\gamma'+|\alpha|-2r/\gamma}. \tag{5.16}$$

If we choose r so large that

$$\gamma'+|\alpha|-2r/\gamma<-n+1, \quad \text{i.e. } 2r>\gamma(\gamma'+|\alpha|+n-1), \tag{5.17}$$

this is an integrable function in a neighbourhood of infinity and hence we get by estimating $\hat{\psi}_r$ in an obvious fashion that

$$(e_\nu) \qquad |e_\nu|\leqslant C_{\alpha,r}|\psi_r|_0\leqslant C_{\alpha,r}R^{-2r}|\phi_\nu^2|_0,$$

where $C_{\alpha,r}$ are finite constants.

Finally, we split the term d into two parts d' and d'',

$$d'=(2\pi)^{1-n}\int\xi'_\alpha\vartheta_1(\xi')\,d\xi'\int_0^{t+R}G^{(j)}(t,s,\xi')\hat{F}^2(s,\xi')\,ds$$

and

$$d''=(2\pi)^{1-n}\int\xi'_\alpha\vartheta_1(\xi')\,d\xi'\int_{t+R}^{\infty}G^{(j)}(t,s,\xi')\hat{F}^2(s,\xi')\,ds.$$

Since $(0,t)\in\Omega'$ we have for every $s\leqslant t+R$

$$F^2(s,x')=0 \text{ if } |x'|<R.$$

Thus we can estimate d' in the same way as e_ν and obtain

$$(d') \qquad |d'|\leqslant C_{\alpha,r}(t+R)R^{-2r}|F^2|_0$$

provided that (5.17) holds.

Finally, to estimate d'' we use the exponential factor in (5.12) which has been neglected until now. We then get

$$|d''|\leqslant C\sup|\hat{F}^2(s,\xi')|\int|\xi'|^{\gamma'+|\alpha|}e^{-cR|\xi'|^{1/\gamma}}\,d\xi'$$

and hence, calculating the integral and estimating $\hat{F}^2$, we get

$$(d'') \qquad |d''|\leqslant C\,\Gamma(\gamma(\gamma'+|\alpha|+n-1))(cR)^{-\gamma(\gamma'+|\alpha|+n-1)}|F^2|_0.$$

Summing up all the estimates denoted by (a), (b), ..., we have now proved that

$$|D'_\alpha u^{(j)}(x)| \leqslant C^{|\alpha|+1}|U|_k + C_\alpha |F^1|_{|\alpha|+\gamma'+n} + C_\alpha \sum_1^\mu |\phi^1_\nu|_{|\alpha|+\gamma'+n} +$$

$$+ C_{\alpha,r} R^{-2r}\left(|F^2|_0 + \sum_1^\mu |\phi^2_\nu|_0\right) + C(2r)!\,(cR)^{-2r}|F^2|_0, \quad x\in\Omega', \quad j\leqslant\sigma, \tag{5.18}$$

where r satisfies (5.17) and we have assumed that $cR<1$, $t+R<1$.

This inequality has been established under the assumption that all derivatives of U of transversal order $\leqslant\sigma$ are continuous. Now let us only assume that $u\in C^k(\Omega\cup\omega)$, hence that $U\in C^k(G_+)$. Let $\psi(x')$ be a function which is non-negative, infinitely differentiable, vanishes when $|x'|\geqslant 1$ and satisfies the condition $\int\psi(x')\,dx'=1$. We form the convolution

$$U_\varepsilon(x) = \int U(x-\varepsilon y')\,\psi(y')\,dy',$$

and define $F^1_\varepsilon, \ldots$ in the same way. We also set $R_\varepsilon = R-\varepsilon$. Since one can write

$$U_\varepsilon(x+x') = \int U(x-y')\,\psi((x'-y')/\varepsilon)\,dx'/\varepsilon^{n-1}$$

and U has continuous derivatives of order $\sigma\leqslant k$, it follows that all derivatives of U_ε of transversal order $\leqslant\sigma$ are continuous. Indeed, one can let the tangential differentiations operate on $\psi(x')$. If $\varepsilon<R$, the inequality (5.18) in thus valid if we replace U by U_ε, F^1 by $F^1_\varepsilon, \ldots$ and R by R_ε. Now we have by an obvious convexity argument for an arbitrary function $K\in C^l_0$ that

$$|K_\varepsilon|_l \leqslant |K|_l$$

for all ε. Hence the right hand side of (5.18) is bounded when $\varepsilon\to 0$ and thus the left hand side is bounded too. Therefore, every derivative of transversal order $\leqslant\sigma$ of U_ε is uniformly bounded in $\overline{\Omega}'$ when $\varepsilon\to 0$, and hence those of transversal order $<\sigma$ are equicontinuous. Letting $\varepsilon\to 0$ through a suitable sequence we can therefore assume that $D_\alpha U_\varepsilon$ is uniformly convergent in $\overline{\Omega}'$ for every differentiation D_α of transversal order $<\sigma$. But this shows that all derivatives of the limit U of transversal order $<\sigma$ exist and are continuous. In virtue of the remarks at the beginning of this section, this completes the proof of Theorem 3.1, except for a verification of Lemma 5.1.

Proof of Lemma 5.1. We first have to rewrite the assumption (3.1) in our present notation. Let A' be the set of those complex ζ' such that the equation

$$P(\zeta'+\tau N)=0$$

has precisely μ zeros with positive imaginary part and none that is real. To simplify the writing we assume, which is no real restriction, that the norm is so chosen that $|\zeta'| = |\dot{\zeta}'|$. It then follows from Theorem 2.2 that A' contains all ζ' satisfying

$$|\operatorname{Im} \zeta'| \leqslant B, \ |\operatorname{Re} \zeta'| \geqslant B'. \tag{5.19}$$

Thus (3.1) means that to any number B there is a number B'' such that

$$|\operatorname{Im} \zeta'| \leqslant B, \ |\operatorname{Re} \zeta'| \geqslant B'' \tag{5.20}$$

implies that $\zeta' \in A'$ and $C(\zeta') \neq 0$. We shall now prove that one can take for B' and B'' sufficiently high powers of B. This is the main step in the proof of Lemma 5.1.

An estimate of B' is contained in Lemma 3.10 in Hörmander [5]. Indeed, this emma can be written in the following way: There are constants $\varrho > 0$ and $C > 0$ such that

$$P(\zeta) \neq 0 \ \text{if} \ |\operatorname{Re} \zeta| \geqslant C(1 + |\operatorname{Im} \zeta|^{\varrho}).$$

As in the proof of Theorems 2.1 and 2.2 it then follows that A' contains all ζ' satisfying the inequality

$$|\operatorname{Re} \zeta'| \geqslant C(1 + |\operatorname{Im} \zeta'|^{\varrho}). \tag{5.19$'$}$$

For if this inequality is fulfilled and τ is real, we have

$$|\operatorname{Re}(\zeta' + \tau N)| \geqslant |\operatorname{Re} \zeta'| \geqslant C(1 + |\operatorname{Im} \zeta'|^{\varrho}) = C(1 + |\operatorname{Im}(\zeta' + \tau N)|),$$

and hence that $P(\zeta' + \tau N) \neq 0$, which obviously implies the assertion.

Also note that if τ is a complex zero of $P(\zeta' + \tau N)$ we get

$$|\operatorname{Re} \zeta'| \leqslant |\operatorname{Re}(\zeta' + \tau N)| \leqslant C(1 + |\operatorname{Im}(\zeta' + \tau N)|^{\varrho}) \leqslant C_1(1 + |\operatorname{Im} \zeta'|^{\varrho} + |\operatorname{Im} \tau|^{\varrho})$$

hence

$$C_1 |\operatorname{Im} \tau|^{\varrho} \geqslant |\operatorname{Re} \zeta'| - C_1(1 + |\operatorname{Im} \zeta'|^{\varrho}). \tag{5.21}$$

This estimate will be useful later. If P is *elliptic*, it follows from Lemma 2.1 that we can take $\varrho = 1$.

LEMMA 5.3. *For sufficiently large M and γ, the set*

$$D = \{\zeta'; \ |\operatorname{Re} \zeta'| \geqslant M(1 + |\operatorname{Im} \zeta'|^{\gamma})\} \tag{5.11}$$

is contained in A' and $C(\zeta') \neq 0$ in D.

Proof. It follows from (5.19)$'$ that the set D is contained in A' for large M and γ; this is of course true if $\gamma = \varrho$ and $M \geqslant C$. It thus remains to study the zeros of $C(\zeta')$. Assuming that $C(\zeta')$ has a zero in A' with $|\mathrm{Re}\,\zeta'|\geqslant t$ for every t — otherwise there is nothing to prove — we write

$$M(t)=\inf|\mathrm{Im}\,\zeta'|$$

where the infimum is taken over all $\zeta'\in A'$ with $|\mathrm{Re}\,\zeta'|\geqslant t$ such that $C(\zeta')=0$. We shall prove that $M(t)$ is a piecewise algebraic function of t. Since we know from (5.20) that $M(t)\to\infty$ when $t\to\infty$, it then follows that $M(t)=ct^{\varepsilon}(1+o(1))$ when $t\to\infty$, with $\varepsilon>0$, $c>0$, if we consider the Puiseux expansion of $M(t)$ at infinity. But then the assertion is proved with $\gamma=\max(\varrho, 1/\varepsilon)$.

That $M(t)$ is piecewise algebraic follows from an elimination theorem of Seidenberg. Indeed, the definition of $M(t)$ may be stated as follows: $M(t)$ is the infimum of all μ such that the following system of equations and inequalities holds:

$$|\mathrm{Re}\,\zeta'|^2\geqslant t^2,\quad |\mathrm{Im}\,\zeta'|^2=\mu^2,\quad \mu>0,\quad P(\zeta'+\tau N)=\sum_1^{\sigma}(\tau-\tau_j),$$

$$\mathrm{Im}\,\tau_1>0,\ \dots,\ \mathrm{Im}\,\tau_\mu>0,\quad \mathrm{Im}\,\tau_{\mu+1}<0,\ \dots,\ \mathrm{Im}\,\tau_\sigma<0,$$

$$k(\tau)=\sum_1^{\mu}(\tau-\tau_j),\quad 0=R(k(\tau);\,Q_1(\zeta'+\tau N),\ \dots,\ Q_\mu(\zeta'+\tau N)).$$

This is in fact a system of polynomial equalities and inequalities involving only real variables, $\mathrm{Re}\,\zeta_j'$, $\mathrm{Im}\,\zeta_j'$, μ, t, $\mathrm{Im}\,\tau_j$, $\mathrm{Re}\,\tau_j$, the real and imaginary parts of the coefficients of k. It thus follows from the results of Seidenberg [14] (Theorem 3) that the system can be satisfied by a suitable choice of the other variables if and only if μ and t satisfy one of a finite number of systems, each composed by a finite number of simultaneous equations and inequalities. Since for fixed t the infimum of all μ with this property is $M(t)$, it follows that $\mu=M(t)$ must satisfy some of the equations or make some of the inequalities to an equality. This implies that $M(t)$ is piecewise algebraic. The details of the argument are precisely the same as in the proof of Lemma 3.9 in Hörmander [5] and need not be repeated.

LEMMA 5.4. *If the set D defined by* (5.11) *is contained in A' and if $C(\zeta')\neq 0$ in D, then there are constants M_1 and c_1 such that*

$$|1/C(\zeta')|\leqslant M_1|\zeta'|^{c_1},\quad \zeta'\in D. \tag{5.22}$$

Proof. As in the proof of Lemma 5.3 we can prove that the supremum of $1/C(\zeta')$ when $\zeta'\in D$ and $|\mathrm{Re}\,\zeta'|=t$ is a piecewise algebraic function of t, and hence the Lemma follows in precisely the same way. The details may be left to the reader.

End of the proof of Lemma 5.1. In virtue of Lemmas 5.3 and 5.4 we can choose the constants M and γ in the definition (5.11) of D so that $D \subset A'$ and

$$|1/C(\zeta')| \leqslant M_1 |\zeta'|^{c_1}, \quad \zeta' \in D. \tag{5.22}$$

We may also assume that $|\operatorname{Re} \zeta'| \geqslant 2C_1(1 + |\operatorname{Im} \zeta'|^{\varrho})$ in D so that in virtue of (5.21) we have, since $\varrho \leqslant \gamma$, if $P(\zeta' + \tau N) = 0$ and $\zeta' \in D$,

$$|\operatorname{Im} \tau| \geqslant C_2 |\operatorname{Re} \zeta'|^{1/\gamma}. \tag*{(5.21)'}$$

Since we may assume that $\gamma \geqslant 1$ (this follows in fact from (5.21)), we can estimate $|\operatorname{Im} \zeta'|$ by $|\operatorname{Re} \zeta'|$ in D and write instead of (5.21)'

$$|\operatorname{Im} \tau| \geqslant C_3 |\zeta'|^{1/\gamma}. \tag*{(5.21)''}$$

We also recall that according to Lemma 2.3 there are constants C and d such that

$$|\tau| \leqslant C(|\zeta'|^d + 1). \tag{5.23}$$

Now we can easily estimate the Poisson kernels and their derivatives. We have by (1.14), with the notations of section 2, p. 240,

$$H_\nu^{(j)}(t, \zeta') = C(\zeta')^{-1} R(k_{\zeta'}; q_{\zeta'}^1, \ldots, q_{\zeta'}^{\nu-1}, (i\tau)^j e^{it\tau}, q_{\zeta'}^{\nu+1}, \ldots, q_{\zeta'}^{\mu}).$$

To estimate $H_\nu^{(j)}$ we now only have to use the estimate (5.22) and inequality (1.9). In fact, the inequalities (5.21)'' and (5.23) show that the convex hull K of the zeros of k is contained in the circle $|\tau| \leqslant C(|\zeta'|^d + 1)$ and also in the half plane $\operatorname{Im} \tau \geqslant C_3 |\zeta'|^{1/\gamma}$. We can therefore estimate the polynomials $q_{\zeta'}^j$ and their derivatives in K by a power of $|\zeta'|$, and noting that $|e^{it\tau}| \leqslant e^{-tC_3|\zeta'|^{1/\gamma}}$ in K, we obtain the estimate (5.13) with suitable M' and γ'.

Let $g_0(t, \zeta')$ be the fundamental solution of $P(\zeta' + \delta N)$ as given by (1.12). Since the estimates (5.21)'' and (5.23) ase valid for the zeros of $P(\zeta' + \delta N)$, it follows from (1.16) that

$$|g_0^{(j)}(t-s, \zeta')| \leqslant 2^{\sigma+j}(C(|\zeta'|^d + 1))^j e^{-C_4|\zeta'|^{1/\gamma}|t-s|}, \; t \neq s,$$

if $\zeta' \in D$. Here C_4 is a positive constant. But in virtue of (1.13) we have

$$g^{(j)}(t, s, \zeta') = g_0^{(j)}(t-s, \zeta') - \sum_1^{\mu} (Q_\nu(\zeta' + \delta N) g_0)(-s, \zeta') H_\nu^{(j)}(t, \zeta'),$$

and hence the desired estimate (5.12) follows from (5.13) and the above estimate of $g_0^{(j)}$. The proof is complete.

6. Sufficiency of the condition for ellipticity

In this section we assume throughout that the condition (3.2) of Theorem 3.2 is fulfilled and we shall prove that a solution $u \in C^k(\Omega \cup \omega)$ of the equations

$$P(D)u = 0 \text{ in } \Omega, \qquad Q_\nu(D)u = 0 \text{ in } \omega, \quad \nu = 1, \dots, \mu, \tag{6.1}$$

is then analytic in $\Omega \cup \omega$. As in section 5 we could have studied the inhomogeneous case also, but since this can be reduced to the homogeneous case by means of the Cauchy-Kovalevsky theorem, we shall not do so.

From the results of section 5 we know already that u is infinitely differentiable. Let Ω' be a domain such as in section 5.

LEMMA 6.1. *If for a solution u of the equation*

$$P(D)u = 0 \tag{6.2}$$

we have
$$|D'_\alpha u^{(j)}(x)| \leqslant C^{|\alpha|+1}|\alpha|!, \quad x \in \Omega', \ 0 \leqslant j < \sigma, \tag{6.3}$$

it follows that u is analytic in a neighbourhood of $\bar{\Omega}'$.

Proof. As observed at the beginning of section 5, the equation (6.2) can be written

$$u^{(\sigma)} = \sum_{j<\sigma} P_j(D')u^{(j)}, \tag{6.2$'$}$$

where $P_j(D')$ is a tangential differential operator of order at most $\sigma - j$. Let K be a bound for the sum of the absolute values of the coefficients in the operators P_j. Assuming as we may that $C \geqslant 1$ and $K \geqslant 1$, we shall prove

$$|D'_\alpha u^{(j)}(x)| \leqslant C^{|\alpha|+j+1}K^j(|\alpha|+j)!, \quad x \in \Omega'. \tag{6.4}$$

This follows from (6.3) when $j < \sigma$. Assume that the inequality has already been proved for $j < J + \sigma$, where $J \geqslant 0$. We shall prove it for $j = J + \sigma$. Differentiating (6.2)$'$ we obtain

$$D'_\alpha u^{(J+\sigma)}(x) = \sum_{j<\sigma} D'_\alpha P_j(D')u^{(j+J)}(x).$$

By assumption we can use (6.4) to estimate the terms in the sum on the right. This gives

$$|D'_\alpha u^{(J+\sigma)}(x)| \leqslant K\,C^{|\alpha|+J+\sigma+1}K^{J+\sigma-1}(|\alpha|+J+\sigma)!, \quad x \in \Omega',$$

which proves that (6.4) holds.

From (6.4) it follows that the Taylor series expansion of u at a point x in Ω' is convergent in a sphere with a radius independent of x and that it converges to u in Ω'. Hence it follows that u is analytic in a neighbourhood of $\bar{\Omega}'$, which proves the Lemma.

It now only remains to prove that the estimates of the previous section can be improved so that (6.3) follows. According to our present assumptions we can take $\gamma=1$ in Lemma 5.3 and hence also in Lemma 5.1. This is the fact which gives rise to estimates of the form (6.3).

First, the estimate (a) of the term a is even better than that required since it does not contain any factorial on the right. Also, the estimate (d'') has obviously the desired form. Since we only consider the homogeneous equations (6.1) the terms b and c vanish so that it only remains to study d' and e_ν. These terms have to be considered more carefully, however. Indeed, in estimating them we have differentiated repeatedly on the non analytic "cut off" function ϑ_1, and this has to be avoided if we want to obtain an estimate of the form (6.3).

We first study e_ν. Writing

$$K(\xi')=\xi'_\alpha H_\nu^{(j)}(t,\xi')(2\pi)^{1-n}, \quad j<\sigma,$$

we have

$$e_\nu=\int(\Delta'^r\hat{\psi}_r(\xi'))K(\xi')\vartheta_1(\xi')d\xi'.$$

Instead of integrating by parts, which leads to repeated differentiations of ϑ_1, we note that

$$(\Delta'^r\hat{\psi}_r)K-\hat{\psi}_r\Delta'^rK=\operatorname{div}' V,$$

where

$$V=\sum_0^{r-1}((\Delta'^jK)(\operatorname{grad}'\Delta'^{r-j-1}\hat{\psi}_r)-(\Delta'^{r-j-1}\hat{\psi}_r)(\operatorname{grad}'\Delta'^jK)).$$

Here we have denoted by div′ and grad′ the operations of divergence and gradiend with respect to the boundary variables. Using this identity and integrating by parts only once we now obtain

$$e_\nu=\int\hat{\psi}_r(\Delta'^rK)\vartheta_1d\xi'-\int(\operatorname{grad}'\vartheta_1,V)d\xi'. \tag{6.5}$$

We now use inequality (5.16). If r is the smallest integer such that (5.17) holds, it follows that the first integral in (6.5) can be estimated by

$$(2r)!\,C_1^{|\alpha|+1}|\psi_r|_0\leqslant(2r)!\,C_1^{|\alpha|+1}R^{-2r}|\phi_\nu^2|_0,$$

where C_1 is a constant. Next consider the second integral in (6.5). We have

$$|\Delta'^{r-j-1}\hat{\psi}_r| = |\hat{\psi}_{j+1}| \leqslant C R^{-2(j+1)} |\phi_\nu^2|_0$$

and

$$|\operatorname{grad} \Delta'^{r-j-1}\hat{\psi}_r| \leqslant C R^{-2j-1} |\phi_\nu^2|_0.$$

From (5.16) it follows that there is a constant C_2 such that with r chosen as above we have in the support of grad ϑ_1, which is compact,

$$|\Delta'^j K| \leqslant C_2^{2r}(2r)!,\ j \leqslant r,$$

and recalling the proof of (5.16) we find that if C_2 is large enough we have also

$$|\operatorname{grad}' \Delta'^j K| \leqslant C_2^{2r}(2r)!,\ j < r,$$

in the support of grad ϑ_1. The second integral in (6.5) can therefore be estimated by

$$C_3 C_2^{2r}(2r)! \sum_{j<2r} R^{-j} |\phi_\nu^2|_0.$$

If we now recall that r is the smallest integer such that (5.17) holds and if we sum the geometric series, it follows that with a constant C, depending on R but not on α and u, this can be estimated by

$$C^{|\alpha|+1}|\alpha|!\ |u|_k.$$

It is obvious that d' can be estimated in the same way as we have estimated e_ν. Thus there is a constant C, depending on Ω and Ω' but independent of u and α such that if u satisfies (6.1) we have

$$|D'_\alpha u^{(j)}(x)| \leqslant C^{|\alpha|+1}|\alpha|!\ |u|_k,\ x \in \Omega',\ \ j < \sigma. \tag{6.6}$$

This completes the proof of Theorem 3.2. Moreover, it follows from the proof of Lemma 6.1 that the solutions of (6.1) can be continued across ω into a domain Ω^* independent of u. In the classical case mentioned in the introduction Ω^* is obtained by geometric reflection of Ω. It would be interesting to investigate more carefully the size of the largest domain Ω^* to which all solutions of (6.1) have analytic continuations. A special case of this question has been answered by F. John [7].

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