ON THE EXTENSIONS OF POSITIVE DEFINITE FUNCTIONS

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1. Introduction

In a beautiful paper [13], M. Krein has given a penetrating analysis of a general problem of moments (see also [17]). This paper is the culmination of a series of notes and papers by Krein stretching over almost a decade (see [13] for a bibliography). His results appear in the form of a theory about a special class of symmetric operators on a Hilbert space (see also [16]). The prototype of an operator in this class may be found in the theory of the classical Hamburger moment problem. The general problem of moments which can be treated by Krein's methods is concerned with conditions on a pre-Hilbert space $\mathfrak{L}$ of analytic functions of a single real variable for which there exists a measure $d\mu(t) \geqslant 0$ so that if $f, g \in \mathfrak{L}$, then

$$(f, g) = \int_{-\infty}^{\infty} f(t)\, \bar{g}(t)\, d\mu(t).$$

The measure $d\mu$ may not be unique and this non-unicity leads to many interesting results.

If one tries to carry over this theory to analytic functions of two real variables (2), one meets rather serious difficulties at the very beginning. In order to try to gain some insight into these multi-variable problems we have, in a series of papers [4, 5,

(1) This research was partially supported by the United States Air Force Office of Scientific Research of the Air Research and Development Command, under contracts No. AF18(600)–1223 and AF18(600)–568. Reproduction in whole or part is permitted for any purpose of the United States Government.

(2) In a recent seminar at Washington University, Professor M. Cotlar has indicated the importance of such a theory in order to unify certain aspects of the theory of singular integrals, multiplier transforms and more general types of integral transforms.

6, 7], and in the present one, investigated some important classical problems. Even in these cases it is not completely clear how much is true, although we suspect a good deal more is true than we have been able to prove. Certain basic similarities appear in the special cases that make it clear that some form of a general theorem exists which cover all of these special cases, but we have not as yet been able to devise a general proof that would cover all of the situations we have considered. Nevertheless, we have felt the special cases to be of sufficient interest to warrant a presentation.

In the present paper we shall be concerned with the problem of extensions of positive definite functions. Suppose $f(x)$ is a continuous complex valued function defined on the interval $(-2a, 2a)$ with the property that for any set $\{x_k\}_1^n \subset (-a, a)$ and $\{\xi_k\}_1^n$ any set of complex numbers,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \xi_j \bar{\xi}_k f(x_j - x_k) \geqslant 0. \tag{1.1}$$

In [10] M. Krein proved that such a function could be extended continuously to the whole real axis so as to retain the positive definite property (1.1) (see also [5] and [15]) and hence, by Bochner's theorem [3; 74], is a Fourier-Stieltjes transform of a non-negative measure. In general such an extension is not unique [9; 22–23]. As a special case of his considerations in [13] Krein obtained an analogous theorem if f is allowed to take values in the space of $n \times n$ matrices.

The same question can be asked when the domain of f is changed in a suitable manner and the range is retained in the complex number field. For example the following could be asked:

Let G be an Abelian, locally compact, topological group and Q a symmetric neighborhood of the identity. Let $f(\mathrm{x})$ be a continuous complex valued function defined on $2Q$ and satisfying (1.1) *for $\{\mathrm{x}_k\} \subset Q$. Is it possible to extend $f(\mathrm{x})$ to all of G so as to retain the positive definite character?*

The answer to this question is in general in the negative since, as we shall show in § 7 by a very simple example, it is already not true for the circle group. For the additive group of integers, with the discrete topology, the question may be very easily answered in the affirmative.

For Euclidean space of any dimension, considered as an additive group under the usual topology, the answer to the above question is open. The problem appears to be a delicate one. By placing additional restrictions on the function f and the neighborhood Q we have been able to answer the question in the affirmative. For simplicity, we shall state our results for only two dimensions.

THEOREM 1. *Let Q be the open rectangle in the (x_1, x_2)-plane given by $|x_k| < a_k$, $k=1, 2$, and $f(\mathbf{x}) = f(x_1, x_2)$ a continuous function on $2Q$ which satisfies (1.1) in Q. If $f(x_1, 0)$ and $f(0, x_2)$ each have unique positive definite extensions along the x_1-axis and x_1-axis respectively, then there exists a unique non-negative measure dF, on the Borel field of the plane, such that*

$$f(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x_1t_1+x_2t_2)}\, dF(t_1, t_2).$$

We can prove a similar theorem for a rectangle of lattice points in the plane having integer components. The proof is considerably simpler.

THEOREM 1′. *Let Q be the rectangle of lattice points (k_1, k_2) where*

$$k_j = 0, \pm 1, \ldots, \pm n_j, \quad j = 1, 2,$$

and Q_1 the points of Q with non-negative components. Further let $f(\mathbf{k}) = f(k_1, k_2)$ be defined on Q and satisfy (1.1) on Q_1. If $f(k_1, 0)$ and $f(0, k_2)$ each has a unique positive definite extension to all of the integers, then there exists a unique measure dF such that

$$f(k_1, k_2) = \int_0^{2\pi} \int_0^{2\pi} e^{i(k_1t_1+k_2t_2)}\, dF(t_1, t_2).$$

The above two theorems are the exact analogues of a theorem proved in [7] on two-parameter moment problems, although the different proofs depend very strongly on the particular situations. In §6 we shall prove two more theorems along the lines of Theorem 1. For example, if Q is as in Theorem 1 and there exists an $\varepsilon > 0$ such that the restriction of $f(x_1, 0)$ to $(-2a_1+\varepsilon, 2a_1-\varepsilon)$ has a *unique* extension, then we may remove all restrictions on $f(0, x_2)$ and be able to prove that $f(x_1, x_2)$ has a positive definite extension to the whole plane. We have not been able to prove analogues of the theorems in §6 for the case of the two-parameter Hamburger moment problem.

The main difficulty encountered in proving these theorems is in the proving of the permutability of the canonical spectral measures of certain unbounded self-adjoint operators. The main part of this paper is devoted to this question and our results are obtained by a very careful examination of the domains of these operators.

The methods developed in this paper and in [6] can be used to considerably simplify certain portions of [4] and [5]. On the other hand, certain basic ideas of the latter papers have been used in the present one. In general, there is very little overlap of results.

We should remark that it is very easy to give necessary and sufficient conditions that a continuous function f defined on a neighborhood Q of the origin may be written as a Fourier-Stieltjes transform of a non-negative measure. Let $\mathfrak{M}$ be the linear space of finite trigonometric sums

$$p(t, \tau) = \sum_{k=1}^{n} \xi_k e^{i(x_k t + y_k \tau)},$$

where $-\infty < x_k, y_k < \infty$ and $-\infty < t, \tau < \infty$. $\mathfrak{M}$ is a partially ordered space by taking $p \geqslant 0$ if $p(t, \tau) \geqslant 0$ for all t, τ. Let $\mathfrak{M}_0$ be the linear subspace of $\mathfrak{M}$ consisting of those trigonometric sums for which $\{(x_k, y_k)\} \subset Q$.

Let L be the linear functional defined on $\mathfrak{M}_0$ by the equation

$$L(p) = \sum_{k=1}^{n} \xi_k f(x_k, y_k).$$

If $L(p) \geqslant 0$ whenever $p \geqslant 0$, then it is known that L may be extended to all of $\mathfrak{M}$ so as to retain this property. The function

$$F(x, y) = L(e^{i(xt+y\tau)})$$

gives a continuous positive definite extension of f to the whole plane.

Conversely, it is clear that the condition $L(p) \geqslant 0$ whenever $p \geqslant 0$ is a necessary condition for f to be a Fourier-Stieltjes transform of a non-negative measure.

2. Preliminaries

To prove our results we shall use the methods of operators in Hilbert space. The fact that functions satisfying (1.1) could be used to construct an inner product on a function space has been a very effective tool in many branches of mathematics. For problems closely allied to those of this paper it has been used by A. Devinatz [5, 6] and M. Krein [11, 13]. General theories concerning non-negative quadratic forms have been constructed by N. Aronszajn [2] and M. Krein [14]. In this paper we shall follow the exposition in [2] as being most suitable for our purposes.

Let E be a set and $K(\mathrm{x}, \mathrm{y})$ a complex valued function defined on $E \times E$ with the property that for any finite set $\{\xi_k\}_1^n$ of complex numbers and points $\{\mathrm{x}_k\}_1^n \subseteq E$,

$$\sum_{j,k=1}^{n} \xi_j \bar{\xi}_k K(\mathrm{x}_k, \mathrm{x}_j) \geqslant 0. \tag{2.1}$$

The main idea in constructing a linear space using the form (2.1) as a norm is as follows. Set

$$g(\mathbf{x})=\sum_{k=1}^{n}\xi_k K(\mathbf{x},\mathbf{x}_k),\ h(\mathbf{x})=\sum_{k=1}^{m}\eta_k K(\mathbf{x},\mathbf{y}_k)$$

and

$$(g,h)=\sum_{k=1}^{n}\sum_{j=1}^{m}\xi_k\bar{\eta}_j K(\mathbf{y}_j,\mathbf{x}_k).$$

In this way we get a pre-Hilbert space which may be completed to a Hilbert space $\mathfrak{F}$ of functions defined on E. An essential property of the space $\mathfrak{F}$ is that if $g\in\mathfrak{F}$ and $K_{\mathbf{y}}(\mathbf{x})=K(\mathbf{x},\mathbf{y})$ then

$$g(\mathbf{y})=(g,K_{\mathbf{y}}).$$

These types of Hilbert spaces have been called by Aronszajn [2] reproducing kernel spaces and the kernels $K(\mathbf{x},\mathbf{y})$, reproducing kernels. Any reproducing kernel space has a unique reproducing kernel.

Another important property enjoyed by these types of Hilbert spaces is that if a sequence of elements converges in the strong topology of $\mathfrak{F}$, then they converge pointwise and even uniformly on those sets for which $K(\mathbf{x},\mathbf{x})$ is bounded. For, we have

$$|g(\mathbf{x})|=|(g,K_{\mathbf{x}})|\leqslant\|g\|\,\|K_{\mathbf{x}}\|=\|g\|\sqrt{K(\mathbf{x},\mathbf{x})}.$$

If $E_1\subseteq E$ and $K_1(\mathbf{x},\mathbf{y})$ is the restriction of $K(\mathbf{x},\mathbf{y})$ to $E_1\times E_1$, then K_1 gives rise to a space $\mathfrak{F}_1$ for which it acts as a reproducing kernel. The pertinent theorem is the following:

THEOREM A [2; 351]. *If K is the reproducing kernel of the space $\mathfrak{F}$ of functions defined on the set E with norm $\|f\|$, then K restricted to the subset $E_1\times E_1\subseteq E\times E$ is the reproducing kernel of the class $\mathfrak{F}_1$ of all restrictions of $\mathfrak{F}$ to the subset E_1. For any such restriction, $f_1\in\mathfrak{F}_1$, the norm $\|f_1\|_1$ is the minimum $\|f\|$ for all $f\in\mathfrak{F}$ whose restriction to E_1 is f_1.*

Finally, we shall have need for the following:

THEOREM B [2; 361]. *$K_1(\mathbf{x}_1,\mathbf{y}_1)$ and $K_2(\mathbf{x}_2,\mathbf{y}_2)$ are reproducing kernels with corresponding spaces $\mathfrak{F}_1$ and $\mathfrak{F}_2$, then $K_1(\mathbf{x}_1,\mathbf{y}_1)K_2(\mathbf{x}_2,\mathbf{y}_2)$ is the reproducing kernel of the direct product of $\mathfrak{F}_1$ and $\mathfrak{F}_2$.*

NOTATION. In the remainder of the paper we shall be working in two dimensional Euclidean space. Real numbers will be denoted by lower case Latin letters. Two dimensional vectors will be denoted by lower case Latin letters in bold face type, their components by the same letters in ordinary type with subscripts; e.g.,

$\mathbf{x} = (x_1, x_2)$. We shall write $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2$, $|\mathbf{x}|^2 = x_1^2 + x_2^2$ and $\mathbf{x} \leqslant \mathbf{y}$ if and only if $x_k \leqslant y_k$, $k = 1,2$. The letters $\mathbf{u}$ and $\mathbf{v}$ will always stand for the special vectors $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$.

If a kernel $K(\mathbf{x}, \mathbf{y})$ satisfies (2.1) we shall write $K(\mathbf{x}, \mathbf{y}) \gg 0$ or simply $K \gg 0$. If $K_2 - K_1 \gg 0$ we shall write $K_1 \ll K_2$.

3. Necessary lemmas

Suppose that Q is an open symmetric neighborhood of the origin in the plane (i.e. $\mathbf{x} \in Q$ implies $(-\mathbf{x}) \in Q$) and $f(\mathbf{x})$ a continuous function defined on $2Q$ such that $f(\mathbf{x}, \mathbf{y}) \gg 0$. Let $\mathfrak{F}$ be the Hilbert space corresponding to the kernel $f(\mathbf{x} - \mathbf{y})$ as discussed in §2. Let $\mathfrak{D}_1^*$ be the linear manifold in $\mathfrak{F}$ such that $\partial g(\mathbf{x})/\partial x_1$ exists and belongs to $\mathfrak{F}$. Define the operator A_1, with domain $\mathfrak{D}_1^*$ by the formula

$$A_1 g(\mathbf{x}) = -i \frac{\partial g(\mathbf{x})}{\partial x_1}.$$

Continuing with definitions and notations, let $\mathfrak{F}'$ be the linear manifold consisting of elements of the form $g(\mathbf{x}) = \sum_1^n \xi_k f(\mathbf{x} - \mathbf{x}_k)$. It is clear from the manner in which $\mathfrak{F}$ is constructed (§2), that $\mathfrak{F}'$ is dense in $\mathfrak{F}$. Since Q is open, for a given $g \in \mathfrak{F}'$, it is always possible to find a vector $\mathbf{r}$, such that $r_1, r_2 \neq 0$ and of sufficiently small norm so that if $|\mathbf{t}| \leqslant |\mathbf{r}|$ then $g_{\mathbf{t}}(\mathbf{x}) = \sum_1^n \xi_k f(\mathbf{x} - \mathbf{x}_k - \mathbf{t})$ is well defined and belongs to $\mathfrak{F}'$. Since $g_{\mathbf{t}}$ is a continuous function of $\mathbf{t}$ in the strong topology of $\mathfrak{F}$, the integral

$$g_{\tilde{\mathbf{r}}} = \frac{1}{r_1 r_2} \int_0^{\mathbf{r}} g_{\mathbf{t}} \, d\mathbf{t} = \frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} g_{\mathbf{t}} \, dt_1 \, dt_2$$

exists and belongs to $\mathfrak{F}$. Further $g_{\tilde{\mathbf{r}}} \boldsymbol{\rightarrow} g$ as $r \rightarrow 0$, where the bold arrow indicates convergence in the strong topology of $\mathfrak{F}$. Let us designate the linear manifold of such elements $g_{\tilde{\mathbf{r}}}$ by $\mathfrak{D}$. It is clear that $\mathfrak{D}$ is dense in $\mathfrak{F}$ since $\mathfrak{F}'$ is dense in $\mathfrak{F}$.

LEMMA 3.1. *$D_1 = A_1^*$ exists, $D_1 \subseteq A_1$ and $D_1^* = A_1$. $\mathfrak{D} \subseteq \mathfrak{D}(D_1)$ (domain of D_1) and the closure of the restriction of D_1 to $\mathfrak{D}$ is D_1.* Further,

$$D_1 g_{\tilde{\mathbf{r}}} = \frac{i}{r_1 r_2} \int_0^{r_2} [g_{r_1\mathbf{u} + t_2\mathbf{v}} - g_{t_2\mathbf{v}}] \, dt_2. \tag{3.1}$$

Proof. Let $\mathbf{s}=(s_1,0)=s_1\mathbf{u}$ with $|\mathbf{s}|$ sufficiently small so that $g_{\mathbf{t}+\mathbf{s}}$ is defined. Consider the element

$$g_{\tilde{\mathbf{s}},\mathbf{r}}=\frac{1}{r_1 r_2}\int_0^{\mathbf{r}} g_{\mathbf{t}+\mathbf{s}}\,d\mathbf{t}.$$

We get

$$\frac{g_{\mathbf{s},\tilde{\mathbf{r}}}-g_{\tilde{\mathbf{r}}}}{s_1}=\frac{1}{r_1 r_2 s_1}\left[\int_0^{r_2}\int_{s_1}^{r_1+s_1} g_{\mathbf{t}}\,d\mathbf{t}-\int_0^{r_2}\int_0^{r_1} g_{\mathbf{t}}\,d\mathbf{t}\right]$$

$$=\frac{1}{r_1 r_2 s_1}\int_0^{r_2}\int_0^{s_1}[g_{\mathbf{t}+r_1\mathbf{u}}-g_{\mathbf{t}}]\,d\mathbf{t}$$

$$\to\frac{1}{r_1 r_2}\int_0^{r_2}[g_{r_1\mathbf{u}+t_2\mathbf{v}}-g_{t_2\mathbf{u}}]\,dt_2 \text{ as } s_1\to 0.$$

If $\mathbf{x}$ and $\mathbf{x}+\mathbf{s}$ both belong to Q, then it is clear $g_{\mathbf{s},\mathbf{r}}(\mathbf{x})=g_{\mathbf{r}}(\mathbf{x}-\mathbf{s})$. Hence, since convergence in the strong topology of $\mathfrak{F}$ implies pointwise convergence we get

$$-i\frac{\partial g_{\tilde{\mathbf{r}}}(\mathbf{x})}{\partial x_1}=A_1 g_{\tilde{\mathbf{r}}}(x)=\frac{i}{r_1 r_2}\int_0^{r_2}[g_{r_1\mathbf{u}+t_2\mathbf{v}}-g_{t_2\mathbf{u}}]\,dt_2.$$

This shows $\mathcal{D}_1^*$ is dense in $\mathfrak{F}$ since $\mathcal{D}$ is dense in $\mathfrak{F}$. Hence $D_1=A_1^*$ exists.(1)

Next let $g\in\mathcal{D}(D_1)$ and for fixed $\mathbf{y}$ set $f_{\mathbf{y}+\mathbf{t}}(\mathbf{x})=f(\mathbf{x}-\mathbf{y}-\mathbf{t})$, provided the latter is defined. Further set

$$f_{\tilde{\mathbf{y}},\mathbf{r}}=\frac{1}{r_1 r_2}\int_0^{\mathbf{r}} f_{\mathbf{y}+\mathbf{t}}\,d\mathbf{t},\quad f_{\tilde{r}_1\mathbf{u}}=\frac{1}{r_1}\int_0^{r_1} f_{\mathbf{y}+t_1\mathbf{u}}\,dt_1.$$

Then
$$(g,\,A_1 f_{\tilde{\mathbf{y}},\mathbf{r}})=\left(g,\,\frac{i}{r_1 r_2}\int_0^{r_2}[f_{\mathbf{y}+r_1\mathbf{u}+t_2\mathbf{v}}-f_{\mathbf{y}+t_2\mathbf{v}}]\,dt_2\right)$$

$$=\frac{-i}{r_1 r_2}\int_0^{r_2}[g(\mathbf{y}+r_1\mathbf{u}+t_2\mathbf{v})-g(\mathbf{y}+t_2\mathbf{v})]\,dt_2$$

$$=(A_1^* g,\,f_{\tilde{\mathbf{y}},\mathbf{r}}).$$

As $r_2\to 0$, $f_{\tilde{\mathbf{y}},\mathbf{r}}\to f_{\tilde{\mathbf{y}},r_1\mathbf{u}}$ and we get

$$(A_1^* g,\,f_{\tilde{\mathbf{y}},r_1\mathbf{u}})=\frac{-i}{r_1}[g(\mathbf{y}+r_1\mathbf{u})-g(\mathbf{y})],$$

(1) We have used here a technique similar to that used in [8].

and as $r_1 \to 0$, $f_{\tilde{y},r_1u} \to f_y$. Hence,

$$A_1^* g(y) = -i \frac{\partial g(y)}{\partial x_1},$$

which shows that $D_1 \subseteq A_1$. Since A_1 is clearly a closed operator (convergence in the norm of $\mathfrak{F}$ implies uniform pointwise convergence!) it follows that $A_1 = A_1^{**} = D_1^*$.

Finally, it remains to prove the second statement of the lemma. Let B_1 be the restriction of A_1 to $\mathcal{D}$ and $g \in D_1^* = \mathcal{D}(A_1)$. If f_{y+t} and $f_{\tilde{y},r}$ are as in the previous paragraph then we have

$$\begin{aligned}(A_1 g, f_{\tilde{y},r}) &= \frac{1}{r_1 r_2} \int_0^r (A_1 g, f_{y+t})\, dt \\ &= \frac{1}{r_1 r_2} \int_0^r -i \frac{\partial g(y+t)}{\partial x_1}\, dt \\ &= \frac{1}{r_1 r_2} \int_0^r \lim_{h \to 0} \left(g, \frac{i}{h} [f_{y+t+hu} - f_{y+t}]\right) dt.\end{aligned}$$

Since the partial derivative of g is continuous (all elements of F are continuous!) we may take the limit outside of the integral sign and then interchange the inner product and the integral sign. We get consequently, after a few manipulations,

$$\begin{aligned}(A_1 g, f_{\tilde{y},r}) &= \lim_{h \to 0} \left(g, \frac{i}{r_1 r_2 h} \int_0^r [f_{y+t+hu} - f_{y+t}]\, dt\right) \\ &= \lim_{h \to 0} \left(g, \frac{i}{r_1 r_2 h} \int_0^h \int_0^{r_2} [f_{y+t+r_1 u} - f_{y+t}]\, dt\right).\end{aligned}$$

As $h \to 0$, the second member of the inner product in the last equation goes strongly to

$$A_1 f_{\tilde{y},r} = \frac{i}{r_1 r_2} \int_0^{r_2} [f_{y+r_1 u+t_2 v} - f_{y+t_2 v}]\, dt_2.$$

Hence,

$$(A_1 g, f_{\tilde{y},r}) = (g, B_1 f_{\tilde{y},r}),$$

which implies $B_1 \subseteq A_1^* = D_1$. On the other hand, by exactly the same method as used to prove $A_1^* \subseteq A_1$ we get $B_1^* \subseteq A_1$. This implies $B_1^* = A_1^{**} = A_1$ and hence $B_1^{**} = A_1^* = D_1$. Since B_1^{**} is the closure of B_1, we have completed the proof of the lemma.

Let r_n be a sequence of non-zero real numbers such that $r_n \to 0$ and $\mathfrak{D}_1$ the manifold of functions of the form

$$g_{\tilde{r}_n} = \frac{1}{r_n} \int_0^{r_n} g_{t\mathbf{u}}\, dt.$$

If we follow through the proof of Lemma 3.1 step by step we arrive at the following.

LEMMA 3.2. *The closure of the restriction of D_1 to $\mathfrak{D}_1$ is D_1 and moreover*

$$D_1 g_{\tilde{r}_n} = \frac{i}{r_n} [g_{r_n \mathbf{u}} - g]. \tag{3.2}$$

If in all of our previous discussion we interchange the subscript 1 with the subscript 2, the corresponding lemmas will be valid for D_2. This will lead to the following lemma.

LEMMA 3.3. $\mathfrak{D} \subseteq \mathfrak{D}(D_1 D_2) \cap \mathfrak{D}(D_2 D_1)$ *and*

$$D_1 D_2 g_{\tilde{\mathbf{r}}} = D_2 D_1 g_{\tilde{\mathbf{r}}} = \frac{-1}{r_1 r_2} [g_{r_1 \mathbf{u} + r_2 \mathbf{v}} - g_{r_1 \mathbf{u}} - g_{r_2 \mathbf{v}} + g].$$

LEMMA 3.4. $aI \leqslant D_1 \leqslant bI$ *if and only if there exists a sequence of non-zero real numbers $r_n \to 0$ such that for $n = 1, 2, \ldots,$*

$$\begin{aligned} a \int_0^{r_n} \int_0^{r_n} f(\mathbf{x}-\mathbf{y}+(s-t)\mathbf{u})\, dt\, ds &\leqslant i \int_0^{r_n} [f(\mathbf{x}-\mathbf{y}-t\mathbf{u}) - f(\mathbf{x}-\mathbf{y}+t\mathbf{u})]\, dt \\ &\leqslant b \int_0^{r_n} \int_0^{r_n} f(\mathbf{x}-\mathbf{y}+(s-t)\mathbf{u})\, dt\, ds. \end{aligned} \tag{3.3}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in Q for which the functions are defined. If a takes on the value $-\infty$ or b the value $+\infty$ the corresponding inequalities are considered redundant.

Proof. We shall first prove the necessity. Let $\{\mathbf{x}_k\}_1^n \subset Q$ and r a real number so that $\mathbf{x}_k + r\mathbf{u} \in Q$. Further, let

$$f_{k,r}(x) = \int_0^r f(\mathbf{x} - \mathbf{x}_k - t\mathbf{u})\, dt$$

and $g = \sum_1^n \xi_k f_{k,r}$. Then from Lemma 3.2 we get

$$(D_1 g, g) = i \sum_{j,k=1}^{n} \xi_j \bar{\xi}_k \int_0^r [f(\mathbf{x}_k - \mathbf{x}_j - (r-t)\mathbf{u}) - f(\mathbf{x}_k - \mathbf{x}_j - t\mathbf{u})]\, dt$$

$$= i \sum_{j,k=1}^{n} \xi_j \bar{\xi}_k \int_0^r [f(\mathbf{x}_k - \mathbf{x}_j - t\mathbf{u}) - f(\mathbf{x}_k - \mathbf{x}_j + t\mathbf{u})]\, dt.$$

Further,
$$(g, g) = \sum_{j,k=1}^{n} \xi_j \bar{\xi}_k \int_0^r \int_0^r f(\mathbf{x}_k - \mathbf{x}_j + (s-t)\mathbf{u})\, dt\, ds.$$

Hence, because of the inequalities satisfied by D_1 we have the conditions (3.3).

To prove the sufficiency we simply note by the computations of the above paragraph that the inequalities (3.3) imply

$$a(g, g) \leqslant (D_1 g, g) \leqslant b(g, g)$$

for any element in the manifold $\mathfrak{D}_1$. By Lemma 3.2 we get this inequality for every element in $\mathfrak{D}(D_1)$ which completes the proof of sufficiency.

Let us now define an operator J on $\mathfrak{J}$ by the formula

$$J g(\mathbf{x}) = \bar{g}(-\mathbf{x}).$$

This is a conjugation operator (see [5; 470]) and clearly permutes with D_1 and D_2. Hence, these operators have self-adjoint extensions.

LEMMA 3.5. *Let H_1 be any self-adjoint extension of D_1, dE_1 its canonical spectral measure and*

$$U(x\mathbf{u}) = \int_{-\infty}^{\infty} e^{ixt}\, dE_1(t), \qquad -\infty < x < \infty.$$

For any $g \in \mathfrak{J}$ we have

$$U(x\mathbf{u})\, g(\mathbf{x}) = g(\mathbf{x} + x\mathbf{u}), \tag{3.4}$$

provided $\mathbf{x} + x\mathbf{u} \in Q$.

Proof. Let $g \in \mathfrak{J}$ such that

$$U(x\mathbf{u}) = \int_{-c}^{c} e^{ixt}\, dE_1(t)\, g,$$

where c is a finite real number. It is clear that $g \in \bigcap_0^\infty \mathfrak{D}(H_1^n)$ and the class of such elements is dense in $\mathfrak{J}$. By expanding e^{ixt} in a Maclaurin series we get

$$U(x\mathbf{u})\, g = \sum_{n=0}^{\infty} \frac{1}{n!} x^n i^n H_1^n g,$$

where convergence is in the strong topology of $\mathfrak{J}$. Since convergence in the strong topology of $\mathfrak{J}$ implies pointwise convergence we have

$$U(x\mathbf{u})\,g(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial g^n(\mathbf{x})}{\partial x_1^n} x^n.$$

Now,

$$\left|\frac{\partial^n g(\mathbf{x})}{\partial x_1^n}\right| = |H_1^n g(\mathbf{x})| = |(H_1^n g, f_{\mathbf{x}})| \leqslant \|H_1^n g\| \sqrt{f(0)} \leqslant c^n \sqrt{f(0)}\, \|g\|,$$

where $f_{\mathbf{x}}(\cdot) = f(\cdot - \mathbf{x})$. This means $g(\mathbf{x})$ is analytic in x_1. Consequently, if $\mathbf{x} + x\mathbf{u} \in Q$, we get (3.4). Since $U(x\mathbf{u})$ is bounded (3.4) must be true for all of $\mathfrak{J}$.

Lemma 3.6. *If* $f_{\mathbf{y}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{y})$ *and* $\mathbf{y} + x\mathbf{u} \in Q$, *then* $U(x\mathbf{u}) f_{\mathbf{y}} = f_{\mathbf{y} - x\mathbf{u}}$.

Proof. By Lemma 3.5, for any $\mathbf{x} \in Q$,

$$U(-x\mathbf{u}) f_{\mathbf{x}}(\mathbf{y}) = f_{\mathbf{x}}(\mathbf{y} - x\mathbf{u}) = (U^*(x\mathbf{u}) f_{\mathbf{x}}, f_{\mathbf{y}})$$
$$= (f_{\mathbf{x}}, f_{\mathbf{y}-x\mathbf{u}}) = (f_{\mathbf{x}}, U(x\mathbf{u}) f_{\mathbf{y}}).$$

Since this is true for every $\mathbf{x} \in Q$ we have our lemma.

If we repeat the arguments of Lemma 3.5 for a self-adjoint extension of D_2 we get a group of unitary operators $U(x\mathbf{v})$ with the same properties as described in the lemma for the second variable. Let us suppose for the moment that for every x_1 and x_2

$$U(x_1\mathbf{u})\,U(x_2\mathbf{v}) = U(x_2\mathbf{v})\,U(x_1\mathbf{u}). \tag{3.5}$$

If we set $U(\mathbf{x}) = U(x_1\mathbf{u})\,U(x_2\mathbf{v})$, then $U(\mathbf{x})$ is a group of unitary operators and for $f_0(\mathbf{x}) = f(\mathbf{x} - \mathbf{0})$ we get

$$F(\mathbf{x}) = (U(\mathbf{x}) f_0, f_0)$$

is a continuous positive definite extension $f(\mathbf{x})$ to the whole plane.

The main difficulty in the problem posed in §1 is the proof of the relation (3.5). The restrictions we put on $f(\mathbf{x})$ in theorem 1 allow us to prove this relation.

If we restrict ourselves to the one-dimensional case the lemmas analogous to those we have been proving in this section would allow us to show that any continuous function defined on $(-2a, 2a)$ such that $f(x-y) \gg 0$ for $x, y \in (-a, a)$ may be extended to a positive definite function on the whole axis. The differential operator, in the one-dimensional case, analogous to D_1 or D_2 we shall simply designate by D.

Lemma 3.7. *If* $f(x)$ *is continuous on the interval* $(-2a, 2a)$, $f(x-y) \gg 0$, $x, y \in (-a, a)$, *then* $f(x)$ *has a unique positive definite extension if and only if* D *is self-adjoint.*

Proof. Suppose first that $f(x)$ has a unique extension. This means there exists only one non-negative bounded measure dF such that

$$f(x) = \int_{-\infty}^{\infty} e^{ixt}\, dF(t).$$

Let H_1 and H_2 be two self-adjoint extensions of D, dE_1 and dE_2 their corresponding spectral measures, and $U_1(x)$ and $U_2(x)$ the corresponding groups of unitary operators as set up in the proof of Theorem 1. By the unicity of the extension of $f(x)$ we must have

$$(E_1(\Delta) f_0, f_0) = (E_2(\Delta) f_0, f_0) = F(\Delta)$$

for any Borel set Δ. Hence for x and y in $(-a, a)$ we have

$$\begin{aligned}(E_1(\Delta) U_1(x) f_0, U_1(y) f_0) &= \int_{\Delta} e^{i(x-y)t}\, d(E_1(t) f_0, f_0)\\ &= \int_{\Delta} e^{i(x-y)t}\, d(E_2(t) f_0, f_0) = (E_2(\Delta) U_2(x) f_0, U_2(y) f_0).\end{aligned}$$

Since the set $\{U_1(x) f_0 = U_2(x) f_0;\ x \in (-a, a)\}$ generates $\mathfrak{F}$, we must have $E_1(\Delta) = E_2(\Delta)$ which in turn implies $H_1 = H_2 = D$.

Conversely, let us suppose that D is self-adjoint and dF is a measure such that

$$f(x) = \int_{-\infty}^{\infty} e^{ixt}\, dF(t).$$

Let $\mathfrak{L}_0$ be the the set of elements in $\mathfrak{L}^2(dF)$ for which $G \in \mathfrak{L}_0$ implies

$$\int_{-\infty}^{\infty} e^{ixt} G(t)\, dF(t) \equiv 0 \quad \text{for } x \in (-a, a).$$

There exists a unitary map U between $\mathfrak{L}_0^{\perp}$ and $\mathfrak{F}$, the correspondence being given by (see [4: 61])

$$h(x) = \int_{-\infty}^{\infty} e^{ixt} H(t)\, dF(t), \quad x \in (-a, a)$$

$$\|h\|^2 = \int_{-\infty}^{\infty} |H(t)|^2\, dF(t).$$

Let $\mathfrak{D} \subseteq \mathfrak{L}_0^{\perp}$ be the class of $H \in \mathfrak{L}_0^{\perp}$ such that $tH(t) \in \mathfrak{L}_0^{\perp}$. Define an operator T on $U\mathfrak{D}$ by the relation

$$Th(x) = \int_{-\infty}^{\infty} e^{ixt}\, t H(t)\, dF(t).$$

It is easy to establish that T is a closed symmetric operator (see [4; 65–66]) and clearly $T \subseteq D$.

Suppose now that

$$g_r^{\sim}(x) = \frac{1}{r}\int_0^r \sum_1^n \xi_k f(x - x_k - y)\,dy = \int_{-\infty}^{\infty} e^{ixt}\,\frac{i}{rt}\left\{\sum_1^n \xi_k e^{-ix_k t}(e^{-itr} - 1)\right\} dF(t).$$

Now, clearly the function

$$H(t) = \frac{1}{r}\int_0^r \sum_1^n \xi_k e^{-ix_k t} e^{-iyt}\,dy = \frac{i}{rt}\left\{\sum_1^n \xi_k e^{-ix_k t}(e^{-itr} - 1)\right\}$$

belongs to $\mathfrak{L}_0^{\downarrow}$ and so also does $tH(t)$. Hence, $g_r^{\sim} \in \mathfrak{D}(T)$. Since $g_r^{\sim} \in \mathfrak{D}(D)$, and by Lemma 3.2 (for the one-dimensional case) D is the closure of its restriction to these elements, and T is closed, we get $D \subseteq T$. This establishes the fact that $T = D$.

If Δ is any Borel set on the line and $H \in \mathfrak{L}_0^{\downarrow}$ set

$$B(\Delta)\,h(x) = \int_{-\infty}^{\infty} e^{ixt} H_{\Delta}(t)\,dF(t),$$

where $H_{\Delta}(t) = H(t)$ for $t \in \Delta$, $H_{\Delta}(t) = 0$ otherwise. It is easy to establish that $B(\Delta)$ is a spectral measure on the Borel field of the line, $(B(\Delta) f_0, f_0) = F(\Delta)$ and

$$(Dg, h) = \int_{-\infty}^{\infty} t\,d(B(t)\,g, h)$$

for any $g \in \mathfrak{D}(D)$ and $h \in \mathfrak{J}$. From this it follows that $dB = dE$, where dE is the canonical spectral measure of D. Hence $dF(t) = d(B(t) f_0, f_0) = d(E(t) f_0, f_0)$. This establishes the unicity of dF.

For explicit details of the proofs of the facts outlined in the previous paragraph see [4; 66–67] where they are given for a similar situation.

4. Proof of Theorem 1

The first point in the proof will be to notice that the operators D_1 and D_2 are self-adjoint. For, the deficiency spaces of D_1, say, are the class of elements in $\mathfrak{J}$ which satisfy $D_1^* g = ig$, $D_1^* h = -ih$, respectively. Let us look at the first of these equations,

$$\frac{\partial g(\mathbf{x})}{\partial x_1} = -g(\mathbf{x}).$$

The solution to this partial differential equation is given by

$$g(x_1, x_2) = e^{-x_1} g(0, x_2).$$

Let $\mathfrak{F}_1$ be the Hilbert space with reproducing kernel $f(x_1 - y_1, 0)$. Since Q is a rectangle, by Theorem A of § 2, $\mathfrak{F}_1$ consists of the restrictions of the elements of $\mathfrak{F}$ to any line parallel to the x_1-axis. (1) Hence, if there exists an x_2 such that $g(0, x_2) \neq 0$, then $e^{-x_1} \in \mathfrak{F}_1$ which by the hypothesis of Theorem 1 and Lemma 3.7 is impossible. Consequently, the deficiency spaces of D_1 contain only the zero element and hence D_1 is self-adjoint and is the closure of its restriction to the class $\mathfrak{D}$ of Lemma 3.1.

The next point in the proof will be to consider the manifold $\mathfrak{M} = (D_1 + iI)\mathfrak{D}$ and show that the closure of the restriction of D_2 to $\mathfrak{M}$ is D_2. Let C_2 be this restriction and $g \in \mathfrak{D}(C_2^*)$. We have then

$$(C_2^* g, (D_1 + iI) \tilde{f}_{\mathbf{x},\mathbf{r}}) = (g, D_2 (D_1 + iI) \tilde{f}_{\mathbf{x},\mathbf{r}}), \tag{4.1}$$

where, as in the proof of Lemma 3.1,

$$\tilde{f}_{\mathbf{x},\mathbf{r}} = \frac{1}{r_1 r_2} \int_0^{\mathbf{r}} f_{\mathbf{x}+\mathbf{t}} \, d\mathbf{t}, \qquad \mathbf{x} + \mathbf{r} \in Q.$$

Let

$$h(\mathbf{x}) = g(\mathbf{x}) - g(\mathbf{x} + r_2 \mathbf{v}) + i \int_0^{r_2} C_2^* g(\mathbf{x} + t\mathbf{v}) \, dt. \tag{4.2}$$

Using the results of Lemmas 3.1 and 3.3 and Equation (4.1) we get

$$\frac{h(\mathbf{x} + r_1 \mathbf{u}) - h(\mathbf{x})}{r_1 r_2} = -\frac{1}{r_1 r_2} \int_0^{r_1} h(\mathbf{x} + t\mathbf{u}) \, dt.$$

Since the limit on the right exists as $r_1 \to 0$ we get

$$\frac{\partial h(\mathbf{x})}{\partial x_1} = -h(\mathbf{x}),$$

and hence

$$h(\mathbf{x}) = e^{-x_1} h(x_2 \mathbf{v}).$$

Putting (4.2) in this last equation and rearranging we get

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(1) The restriction of $f(\mathbf{x} - \mathbf{y})$ to any such line is given by $f(x_1 - y_1, x_2 - x_2) = f(x_1 - y_1, 0)$. Any such line is to be taken as traversing the entire width of Q.

$$\frac{1}{r_2}\int_0^{r_2} C_2^* g(\mathbf{x}+t\mathbf{v})\,dt-\frac{e^{-x_1}}{r_2}\int_0^{r_2} C_2^* g(x_2\mathbf{v}+t\mathbf{v})\,dt$$

$$=-\frac{i}{r_2}\left[g(\mathbf{x}+r_2\mathbf{v})-e^{-x_1}g(x_2\mathbf{v}+r_2\mathbf{v})-g(\mathbf{x})+e^{-x_1}g(x_2\mathbf{v})\right].$$

Letting $r_2\to 0$ we get

$$C_2^* g(\mathbf{x})-e^{-x_1}C_2^* g(x_2\mathbf{v})=-i\,\frac{\partial\left[g(\mathbf{x})-e^{-x_1}g(x_2\mathbf{v})\right]}{\partial x_2}. \tag{4.3}$$

From the facts that $e^{-(x_1+y_1)}\gg 0$ and $f(0, x_2-y_2)\gg 0$ we get [2; 357–361] $e^{-(x_1+y_1)}f(0, x_2-y_2)\gg 0$. Consequently, if $K(\mathbf{x},\mathbf{y})=f(\mathbf{x}-\mathbf{y})+e^{-(x_1+y_1)}f(0, x_2-y_2)$, then $K(\mathbf{x},\mathbf{y})\gg 0$. Let $\mathfrak{F}_2^{\#}$ be the Hilbert space corresponding to the kernel $e^{-(x_1+y_1)}f(0, x_2-y_2)$ and $\mathfrak{F}_3$ the space corresponding to the kernel $K(\mathbf{x},\mathbf{y})$. The elements of $\mathfrak{F}_2^{\#}$ are of the form $e^{-x_1}g(x_2\mathbf{v})$, where $g(\mathbf{x})\in\mathfrak{F}$ and hence the elements of $\mathfrak{F}_3$ are of the form $h(\mathbf{x})+e^{-x_1}g(x_2\mathbf{v})$, where $g, h\in\mathfrak{F}$ [2; 361]. Since D_1 is self-adjoint $\mathfrak{F}$ and $\mathfrak{F}_2^{\#}$ have only the zero element in common and hence $\mathfrak{F}_3=\mathfrak{F}\oplus\mathfrak{F}_2^{\#}$ [2; 352–353]; i.e., $\mathfrak{F}$ and $\mathfrak{F}_2^{\#}$ are orthogonal complements of each other in $\mathfrak{F}_3$.

Let $A_2'=-i\,\partial/\partial x_2$ be the differential operator acting in $\mathfrak{F}_2^{\#}$ and A_2'' the differential operator acting in $\mathfrak{F}_3$. More precisely, the domain of A_2' consists of those elements in $\mathfrak{F}_2^{\#}$ whose partial derivative with respect to x_2 exists and again belongs to $\mathfrak{F}_2^{\#}$, and analogously for A_2''. Since $f(0, x_2)$ has a unique extension, A_2' is self-adjoint.(1) Hence, since $A_2=D_2$ is self-adjoint, the operator $A_2\oplus A_2'$ is self-adjoint, and since the domain of A_2'' consists of those elements in $\mathfrak{F}_3$ whose partial derivative with respect to x_2 exists and again belongs to $\mathfrak{F}_3$, we have

$$A_2''^{*}\subseteq A_2\oplus A_2'\subseteq A_2''.$$

The operator A_2'' is closed, but we do not know, as yet, that it is symmetric. Symmetry, together with the above relation would imply that A_2'' is self-adjoint. We shall now show that the conditions put on f in Theorem 1 imply that A_2'' is self-adjoint.

The operator $A_2''^{*}$ is closed and symmetric and since A_2'' is closed, $A_2''=A_2''^{**}$. In order to show that A_2'' is self-adjoint we shall show that the deficiency spaces connected with the symmetric operator $A_2''^{*}$ consist only of the zero element. Now, for any closed symmetric operator T, the deficiency spaces consist of those elements which satisfy the relations $T^*g=ig$, $T^*h=-ih$, respectively. Hence, in our situa-

(1) A_2' is the tensor product $I\otimes A_2$, where I is the identity operator in the one dimensional space generated by e^{-x_1} and A_2 is the differential operator on $\mathfrak{F}_2$, the space generated by $f(0, x_2-y)$. Hence A_2' is self-adjoint if and only if A_2 is self-adjoint.

tion let us consider, for example, those $g \in \mathfrak{I}_3$ such that $A_2''^{**} g = A_2'' g = i g$. Solving this partial differential equation yields $g(\mathrm{x}) = e^{-x_1} g(x_2 \mathrm{v})$. For any *fixed* x_1, it follows by Theorem A that $g_{x_1}(x_2) = g(x_1, x_2)$ belongs to the space generated by the kernel $K_{x_1}(x_2, y_2) = K((x_1, x_2), (x_1, y_2)) = (1 + e^{-2x_1}) f(0, x_2 - y_2)$. Since $f(0, x_2)$ has a unique positive definitive extension it follows that the function $(1 + e^{-2x_1}) f(0, x_2)$ has a unique positive definite extension. Hence, by Lemmas 3.1 and 3.7, the only element g in the space generated by K_{x_1} which satisfies $-i\,dg/dx_2 = ig$ must be the zero element. Consequently, $g_{x_1}(x_2) = g(x_1, x_2) = 0$. By the same reasoning the only solution in $\mathfrak{I}_3$ to the equation $A_2'' h = -ih$ is the zero element. This shows that A_2'' is self-adjoint and hence

$$A_2'' = A_2 \oplus A_2'.$$

This, in turn, implies $\mathfrak{I}$ and $\mathfrak{I}_2^{\neq}$ reduce A_2''.

Returning now to (4.3) we see that the right side belongs to $\mathfrak{D}(A_2'')$ and hence $g \in \mathfrak{D}(D_2)$ and $e^{-x_1} g(x_2 v) \in \mathfrak{D}(A_2')$. Hence,

$$C_2^* g = D_2 g.$$

This means $C_2^* = D_2$, which is what we set out to prove, namely, that the closure of the restriction of D_2 to $\mathfrak{M}$ is D_2.

Let U_1 be the Cayley transform of D_1. If $g \in \mathfrak{M}$, i.e.,

$g = (D_1 + iI) h$, $h \in \mathfrak{D}$, then $U_1 D_2 g = U_1 (D_1 + iI) D_2 h = (D_1 - iI) D_2 h = D_2 U_1 (D_1 + iI) h$.

Hence,

$$U_1 D_2 g = D_2 U_1 g$$

Since D_2 is the closure of its restriction to $\mathfrak{M}$, it follows that

$$U_1 D_2 \subseteq D_2 U_1.$$

This says D_1 and D_2 permute; i.e., their canonical spectral measures permute.

Let dE_1 and dE_2 be the canonical spectral measures of D_1 and D_2 respectively and $dE = dE_1 dE_2$. By Lemma 3.5 it follows that if

$$U(\mathrm{x}) = \int_{-\infty}^{\infty} e^{\mathrm{x} \cdot \mathrm{t}} dE(\mathrm{t}),$$

then

$$F(\mathrm{x}) = (U(\mathrm{x}) f_0, f_0) = \int_{-\infty}^{\infty} e^{\mathrm{x} \cdot \mathrm{t}} d(E(\mathrm{t}) f_0, f_0)$$

is an extension of $f(\mathrm{x})$.

Finally, the unicity of the extension of $\mathrm{f}(\mathrm{x})$ follows by a two-dimensional analogue of the argument in Lemma 3.7. This completes the proof of Theorem 1.

5. Proof of Theorem I′

We shall start with the proof of two lemmas. In the next two lemmas the symbols j, k, n will be integers.

LEMMA 5.1. *If $\mu(j-k) \gg 0$ for $0 \leqslant j, k \leqslant n$, then $\mu(k)$ has a unique positive definite extension to all of the integers if and only if* $\det\{\mu(j-k)\}_{j,k=0}^{n} = 0$.

Proof. Let $\mathfrak{J}$ be the Hilbert space corresponding to the kernel $\mu(j-k)$. Suppose $\mu(k)$ has a unique positive definite extension; then $\{\mu_k\}_0^{n-1}$ generates $\mathfrak{J}$, where $\mu_k(\cdot) = \mu(\cdot - k)$. For, otherwise consider the operator $U\mu_k = \mu_{k+1}$, $0 \leqslant k \leqslant n-1$. U is an isometric operator defined on a subspace of $\mathfrak{J}$ and clearly this can be extended in many ways to be unitary on $\mathfrak{J}$. Let V be any such unitary extension. Then $V^k \mu_0 = \mu_k$, $0 \leqslant k \leqslant n$, and if $dE(t)$ is the canonical spectral measure of V^* we have for $k \geqslant 0$,

$$\mu(k) = (\mu_0, V^k \mu_0) = \int_0^1 e^{2\pi i k t}\, d(E(t)\mu_0, \mu_0),$$

and since $\mu(-k) = \bar{\mu}(k)$, this formula gives a positive definite extension of $\mu(k)$. Hence every different unitary extension of U will give a different extension which is impossible. Consequently, $\{\mu_k\}_0^{n-1}$ generates $\mathfrak{J}$ which gives the necessity part of our lemma.

Suppose now that $\det\{\mu(j-k)\}_{j,k=0}^{n} = 0$; then $\{\mu_k\}_0^{n-1}$ generates $\mathfrak{J}$ and the translation operator $U\mu_k = \mu_{k+1}$, $0 \leqslant k \leqslant n-1$, is unitary. Suppose $\nu(k)$ is a positive definite extension of $\mu(k)$ to all of the integers, $\mathfrak{J}_1$ the space generated by the kernel $\nu(j-k)$, $-\infty < j, k < \infty$, and V the translation operator $V\nu_k = \nu_{k+1}$. Let $\mathfrak{M}$ be the space generated by $\{\nu_k\}_0^n$; then there exists a $(1-1)$ isometric map between $\mathfrak{M}$ and $\mathfrak{J}$ and hence $\{\nu_k\}_{k=0}^{n-1}$ generates $\mathfrak{M}$ and $V\mathfrak{M} = \mathfrak{M}$. Since V is unitary, $\mathfrak{M}$ reduces V.

Let W be the isometric map from $\mathfrak{J}$ to $\mathfrak{M}$ defined by $W\mu_k = \nu_k$. If $dE(t)$ is the canonical spectral measure of V we have

$$(V\nu_j, \nu_k) = (U\mu_j, \mu_k) = \int_0^1 e^{2\pi i t}\, d(W^{-1}E(t)W\mu_j, \mu_k).$$

Hence $dF = dW^{-1}EW$ is the canonical resolution of the identity of U. Now, for $-n \leqslant k \leqslant n$,

$$\mu(k) = (\mu_0, U^k \mu_0) = (\nu_0, V^k \nu_0) = \int_0^1 e^{-2\pi i k t}\, d(E(t)\nu_0, \nu_0)$$

$$= \int_0^1 e^{-2\pi i k t}\, d(F(t)\mu_0, \mu_0).$$

Hence the measure which gives rise to *any* extension is the same, namely $d(F(t)\mu_0, \mu_0)$ which shows we can have only one extension. This completes the proof of Lemma 1.

In our next lemma the symbols $\mathbf{j}$, $\mathbf{k}$, $\mathbf{n}$ will be the lattice points (j_1, j_2), (k_1, k_2), (n_1, n_2) respectively.

LEMMA 5.2. *Let* $\mu(\mathbf{j}-\mathbf{k}) \gg 0$ *for* $0 \leqslant \mathbf{j}, \mathbf{k} \leqslant \mathbf{n}$ *and* $\mu(k_1 \mathbf{u})$, $\mu(k_2 \mathbf{v})$ *have unique positive definite extensions. If* $\mathfrak{J}$ *is the space generated by the kernel* $\mu(\mathbf{j}-\mathbf{k})$, *then* $\mathfrak{J}$ *is generated by* $\{\mu_{\mathbf{k}};\ 0 \leqslant \mathbf{k} \leqslant (n_1-1, n_2-1)\}$.

Proof. Suppose $g \in \mathfrak{J}$ is perpendicular to the space generated by $\{\mu_{\mathbf{k}};\ 0 \leqslant \mathbf{k} \leqslant (n_1-1, n_2-1)\}$; then $g(\mathbf{k}) = 0$ for $0 \leqslant \mathbf{k} \leqslant (n_1-1, n_2-1)$. By Lemma 1, $g(k_1 \mathbf{u}) = 0$ for $0 \leqslant k_1 \leqslant n_1$, since $\mu(k_1 \mathbf{u})$ has a unique extension. Further, if $1 \leqslant j_2 \leqslant n_2 - 1$, $g(k_1, j_2)$ as a function of k_1 belongs to the space generated by the kernel $\mu(j_1 - k_1, 0)$ and hence again by Lemma 1, $g(k_1, j_2) = 0$ for $0 \leqslant k_1 \leqslant n_1 - 1$ implies $g(n_1, j_2) = 0$. If we now argue on the second variable in the same way we get $g(\mathbf{k}) \equiv 0$. This completes the proof of Lemma 2.

We are now in a position to prove Theorem 1′. The notation in the following is the same as for Lemma 2. By Lemma 2 the set $\{\mu_{\mathbf{k}};\ 0 \leqslant \mathbf{k} \leqslant (n_1-1, n_2-1)\}$ generates $\mathfrak{J}$ and hence the operators $U_1 \mu_{\mathbf{k}} = \mu_{\mathbf{k}+\mathbf{u}}$, $U_2 \mu_{\mathbf{k}} = \mu_{\mathbf{k}+\mathbf{v}}$, $\mathbf{k} \neq \mathbf{n}$, are unitary and clearly permute. If dE_1 and dE_2 are the canonical spectral measures of U_1^* and U_2^* respectively and $dE(\mathbf{t}) = dE_1(t_1) E_2(t_2)$, then

$$\mu(\mathbf{k}) = \int_0^1 e^{2\pi i \mathbf{k}\cdot\mathbf{t}}\, d(E(\mathbf{t})\mu_0, \mu_0),$$

and the right hand integral gives our extension.

The proof of the fact that the measure which appears in the above representation is unique proceeds along the same lines as the second half of the proof of Lemma 5.1. This completes the proof of Theorem 1′.

6. Further theorems

LEMMA 6.1. *Let* $f(x)$ *be defined and continuous on the interval* $(-2a, 2a)$ *and* $f(x-y) \gg 0$ *for* $x, y \in (-a, a)$. *Suppose there exists an* $\varepsilon > 0$ *such that the restriction of* f *to* $(-2a+2\varepsilon,\ 2a-2\varepsilon)$ *has a unique positive definite extension to the whole axis. If* $\mathfrak{J}$ *is the space corresponding to* $f(x-y)$, *for* $x, y \in (-a, a)$, *then* $\mathfrak{J}$ *is generated by the set of elements* $\{f_y(x) = f(x-y);\ -a+\varepsilon < y < a-\varepsilon\}$.

Proof. Let $\mathfrak{J}_\varepsilon$ be the Hilbert space generated by the kernel $f_\varepsilon(x-y)=f(x-y)$ for $x, y \in (-a+\varepsilon, a+\varepsilon)$. By the hypothesis of the lemma and by Lemma 3.7 the differential operator D_ε on $\mathfrak{J}_\varepsilon$ as set up in § 3 is self-adjoint. Hence, we have

$$f(x)=\int_{-\infty}^{\infty} e^{ixt}\, d(E(t) f_{\varepsilon 0}, f_{\varepsilon 0})_\varepsilon, \quad x \in (-2a, 2a),$$

where $f_{\varepsilon 0}(x)=f_\varepsilon(x-0)$, $(\ ,\ )_\varepsilon$ is the inner product in $\mathfrak{J}_\varepsilon$ and dE is the spectral measure of D_ε.

If D is the differential operator set up in the space $\mathfrak{J}$, we also have

$$f(x)=\int_{-\infty}^{\infty} e^{ixt}\, d(F(t) f_0, f_0),$$

where dF is the spectral measure of D. Since f_ε has a unique extension we must have $d(E(t) f_{\varepsilon 0}, f_{\varepsilon 0})_\varepsilon = d(F(t) f_0, f_0)$.

By a general Theorem [4; 61], the space $\mathfrak{J}$ is the set of all functions

$$g(x)=\int_{-\infty}^{\infty} e^{ixt}\, G(t)\, d(F(t) f_0, f_0), \qquad x \in (-a, a)$$

with $G \in \mathfrak{L}^2(d(F(t) f_0, f_0))$, and

$$\|g\|^2=\int_{-\infty}^{\infty} |G(t)|^2\, d(F(t) f_0, f_0).$$

By the same general theorem, $\mathfrak{L}^2(d(F(t) f_0, f_0)) = \mathfrak{L}^2(d(E(t) f_{\varepsilon 0}, f_{\varepsilon 0})_\varepsilon)$ is generated by elements of the form

$$\sum_{k=1}^{n} \xi_k e^{iy_k t},$$

where ξ_k is a complex number and $y_k \in (-a+\varepsilon, a-\varepsilon)$. Hence, the lemma is proved.

THEOREM 2. *Let Q be the open interval in the plane $-\mathbf{a}<\mathbf{x}<\mathbf{a}$ and suppose $f(\mathbf{x})$ defined on $2Q$ is continuous and $f(\mathbf{x}-\mathbf{y}) \gg 0$ for $\mathbf{x}, \mathbf{y} \in Q$. Suppose there exists an $\varepsilon>0$ such that the function $f(x_1, 0)$ restricted to $(-2a_1+2\varepsilon, 2a_1-2\varepsilon)$ has a unique extension to the whole axis; then f may be extended to be positive definite over the whole plane.*

Proof. Let Q_ε be the two dimensional interval defined by the inequalities $|x_1|<a_1-\varepsilon$, $|x_2|<a_2$. Let $\mathfrak{J}_0 \subseteq \mathfrak{J}$ be the class of elements which vanish on Q_ε. If

$g \in \mathfrak{J}_0$, then for fixed x_2, $g(x_1, x_2)$, considered as a function of x_1, belongs, by Theorem A, to the space generated by $f(x_1 - y_1, 0)$ and is orthogonal to the space generated by the elements $f_{y_1}(x_1) = f(x_1 - y_1, 0)$ for $|y_1| < a_1 - \varepsilon$. By Lemma 6.1, this orthogonal space $\mathfrak{J}_0$ consists only of the zero element. Hence $g(x_1, x_2) \equiv 0$. This means $\mathfrak{J}$ is generated by the set of elements $f_{\mathbf{y}}$, for which $|y_1| < a_1 - \varepsilon$, $|y_2| < a_2$.

Let us set up the space $\mathfrak{D}_2'$ of elements of the form

$$g_{\tilde{r}} = \frac{1}{r} \int_0^r g_t \, dt, \tag{6.1}$$

where $g_t(\mathbf{x}) = \sum_1^n \xi_k f(\mathbf{x} - \mathbf{x}_k - t\mathbf{v})$, $|\mathbf{x}_k \cdot \mathbf{u}| < a_1 - \varepsilon$ and $\mathbf{x}_k + r\mathbf{v} \in Q$. We shall show that if D_2' is the restriction of D_2 to $\mathfrak{D}_2'$ then the closure of D_2' is D_2.

As in the Lemmas 3.1 and 3.2 we set up the space $\mathfrak{D}_2$ of elements of form (6.1), where $|\mathbf{x}_k \cdot \mathbf{u}| < a_1$, $|\mathbf{x}_k \cdot \mathbf{v}| < a_2$. The lemmas mentioned proved for us that D_2 is the closure of its restriction to $\mathfrak{D}_2$. Let $h_{\tilde{r}} \in \mathfrak{D}_2$; i.e. there exists an $h_t(\mathbf{x}) = \sum_1^n \xi_k f(\mathbf{x} - \mathbf{y}_k - t\mathbf{v})$ such that

$$h_{\tilde{r}} = \frac{1}{r} \int_0^r h_t \, dt,$$

where $|\mathbf{y}_k \cdot \mathbf{u}| < a_1$, $|\mathbf{y}_k \cdot \mathbf{v}| < a_2$, $\mathbf{y}_k + r\mathbf{v} \in Q$. Let $m = \max |\mathbf{y}_k \cdot \mathbf{v}|$ and $\mathfrak{J}_m$ the subspace of $\mathfrak{J}$ generated by the set $\{f_{\mathbf{y}}; |\mathbf{y} \cdot \mathbf{v}| \leqslant m\}$. The space $\mathfrak{J}_m$ is the orthogonal complement of the set of functions in $\mathfrak{J}$ which vanish on the rectangle $|\mathbf{x} \cdot \mathbf{u}| < a_1$, $|\mathbf{x} \cdot \mathbf{v}| \leqslant m$. By an argument similar to the argument which we have already made previously, $\mathfrak{J}_m$ is generated by the set of elements $\{f_{\mathbf{y}}; |\mathbf{y} \cdot \mathbf{u}| < a_1 - \varepsilon, |\mathbf{y} - \mathbf{v}| < m\}$. Hence, there exist elements $h_{n,t}(\mathbf{x}) = \sum \xi_k^{(n)} f(\mathbf{x} - \mathbf{x}_k^{(n)} - t\mathbf{v})$ with $|\mathbf{x}_k^{(n)} \cdot \mathbf{u}| < a_1 - \varepsilon$, $|\mathbf{x}_k^{(n)} \cdot \mathbf{v}| \leqslant m$ such that $h_{n,0} \to h_0$ as $n \to \infty$.

Let $U(x\mathbf{v})$ be the group of unitary operators as constructed in Lemma 3.5 with any self-adjoint extension of D_2 as its infinitesimal generator. As shown in that lemma, this is a group of translation operators on $\mathfrak{J}$, wherever the translations are defined. Hence, if $|t| < |r|$, since $U^*(t\mathbf{v}) h_{n,0} = h_{n,t}$ and $U^*(t\mathbf{v}) h_0 = h_t$, we have $h_{n,t} \to h_t$ uniformly in t. This means that if we set

$$h_{\tilde{n},r} = \frac{1}{r} \int_0^t h_{n,t} \, dt,$$

then $h_{\tilde{n},r} \to h_{\tilde{r}}$. Now, by Lemma 3.2

$$D_2' h_{n,r}^{\sim} = \frac{i}{r}[h_{n,r} - h_{n,0}] = \frac{i}{r}[U^*(r\mathbf{v}) - I]\, h_{n,0}$$

$$D_2 h_r^{\sim} = \frac{i}{r}[h_r - h_0] = \frac{i}{r}[U^*(r\mathbf{v}) - I]\, h_0.$$

Hence, $D_2' h_{n,r}^{\sim} \to D_2 h_r^{\sim}$; and since D_2 is the closure of its restriction to $\mathcal{D}_2$ it follows that D_2 is the closure of D_2'.

The differential operator D_1 is self-adjoint since $f(x_1 - y_1, 0)$ has a unique extension. Let $U(x\mathbf{u})$ be the group of unitary operators as set up in Lemma 3.5. Choose any x such that $|x| < \varepsilon$ and $g_r^{\sim} \in \mathcal{D}_2'$; i.e.

$$g_r^{\sim} = \frac{1}{r}\int_0^r g_t\, dt,$$

where $g_t(\mathbf{x}) = \sum \xi_k f(\mathbf{x} - \mathbf{x}_k - t\mathbf{v})$, $|\mathbf{x}_k \cdot \mathbf{u}| < a_1 - \varepsilon$ and $\mathbf{x}_k + r\mathbf{v} \in Q$. Further let $h_t(\mathbf{x}) = \sum \xi_k f(\mathbf{x} - \mathbf{x}_k - x\mathbf{u} - t\mathbf{v})$. Then clearly

$$U^*(x\mathbf{u})\, D_2' g_r^{\sim} = \frac{i}{r}[h_r - h_0] = D_2 U^*(x\mathbf{u})\, g_r^{\sim}.$$

Since the closure of D_2' is D_2 and $U(x\mathbf{u})$ is bounded we have

$$U^*(x\mathbf{u})\, D_2 g = D_2\, U^*(x\mathbf{u})\, g$$

for every $g \in \mathcal{D}(D_2)$. Since this is true for every x such that $|x| < \varepsilon$, it follows that the whole group $U(x\mathbf{u})$, $-\infty < x < \infty$, permutes with D_2. Hence D_1 and D_2 permute in the sense that D_2 permutes with the canonical spectral measure of D_1. This means that there exists a sequence of subspaces $\{\mathfrak{M}_n\}$ each of which reduces D_2, and therefore D_2^*, and moreover reduces D_1 to a bounded self-adjoint operator.

As in § 3 define the conjugation operator J by

$$J g(\mathbf{x}) = \bar{g}(-\mathbf{x}).$$

It is clear that J permutes with both D_1 and D_2. Hence $\mathfrak{M}_n$ reduces J and the restrictions of D_1 and D_2 to $\mathfrak{M}_n$ are real with respect to J.

Let D_{1n} and D_{2n} be the restriction to $\mathfrak{M}_n$ of D_1 and D_2 respectively. Since D_{2n} is real with respect to J it has equal deficiency indices and therefore self-adjoint extensions. The deficiency spaces $\mathcal{E}_n^+$ and $\mathcal{E}_n^-$ of D_{2n} are given by $\mathcal{E}_n^+ = [g \mid g \in \mathfrak{M}_n$ and $D_2^* g = i g]$ and $\mathcal{E}_n^- = [g \mid g \in \mathfrak{M}_n$ and $D_2^* g = -i g]$ respectively. Hence $\mathcal{E}_n^+$ and $\mathcal{E}_n^-$ contain only elements of the form $g(x_1, 0)\, e^{-x_2}$ and $g(x_1, 0)\, e^{x_2}$ respectively. On the other hand, any elements of this form which belong to $\mathfrak{M}_n$ belong to $\mathcal{E}_n^+$ and $\mathcal{E}_n^-$ respectively. Since D_{1n} is bounded, the last statement immediately implies that $\mathcal{E}_n^+$ and $\mathcal{E}_n^-$ both reduce D_{1n} and hence both of these manifolds reduce each element of the group $U(x\,\mathbf{u})$.

Let $K_n(\mathbf{x}, \mathbf{y})$ be the reproducing kernel corresponding of $\mathcal{E}_n^+$. If P_n is the projection (1) onto $\mathcal{E}_n^+$, then $K_n(\mathbf{x}, \mathbf{y}) = P_n f(\mathbf{x} - \mathbf{y})$. Further, since every element of the group $U(x\,\mathbf{u})$ is reduced by $\mathcal{E}_n^+$, every element of this group permutes with P_n. Using the fact that $K_n(\mathbf{y}, \mathbf{x}) = \bar{K}_n(\mathbf{x}, \mathbf{y})$ we get

$$K_n(\mathbf{x}, \mathbf{y}) = e^{-(x_2+y_2)} K_n(x_1\,\mathbf{u}, y_1\,\mathbf{u}) = e^{-(x_2+y_2)} P_n\, U(x_1\,\mathbf{u} - y_1\,\mathbf{u}) f_0(0)$$
$$= e^{-(x_2+y_2)}\, U(x_1\,\mathbf{u} - y_1\,\mathbf{u}) P_n f_0(0) = e^{-(x_2+y_2)}\, U(x_1\,\mathbf{u} - y_1\,\mathbf{u}) K_n(0, 0).$$

If we let $K_{1n}(x_1 - y_1) = U(x_1\,\mathbf{u} - y_1\,\mathbf{u}) K_n(0, 0)$ then we get

$$K_n(\mathbf{x}, \mathbf{y}) = K_{1n}(x_1 - y_1)\, e^{-(x_2+y_2)}.$$

That is to say, $K_n(\mathbf{x}, \mathbf{y})$ is the product of two kernels and hence according to theorem B of § 2, $\mathcal{E}_n^+$ is the tensor product of the spaces $\mathfrak{F}_{1n}$ and $\mathfrak{F}_{2n}$, where $\mathfrak{F}_{1n}$ corresponds to the kernel K_{1n} and $\mathfrak{F}_{2n}$ to the kernel $e^{-(x_2+y_2)}$. Hence, if $g(x_1, 0)\, e^{-x_2} \in \mathcal{E}_n^+$, then $g(x_1, 0) \in \mathfrak{F}_{1n}$ and hence $\bar{g}(-x_1, 0) \in \mathfrak{F}_{1n}$. This implies that $\bar{g}(-x_1, 0)\, e^{-x_2} \in \mathcal{E}_n^+$ and

$$\|e^{-x_2} g(x_1, 0\| = \|e^{-x_2}\|_{2n}\, \|g(x_1, 0)\|_{1n} = \|e^{-x_2}\|_{2n}\, \|\bar{g}(-x_1, 0\|_{1n}$$
$$= \|e^{-x_2} \bar{g}(-x_1, 0)\|.$$

Hence $J\bar{g}(-x_1, 0)\, e^{-x_2} = g(x_1, 0)\, e^{x_2} \in \mathcal{E}_n^-$ and $g(x_1, 0)\, e^{-x_2}$ and $g(x_1, 0)\, e^{x_2}$ have the same norm.

Let V_{2n} be the operator defined from $\mathcal{E}_n^+$ onto $\mathcal{E}_n^-$ by the equation.

$$V_{2n}\, e^{-x_2} g(x_1, 0) = e^{x_2} g(x_1, 0)$$

and let V_{2n}' be the Cayley transform of D_{2n}. V_{2n} is an isometric operator which clearly permutes with D_{1n} on $\mathcal{E}_n^+$. On the other hand, $\mathfrak{D}(V_{2n}')$ is given by the set of all elements of the form $h = (D_{2n} + iI)\, g$, where $g \in \mathfrak{D}(D_{2n})$, and hence $V_{2n}' h = (D_{2n} - iI)\, g$. Since D_{1n} is bounded and self-adjoint and permutes with D_{2n} it permutes with V_{2n}' on the orthogonal complement of $\mathcal{E}_n^+$. Consequently, if we set $U_{2n} = V_{2n}' \oplus V_{2n}$, we have that D_{1n} permutes with U_{2n}. If H_{2n} is the self-adjoint extension of D_{2n} whose Cayley transform is U_{2n}, then D_{1n} and H_{2n} permute.

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(1) P_n is the projection from $\mathfrak{F}$ onto $\mathcal{E}_n^+$.

If we construct such a self-adjoint operator H_{2n} for every n and let $H_2 = \sum_1^\infty \oplus H_{2n}$, then H_2 is self-adjoint and D_1 and H_2 permute in the sense that their canonical spectral measures permute. Let dE_2 be the canonical spectral measure of H_2 and set

$$U(x\mathbf{v}) = \int_{-\infty}^{\infty} e^{ixt}\, dE_2(t).$$

Clearly, as $\mathbf{x}$ varies over the plane, the operators $U(\mathbf{x}) = U(x_1\mathbf{u})\, U(x_2\mathbf{v})$ form a group of unitary operators. If $dE = dE_1 E_2$, then

$$F(\mathbf{x}) = (U(\mathbf{x}) f_0, f_0) = \int_{-\infty}^{\infty} e^{i\mathbf{x}\cdot\mathbf{t}}\, d(E(\mathbf{t}) f_0, f_0)$$

is a positive definite extension of $f(\mathbf{x})$. This concludes the proof of Theorem 2.

THEOREM 3. *Let Q be an open symmetric neighbourhood in the plane* (i.e., $\mathbf{x} \in Q$ *implies* $(-\mathbf{x}) \in Q$) *and $f(\mathbf{x})$ a continuous function defined on $2Q$. Necessary and sufficient conditions that there exists a bounded measure $dF \geq 0$ whose support is in the half-plane $\mathbf{t}\cdot\mathbf{u} \geq 0$ and such that*

$$f(\mathbf{x}) = \int_{-\infty}^{\infty} e^{\mathbf{x}\cdot\mathbf{t}}\, dF(\mathbf{t})$$

are:

(a) $$f(\mathbf{x} - \mathbf{y}) \gg 0 \text{ for } \mathbf{x}, \mathbf{y} \in Q.$$

(b) *There exists a sequence of real numbers $r_n \to 0$ such that for $n = 1, 2, \ldots,$*

$$i\int_0^{r_n} [f(\mathbf{x} - \mathbf{y} - t\mathbf{u}) - f(\mathbf{x} - \mathbf{y} + t\mathbf{u}]\, dt \gg 0$$

for all $\mathbf{x}$ and $\mathbf{y}$ in Q for which the functions are defined.

Proof. That these conditions are necessary may be checked immediately by a simple computation.

To prove the sufficiency we set up the space $\mathfrak{J}$ corresponding to $f(\mathbf{x} - \mathbf{y})$. Condition (b) together with Lemma 3.4 tells us that the operator D_1 is non-negative. Let H_1 be any positive self-adjoint extension of D_1, dE_1 its canonical spectral measure and

$$U(x\mathbf{u}) = \int_{-\infty}^{\infty} e^{ixt}\, dE_1(t).$$

Further, let H_2 be any self-adjoint extension of D_2 and U_2 its Cayley transform.

Suppose $h^{\sim} \in D_2$; i.e.,

$$h^{\sim} = \frac{1}{r}\int_0^r h_t\, dt,$$

where $h_t(\mathbf{x}) = \sum \xi_k f(\mathbf{x} - \mathbf{x}_k - t\mathbf{v})$, $\mathbf{x}_k \in Q$, $\mathbf{x}_k + r\mathbf{v} \in Q$. If we let $h' = (H_2 + iI)h^{\sim}$, it is clear that for all sufficiently small x, say $|x| < s$, where $\mathbf{x}_k + r\mathbf{v} - s\mathbf{u} \in Q$,

$$U(x\mathbf{u})\, U_2 h' = U_2\, U(x\mathbf{u})\, h'. \tag{6.2}$$

Suppose $g \in \mathcal{D}(H_1)$ such that

$$H_1 g = \int_0^a t\, dE_1(t)\, g,$$

where a is a finite positive number. We have then

$$\begin{aligned}(U(x\mathbf{u})\, U_2 h', g) &= \int_0^a e^{ixt}\, d(E_1(t)\, U_2 h', g)\\ &= \int_0^\infty e^{ixt}\, d(U_2 E_1(t)\, h', g) = (U_2\, U(x\mathbf{u})\, h', g).\end{aligned} \tag{6.3}$$

Now,

$$F_1(z) = \int_0^a e^{i(x+iy)t}\, d(E_1(t)\, U_2 h', g), \quad z = x + iy,$$

exists for $-\infty < y < \infty$ and is analytic. On the other hand,

$$F_2(z) = \int_0^\infty e^{i(x+iy)t}\, d(U_2 E_1(t)\, h', g)$$

exists for $y \geqslant 0$ and is analytic in the half plane $y > 0$. Further from (6.3)

$$\lim_{y \uparrow 0} F_1(z) = F_1(x) = \lim_{y \downarrow 0} F_2(z).$$

Since $F_1(x)$ is continuous, $F_1(z)$ is an analytic extension of $F_2(z)$ and hence $F_1(z) \equiv F_2(z)$ for $y \geqslant 0$.

By the uniqueness theorem for Laplace-Stieltjes transforms we must have

$$(E_1(t)\,U_2 h', g) = (U_2 E_1(t)\,h', g),$$

since both functions are normalized in the same way. Since the class of g for which this is true is dense in F we have

$$E_1(t)\,U_2 h' = U_2 E_1(t)\,h'. \tag{6.4}$$

Let V_2 be the Cayley transform of D_2. Since $U_2 h' = V_2 h'$, it follows from (6.4) that if we consider U_2 as a generic symbol for the Cayley transform of *any* self-adjoint extension of D_2, the element $U_2 E_1(t)\,h'$ remains constant as U_2 varies. This implies $E_1(t)\,h' \in \mathcal{D}(V_2)$. For, since $U_2 E_1(t)\,h'$ remains constant, $g = (I - U_2)\,E_1(t)\,h'$ remains constant as U_2 varies and is in the domain of every self-adjoint extension of D_2. Hence $g \in \mathcal{D}(D_2)$ (see [7; 494] and [1; 279]). Hence there exists an $h \in \mathcal{D}(V_2)$ such that

$$g = (I - U_2)\,E_1(t)\,h' = (I - V_2)\,h = (I - U_2)\,h.$$

We get therefore

$$E_1(t)\,h' - h = U_2\,[E_1(t)\,h' - h].$$

Since $I - U_2$ has as inverse we must have $h = E_1(t)\,h'$ which proves the fact that $E_1(t)\,h' \in \mathcal{D}(V_2)$. Using this fact and (6.4) we get

$$E_1(t)\,V_2 \subseteq V_2 E_1(t),$$

since in (6.4) h' runs over a dense set in $\mathcal{D}(V_2)$. Hence

$$E_1(t)\,D_2 \subseteq D_2 E_1(t). \tag{6.5}$$

Let $E_n = E_1(n) - E_1(0-)$ and let $\mathfrak{M}_n$ be the range of E_n. From (6.5) it follows that $\mathfrak{M}_n$ reduces D_2 and therefore D_2^*. Further $\mathfrak{M}_n$ reduces H_1 to a bounded self-adjoint. If we now follow through the argument of the latter part of the proof of Theorem 2 we will have completed the proof of Theorem 3.

7. The circle group case

We shall here give an example which shows that the general extension problem formulated in § 1 is not true for the circle group. Let $f(x)$ be an analytic positive definite function defined on the real axis which is not periodic. Let $f_1(x)$ be the restriction of $f(x)$ to the interval $-a < x < a$, where $0 < a < \pi$. Let $F(e^{ix}) = f_1(x)$; then

$F(e^{ix})$ is a positive definite function defined on a symmetric neighborhood of the identity of the circle group in the sense that $F(e^{ix}e^{-iy}) \gg 0$ for $-\frac{1}{2}a < x, y < \frac{1}{2}a$. Now, if $F(e^{ix})$ had a positive definite extension to the whole circle group, then $f_1(x)$ would have a periodic positive definite extension to the real axis. But this is impossible since $f_1(x)$ is analytic and its only positive definite extension is $f(x)$.

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