CONCORDANCE AND THE HARMONIC ANALYSIS OF SEQUENCES

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1. Introduction

This paper introduces a theory which can be regarded as a natural completion of the classical theories of harmonic analysis and of almost periodic functions. This theory is primarily concerned with the harmonic analysis of almost periodic sequences,(1) and it brings to light the phenomenon that such almost periodic sequences necessarily have almost periodic frequencies. I have called this phenomenon "concordance".

The theory is also connected with theories of summation formulae and of meromorphic almost periodic functions, and it explains the nature of the connexion between the prime numbers and the zeros of the Riemann zeta-function.

In the second section various known types of Fourier reciprocities connecting functions and sequences are presented in order to show how a theory of almost periodic sequences would complete the classical schemes of harmonic analysis, and analogies are used to suggest the form such a theory should take.

In the third section the choices of suitable definitions for almost periodic sequences are discussed. The choices of definitions made here are certainly not the only ones possible; they are the choices most expedient in first introducing the idea of almost periodicity for sequences. Detailed reasons for these choices are given in section 11 (2).

In the fourth to eighth sections the basic results of the theory are proved, and the connexions with summation formulae and meromorphic almost periodic functions are established.

In the ninth section it is shown how use of the Dirac δ-function gives a formal unification of the different types of harmonic analysis of the second section.

In the tenth section examples are given, and in the eleventh section the results are discussed, together with their relationship to previous work and possible future results.

(1) Defined in Section 3.

2. Fourier reciprocities and types of harmonic analysis

The classical methods of harmonic analysis can be associated with three types of Fourier reciprocity. Formally, these types are as follows.

(A) THE FOURIER SERIES.

If $\varphi(x)$ *is a periodic function of* x *of period* 1, *then*

$$\varphi(x) = \sum_{n=-\infty}^{\infty} k_n e^{2\pi i n x},$$

where

$$k_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(t)\, e^{-2\pi i n t}\, dt.$$

These two formulae can be regarded as a reciprocity between the periodic function $\varphi(x)$ *and the sequence* $\{k_n\}$.

(B) FOURIER INTEGRALS.

By considering the Fourier series of periodic functions with increasing periods ([27], 1–2) *we can derive the formulae*

$$f(x) = \int_{-\infty}^{\infty} g(t)\, e^{2\pi i x t}\, dt$$

and

$$g(x) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i x t}\, dt.$$

These formulae are a form of the Fourier integral formulae, and can be regarded as a reciprocity between the functions $f(x)$ *and* $g(x)$.

(C) THE FINITE FOURIER SERIES.

If $\{A_n\}$ $(n = 0, \pm 1, \pm 2, \ldots)$ *is a periodic sequence of period* N,(1) *then there is a finite Fourier series for* A_n *which can be written*(2)

$$A_n = N^{-\frac{1}{2}} \sum_{m=1}^{N} B_m e^{2\pi i m n/N},$$

where

$$B_n = N^{-\frac{1}{2}} \sum_{m=1}^{N} A_m e^{-2\pi i m n/N}.$$

These two formulae can be regarded as a reciprocity between the two periodic sequences $\{A_n\}$ *and* $\{B_n\}$.

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(1) That is $A_{n+N} = A_n$ for all n.

(2) Alternatively, the sums may be taken over any complete set of residues modulo N.

Thus we have three types of Fourier reciprocity:

(A) between a function and a sequence,

(B) between two functions,

(C) between two sequences.

Now the reciprocity (A) was extended by Bohr [7] and others to give the theory of almost periodic functions. The basic formulae of this theory give the following extension of the reciprocity (A).

(A′) *If $f(x)$ is an almost periodic function of x, then it is associated with a Fourier series*

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n x},$$

where

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, e^{-i\lambda t}\, dt = \begin{cases} c_n & (\lambda = \lambda_n), \\ 0 & \textit{elsewhere}. \end{cases}$$

These two formulae can be regarded as a reciprocity between the almost periodic function $f(x)$ and the weighted sequence(1) $\{c_n, \lambda_n\}$.

Thus the reciprocity (A′) is that extension of the reciprocity (A) which is obtained by allowing arbitrary real frequencies λ_n instead of restricting the frequencies to integral multiples of a fundamental frequency, as in (A).

We can now ask if there exist analogous extensions of the reciprocities (B) and (C). In the case of the reciprocity (B) the frequencies involved already range over all real values, so there is no analogous extension. However, for later convenience, let us note the form of reciprocity below in which the factor 2π does not occur in the exponent.

(B′) *If*

$$F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} G(t)\, e^{ixt}\, dt,$$

then

$$G(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} F(t)\, e^{-ixt}\, dt.$$

In the case of the reciprocity (C), no extension analogous to (A′) has yet been discussed. The form which such an extension might be expected to take is:

(C′) *If $\{a_n, \alpha_n\}$ is, in some sense, an almost periodic weighted sequence, then there exists a weighted sequence $\{b_n, \beta_n\}$ such that*

(1) We use the term "weighted sequence" and the notation $\{c_n, \lambda_n\}$ to denote a sequence of values $\dots \lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \dots$ in which each λ_n is associated with a corresponding weight c_n. The sequence $\{\lambda_n\}$ will be called the "basis" of the weighted sequence $\{c_n, \lambda_n\}$.

$$(2\pi)^{\frac{1}{2}} \lim_{T\to\infty} \frac{1}{2T} \sum_{-T<\beta_n<T} b_n\, e^{i\beta_n x} = \begin{cases} a_n & (x=\alpha_n), \\ 0 & \textit{elsewhere}, \end{cases}$$

and

$$(2\pi)^{\frac{1}{2}} \lim_{T\to\infty} \frac{1}{2T} \sum_{-T<\alpha_n<T} a_n\, e^{-i\alpha_n x} = \begin{cases} b_n & (x=\beta_n), \\ 0 & \textit{elsewhere}. \end{cases}$$

These formulae could be regarded as a reciprocity between the weighted sequences $\{a_n, \alpha_n\}$ *and* $\{b_n, \beta_n\}$, *and it is to be expected that the weighted sequence* $\{b_n, \beta_n\}$ *should also be almost periodic in some sense.*

This reciprocity includes the reciprocity (C) as the special case in which the sequences of weights $\{a_n\}$ and $\{b_n\}$ are periodic with period N, and the bases $\{\alpha_n\}$ and $\{\beta_n\}$ are equally spaced sequences given by(1)

$$\alpha_n = \beta_n = n(2\pi/N)^{\frac{1}{2}}.$$

Further trivial examples of weighted sequences satisfying reciprocities of the form (C′) are easily constructed from finite combinations of periodic weighted sequences.(2) For instance if $a_n = 1$ and $\{\alpha_n\}$ consists of the sequence of integers and of integral multiples of $\sqrt{2}$, then the reciprocity (C′) holds with $\{\beta_n\}$ the sequence of integral multiples of 2π and of $\pi\sqrt{2}$, where $b_n = (2\pi)^{\frac{1}{2}}$ in the former case, $b_n = \pi^{\frac{1}{2}}$ in the latter.

Now the "Parseval equations" play an important part in developing the theories of Fourier series and Fourier integrals. Let us therefore consider their possible forms for the present reciprocities. For the reciprocity (B′), if $F_1(x)$, $G_1(x)$ and $F_2(x)$, $G_2(x)$ are two pairs of functions, each connected by the reciprocity, then various forms of the Parseval equations are

$$\int_{-\infty}^{\infty} |F_1(x)|^2\, dx = \int_{-\infty}^{\infty} |G_1(x)|^2\, dx,$$

$$\int_{-\infty}^{\infty} F_1(x)\, F_2(x)\, dx = \int_{-\infty}^{\infty} G_1(x)\, G_2(-x)\, dx,$$

$$\int_{-\infty}^{\infty} F_1(x)\, G_2(x)\, dx = \int_{-\infty}^{\infty} F_2(x)\, G_1(x)\, dx,$$

and other obvious variations of these.

Similarly if $f_1(x)$ and $f_2(x)$ are two almost periodic functions as in the reciprocity (A′), and if their Fourier series are

(1) Such weighted sequences will be said to be periodic.

(2) In the same way finite combinations of periodic functions, such as $\sin x + \sin x\sqrt{2}$, are used to introduce almost periodic functions. Cf. [2], ix.

$$\sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n x}, \qquad \sum_{n=-\infty}^{\infty} d_n e^{i\mu_n x}$$

respectively, then the usual Parseval equations ([6], 60–67) are

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f_1(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{T} f_1(t)\, f_2(t)\, dt = \sum_{\lambda_m=-\mu_n} c_m d_n.$$

Another result which could also be described as a Parseval equation in this case is(1)

$$\sum_{n=-\infty}^{\infty} c_n f_2(\lambda_n) = \sum_{n=-\infty}^{\infty} d_n f_1(\mu_n).$$

For the conjectured reciprocity (C′) we have, formally,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T} \sum_{-T<\alpha_n<T} |a_n|^2 &= (2\pi)^{\frac{1}{2}} \lim_{T\to\infty} \frac{1}{2T} \sum_{-T<\alpha_n<T} \bar{a}_n \lim_{U\to\infty} \frac{1}{2U} \sum_{-U<\beta_m<U} b_m e^{i\beta_m\alpha_n} \\ &= (2\pi)^{\frac{1}{2}} \lim_{U\to\infty} \frac{1}{2U} \sum_{-U<\beta_m<U} b_m \lim_{T\to\infty} \frac{1}{2T} \sum_{-T<\alpha_n<T} \bar{a}_n e^{i\beta_m\alpha_n} \\ &= \lim_{U\to\infty} \frac{1}{2U} \sum_{-U<\beta_m<U} |b_m|^2.\end{aligned}$$

That is, we would expect a Parseval equation of the form

$$\lim_{T\to\infty} \frac{1}{2T} \sum_{-T<\alpha_n<T} |a_n|^2 = \lim_{T\to\infty} \frac{1}{2T} \sum_{-T<\beta_n<T} |b_n|^2.$$

Further, if $\{c_n, \gamma_n\}$, $\{d_n, \delta_n\}$ is another pair of weighted sequences which are Fourier transforms(2) of the type (C′), then a similar formal argument leads to the Parseval equation

(1) I do not know of any discussion of such an equation. Formally, each side is equal to the double sum

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m d_n e^{i\lambda_m \mu_n}.$$

(2) By analogy with the term "Fourier transforms" as applied to functions, we also describe a pair of weighted sequences connected by the reciprocity (C′) as Fourier transforms of each other.

$$\lim_{T\to\infty} \frac{1}{2T} \sum_{-T<\alpha_m=\delta_n<T} a_m d_n = \lim_{T\to\infty} \frac{1}{2T} \sum_{-T<\beta_m=\gamma_n<T} b_m c_n.$$

Now the existence of all these Parseval equations suggests that there may also be "mixed Parseval equations" involving pairs of transforms of different types. For example, if $\varphi(x)$, $\{k_n\}$ are connected by the reciprocity (A), and $f(x)$, $g(x)$ by the reciprocity (B), then, formally(¹),

$$\int_{-\infty}^{\infty} g(x)\,\varphi(x)\,dx = \sum_{n=-\infty}^{\infty} k_n \int_{-\infty}^{\infty} g(x)\,e^{2\pi i n x}\,dx = \sum_{n=-\infty}^{\infty} k_n f(n).$$

For a function and sequences connected by the reciprocities (A) and (C) a similar formal argument gives

$$N^{-\frac{1}{2}} \sum_{n=1}^{N} B_n \varphi(n/N) = \sum_{n=-\infty}^{\infty} A_n k_n.$$

For functions and sequences connected by the reciprocities (B) and (C) it does not seem that a direct formal argument can give a Parseval equation. Nevertheless the formula (cf. [11], [25])

$$\sum_{n=-\infty}^{\infty} B_n f(n N^{-\frac{1}{2}}) = \sum_{n=-\infty}^{\infty} A_n g(n N^{-\frac{1}{2}})$$

is an extension of Poisson's summation formula, and is of the form to be expected for a "mixed Parseval equation" for transforms of types (B) and (C). It can be deduced formally from Poisson's summation formula as follows.

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} B_n f(n N^{-\frac{1}{2}}) &= \sum_{m=-\infty}^{\infty} \sum_{r=1}^{N} B_{mN+r} f(N^{\frac{1}{2}} m + N^{-\frac{1}{2}} r) \\
&= \sum_{r=1}^{N} B_r \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(N^{\frac{1}{2}} t + N^{-\frac{1}{2}} r)\, e^{-2\pi i m t}\, dt \\
&= N^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} g(m N^{-\frac{1}{2}}) \sum_{r=1}^{N} B_r e^{2\pi i m r/N} \\
&= \sum_{m=-\infty}^{\infty} A_m g(m N^{-\frac{1}{2}}).
\end{aligned} \tag{2.1}$$

For functions and sequences connected by the reciprocities (B′) and (C′) we might expect a "mixed Parseval equation" or "summation formula" of the form

$$\sum_{n=-\infty}^{\infty} b_n F(\beta_n) = \sum_{n=-\infty}^{\infty} a_n G(\alpha_n). \tag{2.2}$$

(¹) Such results have been discussed, but they were not regarded as Parseval equations. Cf. [1], [19].

In this case we can no longer deduce the result from Poisson's summation formula. However, the formula (2.2) plays an essential part in the present development of the theory of almost periodic weighted sequences, and we establish conditions for the formula in the sixth section.

3. The definition of an almost periodic weighted sequence

In order to prove any form of the reciprocity (C′) we must decide on a suitable definition for almost periodicity of a weighted sequence. We begin by using the definition of a uniformly almost periodic function as a guide, and then we make successive modifications of the definition until we arrive at a form of definition more suitable for discussing the reciprocity (C′).

A function $f(x)$ is said to be a uniformly almost periodic (u.a.p.) function ([6], p. 32) of x if

(i) for any given $\varepsilon > 0$ there exists a relatively dense set of translation numbers $\tau = \tau(\varepsilon)$ for which
$$|f(x+\tau) - f(x)| < \varepsilon$$
for all x, and

(ii) $f(x)$ is continuous.

If we regard the weighted sequence $\{a_n, \alpha_n\}$ as equivalent to a function $H(x)$ which takes the value a_n when $x = \alpha_n$ and vanishes when x is not equal to any value of α_n, then part (i) of the definition of a u.a.p. function can only apply to $H(x)$ if the α_n are equally spaced, and the weights a_n are the values of some u.a.p. function of x at the points $x = \alpha_n$. Sequences which are almost periodic in this sense have been considered by Walther, but such a definition is too restricted for our purposes.(1)

Instead of associating the weighted sequence $\{a_n, \alpha_n\}$ with the function $H(x)$, we associate it with a function $K(x)$ obtained by choosing some suitable function $k(x)$, preferably vanishing outside a finite range of values of x, and then making each α_n contribute a term $a_n k(x - \alpha_n)$ to $K(x)$. That is, we put
$$K(x) = \sum_{n=-\infty}^{\infty} a_n k(x - \alpha_n), \tag{3.1}$$
and we use the following preliminary definition.

DEFINITION 1. *A weighted sequence $\{a_n, \alpha_n\}$ in which the α_n are real and arranged in increasing order of magnitude is said to be uniformly almost periodic (u.a.p.) for $k(x)$ if the function $K(x)$ defined by* (3.1) *exists for all real x and is uniformly almost periodic.*

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(1) [28], [29]. Walther's definition does not cover most of the examples of section 10.

If $K(x)$ is to be u.a.p. then we must chose $k(x)$ continuous to get continuity of $K(x)$. Let us take

$$k(x) = \begin{cases} p+q+x & (-p-q \leqslant x \leqslant -q), \\ p & (-q \leqslant x \leqslant q), \\ p+q-x & (q \leqslant x \leqslant p+q), \\ 0 & \text{elsewhere}. \end{cases} \tag{3.2}$$

That is, the graph of $k(x)$ is a trapezoid with the sloping sides at $\pm 45^\circ$ to the x axis, the height p, and the length of the top $2\,q$. We then take the following special case of Definition 1.

Definition 2. *A weighted sequence $\{a_n, \alpha_n\}$ in which the α_n are real and arranged in increasing order of magnitude is said to be uniformly almost periodic for trapezoidal functions if the functions $K(x)$ defined by* (3.1) *with $k(x)$ defined by* (3.2) *are continuous uniformly for all positive p and q, are uniformly almost periodic and have relatively dense common translation numbers $\tau = \tau(\varepsilon)$ for any $\varepsilon > 0$ and all p, $q \geqslant 0$. That is*

$$|K(x+\tau) - K(x)| < \varepsilon$$

for all x and all p, $q \geqslant 0$.

It is more convenient in the sequel to use an equivalent definition which is less direct, but which does not depend on the somewhat arbitrary choice of the function $k(x)$ made in Definition 2. This definition is:

Definition 3. *A weighted sequence $\{a_n, \alpha_n\}$ in which the α_n are real and arranged in increasing order of magnitude and $\alpha_0 = 0$ is said to be uniformly almost periodic with respect to $A(x)$ if the function defined by*

$$\left.\begin{aligned} &\tfrac{1}{2} a_0 x + \sum_{0<\alpha_n<x} a_n (x-\alpha_n) - \int_0^x A(t)\,dt \quad (x \geqslant 0), \\ &-\tfrac{1}{2} a_0 x + \sum_{x<\alpha_n<0} a_n (\alpha_n - x) + \int_x^0 A(t)\,dt \quad (x \leqslant 0), \end{aligned}\right\} \tag{3.3}$$

is uniformly almost periodic.(1)

We now need the following lemmas in order to prove a theorem connecting Definitions 2 and 3.

Lemma α. *If $\psi(x)$ is defined for all real x, and the functions $\psi(x) - \psi(x-p)$ are continuous uniformly for all p, and are uniformly almost periodic and have relatively dense*

(1) Putting $\alpha_0 = 0$ is only a matter of arranging the notation for the basis conveniently. If zero does not occur in the basis then we can still write $\alpha_0 = 0$ with $a_0 = 0$.

common translation numbers for all p, then there exists a function $c(x)$, satisfying the functional equation

$$c(x) + c(y) = c(x + y),$$

such that $\psi(x) - c(x)$ is uniformly almost periodic. Further, any common translation number belonging to $\frac{1}{2}\varepsilon$ for the functions $\psi(x) - \psi(x - p)$ is a translation number belonging to ε for $\psi(x) - c(x)$.

Proof. Put

$$c(p) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{\psi(t) - \psi(t - p)\}\, dt.$$

The limit exists for all p since $\psi(x) - \psi(x - p)$ is u.a.p. It follows by addition that

$$c(p) + c(q) = c(p + q) \tag{3.4}$$

for all p and q.

Let $\tau = \tau(\frac{1}{2}\varepsilon)$ be a common translation number belonging to $\frac{1}{2}\varepsilon$ for the functions $\psi(x) - \psi(x - p)$. Then for all x and p

$$|\{\psi(x + \tau) - \psi(x + \tau - p)\} - \{\psi(x) - \psi(x - p)\}| < \tfrac{1}{2}\varepsilon.$$

Putting $p = x$ we have

$$|\{\psi(x + \tau) - \psi(x)\} - \{\psi(\tau) - \psi(0)\}| < \tfrac{1}{2}\varepsilon \tag{3.5}$$

for all x. Now the function within the modulus sign is itself a u.a.p. function of x, so its mean value exists, and is

$$c(\tau) - \{\psi(\tau) - \psi(0)\}.$$

Hence by (3.5)
$$|c(\tau) - \{\psi(\tau) - \psi(0)\}| < \tfrac{1}{2}\varepsilon, \tag{3.6}$$

and therefore

$$|\psi(x + \tau) - \psi(x) - c(\tau)| \leqslant |\{\psi(x + \tau) - \psi(x)\} - \{\psi(\tau) - \psi(0)\}| + |\psi(\tau) - \psi(0) - c(\tau)| < \varepsilon$$

by (3.5) and (3.6). Now by (3.4)

$$c(x + \tau) - c(x) = c(\tau).$$

Hence
$$|\{\psi(x + \tau) - c(x + \tau)\} - \{\psi(x) - c(x)\}| < \varepsilon$$

for all x and relatively dense τ. That is, the function $\psi(x) - c(x)$ satisfies the translation properties in the definition of a u.a.p. function.

Since $\psi(x) - \psi(x - p)$ is continuous uniformly in p, we can find an $\eta = \eta(\frac{1}{2}\varepsilon)$ such that for all $|\delta| < \eta$

$$|\{\psi(x+\delta)-\psi(x+\delta-p)\} - \{\psi(x)-\psi(x-p)\}| < \tfrac{1}{2}\varepsilon$$

for all x and p. A similar argument then proves that $\psi(x)-c(x)$ is continuous.

Hence $\psi(x)-c(x)$ is u.a.p., as required.

LEMMA β (ref. [18], [20]). *If $c(x)$ satisfies the functional equation $c(x+y)=c(x)+c(y)$ for all real x and y, and is continuous at some point, then there exists a constan t C for which $c(x)=Cx$.*

We can now deduce the following result:

THEOREM 1. *If the weighted sequence $\{a_n, \alpha_n\}$ is uniformly almost periodic for trapezoidal functions then there exist constants A, B for which it is uniformly almost periodic with respect to the function $Ax+B$.*

Proof. Consider the function defined by

$$\psi(x)=\begin{cases} \tfrac{1}{2}a_0x+\sum\limits_{0<\alpha_n<x} a_n(x-\alpha_n) & (x>0), \\ 0 & (x=0), \\ -\tfrac{1}{2}a_0x+\sum\limits_{x<\alpha_n<0} a_n(\alpha_n-x) & (x<0). \end{cases} \tag{3.7}$$

In the function

$$\psi(x+p+q)-\psi(x+q)-\psi(x-q)+\psi(x-p-q) \tag{3.8}$$

the term in a_n contributes a term $a_n\,k(x-\alpha_n)$ to (3.8), where $k(x)$ is the trapezoidal function of (3.2). Hence by Definition 2 the function (3.8) is u.a.p. for all p, $q>0$. Writing (3.8) as

$$\{\psi(x+p+q)-\psi(x-q)\} - \{\psi(x+q)-\psi(x-p-q)\}$$

it follows from Lemma α that there exists a function $c_1(x)$ satisfying (3.4) for which

$$\psi(x+p+q)-\psi(x-q)-c_1(x) \tag{3.9}$$

is u.a.p. Now $\psi(x)$ is continuous from its definition (3.7), and (3.9) is continuous since it is u.a.p. Hence $c_1(x)$ is continuous, and must be of the form Ax, say, by Lemma β. That is, for constants A depending on p and q only

$$\psi(x+p+q)-\psi(x-q)-Ax$$

is u.a.p. and has common translation numbers for all p, $q\geqslant 0$. Alternatively, if we put $r=p+2q$, then for appropriate constants $A(r)$ depending on r only

$$\psi(x+r)-\psi(x)-xA(r) \tag{3.10}$$

is u.a.p. and has common translation numbers for all $r\geqslant 0$. Hence both

$$\psi(x+r+s)-\psi(x+r)-xA(s) \tag{3.11}$$

and

$$\psi(x+r+s)-\psi(x)-xA(r+s) \tag{3.12}$$

are also u.a.p. for $r, s \geqslant 0$. Subtracting (3.12) from the sum of (3.10) and (3.11), it follows that

$$x\{A(r+s)-A(r)-A(s)\}$$

is u.a.p. This can only be true if it is zero, hence $A(r)$ is a solution of (3.4).

Let $\tau > 1$ be a common translation number belonging to $\frac{1}{4}\varepsilon$ for the functions (3.10). Then for any r, s

$$|\psi(x+\tau+r)-\psi(x+\tau)-\psi(x+r)+\psi(x)-\tau A(r)| < \tfrac{1}{4}\varepsilon,$$

and

$$|\psi(x+\tau+s)-\psi(x+\tau)-\psi(x+s)+\psi(x)-\tau A(s)| < \tfrac{1}{4}\varepsilon.$$

Since $\psi(x)$ is continuous we can choose x and then find an $\eta = \eta(\varepsilon)$ such that for $|r-s| < \eta$ we have

$$|\psi(x+\tau+r)-\psi(x+\tau+s)| < \tfrac{1}{4}\varepsilon,$$

and

$$|\psi(x+r)-\psi(x+s)| < \tfrac{1}{4}\varepsilon.$$

Then

$$\begin{aligned}
&|\tau\{A(r)-A(s)\}|\\
&\quad = |\{\psi(x+\tau+s)-\psi(x+\tau)-\psi(x+s)+\psi(x)-\tau A(s)\} -\\
&\qquad - \{\psi(x+\tau+r)-\psi(x+\tau)-\psi(x+r)+\psi(x)-\tau A(r)\} +\\
&\qquad + \{\psi(x+\tau+r)-\psi(x+\tau+s)\} - \{\psi(x+r)-\psi(x+s)\}|\\
&\quad < \tfrac{1}{4}\varepsilon+\tfrac{1}{4}\varepsilon+\tfrac{1}{4}\varepsilon+\tfrac{1}{4}\varepsilon = \varepsilon.
\end{aligned}$$

Since $\tau > 1$ this gives

$$|A(r)-A(s)| < \varepsilon.$$

Hence $A(r)$ is continuous, and by Lemma β $A(r) = Ar$ for some constant A. That is, the functions

$$\psi(x+r)-\psi(x)-xAr \tag{3.13}$$

are u.a.p. for all r, with common translation numbers, and are continuous uniformly in r. Hence the functions

$$\{\psi(x+r)-\tfrac{1}{2}A(x+r)^2\} - \{\psi(x)-\tfrac{1}{2}Ax^2\},$$

which differ from (3.13) only by $\frac{1}{2}Ar^2$, satisfy the conditions of Lemma α. Hence

$$\psi(x)-\tfrac{1}{2}Ax^2-B(x)$$

is u.a.p. for some solution $B(x)$ of (3.4). Since $\psi(x) - \frac{1}{2}Ax^2$ is continuous it follows that $B(x)$ is continuous, and, by Lemma β, that $B(x) = Bx$ for some constant B. That is

$$\psi(x) - \tfrac{1}{2}Ax^2 - Bx$$

is u.a.p., and by Definition 3 the weighted sequence $\{a_n, \alpha_n\}$ is u.a.p. with respect to the function $Ax + B$, and Theorem 1 is proved.

Now the function (3.3) of Definition 3 is equal to

$$\int_0^x \Big\{ \sum_{0 \leqslant \alpha_n \leqslant t}' a_n - A(t) \Big\} dt,$$

where the dash indicates that the terms a_n corresponding to $\alpha_n = 0$ or $\alpha_n = t$ are to be halved if they occur, and if t is negative the sum is to be interpreted as

$$- \sum_{t \leqslant \alpha_n \leqslant 0}' a_n.$$

The function

$$\sum_{0 \leqslant \alpha_n \leqslant x}' a_n - A(x) \tag{3.14}$$

cannot be u.a.p. since it is discontinuous, but the result of Theorem 1 suggests that we could make a more direct definition of almost periodicity for weighted sequences if we use a more general class of almost periodic functions, thus:

DEFINITION 4. *A weighted sequence $\{a_n, \alpha_n\}$ in which the α_n are real and arranged in increasing order of magnitude is said to be B^2 almost periodic (B^2 a.p.) with respect to $A(x)$ if the function* (3.14) *is B^2 almost periodic. The function $A(x)$ is then said to be the distribution function of the weighted sequence $\{a_n, \alpha_n\}$.*

It should be noted that we cannot prove strict analogues of Lemma α and Theorem 1 for B^2 almost periodicity since $\psi(x) - \psi(x-k)$ can be a B^2 null-function(1) for all real k even when $\psi(x)$ is not itself a B^2 null-function. A simple example is $\psi(x) = x^{\frac{1}{2}}$.

4. A pair of complex Hankel transforms

I have shown elsewhere how the Parseval theorem for Hankel transforms can be used to prove Poisson's summation formula (see [11]). We can also use this method in the present case with the following complex form of the Hankel inversion formula.

(1) If

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |R(x)|^2 \, dx = 0$$

then $R(x)$ is said to be a B^2 null-function.

LEMMA γ. *If $f(x)$ belongs to $L^2(-\infty,\infty)$ then*

$$g(x) = \underset{T\to\infty}{\text{l.i.m.}}\,(2\pi)^{-\frac{1}{2}} \int_{-T}^{T} f(t)\left\{-i\left(\frac{1-e^{-ixt}}{xt}\right) - e^{-ixt}\right\} dt \tag{4.1}$$

converges in mean square, $g(x)$ belongs to $L^2(-\infty,\infty)$, and

$$f(x) = \underset{T\to\infty}{\text{l.i.m.}}\,(2\pi)^{-\frac{1}{2}} \int_{-T}^{T} g(t)\left\{i\left(\frac{1-e^{ixt}}{xt}\right) - e^{ixt}\right\} dt \tag{4.2}$$

almost everywhere. Further, if $f_1(x)$, $g_1(x)$ and $f_2(x)$, $g_2(x)$ are two pairs of such transforms then

$$\int_{-\infty}^{\infty} f_1(x)\, g_2(x)\, dx = \int_{-\infty}^{\infty} f_2(x)\, g_1(x)\, dx. \tag{4.3}$$

Proof. The kernel function in (4.2) is

$$\begin{aligned} j(x) &= (2\pi)^{-\frac{1}{2}}\left\{i\left(\frac{1-e^{ix}}{x}\right) - e^{ix}\right\} \\ &= (2\pi)^{-\frac{1}{2}}\left\{\left(\frac{\sin x}{x} - \cos x\right) + i\left(\frac{1-\cos x}{x} - \sin x\right)\right\} \\ &= \tfrac{1}{2}\, x^{\frac{1}{2}} J_{3/2}(x) + \tfrac{1}{2}\, i\left\{x^{\frac{1}{2}} J_{-3/2}(x) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{x}\right\}. \end{aligned}$$

The functions $x^{\frac{1}{2}} J_{3/2}(x)$ and $x^{\frac{1}{2}} J_{-3/2}(x) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{x}$ are both Hankel kernels (see [27], 214–215). The first is an even function of x and the second an odd function of x. That is

$$j(x) = \tfrac{1}{2} j_e(x) + \tfrac{1}{2} i\, j_0(x)$$

say. Lemma γ can be deduced from the inversion formulae for the separate kernels $j_e(x)$ and $j_0(x)$ by splitting $f(x)$ into even and odd parts $\frac{1}{2}\{f(x) + f(-x)\}$ and $\frac{1}{2}\{f(x) - f(-x)\}$; just as the complex Fourier inversion formulae (B′) can be deduced from the Fourier cosine and sine inversions.(1)

LEMMA δ.(2) *If $F(x)$ is an integral, tends to zero as x tends to $\pm\infty$, and $x\,F'(x)$ belongs to $L^2(-\infty,\infty)$, then $F(x)$ belongs to $L^2(-\infty,\infty)$ and $x^{\frac{1}{2}}\, F(x)$ tends to zero as x tends to zero or to $\pm\infty$.*

(1) By the argument of [27], 1–3, for instance.

(2) Results corresponding to Lemmas δ and ε for functions over the range $(0, \infty)$ are given in [11], Lemmas 2 and 4. The present lemmas are proved in the same way if we split $F(x)$ into even and odd parts. See also [24] for more extended results of this kind.

Lemma ε. *If $F(x)$ satisfies the conditions of Lemma δ then it has a complex Fourier transform $G(x)$ for which*

$$G(x) = \lim_{T\to\infty} (2\pi)^{-\frac{1}{2}} \int_{-T}^{T} F(t)\, e^{-ixt}\, dt$$

and

$$F(x) = \lim_{T\to\infty} (2\pi)^{-\frac{1}{2}} \int_{-T}^{T} G(t)\, e^{ixt}\, dt.$$

Further, $G(x)$ satisfies the same conditions as $F(x)$, and $xF'(x)$, $xG'(x)$ is a pair of transforms with respect to the transformation of Lemma γ.

Now suppose that the weighted sequence $\{a_n, \alpha_n\}$ is B^2 a.p. with respect to Ax, and that the Fourier series of (3.14) is given by

$$\sum_{0\leqslant\alpha_n\leqslant x}{}' a_n - Ax \sim c_0 - i(2\pi)^{-\frac{1}{2}} \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{b_n}{\beta_n} e^{i\beta_n x}, \tag{4.4}$$

where the β_n are arranged in increasing order of magnitude, with $\beta_0 = 0$. Consider

$$\int_{-T}^{T} \left\{\sum_{0<\beta_n\leqslant t}{}' b_n - Bt\right\} j(xt) \frac{dt}{t} = (2\pi)^{-\frac{1}{2}} i \int_{-T}^{T} \left\{\sum_{0<\beta_n\leqslant t}{}' b_n - Bt\right\} \frac{d}{dt}\left(\frac{e^{ixt}-1}{xt}\right) dt \tag{4.5}$$

$$= (2\pi)^{-\frac{1}{2}} i \left[\left\{\sum_{0<\beta_n\leqslant t}{}' b_n - Bt\right\}\left\{\frac{e^{ixt}-1}{xt}\right\}\right]_{-T}^{T} - (2\pi)^{-\frac{1}{2}} i \int_{-T}^{T} \frac{e^{ixt}-1}{xt}\, d\left\{\sum_{0<\beta_n\leqslant t} b_n - Bt\right\}. \tag{4.6}$$

If we assume that

$$\sum_{0<\beta_n\leqslant t}{}' b_n - Bt = O(t^{\delta}) \tag{4.7}$$

for some B and some δ in $0 < \delta < \frac{1}{2}$ as $t \to \pm\infty$, then the integrated terms in (4.6) are $O(T^{\delta-1})$, so (4.5) is equal to

$$O(T^{\delta-1}) - (2\pi)^{-\frac{1}{2}} \frac{i}{x} \sum_{\substack{-T\leqslant\beta_n\leqslant T\\ \beta_n\neq 0}}{}' \frac{b_n}{\beta_n}(e^{i\beta_n x} - 1) + (2\pi)^{-\frac{1}{2}} \frac{iB}{x} \int_{-T}^{T} \frac{e^{ixt}-1}{t}\, dt. \tag{4.8}$$

Now
$$\int_{-T}^{T} \frac{e^{ixt}-1}{t}\, dt = i \int_{-T}^{T} \frac{\sin xt}{t}\, dt = \pi i \operatorname{sgn} x + O(T^{-1}).$$

Also
$$\lim_{T\to\infty} \sum_{\substack{-T\leqslant\beta_n\leqslant T\\ \beta_n\neq 0}} \frac{b_n}{\beta_n} = L \tag{4.9}$$

exists since

$$\sum_{1 \leqslant \beta_n \leqslant T}' \frac{b_n}{\beta_n} = \int_1^T t^{-1}\, d\left\{\sum_{0<\beta_n\leqslant t}' b_n - B t\right\} + B\int_1^T t^{-1}\, dt$$

$$= \left[t^{-1}\left\{\sum_{0<\beta_n\leqslant t}' b_n - B t\right\}\right]_1^T + \int_1^T t^{-2}\left\{\sum_{0<\beta_n\leqslant t}' b_n - B t\right\} dt + B \log T$$

$$= -\left\{\sum_{0<\beta_n\leqslant 1}' b_n - B\right\} + \int_1^\infty t^{-2}\left\{\sum_{0<\beta_n\leqslant t}' b_n - B t\right\} dt + O(T^{\delta-1}) + B\log T$$

$$= L_1 + O(T^{\delta-1}) + B \log T,$$

say. Similarly

$$\sum_{-T\leqslant\beta_n\leqslant -1}' \frac{b_n}{\beta_n} = L_2 + O(T^{\delta-1}) - B\log T,$$

and (4.9) follows by addition, with $L = L_1 + L_2$. Hence (4.8) becomes

$$-(2\pi)^{-\frac{1}{2}} \frac{i}{x} \sum_{\substack{-T\leqslant\beta_n\leqslant T \\ \beta_n\neq 0}}' \frac{b_n}{\beta_n} e^{i\beta_n x} + (2\pi)^{-\frac{1}{2}} \frac{i\,L}{x} - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{B}{x} \operatorname{sgn} x + O(T^{\delta-1}). \tag{4.10}$$

Now by (4.7) the function

$$\mathcal{B}(x) = x^{-1}\left\{\sum_{0<\beta_n\leqslant x}' b_n - B x\right\} \tag{4.11}$$

belongs to $L^2(-\infty,\infty)$. Hence (4.5) converges in mean square as $T\to\infty$ by Lemma γ, and (4.10) gives

$$\underset{T\to\infty}{\text{l.i.m.}} \int_{-T}^{T} \mathcal{B}(t)\; j(x t)\, dt$$

$$= x^{-1}\left\{\sum_{0\leqslant\alpha_n\leqslant x}' a_n - A x - c_0 + R(x)\right\} + (2\pi)^{-\frac{1}{2}} \frac{i\,L}{x} - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{B}{x} \operatorname{sgn} x, \tag{4.12}$$

where $R(x)$ is a B^2 null-function. If we put

$$a_0 = (2\pi)^{\frac{1}{2}} B, \qquad A_1 = c_0 - (2\pi)^{-\frac{1}{2}} i L,$$

then (4.12) can be written

$$\mathcal{A}(x) = x^{-1}\left\{\sum_{0<\alpha_n\leqslant x}' a_n - A x - A_1 + R(x)\right\}, \tag{4.13}$$

say. Thus we have:

THEOREM 2. *If* $\{a_n, \alpha_n\}$ *is a* B^2 *a.p. weighted sequence with respect to* Ax, *and* $\{b_n, \beta_n\}$ *is the weighted sequence derived from the Fourier series* (4.4), *and* (4.7) *is satisfied, and the sequence* $\{\beta_n\}$ *is discrete*(1) *and arranged in increasing order of magnitude, then there exists a constant* A_1 *and a* B^2 *null-function* $R(x)$ *such that the functions* $\mathcal{A}(x)$, $\mathcal{B}(x)$ *of* (4.13) *and* (4.11) *are transforms of the type of Lemma* γ.

This result, though it corresponds to some of the examples given later in section 10, is unsatisfactory in that it involves the unspecified B^2 null-function $R(x)$. If we assume further that the series (4.4) converges in mean square over any finite range of x to

$$\sum_{0\leqslant\alpha_n\leqslant x}{}' a_n - A x \tag{4.14}$$

then our argument gives

$$\mathcal{A}(x) = x^{-1}\left\{\sum_{0<\alpha_n\leqslant x}{}' a_n - A x - A_1\right\}$$

as the complex Hankel transform of $\mathcal{B}(x)$. Since $\mathcal{B}(x)$ belongs to $L^2(-\infty,\infty)$ so does $\mathcal{A}(x)$, by Lemma γ. But in the neighbourhood of $x=0$ the function $\mathcal{A}(x)$ behaves like A_1/x, and thus $\mathcal{A}(x)$ can only belong to $L^2(-\infty,\infty)$ if $A_1 = 0$. Thus we have:

THEOREM 3. *If* $\{a_n, \alpha_n\}$ *is a* B^2 *a.p. weighted sequence, and the conditions of Theorem* 2 *are satisfied, and the Fourier series* (4.4) *converges in mean square to* (4.14) *over any finite range of* x, *then the functions*

$$x^{-1}\left\{\sum_{0<\alpha_n\leqslant x}{}' a_n - A x\right\}, \qquad x^{-1}\left\{\sum_{0<\alpha_n\leqslant x}{}' b_n - B x\right\}$$

are complex Hankel transforms of the type of Lemma γ.

5. Concordance

We can use a method of Titchmarsh(2) to deduce the convergence of the integrals in Theorem 3.

LEMMA ζ. *If* $f(x)$, $g(x)$ *is a pair of transforms as in Lemma* γ, *and* $g(y)$ *is of bounded variation in some neighbourhood of* $y = x$, *then*

$$\tfrac{1}{2}\{g(x+0)+g(x-0)\} = \lim_{T\to\infty}\int_{-T}^{T} f(t)\,\bar{\jmath}(xt)\,dt.$$

If $f(x)$ *is of bounded variation a similar inverse result holds with* $\bar{\jmath}(x)$ *replaced by* $\jmath(x)$.

---

(1) The last two examples of the tenth section show that cases where $\{\beta_n\}$ is not discrete can arise, but they will not be considered in this paper.

(2) [27], 83 and 266, or [15], 111. Lemma ζ is the extension to complex Hankel transforms, and is proved in the same way.

Applying this lemma to the transforms of Theorem 3, and evaluating the integrals concerned as in section 4, we have:

THEOREM 4. *With the conditions of Theorem* 3

$$\sum_{0\leqslant\alpha_n\leqslant x}' a_n - A\,x = -(2\pi)^{-\frac{1}{2}}\, i \lim_{T\to\infty} \sum_{\substack{-T\leqslant\beta_n\leqslant T\\ \beta_n\neq 0}} \frac{b_n}{\beta_n}(e^{i\beta_n x}-1). \tag{5.1}$$

If, in addition

$$\sum_{0\leqslant\alpha_n\leqslant x}' a_n - A\,x = o(x)$$

as $x\to\pm\infty$, *and the sequence* $\{\beta_n\}$ *has no finite points of accumulation,*(1) *then*

$$\sum_{0\leqslant\beta_n\leqslant x}' b_n - B\,x = (2\pi)^{-\frac{1}{2}}\, i \lim_{T\to\infty} \sum_{\substack{-T\leqslant\alpha_n\leqslant T\\ \alpha_n\neq 0}} \frac{a_n}{\alpha_n}(e^{-i\alpha_n x}-1). \tag{5.2}$$

If the series

$$\sum_{\alpha_n\neq 0}\left|\frac{a_n}{\alpha_n}\right|^2$$

is convergent, then the formula (5.2) shows that the function

$$\sum_{0\leqslant\beta_n\leqslant x}' b_n - B\,x$$

is B^2 a.p. That is, the weighted sequence $\{b_n, \beta_n\}$ is B^2 a.p. with respect to $B\,x$, and we have the following result, which may be regarded as a "theorem of concordance" for B^2 a.p. weighted sequences.

THEOREM 5. *If* $\{a_n, \alpha_n\}$ *is* B^2 *a.p. with respect to* $A\,x$, *and the conditions of Theorems* 2, 3, 4 *are satisfied, and*

$$\sum_{\alpha_n\neq 0}\left|\frac{a_n}{\alpha_n}\right|^2$$

is convergent, then the weighted sequence $\{b_n, \beta_n\}$ *is* B^2 *a.p. with respect to* Bx.

Theorem 5 shows that the frequencies β_n associated with an almost periodic weighted sequence $\{a_n, \alpha_n\}$ automatically form an almost periodic weighted sequence when they are associated with appropriate weights b_n. We use the term "concordance" to describe this phenomenon, and in the sequel we will describe as "concordant" any almost periodic weighted sequence or almost periodic function whose frequencies can be the basis of an almost periodic weighted sequence. The name "concordance" is chosen as indicating that

(1) This condition is needed to ensure that the function $\sum_{0\leqslant\beta_n\leqslant x}' b_n$ is of bounded variation. Example (9) of section 10 shows that some such condition is needed.

the frequencies concerned are, in a sense, in some concord with one another (cf. [17]). Concordant functions thus constitute a class of functions intermediate between the class of periodic functions, whose frequencies are necessarily periodic, and the class of almost periodic functions, whose frequencies are quite unrelated to each other.

Theorem 5 can be regarded as asserting that, under certain conditions, all almost periodic weighted sequences are concordant.

6. The summation formula

If $F(x)$ and $G(x)$ are complex Fourier transforms satisfying the conditions of Lemma ε, then we can apply the Parseval equation (4.2) for complex Hankel transforms to the pairs of functions $xF'(x)$, $xG'(x)$ and $\mathcal{A}(x)$, $\mathcal{B}(x)$ of Theorem 3. That is

$$\int_{-\infty}^{\infty} x^{-1}\left\{\sum_{0<\alpha_n\leqslant x} a_n - A\,x\right\}\cdot x\,G'(x)\,dx = \int_{-\infty}^{\infty} x^{-1}\left\{\sum_{0<\beta_n\leqslant x} b_n - B\,x\right\}\cdot x\,F'(x)\,dx. \tag{6.1}$$

The left-hand side is equal to

$$\left[\left\{\sum_{0<\alpha_n\leqslant x}{}' a_n - A\,x\right\} G(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} G(x)\,d\left\{\sum_{0<\alpha_n\leqslant x}{}' a_n - A\,x\right\}. \tag{6.2}$$

Now $G(x) = o(x^{-\frac{1}{2}})$ as $x \to \pm\infty$ by Lemmas δ and ε. Hence the integrated term in (6.2) vanishes if

$$\sum_{0<\alpha_n\leqslant x}{}' a_n - A\,x = O(x^{\eta}) \tag{6.3}$$

and either $\eta \leqslant \frac{1}{2}$ or $x^{\eta} G(x) \to 0$ as $x \to \pm\infty$. The integral term in (6.2) is then equal to

$$-\lim_{T,\,U\to\infty}\left\{\sum_{\substack{-U<\alpha_n<T\\ \alpha_n\neq 0}} a_n\,G(\alpha_n) - A\int_{-U}^{T} G(x)\,dx\right\}.$$

Treating the right-hand side of (6.1) in the same way we have:

THEOREM 6. *If*

(i) *$F(x)$ satisfies the conditions of Lemma ε, and $G(x)$ is its complex Fourier transform,*

(ii) *the weighted sequence $\{a_n, \alpha_n\}$ is B^2 a.p. with respect to $A\,x$ and satisfies the conditions of Theorem* 3,

(iii) *with δ and η as in* (4.7) *and* (6.3) *and either* (a) *$\delta \leqslant \frac{1}{2}$ and $\eta \leqslant \frac{1}{2}$, or* (b) *$x^{\delta} F(x)$ and $x^{\eta} G(x)$ both tend to zero as x tends to $\pm\infty$,* *then*

$$\lim_{T,U\to\infty}\left\{\sum_{\substack{-U<\alpha_n<T\\ \alpha_n\neq 0}} a_n G(\alpha_n) - A\int_{-U}^{T} G(x)\,dx\right\} = \lim_{T,U\to\infty}\left\{\sum_{\substack{-U<\beta_n<T\\ \beta_n\neq 0}} b_n F(\beta_n) - B\int_{-T}^{T} F(x)\,dx\right\}. \tag{6.4}$$

If $F(x)$ belongs to $L(-\infty, \infty)$, then the integral

$$G(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} F(t)\, e^{-ixt}\, dt$$

converges absolutely for all x, including $x = 0$, and

$$G(0) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} F(t)\, dt.$$

Since we have taken $\alpha_0 = \beta_0 = 0$, if we put $A = (2\pi)^{\frac{1}{2}} b_0$, $B = (2\pi^{\frac{1}{2}}) a_0$, then we have:

THEOREM 7. *If the conditions of Theorem* 6 *are satisfied and both* $F(x)$ *and* $G(x)$ *belong to* $L(-\infty, \infty)$ *then*

$$\sum_{n=-\infty}^{\infty} a_n G(\alpha_n) = \sum_{n=-\infty}^{\infty} b_n F(\beta_n). \tag{6.5}$$

If we apply Theorem 7 to the sum

$$K(x) = \sum_{n=-\infty}^{\infty} a_n k(x - \alpha_n) \tag{6.6}$$

of (3.1) then we obtain, formally,

$$K(x) = \sum_{n=-\infty}^{\infty} b_n l(\beta_n)\, e^{i\beta_n x}, \tag{6.7}$$

where
$$l(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} k(t)\, e^{ixt}\, dt$$

is the complex Fourier transform of $k(x)$. Hence if $l(x)$ decreases rapidly enough as $x \to \pm\infty$ then $K(x)$ is an almost periodic function of x by (6.7).

That is to say the series (6.6), used in the preliminary definition of almost periodicity for weighted sequences, defines an almost periodic function of x for a wide class functions $k(x)$ of which the trapezoidal functions (3.2) are special cases.

Further, the functions $K(x)$ defined by (6.6) have frequencies β_n which form the basis of the almost periodic weighted sequence $\{b_n, \beta_n\}$. That is $K(x)$ is a concordant function of x if $k(x)$, $l(x)$ decrease rapidly enough to ensure the absolute convergence of (6.6) and (6.7).

7. Inversion formulae

We can now deduce the inversion formulae of (C′) from the summation formula. If we apply Theorem 7 to the function[1]

$$F(x)=\begin{cases} e^{izx}-\cos zT-\dfrac{ix}{T}\sin zT & (-T<x<T),\\ 0 & \text{elsewhere,}\end{cases}$$

with $z \neq 0$, then

$$G(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left\{\frac{z\sin(z-x)T}{x(z-x)}-\frac{\sin zT\sin xT}{x^2T}\right\}$$

when $x\neq 0$ or z, and

$$G(z)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left\{T-\frac{\sin^2 zT}{z^2T}\right\},$$

$$G(0)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left\{\frac{\sin zT}{z}-T\cos zT\right\}.$$

Theorem 7 then gives

$$\sum_{-T<\beta_n<T} b_n\left\{e^{i\beta_n z}-\cos zT-\frac{i\beta_n}{T}\sin zT\right\}$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sum_{\alpha_n\neq 0,z} a_n\left\{\frac{z\sin(z-\alpha_n)T}{\alpha_n(z-\alpha_n)}-\frac{\sin zT\sin\alpha_nT}{\alpha_n^2T}\right\}+$$

$$+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}a_0\left\{\frac{\sin zT}{z}-T\cos zT\right\}+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left\{T-\frac{\sin^2 zT}{z^2T}\right\}\begin{Bmatrix} a_n & (z=\alpha_n)\\ 0 & \text{elsewhere}\end{Bmatrix}. \tag{7.1}$$

Now

$$\sideset{}{'}\sum_{-T\leqslant\beta_n\leqslant T} b_n\beta_n=\int_{-T}^{T} t\,d\left\{\sideset{}{'}\sum_{0\leqslant\beta_n\leqslant t} b_n-Bt\right\}$$

$$=\left[t\left\{\sideset{}{'}\sum_{0\leqslant\beta_n\leqslant t} b_n-Bt\right\}\right]_{-T}^{T}-\int_{-T}^{T}\left\{\sideset{}{'}\sum_{0\leqslant\beta_n\leqslant t} b_n-Bt\right\}dt$$

$$=O(T^{\delta+1}) \tag{7.2}$$

by (4.7). If the series

$$\sum_{\alpha_n\neq 0}\left|\frac{a_n}{\alpha_n^2}\right| \tag{7.3}$$

is convergent, then (7.1) gives, with $a_0=(2\pi)^{\frac{1}{2}}B$,

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[1] The result would follow more directly if we could put $F(x)=e^{izx}$ $(-T<x<T)$, $F(x)=0$ elsewhere, but the conditions of Theorem 7 do not cover such a function.

$$\sum_{-T\leq\beta_n\leq T}' b_n e^{i\beta_n z} - \cos z T \left\{ \sum_{-T\leq\beta_n\leq T}' b_n - 2BT \right\} + O(T^\delta)$$

$$= O(1) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} T \begin{cases} a_n & (z=\alpha_n) \\ 0 & \text{elsewhere} \end{cases}$$

as $T\to\infty$. By (4.7) this gives

$$\frac{(2\pi)^{\frac{1}{2}}}{2T} \sum_{-T\leq\beta_n\leq T}' b_n e^{i\beta_n z} = O(T^{\delta-1}) + \begin{cases} a_n & (z=\alpha_n) \\ 0 & \text{elsewhere} \end{cases}$$

for $z \neq 0$. For $z=0$ the result follows immediately from (4.7).

If all the weights a_n are real and have the same sign, then the convergence of (7.3) follows from (6.3) by the method of (7.2). Making $T\to\infty$, and reversing the roles of $F(x)$ and $G(x)$ to prove the inverse result, we have:

THEOREM 8. *If*

(i) $\{a_n, \alpha_n\}$ *is a* B^2 *a.p. weighted sequence with respect to* Ax,

(ii) *the conditions of Theorem* 6 *are satisfied,*

(iii) *either* (a) *sequences of weights* $\{a_n\}$ *and* $\{b_n\}$ *are both real and each is of constant sign, or* (b) *the series*

$$\sum_{\alpha_n\neq 0} \left|\frac{a_n}{\alpha_n^2}\right|, \qquad \sum_{\beta_n\neq 0} \left|\frac{b_n}{\beta_n^2}\right|$$

converge,

then for all real z

$$(2\pi)^{\frac{1}{2}} \lim_{T\to\infty} \frac{1}{2T} \sum_{-T\leq\beta_n\leq T}' b_n e^{i\beta_n z} = \begin{cases} a_n & (z=\alpha_n), \\ 0 & \textit{elsewhere}, \end{cases}$$

and

$$(2\pi)^{\frac{1}{2}} \lim_{T\to\infty} \frac{1}{2T} \sum_{-T\leq\alpha_n\leq T}' a_n e^{-i\alpha_n z} = \begin{cases} b_n & (z=\beta_n), \\ 0 & \textit{elsewhere}. \end{cases}$$

As explained in the second section, Theorem 8 or (C′) is the analogue for almost periodic weighted sequences of the Fourier integral inversion (B′). In the theory of Fourier integrals it is often more convenient to consider the formulae in an integrated form, such as

$$\int_0^x G(t)\,dt = (2\pi)^{-\frac{1}{2}} i \int_{-\infty}^{\infty} \frac{F(t)}{t} (e^{-ixt}-1)\,dt,$$

and

$$\int_0^x F(t)\,dt = -(2\pi)^{-\frac{1}{2}} i \int_{-\infty}^{\infty} \frac{G(t)}{t} (e^{ixt}-1)\,dt.$$

The formulae (5.1) and (5.2) of Theorem 4 can be regarded as analogues of this integrated form of the Fourier integral inversion.

Another form in which the inversion formulae can be expressed is

$$(2\pi)^{\frac{1}{2}} \lim_{T\to\infty} T \sum_{n=-\infty}^{\infty} b_n\, e^{-\frac{1}{2}\beta_n^2 T^2 + i\beta_n z} = \begin{cases} a_n & (z=\alpha_n), \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$(2\pi)^{\frac{1}{2}} \lim_{T\to\infty} T \sum_{n=-\infty}^{\infty} a_n\, e^{-\frac{1}{2}\alpha_n^2 T^2 - i\alpha_n z} = \begin{cases} b_n & (z=\beta_n), \\ 0 & \text{elsewhere.} \end{cases}$$

This form is readily deduced from the summation formula (6.5), and is analogous to a summability theorem for Fourier integrals ([27], 26–40).

8. Meromorphic almost periodic functions

The function

$$G(x) = \frac{1}{z-x}$$

with z complex, $\mathrm{Im}(z) \neq 0$, satisfies the conditions of Theorem 6, and we then have

$$F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{ixt}}{z-t}\, dt$$

$$= \begin{cases} (2\pi)^{\frac{1}{2}}\, i\, e^{ixz} & \{\mathrm{Im}(z) < 0,\ x < 0\}, \\ 0 & \{\mathrm{Im}(z) < 0,\ x > 0\}, \\ -(2\pi)^{\frac{1}{2}}\, i\, e^{ixz} & \{\mathrm{Im}(z) > 0,\ x > 0\}, \\ 0 & \{\mathrm{Im}(z) > 0,\ x < 0\}. \end{cases}$$

Theorem 6 then gives, for $\mathrm{Im}(z) > 0$, taking $T = U$

$$\lim_{T\to\infty} \sum_{-T<\alpha_n<T} \frac{a_n}{z-\alpha_n} = -(2\pi)^{\frac{1}{2}}\, i \left\{ \tfrac{1}{2} b_0 + \sum_{\beta_n>0} b_n\, e^{i\beta_n z} \right\} \tag{8.1}$$

For $\mathrm{Im}(z) < 0$ we get

$$\lim_{T\to\infty} \sum_{-T<\alpha_n<T} \frac{a_n}{z-\alpha_n} = (2\pi)^{\frac{1}{2}}\, i \left\{ \tfrac{1}{2} b_0 + \sum_{\beta_n<0} b_n\, e^{i\beta_n z} \right\}. \tag{8.2}$$

Now the series

$$\sum_{\alpha_n \neq 0} \frac{a_n}{\alpha_n^2}$$

converges, as indicated in Theorem 8, and hence the function $\varphi(z)$ defined by

$$\varphi(z) = \frac{a_0}{z} + \sum_{\alpha_n \neq 0} a_n \left(\frac{1}{z-\alpha_n} + \frac{1}{\alpha_n} \right) + C,$$

where
$$C = -\lim_{T\to\infty} \sum_{\substack{-T<\alpha_n<T \\ \alpha_n \neq 0}} \frac{a_n}{\alpha_n},$$

defines a function of z meromorphic in the whole z plane with simple poles of residues a_n at the points $z = \alpha_n$. Further, the series (8.1) is uniformly convergent in any half-plane $\operatorname{Im}(z) > \sigma > 0$, and hence $\varphi(z)$ is analytic almost periodic in this half-plane. Similarly, from (8.2), $\varphi(z)$ is analytic almost periodic in $\operatorname{Im}(z) < -\sigma < 0$.

This example suggests that it should be possible to develop a theory of meromorphic almost periodic functions in which sequences of poles and residues form almost periodic weighted sequences. Let us introduce the definition:

DEFINITION 5.(1) *A function $\varphi(z)$, meromorphic in a strip $\sigma < \operatorname{Im}(z) < \sigma_1$, is said to be meromorphic almost periodic in that strip if*
$$\varphi(z) = \frac{p(z)}{q(z)}, \tag{8.3}$$
where $p(z)$ and $q(z)$ are both analytic almost periodic in that strip, and $q(z)$ is not identically zero.

Suppose that $\varphi(z)$ of (8.3) is meromorphic almost periodic in a strip $-\sigma < \operatorname{Im}(z) < \sigma$, $(\sigma > 0)$, and that all its poles in this strip are simple poles within the strip $-\varrho < \operatorname{Im}(z) < \varrho$, $0 < \varrho < \sigma$. Let $\operatorname{Re}(z) = Z$ be a line crossing these strips which does not pass through any zero of $q(z)$. Then $p(z)/q(z)$ is analytic at all points of this line within the strips, and is therefore bounded on this part of the line. Let
$$|q(z)| > Q > 0$$
and
$$|p(z)| < P$$
on
$$z = Z + iv, \quad -\sigma < v < \sigma.$$

Now $p(z)$ and $q(z)$ have relatively dense common translation numbers $\tau = \tau(\varepsilon)$ belonging to ε within the strip. Choose $0 < \varepsilon < \frac{1}{2}Q$, so we have
$$|p(z)| < P + \tfrac{1}{2}Q, \qquad |q(z)| > \tfrac{1}{2}Q$$
on
$$z = Z + \tau + iv, \quad -\sigma < v < \sigma.$$

Hence on any such line
$$|\varphi(z)| < \frac{P + \frac{1}{2}Q}{\frac{1}{2}Q} = M$$

(1) This definition is suggested by a remark of Bohr, [6], 103–104.

say. Consequently there exists a relatively dense set of lines crossing the strip on which $|\varphi(z)| < M$. Let the lines be

$$\operatorname{Re}(z) = T_n \quad (n = \pm 1, \pm 2, \ldots)$$

with
$$\ldots < T_{-2} < T_{-1} < 0 < T_1 < T_2 \ldots$$

and
$$T_n \to \pm\infty \quad \text{as} \quad n \to \pm\infty.$$

Let $F(z)$ be a function regular in $-\sigma < \operatorname{Im}(z) < \sigma$ for which $F(z) \to 0$ as $\operatorname{Re}(z) \to \pm\infty$ uniformly in the strip. Consider

$$\int_R \varphi(z) F(z)\, dz, \tag{8.4}$$

where R is the rectangle $T_m + iK$, $T_{-m} + iK$, $T_{-m} - iK$, $T_m - iK$, where $0 < \varrho < K < \sigma$. If the poles of $\varphi(z)$ are at the points $z = \beta_n$ with residues b_n, the integral (8.4) is equal to

$$2\pi i \sum_{T_{-m} < \beta_n < T_m} b_n F(\beta_n).$$

As $T_m \to \infty$ and $T_{-m} \to -\infty$

$$\left| \int_{T_m - iK}^{T_m + iK} \varphi(z) F(z)\, dz \right| < M \int_{T_m - iK}^{T_m + iK} |F(z)|\, dz \to 0,$$

and similarly
$$\int_{T_{-m}+iK}^{T_{-m}-iK} \varphi(z) F(z)\, dz \to 0.$$

By a theorem of Bohr ([6], 103), $\varphi(z)$ is analytic almost periodic in the strip $\varrho < \operatorname{Im}(z) < \sigma$. Hence it has a Fourier series in this strip, say

$$\varphi(z) \sim \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n z}.$$

If we assume that this series is uniformly convergent on $\operatorname{Im}(z) = K$ and that

$$\int_{-\infty}^{\infty} |F(x + iK)|\, dx$$

converges, then

$$\begin{aligned}
\lim_{m\to\infty} \int_{T_m+iK}^{T_{-m}+iK} \varphi(z) F(z)\, dz &= - \int_{-\infty+iK}^{\infty+iK} F(z) \left\{ \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n z} \right\} dz \\
&= - \sum_{n=-\infty}^{\infty} c_n \int_{-\infty+iK}^{\infty+iK} F(z) e^{i\lambda_n z}\, dz \\
&= - \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} F(z) e^{i\lambda_n z}\, dz \\
&= -(2\pi)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} c_n G(-\lambda_n).
\end{aligned}$$

Similarly, if
$$\sum_{n=-\infty}^{\infty} d_n e^{i\mu_n z}$$
is the Fourier series of $\varphi(z)$ in $-\sigma < \operatorname{Im}(z) < -\varrho$, and assumptions similar to those above are made, then
$$\lim_{m\to\infty} \int_{T_{-m}-iK}^{T_m - iK} \varphi(z) F(z)\, dz = (2\pi)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} d_n G(-\mu_n).$$

Hence, as $m \to \infty$, (8.4) gives
$$2\pi i \sum_{n=-\infty}^{\infty} b_n F(\beta_n) = (2\pi)^{\frac{1}{2}} \left\{ \sum_{n=-\infty}^{\infty} d_n G(-\mu_n) - \sum_{n=-\infty}^{\infty} c_n G(-\lambda_n) \right\}.$$

Changing the notation this can be written
$$\sum_{n=-\infty}^{\infty} b_n F(\beta_n) = \sum_{n=-\infty}^{\infty} a_n G(\alpha_n),$$
where the weighted sequence $\{a_n, \alpha_n\}$ is obtained as the difference of the weighted sequences
$$\{-(2\pi)^{-\frac{1}{2}} i\, d_n,\ -\mu_n\},\ \{-(2\pi)^{-\frac{1}{2}} i\, c_n,\ -\lambda_n\}. \tag{8.5}$$
Thus we have:

THEOREM 9. *If*

(i) $\varphi(z)$ *is meromorphic almost periodic in a strip* $-\sigma < \operatorname{Im}(z) < \sigma$,

(ii) *the only singularities of* $\varphi(z)$ *in this strip are simple poles at* $z = \beta_n$ *with residues* b_n *and all* β_n *lie within the strip* $-\sigma < -\varrho < \operatorname{Im}(z) < \varrho < \sigma$,

(iii) *the Fourier series for* $\varphi(z)$ *in the strip* $\varrho < \operatorname{Im}(z) < \sigma$ *is uniformly convergent on some line* $\operatorname{Im}(z) = K$, $\varrho < K < \sigma$, *and similarly for the Fourier series for* $\varphi(z)$ *in* $-\sigma < \operatorname{Im}(z) < -\varrho$,

(iv) $F(z)$ *is regular in* $-\sigma < \operatorname{Im}(z) < \sigma$, *tends to zero uniformly in this strip as* $\operatorname{Re}(z) \to \pm\infty$, *and the integrals*
$$\int_{-\infty}^{\infty} |F(x \pm iK)|\, dx$$
converge,

then
$$\sum_{n=-\infty}^{\infty} b_n F(\beta_n) = \sum_{n=-\infty}^{\infty} a_n G(\alpha_n),$$
where $\{a_n, \alpha_n\}$ *is the difference of the weighted sequences* (8.5) *and*
$$G(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} F(t) e^{ixt}\, dt.$$

Theorem 9 shows that certain meromorphic almost periodic functions give rise to summation formulae of the same form as those associated with almost periodic weighted sequences. However, in Theorem 9 the β_n need not be real. Consequently we can only expect the weighted sequences $\{a_n, \alpha_n\}$, $\{b_n, \beta_n\}$ to be almost periodic in the case when all the β_n lie on a single line along the strip. In order to show that the weighted sequences are then almost periodic in the sense of Definition 4 we need to prove the summation formula for a wider class of functions than is covered by Theorem 9.

If
$$G(x) = e^{-\frac{1}{2}sx^2}, \quad \operatorname{Re}(s) > 0$$
in Theorem 9, then
$$F(x) = s^{-\frac{1}{2}} e^{-\frac{1}{2}x^2/s},$$
and these functions satisfy the conditions of the theorem. Further
$$x F'(x) = -s^{-\frac{3}{2}} x e^{-\frac{1}{2}x^2/s},$$
$$x G'(x) = -s x e^{-\frac{1}{2}sx^2}$$
are transforms of the type of Lemma γ, and the summation formula can be written
$$\int_{-\infty}^{\infty} x^{-1} \left\{ \sum_{0<\alpha_n\leqslant x}' a_n - A x \right\} \{-s x e^{-\frac{1}{2}sx^2}\} \, dx$$
$$= \int_{-\infty}^{\infty} x^{-1} \left\{ \sum_{0<\beta_n\leqslant x}' b_n - B x \right\} \{-s^{-\frac{3}{2}} x e^{-\frac{1}{2}x^2/s}\} \, dx \tag{8.6}$$
as in section 6. If we assume (4.7), so that $\mathcal{B}(x)$ of (4.11) belongs to $L^2(-\infty,\infty)$, then it has a transform $\mathcal{A}_1(x)$ say, with respect to the kernel $j(x)$, for which the right-hand side of (8.6) is equal to
$$\int_{-\infty}^{\infty} \mathcal{A}_1(x) \{-s x e^{-\frac{1}{2}sx^2}\} \, dx$$
by (4.3). Hence, by subtraction, with
$$\mathcal{A}(x) = x^{-1} \left\{ \sum_{0<\alpha_n\leqslant x}' a_n - A x \right\}$$
and
$$\mathcal{D}(x) = \mathcal{A}(x) - \mathcal{A}_1(x)$$
we have
$$\int_{-\infty}^{\infty} \mathcal{D}(x) \, x e^{-\frac{1}{2}sx^2} \, dx = 0 \tag{8.7}$$
for all Re $(s) > 0$. If we divide $\mathcal{D}(x)$ into even and odd parts
$$\mathcal{D}_e(x) = \tfrac{1}{2}\{\mathcal{D}(x) + \mathcal{D}(-x)\},$$
$$\mathcal{D}_0(x) = \tfrac{1}{2}\{\mathcal{D}(x) - \mathcal{D}(-x)\},$$
then (8.7) gives

$$\int_0^\infty \mathcal{D}_0(x)\, x\, e^{-\frac{1}{2}sx^2}\, dx = 0,$$

or, putting $s = t^{\frac{1}{2}}$,

$$\int_0^\infty \mathcal{D}_0(t^{\frac{1}{2}})\, e^{-\frac{1}{2}st}\, dt = 0.$$

Hence, by the uniqueness theorem for Laplace transforms, [8], $\mathcal{D}_0(x) = 0$ almost everywhere.

A similar argument with

$$G(x) = s^{\frac{1}{2}}\, x\, e^{-\frac{1}{2}sx^2}, \qquad F(x) = i\, s^{-1}\, x\, e^{-\frac{1}{2}x^2/s}$$

shows that $\mathcal{D}_e(x) = 0$ almost everywhere, so $\mathcal{D}(x) = \mathcal{D}_0(x) + \mathcal{D}_e(x) = 0$ almost everywhere. That is $\mathcal{A}(x)$ and $\mathcal{B}(x)$ are transforms of the type of Lemma γ, and we can again show as in Theorems 4 and 5 that $\{b_n, \beta_n\}$ is B^2 a.p. We have:

THEOREM 10. *If*

(i) $\varphi(z)$ *is meromorphic almost periodic in the strip* $-\sigma < \operatorname{Im}(z) < \sigma$ *and satisfies the conditions* (ii) *and* (iii) *of Theorem* 9,

(ii) *all the* β_n *are real,*

(iii) $\sum'_{0<\beta_n\leqslant x} b_n - Bx = O(x^\delta)$ *for some* $0 < \delta < 1$ *as* $x \to \pm\infty$,

(iv) $\sum_{\alpha_n \neq 0} \left|\frac{a_n}{\alpha_n}\right|^2$ *converges,*

then the weighted sequence $\{b_n, \beta_n\}$ *is* B^2 *a.p. with respect to* Bx.

9. Formal unification of the three types of reciprocity by use of the Dirac δ-function

The three types of reciprocity (A′), (B′), and (C′) of the second section can all be regarded as cases of the Fourier integral reciprocity (B′) if we use the Dirac δ-function.

The reciprocity (A′) corresponds to the pair of Fourier complex transforms

$$f(x), \qquad \sum_{n=-\infty}^{\infty} c_n\, \delta(x - \lambda_n),$$

and the reciprocity (C′) corresponds to the pair of Fourier complex transforms(1)

$$\sum_{n=-\infty}^{\infty} a_n\, \delta(x - \alpha_n), \qquad \sum_{n=-\infty}^{\infty} b_n\, \delta(x - \beta_n). \tag{9.1}$$

If $F(x)$, $G(x)$ is a pair of Fourier complex transforms, as in (B′), then the Parseval equation with the pair (9.1) gives

(1) Applications of this method in the periodic case are given in [22].

$$\int_{-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty}\left\{b_n\,\delta\,(x-\beta_n)\right\}F(x)\,dx=\int_{-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty}a_n\,\delta\,(x-\alpha_n)\right\}G(x)\,dx,\right.\right.$$

and this is formally equivalent to the summation formula

$$\sum_{n=-\infty}^{\infty}b_n\,F(\beta_n)=\sum_{n=-\infty}^{\infty}a_n\,G(\alpha_n).$$

That is, the summation formula (9.2) can be regarded directly as a Parseval equation for Fourier complex transforms if we use the Dirac δ-function.

10. Examples

Many examples of summation formulae are known in which the kernels involved are Bessel functions or combinations of Bessel functions ([9], [10], [14]). In order to find examples of almost periodic weighted sequences we can use summation formulae which involve either the Fourier cosine or the Fourier sine kernel.

Thus, if we have a summation formula involving the Fourier cosine kernel

$$p_0\,g(0)+\sum_{n=1}^{\infty}p_n\,g(\alpha_n)=q_0\,f(0)+\sum_{n=1}^{\infty}q_n\,f(\beta_n),$$

where

$$g(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}f(t)\cos x\,t\,d\,t, \tag{10.1}$$

then if we put

$$\alpha_{-n}=-\alpha_n,\quad \beta_{-n}=-\beta_n,\quad \alpha_0=\beta_0=0,\quad a_0=2\,p_0,\quad b_0=2\,q_0,$$
$$a_n=a_{-n}=p_n,\quad b_n=b_{-n}=q_n$$

we have

$$\sum_{n=-\infty}^{\infty}a_n\,G(\alpha_n)=\sum_{n=-\infty}^{\infty}b_n\,F(\beta_n),$$

with $F(x)$, $G(x)$ a pair of Fourier complex transforms as in (B′).

Similarly for the Fourier sine kernel, if

$$\sum_{n=1}^{\infty}p_n\,g(\alpha_n)=\sum_{n=1}^{\infty}q_n\,f(\beta_n),$$

where

$$g(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}f(t)\sin x\,t\,d\,t, \tag{10.2}$$

then putting

$$\alpha_{-n} = -\alpha_n, \quad \beta_{-n} = -\beta_n, \quad \alpha_0 = \beta_0 = 0 = a_0 = b_0, \quad a_n = -a_{-n} = p_n, \quad b_n = -b_{-n} = -iq_n$$

we again have
$$\sum_{n=-\infty}^{\infty} a_n G(\alpha_n) = \sum_{n=-\infty}^{\infty} b_n F(\beta_n).$$

Various examples of summation formulae are given below. For brevity no attempt is made to give conditions on $F(x)$ and $G(x)$ for each summation formula, but each example is associated with a pair of almost periodic weighted sequences of some kind.

1. Poisson's formula and its extensions.

$$\sum_{n=-\infty}^{\infty} F(n\sqrt{2\pi}) = \sum_{n=-\infty}^{\infty} G(n\sqrt{2\pi}), \tag{10.3}$$

$$\sum_{n=-\infty}^{\infty} B_n F\left\{n\left(\frac{2\pi}{N}\right)^{\frac{1}{2}}\right\} = \sum_{n=-\infty}^{\infty} A_n G\left\{n\left(\frac{2\pi}{N}\right)^{\frac{1}{2}}\right\}, \tag{10.4}$$

where A_n, B_n are periodic sequences of period N as in (C) of section 2.(1)

2. The prime number summation formula ([15]).

If the Riemann hypothesis is true and $f(x)$, $g(x)$ are Fourier cosine transforms as in (10.1), then

$$\lim_{T\to\infty}\left\{\sum_{0<m\log p<T} \frac{\log p}{p^{\frac{1}{2}m}} f(m\log p) - \int_0^T e^{\frac{1}{2}t} f(t)\,dt\right\} - \frac{1}{2}\int_0^\infty f(t)\left(\frac{1}{t} - \frac{e^{-\frac{3}{2}t}}{\sinh t}\right)dt$$
$$= -(2\pi)^{\frac{1}{2}} \lim_{T\to\infty}\left\{\sum_{0<\gamma<T} g(\gamma) - \frac{1}{2\pi}\int_0^T g(t)\log\frac{t}{2\pi}\,dt\right\}, \tag{10.5}$$

where p^m runs through the positive powers of primes p and γ through the non-trivial zeros of $\zeta(\frac{1}{2}+it)$.

3. Further summation formulae of the type (10.5) are associated with Dirichlet L-functions, and can be proved in the same way if we assume that the Riemann hypothesis is true for Dirichlet L-functions.

If
$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s},$$

and $\chi(n)$ is a real even primitive character modulo k ($k > 1$), and δ runs through the non-trivial zeros of $L(\frac{1}{2}+it, \chi)$, then with the notation of example (2)

(1) These are the formulae (2.1) modified for transforms of the form (B′).

$$\sum_{p,m} \chi(p^m) \frac{\log p}{p^{\frac{1}{2}m}} f(m \log p) - \tfrac{1}{2} \int_0^\infty f(t) \left(\frac{1}{t} - \frac{e^{-\frac{3}{2}t}}{\sinh t}\right) dt$$

$$= -(2\pi)^{\frac{1}{2}} \lim_{T \to \infty} \left\{ \sum_{0 < \delta < T} g(\delta) - \frac{1}{2\pi} \int_0^T g(t) \log \frac{k t}{2\pi} dt \right\}. \quad (10.6)$$

4. If $\chi_1(n)$ and $\chi_2(n)$ are two different, real, even, primitive characters modulo k_1 and k_2 respectively, then examples of summation formulae without the integral terms in $f(t)$ of (10.6) can be constructed by subtraction, thus:

$$\sum_{p,m} \frac{\chi_1(p^m) - \chi_2(p^m)}{p^{\frac{1}{2}m}} \log p \, f(m \log p)$$

$$= -(2\pi)^{\frac{1}{2}} \lim_{T \to \infty} \left\{ \sum_{0 < \delta_1 < T} g(\delta_1) - \sum_{0 < \delta_2 < T} g(\delta_2) - \frac{1}{2\pi} \log \frac{k_1}{k_2} \int_0^T g(t) \, dt \right\},$$

where δ_1 and δ_2 run through the non-trivial zeros of $L(\frac{1}{2} + it, \chi_1)$ and $L(\frac{1}{2} + it, \chi_2)$ respectively, $f(x)$ and $g(x)$ are Fourier cosine transforms, and the Riemann hypothesis is assumed for $L(s, \chi_1)$ and $L(s, \chi_2)$.

5. Various writers have shown how summation formulae can be deduced from modular equations ([5], [14]). For example

$$\left\{ \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} \right\}^p = \sum_{n=0}^{\infty} r_p(n) e^{-\pi n x}$$

$$= \left\{ x^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/x} \right\}^p$$

$$= x^{-\frac{1}{2}p} \sum_{n=0}^{\infty} r_p(n) e^{-\pi n/x},$$

where p is a positive integer and $r_p(n)$ is the number of ways of expressing n as the sum of p squares. From this we can deduce the summation formula

$$\sum_{n=1}^{\infty} r_p(n) n^{\frac{1}{2} - \frac{1}{4}p} f_1(n) - \frac{\pi^{\frac{1}{2}p}}{\Gamma(\frac{1}{2}p)} \int_0^\infty t^{\frac{1}{4}p - \frac{1}{2}} f_1(t) \, dt$$

$$= \sum_{n=1}^{\infty} r_p(n) n^{\frac{1}{2} - \frac{1}{4}p} g_1(n) - \frac{\pi^{\frac{1}{2}p}}{\Gamma(\frac{1}{2}p)} \int_0^\infty t^{\frac{1}{4}p - \frac{1}{2}} g_1(t) \, dt, \quad (10.7)$$

where
$$g_1(x) = \pi \int_0^\infty f_1(t) J_{\frac{1}{2}p-1}(2\pi x^{\frac{1}{2}} t^{\frac{1}{2}}) \, dt. \quad (10.8)$$

If $p=1$ or $p=3$ then (10.8) is equivalent to a Fourier cosine or sine transformation. The case $p=1$ reduces to Poisson's summation formula. For $p=3$, if we put

$$f(x)=x^{\frac{1}{2}} f_1\left(\frac{x^2}{2\pi}\right), \qquad g(x)=x^{\frac{1}{2}} g_1\left(\frac{x^2}{2\pi}\right)$$

then (10.8) gives
$$g(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(t)\sin xt\,dt,$$

and (10.7) becomes

$$\sum_{n=1}^{\infty} r_3(n)\, n^{-\frac{1}{2}} f\{2\pi n)^{\frac{1}{2}}\} - 2^{\frac{3}{4}}\pi^{-\frac{1}{4}} \int_0^\infty t f(t)\,dt = \sum_{n=1}^{\infty} r_3(n)\, n^{-\frac{1}{2}} g\{(2\pi n)^{\frac{1}{2}}\} - 2^{\frac{3}{4}}\pi^{-\frac{1}{4}} \int_0^\infty t g(t)\,dt.$$

This summation formula corresponds to a weighted sequence $\{a_n, \alpha_n\}$ with $a_0=\alpha_0=0$, $\alpha_n=-\alpha_{-n}=(2\pi n)^{\frac{1}{2}}$, $a_n=-a_{-n}=r_3(n)n^{-\frac{1}{2}}$ for $n\geqslant 1$, and a distribution function $A(x) =2^{-\frac{1}{4}}\pi^{-\frac{1}{4}}x^2$. The function (3.14) is then $O(x^{14/29+\varepsilon})$ as x tends to $\pm\infty$ ([27], 267).

6. If $\chi(n)$ is the odd character modulo 3, $\chi(1)=1$, $\chi(2)=-1$, $\chi(3)=0$, then[1]

$$\sum_{n=1}^{\infty} n\,\chi(n)\, e^{-\frac{1}{3}\pi n^2 a^2} = a^{-3} \sum_{n=1}^{\infty} n\,\chi(n)\, e^{-\frac{1}{3}\pi n^2/a^2} \tag{10.9}$$

If the coefficients i_n $(n=0, 1, 2, \ldots)$ are defined as the real solutions of

$$x\left\{\sum_{n=0}^{\infty} i_n x^{3n}\right\}^3 = \sum_{n=1}^{\infty} n\,\chi(n)\, x^{n^2},$$

then the cube root of (10.9) gives

$$\sum_{n=0}^{\infty} i_n\, e^{-\pi a^2(n+1/9)} = a^{-1} \sum_{n=0}^{\infty} i_n\, e^{-\pi(n+1/9)/\alpha^2}.$$

This leads to a summation formula

$$\sum_{n=0}^{\infty} i_n f\{(2\pi)^{\frac{1}{2}}(n+\tfrac{1}{9})^{\frac{1}{2}}\} = \sum_{n=0}^{\infty} i_n g\{(2\pi)^{\frac{1}{2}}(n+\tfrac{1}{9})^{\frac{1}{2}}\},$$

where $f(x)$, $g(x)$ are Fourier cosine transforms as in (10.1).

By computation we find

$$i_0=1,\quad i_1=-\tfrac{2}{3},\quad i_2=-\tfrac{4}{9},\quad i_3=-\tfrac{40}{81},\quad i_4=-\tfrac{160}{243},\quad i_5=\tfrac{268}{729},\quad i_6=\tfrac{1808}{6561},\ \ldots .$$

This summation formula corresponds to the weighted sequence $\{a_n, \alpha_n\}$ with $a_0=\alpha_0=0$, $\alpha_n=-\alpha_{-n}=(2\pi)^{\frac{1}{2}}(n-8/9)^{\frac{1}{2}}$, $a_n=a_{-n}=i_{n-1}$ for $n\geqslant 1$, and zero for distribution function.

[1] [21], 486–494, or (10.4) above with $N=3$, $A_n=B_n=\chi(n)$.

7. More direct examples of almost periodic weighted sequences can also be constructed. If $a_n = e^{2\pi i n \vartheta}$, $\alpha_n = n$, ϑ irrational, then $\{a_n\}$ is an almost periodic sequence in Walther's sense ([28], [29]). Poisson's summation formula then gives the summation formula (6.5) with $b_n = (2\pi)^{\frac{1}{2}}$, $\beta_n = 2\pi(n+\vartheta)$.

Alternatively we can produce an almost periodic sequence by varying the spacings between successive terms of the basis $\{\alpha_n\}$. For example, put $a_n = 1$, $\alpha_n = n + \frac{1}{3}\sin 2\pi n\vartheta$, ϑ irrational. Then it can be shown that $\{a_n, \alpha_n\}$ is uniformly almost periodic in the sense of Definition 3, but there does not appear to be any simple method of finding the transform $\{b_n, \beta_n\}$ explicitly.

8. Examples of summation formulae in which one or both of the bases $\{\alpha_n\}$, $\{\beta_n\}$ is everywhere dense can also be constructed. Such summation formulae are not associated with almost periodic weighted sequences in the sense of our definitions. For instance, by Poisson's summation formula

$$p^{-2} \sum_{n=-\infty}^{\infty} G(np) = (2\pi)^{\frac{1}{2}} p^{-3} \sum_{m=-\infty}^{\infty} F(2\pi m/p).$$

Summing over all prime numbers p we get

$$\sum_{n=-\infty}^{\infty} a_n G(n) = (2\pi)^{\frac{1}{2}} \sum_{p \geqslant 2} \sum_{m=-\infty}^{\infty} p^{-3} F(2\pi m/p),$$

where
$$a_0 = \sum_{p \geqslant 2} p^{-2}, \quad a_n = a_{-n} = \sum_{p/n} p^{-2}.$$

In this case the sequence $\{\alpha_n\}$ with $\alpha_n = n$ is discrete but the other basis $\{\beta_n\}$ consists of all numbers of the form $2\pi m/p$, and is everywhere dense on the real axis.

9. An example of a summation formula in which both bases $\{\alpha_n\}$ and $\{\beta_n\}$ are everywhere dense is obtained by multiplying together the modular equations

$$\sum_{m=-\infty}^{\infty} \exp\{-\tfrac{1}{2}\pi(z - m\sqrt{2})^2\} = \sum_{m=-\infty}^{\infty} \exp\{-\pi m^2 + \pi i m z \sqrt{2}\}$$

and
$$\sum_{n=-\infty}^{\infty} \exp\{-\tfrac{1}{2}\pi(z - n\vartheta\sqrt{2})^2\} = \varphi \sum_{n=-\infty}^{\infty} \exp\{-\pi n^2 \varphi^2 + \pi i n z \varphi \sqrt{2}\},$$

where $\vartheta\varphi = 1$. This gives

$$\vartheta^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}\pi(m - n\vartheta)^2 - \pi\left(z - \frac{m+n\vartheta}{\sqrt{2}}\right)^2\right\}$$
$$= \varphi^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}\pi(m - n\varphi)^2 - \tfrac{1}{2}\pi(m + n\varphi)^2 + 2\pi i z\left(\frac{m+n\varphi}{\sqrt{2}}\right)\right\}.$$

This corresponds to a summation formula

$$\vartheta^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp\{-\tfrac{1}{2}\pi(m-n\vartheta)^2\}\, G\{\pi^{\frac{1}{2}}(m+n\vartheta)\}$$
$$= \varphi^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp\{-\tfrac{1}{2}\pi(m-n\varphi)^2\}\, F\{\pi^{\frac{1}{2}}(m+n\varphi)\}.$$

If ϑ is irrational then φ is also irrational, and the bases $\{\pi^{\frac{1}{2}}(m+n\vartheta)\}$, $\{\pi^{\frac{1}{2}}(m+n\varphi)\}$ are both everywhere dense on the real axis.

11. Remarks

1. *Other types of almost periodic weighted sequences.*

Several of the examples of section 10 correspond to summation formulae of the form

$$\sum_{n=-\infty}^{\infty} a_n G(\alpha_n) - \int_{-\infty}^{\infty} G(t)\, dA(t) = \sum_{n=-\infty}^{\infty} b_n F(\beta_n) - \int_{-\infty}^{\infty} F(t)\, dB(t), \tag{11.1}$$

where the functions $A(t)$ and $B(t)$ are elementary functions of t. For simplicity the present paper only deals in detail with the case $A(t) = At$, $B(t) = Bt$, but the methods can be extended to cover more general distribution functions $A(t)$ and $B(t)$.

For example, consider the prime number summation formula (10.5). It can be written

$$\sum_{p,m} \frac{\log p}{p^{\frac{1}{2}m}} G(-m\log p) + \sum_{p,m} \frac{\log p}{p^{\frac{1}{2}m}} G(m\log p) -$$
$$- 2\int_{-\infty}^{\infty} G(t)\cosh \tfrac{1}{2}t\, dt + \tfrac{1}{4}\int_{-\infty}^{\infty} G(t)\,\mathrm{sech}\,\tfrac{1}{2}t\, dt -$$
$$- \tfrac{1}{2}\int_{-\infty}^{\infty} G(t)\left|\frac{1}{t} - \tfrac{1}{2}\,\mathrm{cosech}\,\tfrac{1}{2}t\right| dt$$
$$= -(2\pi)^{\frac{1}{2}}\sum_{\gamma} F(\gamma) + (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} F(t)\log\left|\frac{t}{2\pi}\right| dt. \tag{11.2}$$

If we put $G(x) = k(z-x)$ with

$$k(x) = \begin{cases} 1 & (-3\delta < x < -\delta \text{ or } \delta < x < 3\delta), \\ -2\cosh\delta & (-\delta < x < \delta), \\ 0 & \text{elsewhere}, \end{cases} \tag{11.3}$$

then the term
$$2\int_{-\infty}^{\infty} G(t)\cosh\tfrac{1}{2}t\, dt$$

in (11.2) vanishes for all z and all positive δ, and all the other integrals tend to zero as z tends to $\pm\infty$. Also

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{izx}}{x} \{\sin 3\,\delta\, x - (1 + 2\cosh\delta)\sin\delta\, x\}.$$

Consequently the series

$$\sum_{\gamma} F(\gamma) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{\gamma} \frac{e^{iz\gamma}}{\gamma} \{\sin 3\,\delta\,\gamma - (1 + 2\cosh\delta)\sin\delta\,\gamma\}$$

in (11.2) is the Fourier series of a B^2 a.p. function of z, since the series

$$\sum_{\gamma} \frac{1}{\gamma^2}$$

is convergent.

That is, if the Riemann hypothesis is true, then the weighted sequence

$$\left\{\frac{\log p}{p^{\frac{1}{2}m}}, \pm m \log p\right\}$$

is B^2 a. p. for the functions $k(x)$ of (11.3).

2. *Conditions for the reciprocity* (C′) *and reasons for the choice of definitions.*

The theories of Fourier series, Fourier integrals, and almost periodic functions are, in a sense, completed by "mean square" theories; that is by the L^2 and B^2 a.p. theories. For the reciprocity (C′) it is probable that a corresponding theory could be developed for weighted sequences almost periodic in some sense and for which

$$\overline{\lim_{T\to\infty}} \frac{1}{2\,T} \sum_{-T<\alpha_n<T} |a_n|^2$$

is finite. This has not been attempted in the present introductory paper, but it has nevertheless been necessary to use the B^2 theory of almost periodic functions in order to have enough symmetry to prove a form of the theorem of concordance (Theorem 5). This theorem asserts that, with certain additional conditions, any weighted sequence $\{a_n, \alpha_n\}$ which is almost periodic in a certain sense has a transform $\{b_n, \beta_n\}$ which is almost periodic in the same sense. Consequently the use of a type of almost periodicity with some symmetry is essential.

The results of the paper could have been proved more briefly if we had started with Definition 4 for an almost periodic weighted sequence, but it would have been somewhat unsatisfactory to have introduced Definition 4 without indicating the connexions with simpler definitions and with the idea of almost periodic functions. Definitions 1, 2, and 3 were discussed in order to show these connexions.

3. *Relationship to the Riemann hypothesis, and the connexion between the prime numbers and the zeros of the Riemann zeta-function.*

The present investigation arose from earlier work on the connexion between the prime numbers and the non-trivial zeros of the Riemann zeta-function ([12], [13], [15,] [16], [17]). In the present theory the Riemann hypothesis is equivalent to the hypothesis that the weighted sequence

$$\left\{\frac{\log p}{p^{\frac{1}{2}m}},\ \pm m \log p\right\} \tag{11.4}$$

is B^2 a.p. with respect to $4 \sinh \frac{1}{2} x$. Now for each p the sub-sequence $\{\pm m \log p\}$ of the basis is periodic with period $\log p$. Thus the basis of the weighted sequence (11.4) is a combination of periodic sequences; this is far from showing that (11.4) is almost periodic, but it does help to make such a conjecture plausible.

The problem of the nature of the connexion between the prime numbers and the zeros of the Riemann zeta-function was first raised by Landau ([21], 367–368), who conjectured that some arithmetical relationship exists. The present theory shows that, if the Riemann hypothesis is true, then the connexion consists in that the weighted sequence

$$\{-(2\pi)^{\frac{1}{2}},\ \gamma\} \tag{11.5}$$

is almost periodic with respect to the distribution function

$$-(2\pi)^{-\frac{1}{2}}\, x \log \left|\frac{x}{2\pi}\right| + (2\pi)^{-\frac{1}{2}}\, x$$

and (11.5) is the Fourier transform of the weighted sequence (11.4).

4. *Related work and almost elliptic functions.*

In addition to the work of Walther on a more restricted type of almost periodic sequence, as mentioned in section 3, both Bessonof and Métral [3], [4], [23] have written papers on doubly almost periodic functions of a complex variable (or almost elliptic functions) in which they attempt to define such functions by their translation properties. Norgil [26] has shown that such an approach leads to difficulties. Hence it has seemed better to use the indirect Definition 5 for meromorphic almost periodic functions, as suggested by Bohr's remark ([6], 103–104).

Almost elliptic functions have not been considered above, but if $\{a_n, \alpha_n\}$, $\{c_n, \gamma_n\}$ are two uniformly almost periodic weighted sequences with respect to Ax and Cx respectively, then we can define an analogue of the Weierstrass $\wp$-function by the double series

$$\wp_1(z) = \frac{a_0 c_0}{z^2} + \sum \sum a_m c_n \left\{(z - \omega \alpha_m - \omega' \gamma_n)^{-2} - (\omega \alpha_m + \omega' \gamma_n)^{-2}\right\},$$

where $\alpha_0 = \gamma_0 = 0$, the double sum is taken over all m, n not both zero, and the ratio ω'/ω

is not real. It can then be shown that $\wp_1(z)$ is meromorphic almost periodic in any strip of finite width parallel either to the line joining 0 and ω or to the line joining 0 and ω'. Conversely we could define an almost elliptic function as a function which is doubly meromorphic almost periodic in this way, and make this definition the starting point of a theory of almost elliptic functions.

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