# ON THE EXTENSIONS OF POSITIVE DEFINITE FUNCTIONS 

BY<br>ALLEN DEVINATZ ${ }^{(1)}$<br>Washington University, Saint Louis, Missouri

## 1. Introduction

In a beautiful paper [13], M. Krein has given a penetrating analysis of a general problem of moments (see also [17]). This paper is the culmination of a series of notes and papers by Krein stretching over almost a decade (see [13] for a bibliography). His results appear in the form of a theory about a special class of symmetric operators on a Hilbert space (see also [16]). The prototype of an operator in this class may be found in the theory of the classical Hamburger moment problem.

The general problem of moments which can be treated by Krein's methods is concerned with conditions on a pre-Hilbert space $\mathcal{L}$ of analytic functions of a single real variable for which there exists a measure $d \mu(t) \geqslant 0$ so that if $f, g \in \mathcal{L}$, then

$$
(f, g)=\int_{-\infty}^{\infty} f(t) \bar{g}(t) d \mu(t) .
$$

The measure $d \mu$ may not be unique and this non-unicity leads to many interesting results.

If one tries to carry over this theory to analytic functions of two real variables $\left({ }^{2}\right)$, one meets rather serious difficulties at the very beginning. In order to try to gain some insight into these multi-variable problems we have, in a series of papers [4, 5,

[^0]$6,7]$, and in the present one, investigated some important classical problems. Even in these cases it is not completely clear how much is true, although we suspect a good deal more is true than we have been able to prove. Certain basic similarities appear in the special cases that make it clear that some form of a general theorem exists which cover all of these special cases, but we have not as yet been able to devise a general proof that would cover all of the situations we have considered. Nevertheless, we have felt the special cases to be of sufficient interest to warrant a presentation.

In the present paper we shall be concerned with the problem of extensions of positive definite functions. Suppose $f(x)$ is a continuous complex valued function defined on the interval ( $-2 a, 2 a$ ) with the property that for any set $\left\{x_{k}\right\}_{1}^{n} \subset(-a, a)$ and $\left\{\xi_{k}\right\}_{1}^{n}$ any set of complex numbers,

$$
\begin{equation*}
\sum_{j=1}^{n} \sum_{k=1}^{n} \xi_{j} \xi_{k} f\left(x_{j}-x_{k}\right) \geqslant 0 . \tag{1.1}
\end{equation*}
$$

In [10] M. Krein proved that such a function could be extended continuously to the whole real axis so as to retain the positive definite property (1.1) (see also [5] and [15]) and hence, by Bochner's theorem [3; 74], is a Fourier-Stieltjes transform of a non-negative measure. In general such an extension is not unique [9; 22-23]. As a special case of his considerations in [13] Krein obtained an analogous theorem if $f$ is allowed to take values in the space of $n \times n$ matrices.

The same question can be asked when the domain of $f$ is changed in a suitable manner and the range is retained in the complex number field. For example the following could be asked:

Let $G$ be an Abelian, locally compact, topological group and $Q$ a symmetric neighborhood of the identity. Let $f(\mathbf{x})$ be a continuous complex valued function defined on $2 Q$ and satisfying (1.1) for $\left\{\mathbf{x}_{k}\right\} \subset Q$. Is it possible to extend $f(\mathbf{x})$ to all of $G$ so as to retain the positive definite character?

The answer to this question is in general in the negative since, as we shall show in $\S 7$ by a very simple example, it is already not true for the circle group. For the additive group of integers, with the discrete topology, the question may be very easily answered in the affirmative.

For Euclidean space of any dimension, considered as an additive group under the usual topology, the answer to the above question is open. The problem appears to be a delicate one. By placing additional restrictions on the function $f$ and the neighborhood $Q$ we have been able to answer the question in the affirmative. For simplicity, we shall state our results for only two dimensions.

Theorem 1. Let $Q$ be the open rectangle in the ( $x_{1}, x_{2}$ )-plane given by $\left|x_{k}\right|<a_{k}$, $k=1,2$, and $f(\mathbf{x})=f\left(x_{1}, x_{2}\right)$ a continuous function on $2 Q$ which satisfies (1.1) in $Q$. If $f\left(x_{1}, 0\right)$ and $f\left(0, x_{2}\right)$ each have unique positive definite extensions along the $x_{1}$-axis and $x_{1}$-axis respectively, then there exists a unique non-negative measure $d F$, on the Borel field of the plane, such that

$$
f\left(x_{1}, x_{2}\right)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t\left(x_{1} t_{1}+x_{2} t_{2}\right)} d F^{\prime}\left(t_{1}, t_{2}\right) .
$$

We can prove a similar theorem for a rectangle of lattice points in the plane having integer components. The proof is considerably simpler.

Theorem 1'. Let $Q$ be the rectangle of lattice points ( $k_{1}, k_{2}$ ) where

$$
k_{i}=0, \pm 1, \ldots, \pm n_{j}, \quad j=1,2
$$

and $Q_{1}$ the points of $Q$ with non-negative components. Further let $f(\mathbf{k})=f\left(k_{1}, k_{2}\right)$ be defined on $Q$ and satisfy (1.1) on $Q_{1}$. If $f\left(k_{1}, 0\right)$ and $f\left(0, k_{2}\right)$ each has a unique positive definite extension to all of the integers, then there exists a unique measure $d F$ such that

$$
f\left(k_{1}, k_{2}\right)=\int_{0}^{2 \pi} \int_{0}^{2 \pi} e^{i\left(k_{k_{2}} t_{+}+k_{2} t_{2}\right)} d F^{\prime}\left(t_{1}, t_{2}\right) .
$$

The above two theorems are the exact analogues of a theorem proved in [7] on two-parameter moment problems, although the different proofs depend very strongly on the particular situations. In $\S 6$ we shall prove two more theorems along the lines of Theorem 1. For example, if $Q$ is as in Theorem 1 and there exists an $\varepsilon>0$ such that the restriction of $f\left(x_{1}, 0\right)$ to $\left(-2 a_{1}+\varepsilon, 2 a_{1}-\varepsilon\right)$ has a unique extension, then we may remove all restrictions on $f\left(0, x_{2}\right)$ and be able to prove that $f\left(x_{1}, x_{2}\right)$ has a positive definite extension to the whole plane. We have not been able to prove analogues of the theorems in $\S 6$ for the case of the two-parameter Hamburger moment problem.

The main difficulty encountered in proving these theorems is in the proving of the permutability of the canonical spectral measures of certain unbounded self-adjoint operators. The main part of this paper is devoted to this question and our results are obtained by a very careful examination of the domains of these operators.

The methods developed in this paper and in [6] can be used to considerably simplify certain portions of [4] and [5]. On the other hand, certain basic ideas of the latter papers have been used in the present one. In general, there is very little overlap of results.

We should remark that it is very easy to give necessary and sufficient conditions that a continuous function $f$ defined on a neighborhood $Q$ of the origin may be written as a Fourier-Stieltjes transform of a non-negative measure. Let $m$ be the linear space of finite trigonometric sums

$$
p(t, \tau)=\sum_{k=1}^{n} \xi_{k} e^{i\left(x_{k} t+y_{k} \tau\right)}
$$

where $-\infty<x_{k}, y_{k}<\infty$ and $-\infty<t, \tau<\infty . ~ M$ is a partially ordered space by taking $p \geqslant 0$ if $p(t, \tau) \geqslant 0$ for all $t, \tau$. Let $m_{0}$ be the linear subspace of $m$ consisting of those trigonometric sums for which $\left\{\left(x_{k}, y_{k}\right)\right\} \subset Q$.

Let $L$ be the linear functional defined on $m_{0}$ by the equation

$$
L(p)=\sum_{k=1}^{n} \xi_{k} f\left(x_{k}, y_{k}\right) .
$$

If $L(p) \geqslant 0$ whenever $p \geqslant 0$, then it is known that $L$ may be extended to all of $m$ so as to retain this property. The function

$$
F(x, y)=L\left(e^{i(x t+y t)}\right)
$$

gives a continuous positive definite extension of $f$ to the whole plane.
Conversely, it is clear that the condition $L(p) \geqslant 0$ whenever $p \geqslant 0$ is a necessary condition for $f$ to be a Fourier-Stieltjes transform of a non-negative measure.

## 2. Preliminaries

To prove our results we shall use the methods of operators in Hilbert space. The fact that functions satisfying (1.1) could be used to construct an inner product on a function space has been a very effective tool in many branches of mathematics. For problems closely allied to those of this paper it has been used by A. Devinatz [5, 6] and M. Krein [11, 13]. General theories concerning non-negative quadratic forms have been constructed by N. Aronszajn [2] and M. Krein [14]. In this paper we shall follow the exposition in [2] as being most suitable for our purposes.

Let $E$ be a set and $K(x, y)$ a complex valued function defined on $E \times E$ with the property that for any finite set $\left\{\xi_{k}\right\}_{1}^{n}$ of complex numbers and points $\left\{\mathbf{x}_{k}\right\}_{1}^{n} \subseteq E$,

$$
\begin{equation*}
\sum_{i, k=1}^{n} \xi_{j} \bar{\xi}_{k} K\left(\mathbf{x}_{k}, \mathbf{x}_{j}\right) \geqslant 0 \tag{2.1}
\end{equation*}
$$

The main idea in constructing a linear space using the form (2.1) as a norm is as follows. Set

$$
\begin{gathered}
g(\mathbf{x})=\sum_{k=1}^{n} \xi_{k} K\left(\mathbf{x}, \mathbf{x}_{k}\right), h(\mathbf{x})=\sum_{k=1}^{m} \eta_{k} K\left(\mathbf{x}, \mathbf{y}_{k}\right) \\
(g, h)=\sum_{k=1}^{n} \sum_{j=1}^{m} \xi_{k} \bar{\eta}_{j} K\left(\mathbf{y}_{j}, \mathbf{x}_{k}\right) .
\end{gathered}
$$

In this way we get a pre-Hilbert space which may be completed to a Hilbert space $\mathcal{F}$ of functions defined on $E$. An essential property of the space $\mathcal{F}$ is that if $g \in \mathcal{F}$ and $K_{\mathbf{y}}(\mathbf{x})=K(\mathbf{x}, \mathbf{y})$ then

$$
g(\mathbf{y})=\left(g, K_{\mathbf{y}}\right) .
$$

These types of Hilbert spaces have been called by Aronszajn [2] reproducing kernel spaces and the kernels $K(x, y)$, reproducing kernels. Any reproducing kernel space has a unique reproducing kernel.

Another important property enjoyed by these types of Hilbert spaces is that if a sequence of elements converges in the strong topology of $\mathcal{F}$, then they converge pointwise and even uniformly on those sets for which $K(\mathbf{x}, \mathbf{x})$ is bounded. For, we have

$$
|g(\mathbf{x})|=\left|\left(g, K_{\mathbf{x}}\right)\right| \leqslant\|g\|\left\|K_{\mathbf{x}}\right\|=\|g\| \sqrt{K(\mathbf{x}, \mathbf{x})} .
$$

If $E_{1} \subseteq E$ and $K_{1}(\mathbf{x}, \mathrm{y})$ is the restriction of $K(\mathbf{x}, \mathrm{y})$ to $E_{1} \times E_{1}$, then $K_{1}$ gives rise to a space $\mathcal{F}_{1}$ for which it acts as a reproducing kernel. The pertinent theorem is the following:

Theorem A [2; 351]. If $K$ is the reproducing kernel of the space $\mathcal{F}$ of functions defined on the set $E$ with norm $\|f\|$, then $K$ restricted to the subset $E_{1} \times E_{1} \subseteq E \times E$ is the reproducing kernel of the class $\mathcal{F}_{1}$ of all restrictions of $\mathcal{F}$ to the subset $E_{1}$. For any such restriction, $f_{1} \in \mathcal{F}_{1}$, the norm $\left\|f_{1}\right\|_{1}$ is the minimum $\|f\|$ for all $f \in \mathcal{F}$ whose restriction to $E_{1}$ is $f_{1}$.

Finally, we shall have need for the following:
Theorem B [2; 361]. $K_{1}\left(\mathbf{x}_{1}, \mathbf{y}_{1}\right)$ and $K_{2}\left(\mathbf{x}_{2}, \mathbf{y}_{2}\right)$ are reproducing kernels with corresponding spaces $\mathfrak{F}_{1}$ and $\mathfrak{F}_{2}$, then $K_{1}\left(\mathbf{x}_{1}, \mathbf{y}_{1}\right) K_{2}\left(\mathbf{x}_{2}, \mathbf{y}_{2}\right)$ is the reproducing kernel of the direct product of $\mathcal{F}_{1}$ and $\mathfrak{F}_{2}$.

Notation. In the remainder of the paper we shall be working in two dimensional Euclidean space. Real numbers will be denoted by lower case Latin letters. Two dimensional vectors will be denoted by lower case Latin letters in bold face type, their components by the same letters in ordinary type with subscripts; e.g., 8-593804. Acta mathematica. 102. Imprimé le 28 septembre 1959
$\mathbf{x}=\left(x_{1}, x_{2}\right)$. We shall write $\mathbf{x} \cdot \mathbf{y}=x_{1} y_{1}+x_{2} y_{2},|\mathbf{x}|^{2}=x_{1}^{2}+x_{2}^{2}$ and $\mathbf{x} \leqslant \mathbf{y}$ if and only if $x_{k} \leqslant y_{k}, k=1,2$. The letters $\mathbf{u}$ and $\mathbf{v}$ will always stand for the special vectors $\mathbf{u}=(1,0)$, $\mathrm{v}=(0,1)$.

If a kernel $K(\mathbf{x}, \mathbf{y})$ satisfies (2.1) we shall write $\mathrm{K}(\mathbf{x}, \mathbf{y}) \gg 0$ or simply $K \gg 0$. If $K_{2}-K_{1}>0$ we shall write $K_{1}<K_{2}$.

## 3. Necessary lemmas

Suppose that $Q$ is an open symmetric neighborhood of the origin in the plane (i.e. $x \in Q$ implies $(-x) \in Q$ ) and $f(\mathbf{x})$ a continuous function defined on $2 Q$ such that $f(\mathbf{x}, \mathbf{y})>\mathbf{0}$. Let $\mathcal{F}$ be the Hilbert space corresponding to the kernel $f(\mathbf{x}-\mathbf{y})$ as discussed in §2. Let $\mathcal{D}_{1}^{*}$ be the linear manifold in $\mathcal{F}$ such that $\partial g(\mathbf{x}) / \partial x_{1}$ exists and belongs to $\mathcal{F}$. Define the operator $A_{1}$, with domain $\mathcal{D}_{1}^{*}$ by the formula

$$
A_{1} g(\mathbf{x})=-i \frac{\partial g(\mathbf{x})}{\partial x_{1}}
$$

Continuing with definitions and notations, let $\mathcal{F}^{\prime}$ be the linear manifold consisting of elements of the form $g(\mathbf{x})=\sum_{1}^{n} \xi_{k} f\left(\mathbf{x}-\mathbf{x}_{k}\right)$. It is clear from the manner in which $\mathcal{F}$ is constructed (§2), that $\mathcal{F}^{\prime}$ is dense in $\mathcal{F}$. Since $Q$ is open, for a given $g \in \mathcal{F}^{\prime}$, it is always possible to find a vector $r$, such that $r_{1}, r_{2} \neq 0$ and of sufficiently small norm so that if $|\mathbf{t}| \leqslant|\mathbf{r}|$ then $g_{\mathbf{t}}(\mathbf{x})=\sum_{1}^{n} \xi_{k} f\left(\mathbf{x}-\mathbf{x}_{k}-\mathbf{t}\right)$ is well defined and belongs to $\mathcal{F}^{\prime}$. Since $g_{t}$ is a continuous function of $t$ in the strong topology of $\mathfrak{F}$, the integral

$$
g_{\mathrm{r}}^{\mathrm{r}}=\frac{1}{r_{1} r_{2}} \int_{0}^{\mathrm{r}} g_{\mathrm{t}} d \mathrm{t}=\frac{1}{r_{1} r_{2}} \int_{0}^{r_{2}} \int_{0}^{r_{1}} g_{\mathrm{t}} d t_{1} d t_{2}
$$

exists and belongs to $\mathcal{F}$. Further $g_{r} \rightarrow g$ as $r \rightarrow 0$, where the bold arrow indicates convergence in the strong topology of $\mathcal{F}$. Let us designate the linear manifold of such elements $\boldsymbol{g}_{\mathbf{r}}$ by $\mathcal{D}$. It is clear that $\mathcal{D}$ is dense in $\mathcal{F}$ since $\mathcal{F}^{\prime}$ is dense in $\mathcal{F}$.

Lemma 3.1. $D_{1}=A_{1}^{*}$ exists, $D_{1} \subseteq A_{1}$ and $D_{1}^{*}=A_{1} . \mathcal{D} \subseteq \mathcal{D}\left(D_{1}\right)$ (domain of $D_{1}$ ) and the closure of the restriction of $D_{1}$ to $D$ is $D_{1}$. Further,

$$
\begin{equation*}
D_{1} g_{\mathrm{r}}=\frac{i}{r_{1} r_{2}} \int_{0}^{r_{2}}\left[g_{r_{1} \mathbf{u}+t_{2} v}-g_{t_{\mathbf{v}} \mathrm{v}}\right] d t_{2} \tag{3.1}
\end{equation*}
$$

Proof. Let $\mathrm{s}=\left(s_{1}, 0\right)=s_{1} \mathbf{u}$ with $|\mathrm{s}|$ sufficiently small so that $g_{\mathrm{t}+\mathrm{s}}$ is defined. Consider the element

$$
g_{\mathrm{s}, \mathrm{r}}=\frac{1}{r_{1} r_{2}} \int_{0}^{\mathrm{r}} g_{\mathrm{t}+\mathrm{s}} d t
$$

We get

$$
\begin{aligned}
\frac{g_{\mathrm{s}, \mathrm{r}}-g_{\mathrm{r}}}{s_{1}} & =\frac{1}{r_{1} r_{2} s_{1}}\left[\int_{0}^{r_{2}} \int_{s_{1}}^{r_{1}+s_{1}} g_{\mathrm{t}} d \mathrm{t} \cdots \int_{0}^{r_{2}} \int_{0}^{r_{1}} g_{\mathrm{t}} d \mathrm{t}\right] \\
& =\frac{1}{r_{1} r_{2} s_{1}} \int_{0}^{r_{2}} \int_{0}^{s_{1}}\left[g_{\mathrm{t}+r_{1} \mathbf{u}}-g_{\mathrm{t}}\right] d \mathrm{t} \\
& \rightarrow \frac{1}{r_{1} r_{2}} \int_{0}^{r_{2}}\left[g_{r_{1} \mathbf{u}+t_{\mathrm{z}} \mathrm{v}}-g_{t_{s} \mathbf{u}}\right] d t_{2} \text { as } s_{1} \rightarrow 0 .
\end{aligned}
$$

If $x$ and $x+s$ both belong to $Q$, then it is clear $g_{s, r}(\mathbf{x})=g_{r}(x-s)$. Hence, since convergence in the strong topology of $\mathcal{F}$ implies pointwise convergence we get

$$
-i \frac{\partial g_{\tilde{\mathbf{r}}}(\mathrm{x})}{\partial x_{1}}=A_{1} g_{\mathbf{r}}^{\tilde{\mathrm{r}}}(x)=\frac{i}{r_{1} r_{2}} \int_{0}^{r_{\mathbf{2}}}\left[g_{r_{1} \mathbf{u}+t_{2} \mathbf{v}}-g_{t_{\mathbf{2}} \mathrm{u}}\right] d t_{2}
$$

This shows $D_{1}^{*}$ is dense in $\mathcal{F}$ since $\mathcal{D}$ is dense in $\mathcal{F}$. Hence $D_{1}=A_{1}^{*}$ exists. $\left({ }^{1}\right)$
Next let $g \in \mathcal{D}\left(D_{1}\right)$ and for fixed $\mathbf{y}$ set $f_{\mathbf{y}+\mathrm{t}}(\mathbf{x})=f(\mathbf{x}-\mathbf{y}-\mathbf{t})$, provided the latter is defined. Further set

$$
f_{\mathbf{y}, \mathrm{r}}=\frac{1}{r_{1} r_{2}} \int_{0}^{\mathbf{r}} f_{\mathbf{y}+\mathrm{t}} d \mathrm{t}, f_{r_{1} \mathbf{u}}=\frac{1}{r_{1}} \int_{0}^{r_{1}} f_{\mathbf{y}+\ell_{1} \mathbf{u}} d t_{1}
$$

Then

$$
\begin{aligned}
\left(g, A_{1} f_{\tilde{y}, \mathrm{r}}\right) & =\left(g, \frac{i}{r_{1} r_{2}} \int_{0}^{r_{2}}\left[f_{\mathbf{y}+r_{1} \mathbf{u}+t_{\mathbf{1}} \mathrm{v}}-f_{\mathbf{y}+t_{2} \mathrm{v}}\right] d t_{2}\right) \\
& =\frac{-i}{r_{1} r_{2}} \int_{0}^{r_{1}}\left[g\left(\mathbf{y}+r_{1} \mathbf{u}+t_{2} \mathbf{v}\right)-g\left(\mathbf{y}+t_{2} \mathbf{v}\right)\right] d t_{\mathbf{2}} \\
& =\left(A_{1}^{*} g, f_{\mathbf{y}, \mathbf{r}}\right) .
\end{aligned}
$$

As $r_{2} \rightarrow 0, f_{\mathbf{y}, \mathrm{r}} \rightarrow \tilde{f}_{\boldsymbol{y}, r_{1} \mathbf{u}}$ and we get

$$
\left(A_{1}^{*} g, f_{\mathbf{y}, r_{1} \mathbf{u}}\right)=\frac{-i}{r_{\mathbf{1}}}\left[g\left(\mathbf{y}+r_{1} \mathbf{u}\right)-g(\mathbf{y})\right]
$$

${ }^{(1)}$ We have used here a technique similar to that used in [8].
and as $r_{1} \rightarrow 0, \tilde{f}_{\tilde{y}, r_{1} \mathbf{u}} \rightarrow f_{\mathbf{y}}$. Hence,

$$
A_{1}^{*} g(\mathbf{y})=-i \frac{\partial g(\mathbf{y})}{\partial x_{1}}
$$

which shows that $D_{1} \subseteq A_{1}$. Since $A_{1}$ is clearly a closed operator (convergence in the norm of $\mathcal{F}$ implies uniform pointwise convergence!) it follows that $A_{1}=A_{1}^{* *}=D_{1}^{*}$.

Finally, it remains to prove the second statement of the lemma. Let $B_{1}$ be the restriction of $A_{1}$ to $\mathcal{D}$ and $g \in D_{1}^{*}=\mathcal{D}\left(A_{1}\right)$. If $f_{\mathbf{y}+\mathrm{t}}$ and $f_{\tilde{y}, \mathrm{r}}$ are as in the previous paragraph then we have

$$
\begin{aligned}
\left(A_{1} g, f_{\mathbf{y}, \mathbf{r}}\right) & =\frac{\mathbf{1}}{r_{1} r_{2}} \int_{0}^{\mathbf{r}}\left(A_{1} g, f_{\mathbf{y}+\mathbf{t}}\right) d \mathbf{t} \\
& =\frac{1}{r_{1} r_{2}} \int_{0}^{\mathbf{r}}-i \frac{\partial g(\mathbf{y}+\mathbf{t})}{\partial x_{1}} \cdot d \mathbf{t} \\
& =\frac{1}{r_{1} r_{2}} \int_{0}^{\mathbf{r}} \lim _{h \rightarrow 0}\left(g, \frac{i}{h}\left[f_{\mathbf{y}+\mathbf{t}+h \mathbf{u}}-f_{\mathbf{y}+\mathbf{t}}\right]\right) d \mathbf{t}
\end{aligned}
$$

Since the partial derivative of $g$ is continuous (all elements of $\mathbf{F}$ are continuous!) we may take the limit outside of the integral sign and then interchange the inner product and the integral sign. We get consequently, after a few manipulations,

$$
\begin{aligned}
\left(A_{1} g, f_{\mathbf{y}, \mathbf{r}}\right) & =\lim _{h \rightarrow 0}\left(g, \frac{i}{r_{1} r_{2} h} \int_{0}^{\mathbf{r}}\left[f_{\mathbf{y}+\mathbf{t}+h \mathbf{u}}-f_{\mathbf{y}+\mathbf{t}}\right] d \mathbf{t}\right) \\
& =\lim _{h \rightarrow 0}\left(g, \frac{i}{r_{1} r_{2} h} \int_{0}^{n} \int_{0}^{r_{\mathbf{3}}}\left[f_{\mathbf{y}+\mathbf{t}+r_{\mathbf{1}} \mathbf{u}}-f_{\mathbf{y}+\mathbf{t}}\right] d \mathbf{t}\right) .
\end{aligned}
$$

As $h \rightarrow 0$, the second member of the inner product in the last equation goes strongly to

$$
A_{1} f_{\tilde{y}, \mathrm{r}}=\frac{i}{r_{1} r_{2}} \int_{0}^{r_{1}}\left[t_{\mathrm{y}+r_{1} \mathrm{u}+t_{\mathbf{2}} \mathrm{v}}-f_{\mathrm{y}+t_{\mathbf{x}} \mathrm{v}}\right] d t_{2}
$$

Hence,

$$
\left(A_{1} g, f_{\mathbf{y}, \mathbf{r}}\right)=\left(g, B_{1} f_{\mathbf{y}, \mathbf{r}}\right)
$$

which implies $B_{1} \subseteq A_{1}^{*}=D_{1}$. On the other hand, by exactly the same method as used to prove $A_{1}^{*} \subseteq A_{1}$ we get $B_{1}^{*} \subseteq A_{1}$. This implies $B_{1}^{*}=A_{1}^{* *}=A_{1}$ and hence $B_{1}^{* *}=A_{1}^{*}=D_{1}$. Since $B_{1}^{* *}$ is the closure of $B_{1}$, we have completed the proof of the lemma.

Let $r_{n}$ be a sequence of non-zero real numbers such that $r_{n} \rightarrow 0$ and $\mathcal{D}_{1}$ the manifold of functions of the form

$$
g_{r_{n}}=\frac{1}{r_{n}} \int_{0}^{r_{n}} g_{\text {tu }} d t
$$

If we follow through the proof of Lemma 3.1 step by step we arrive at the following.
Lemma 3.2. The closure of the restriction of $D_{1}$ to $D_{1}$ is $D_{1}$ and moreover

$$
\begin{equation*}
D_{1} g_{r_{n}}=\frac{i}{r_{n}}\left[g_{r_{n} \mathbf{u}}-g\right] . \tag{3.2}
\end{equation*}
$$

If in all of our previous discussion we interchange the subscript 1 with the subscript 2, the corresponding lemmas will be valid for $D_{2}$. This will lead to the following lemma.

Lemma 3.3. $\mathcal{D} \subseteq \mathcal{D}\left(D_{1} D_{2}\right) \cap \mathcal{D}\left(D_{2} D_{1}\right)$ and

$$
D_{1} D_{2} g_{\mathbf{r}}^{\tilde{\mathbf{r}}}=D_{2} D_{1} g_{\mathbf{r}}=\frac{-1}{r_{1} r_{2}}\left[g_{r_{1} \mathbf{u}+r_{2} \mathbf{v}}-g_{r_{1} \mathbf{u}}-g_{r_{2} \mathbf{v}}+g\right]
$$

Lemma 3.4. $a I \leqslant D_{1} \leqslant b I$ if and only if there exists a sequence of non-zero real numbers $r_{n} \rightarrow 0$ such that for $n=1,2, \ldots$,

$$
\begin{align*}
a \int_{0}^{r_{n}} \int_{0}^{r_{n}} f(\mathbf{x}-\mathbf{y}+(s-t) \mathbf{u}) d t d s & <i \int_{0}^{r_{n}}[f(\mathbf{x}-\mathbf{y}-t \mathbf{u})-f(\mathbf{x}-\mathbf{y}+t \mathbf{u})] d t \\
& <b \int_{0}^{r_{n}} \int_{0}^{r_{n}} f(\mathbf{x}-\mathbf{y}+(s-t) \mathbf{u}) d t d s . \tag{3.3}
\end{align*}
$$

for all $\mathbf{x}$ and $\mathbf{y}$ in $Q$ for which the functions are defined. If a takes on the value $-\infty$ or $b$ the value $+\infty$ the corresponding inequalities are considered redundant.

Proof. We shall first prove the necessity. Let $\left\{\mathrm{x}_{k}\right\}_{1}^{n} \subset Q$ and $r$ a real number so that $\mathbf{x}_{k}+r \mathbf{u} \in Q$. Further, let

$$
f_{k, r}(x)=\int_{0}^{r} f\left(\mathbf{x}-\mathbf{x}_{k}-t \mathbf{u}\right) d t
$$

and $g=\sum_{1}^{n} \xi_{k} f_{k, r}$. Then from Lemma 3.2 we get

$$
\begin{aligned}
\left(D_{1} g, g\right)= & i \sum_{j, k=1}^{n} \xi_{,} \xi_{k} \int_{0}^{r}\left[f\left(\mathbf{x}_{k}-\mathbf{x}_{j}-(r-t) \mathbf{u}\right)-f\left(\mathbf{x}_{k}-\mathbf{x}_{j}-t \mathbf{u}\right)\right] d t \\
& =i \sum_{j, k=1}^{n} \xi_{j} \xi_{k} \int_{0}^{r}\left[f\left(\mathbf{x}_{k}-\mathbf{x}_{\boldsymbol{j}}-t \mathbf{u}\right)-f\left(\mathbf{x}_{k}-\mathbf{x}_{j}+t \mathbf{u}\right)\right] d t \\
& (g, g)=\sum_{j, k=1}^{n} \xi_{j} \bar{\xi}_{k} \int_{0}^{T} \int_{0}^{r} f\left(\mathbf{x}_{k}-\mathbf{x}_{j}+(s-t) \mathbf{u}\right) d t d s
\end{aligned}
$$

Further,

Hence, because of the inequalities satisfied by $D_{1}$ we have the conditions (3.3).
To prove the sufficiency we simply note by the computations of the above paragraph that the inequalities (3.3) imply

$$
a(g, g) \leqslant\left(D_{1} g, g\right) \leqslant b(g, g)
$$

for any element in the manifold $D_{1}$. By Lemma 3.2 we get this inequality for every element in $\mathcal{D}\left(D_{1}\right)$ which completes the proof of sufficiency.

Let us now define an operator $J$ on $\mathcal{F}$ by the formula

$$
J g(\mathbf{x})=\bar{g}(-\mathbf{x}) .
$$

This is a conjugation operator (see $[5 ; 470]$ ) and clearly permutes with $D_{1}$ and $D_{2}$. Hence, these operators have self-adjoint extensions.

Lemma 3.5. Let $H_{1}$ be any self-adjoint extension of $D_{1}, d E_{1}$ its canonical spectral measure and

$$
U(x \mathbf{u})=\int_{-\infty}^{\infty} e^{12 t} d E_{1}(t), \quad-\infty<x<\infty .
$$

For any $g \in \mathcal{F}$ we have

$$
\begin{equation*}
U(x \mathbf{u}) g(\mathbf{x})=g(\mathbf{x}+x \mathbf{u}) \tag{3.4}
\end{equation*}
$$

provided $\mathbf{x}+x \mathbf{u} \in Q$.
Proof. Let $g \in \mathcal{F}$ such that

$$
U(x \mathbf{u})=\int_{-c}^{c} e^{i x t} d E_{1}(t) g
$$

where $c$ is a finite real number. It is clear that $g \in \bigcap_{0}^{\infty} \mathcal{D}\left(H_{1}^{n}\right)$ and the class of such elements is dense in $\mathcal{F}$. By expanding $e^{i x t}$ in a Maclaurin series we get

$$
U(x \mathbf{u}) g=\sum_{n=0}^{\infty} \frac{1}{n!} x^{n} i^{n} H_{1}^{n} g
$$

where convergence is in the strong topology of $\mathcal{F}$. Since convergence in the strong topology of $\mathcal{F}$ implies pointwise convergence we have

$$
U(x \mathbf{u}) g(\mathbf{x})=\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial g^{n}(\mathbf{x})}{\partial x_{1}^{n}} x^{n}
$$

Now,

$$
\left|\frac{\partial^{n} g(\mathbf{x})}{\partial x_{1}^{n}}\right|=\left|H_{1}^{n} g(\mathbf{x})\right|=\left|\left(H_{1}^{n} g, f_{\mathbf{x}}\right)\right| \leqslant\left\|H_{1}^{n} g\right\| \sqrt{f(0)} \leqslant c^{n} \sqrt{f(0)}\|g\|,
$$

where $f_{\mathbf{x}}(\cdot)=f(\cdot-\mathbf{x})$. This means $g(\mathbf{x})$ is analytic in $x_{1}$. Consequently, if $\mathbf{x}+x \mathbf{u} \in Q$, we get (3.4). Since $U(x \mathbf{u})$ is bounded (3.4) must be true for all of $\mathcal{F}$.

Lemma 3.6. If $f_{\mathbf{y}}(\mathbf{x})=f(\mathbf{x}-\mathbf{y})$ and $\mathbf{y}+x \mathbf{u} \in Q$, then $U(x \mathbf{u}) f_{\mathbf{y}}=f_{\mathbf{y}-x \mathbf{u}}$.
Proof. By Lemma 3.5, for any $\mathbf{x} \in Q$,

$$
\begin{aligned}
U(-x \mathbf{u}) f_{\mathbf{x}}(\mathbf{y}) & =f_{\mathbf{x}}(\mathbf{y}-x \mathbf{u})=\left(U^{*}(x \mathbf{u}) f_{\mathbf{x}}, f_{\mathbf{y}}\right) \\
& =\left(f_{\mathbf{x}}, f_{\mathbf{y}-x \mathbf{u}}\right)=\left(f_{\mathbf{x}}, U(x \mathbf{u}) f_{\mathbf{y}}\right) .
\end{aligned}
$$

Since this is true for every $x \in Q$ we have our lemma.
If we repeat the arguments of Lemma 3.5 for a self-adjoint extension of $D_{2}$ we get a group of unitary operators $U(x \mathbf{v})$ with the same properties as described in the lemma for the second variable. Let us suppose for the moment that for every $x_{1}$ and $x_{2}$

$$
\begin{equation*}
U\left(x_{1} \mathbf{u}\right) U\left(x_{2} \mathbf{v}\right)=U\left(x_{2} \mathbf{v}\right) U\left(x_{1} \mathbf{u}\right) \tag{3.5}
\end{equation*}
$$

If we set $U(\mathbf{x})=U\left(x_{1} \mathbf{u}\right) U\left(x_{2} \mathbf{v}\right)$, then $U(\mathbf{x})$ is a group of unitary operators and for $f_{0}(\mathbf{x})=f(\mathbf{x}-\mathbf{0})$ we get

$$
F(\mathbf{x})=\left(U(\mathbf{x}) f_{0}, f_{0}\right)
$$

is a continuous positive definite extension $f(\mathrm{x})$ to the whole plane.
The main difficulty in the problem posed in $\S 1$ is the proof of the relation (3.5). The restrictions we put on $f(\mathbf{x})$ in theorem 1 allow us to prove this relation.

If we restrict ourselves to the one-dimensional case the lemmas analogous to those we have been proving in this section would allow us to show that any continuous function defined on $(-2 a, 2 a)$ such that $f(x-y)>0$ for $x, y \in(-a, a)$ may be extended to a positive definite function on the whole axis. The differential operator, in the one-dimensional case, analogous to $D_{1}$ or $D_{2}$ we shall simply designate by $D$.

Lemma 3.7. If $f(x)$ is continuous on the interval $(-2 a, 2 a), f(x-y) \gg 0$, $x, y \in(-a, a)$, then $f(x)$ has a unique positive definite extension if and only if $D$ is self-adjoint.

Proof. Suppose first that $f(x)$ has a unique extension. This means there exists only one non-negative bounded measure $d F$ such that

$$
f(x)=\int_{-\infty}^{\infty} e^{t z t} d F(t) .
$$

Let $H_{1}$ and $H_{\mathbf{2}}$ be two self-adjoint extensions of $D, d E_{1}$ and $d E_{2}$ their corresponding spectral measures, and $U_{1}(x)$ and $U_{2}(x)$ the corresponding groups of unitary operators as set up in the proof of Theorem 1. By the unicity of the extension of $f(x)$ we must have

$$
\left(E_{1}(\Delta) f_{0}, f_{0}\right)=\left(E_{2}(\Delta) f_{0}, f_{0}\right)=F(\Delta)
$$

for any Borel set $\Delta$. Hence for $x$ and $y$ in $(-a, a)$ we have

$$
\begin{aligned}
\left(E_{1}(\Delta) U_{1}(x) f_{0}, U_{1}(y) f_{0}\right) & =\int_{\Delta} e^{i(x-y) t} d\left(E_{1}(t) f_{0}, f_{0}\right) \\
& =\int_{\Delta}^{i(x-y) t} d\left(E_{2}(t) f_{0}, f_{0}\right)=\left(E_{2}(\Delta) U_{2}(x) f_{0}, U_{2}(y) f_{0}\right) .
\end{aligned}
$$

Since the set $\left\{U_{1}(x) f_{0}=U_{2}(x) f_{0} ; x \in(-a, a)\right\}$ generates $\mathcal{F}$, we must have $E_{1}(\Delta)=E_{2}(\Delta)$ which in turn implies $H_{1}=H_{2}=D$.

Conversely, let us suppose that $D$ is self-adjoint and $d F$ is a measure such that

$$
f(x)=\int_{-\infty}^{\infty} e^{i x t} d F(t)
$$

Let $\mathscr{R}_{0}$ be the the set of elements in $\mathfrak{Z}^{2}(d F)$ for which $G \in \mathcal{Z}_{0}$ implies

$$
\int_{-\infty}^{\infty} e^{i x t} G(t) d F(t) \equiv 0 \quad \text { for } x \in(-a, a)
$$

There exists a unitary map $U$ between $\mathcal{Q}_{0}$ and $\mathcal{F}$, the correspondence being given by (see [4:61])

$$
\begin{aligned}
& h(x)=\int_{-\infty}^{\infty} e^{i x t} H(t) d F(t), \quad x \in(-a, a) \\
& \|h\|^{2}=\int_{-\infty}^{\infty}|H(t)|^{2} d F(t) .
\end{aligned}
$$

Let $\mathfrak{D} \subseteq \mathcal{Q}_{\mathbf{1}}^{1}$ be the class of $H \in \mathcal{Q}_{\mathbf{1}}^{1}$ such that $t H(t) \in \mathcal{Q}_{\mathbf{1}}^{1}$. Define an operator $T$ on $U \mathfrak{D}$ by the relation

$$
T h(x)=\int_{-\infty}^{\infty} e^{i x t} t H(t) d F(t) .
$$

It is easy to establish that $T$ is a closed symmetric operator (see $[4 ; 65-66]$ ) and clearly $T \subseteq D$.

Suppose now that

$$
g_{r}^{\sim}(x)=\frac{1}{r} \int_{0}^{r} \sum_{1}^{n} \xi_{k} f\left(x-x_{k}-y\right) d y=\int_{-\infty}^{\infty} e^{i x t} \frac{i}{r t}\left\{\sum_{1}^{n} \xi_{k} e^{-i x_{k} t}\left(e^{-i t r}-1\right)\right\} d F^{\prime}(t)
$$

Now, clearly the function

$$
H(t)=\frac{1}{r} \int_{0}^{r} \sum_{1}^{n} \xi_{k} e^{-i x_{k} t} e^{-i y t} d y=\frac{i}{r t}\left\{\sum_{1}^{n} \xi_{k} e^{-i x_{k} t}\left(e^{-i t r}-1\right)\right\}
$$

belongs to $\mathcal{L}_{0}$ and so also does $t H(t)$. Hence, $g_{\tilde{r}} \in \mathbb{D}(T)$. Since $g_{r} \in \mathcal{D}(D)$, and by Lemma 3.2 (for the one-dimensional case) $D$ is the closure of its restriction to these elements, and $T$ is closed, we get $D \subseteq T$. This establishes the fact that $T=D$.

If $\Delta$ is any Borel set on the line and $H \in \Omega_{0}^{1}$ set

$$
B(\Delta) h(x)=\int_{-\infty}^{\infty} e^{i x t} H_{\Delta}(t) d F^{\prime}(t)
$$

where $H_{\Delta}(t)=H(t)$ for $t \in \Delta, H_{\Delta}(t)=0$ otherwise. It is easy to establish that $B(\Delta)$ is a spectral measure on the Borel field of the line, $\left(B(\Delta) f_{0}, f_{0}\right)=F(\Delta)$ and

$$
(D g, h)=\int_{-\infty}^{\infty} t d(B(t) g, h)
$$

for any $g \in \mathcal{D}(D)$ and $h \in \mathcal{F}$. From this it follows that $d B=d E$, where $d E$ is the canonical spectral measure of $D$. Hence $d F(t)=d\left(B(t) f_{0}, f_{0}\right)=d\left(E(t) f_{0}, f_{0}\right)$. This establishes the unicity of $d F$.

For explicit details of the proofs of the facts outlined in the previous paragraph see $[4 ; 66-67]$ where they are given for a similar situation.

## 4. Proof of Theorem 1

The first point in the proof will be to notice that the operators $D_{1}$ and $D_{2}$ are self-adjoint. For, the deficiency spaces of $D_{1}$, say, are the class of elements in $\mathcal{F}$ which satisfy $D_{1}^{*} g=i g, D_{1}^{*} h=-i h$, respectively. Let us look at the first of these equations,

$$
\frac{\partial g(\mathbf{x})}{\partial x_{1}}=-g(\mathbf{x})
$$

The solution to this partial differential equation is given by

$$
g\left(x_{1}, x_{2}\right)=e^{-x_{1}} g\left(0, x_{2}\right) .
$$

Let $\mathcal{F}_{1}$ be the Hilbert space with reproducing kernel $f\left(x_{1}-y_{1}, 0\right)$. Since $Q$ is a rectangle, by Theorem A of $\S 2, \mathcal{F}_{1}$ consists of the restrictions of the elements of $\mathcal{F}$ to any line parallel to the $x_{1}$-axis. ( ${ }^{1}$ ) Hence, if there exists an $x_{2}$ such that $g\left(0, x_{2}\right) \neq 0$, then $e^{-x_{1}} \in \mathcal{F}_{1}$ which by the hypothesis of Theorem 1 and Lemma 3.7 is impossible. Consequently, the deficiency spaces of $D_{1}$ contain only the zero element and hence $D_{1}$ is self-adjoint and is the closure of its restriction to the class $\mathcal{D}$ of Lemma 3.1.

The next point in the proof will be to consider the manifold $m=\left(D_{1}+i I\right) \mathcal{D}$ and show that the closure of the restriction of $D_{2}$ to $m$ is $D_{2}$. Let $C_{2}$ be this restriction and $g \in \mathcal{D}\left(C_{2}^{*}\right)$. We have then

$$
\begin{equation*}
\left(C_{2}^{*} g,\left(D_{1}+i I\right) f_{\mathbf{x}, \mathbf{r}}\right)=\left(g, D_{2}\left(D_{1}+i I\right) f_{\tilde{\mathbf{x}}, \mathrm{r}}\right) \tag{4.1}
\end{equation*}
$$

where, as in the proof of Lemma 3.1,

$$
f_{\mathbf{x}, \mathbf{r}}=\frac{\mathbf{1}}{r_{1} r_{2}} \int_{0}^{\mathbf{r}} f_{\mathbf{x}+\mathbf{t}} d \mathbf{t}, \quad \mathbf{x}+\mathbf{r} \in Q
$$

Let

$$
\begin{equation*}
h(\mathbf{x})=g(\mathbf{x})-g\left(\mathbf{x}+r_{2} \mathbf{v}\right)+i \int_{0}^{r_{2}} C_{2}^{*} g(\mathbf{x}+t \mathbf{v}) d t . \tag{4.2}
\end{equation*}
$$

Using the results of Lemmas 3.1 and 3.3 and Equation (4.1) we get

$$
\frac{h\left(\mathbf{x}+r_{1} \mathbf{u}\right)-h(\mathbf{x})}{r_{1} r_{2}}=-\frac{1}{r_{1} r_{2}} \int_{0}^{r_{2}} h(\mathbf{x}+t \mathbf{u}) d t .
$$

Since the limit on the right exists as $r_{1} \rightarrow 0$ we get

$$
\frac{\partial h(\mathbf{x})}{\partial x_{1}}=-h(\mathbf{x})
$$

and hence

$$
h(\mathbf{x})=e^{-x_{1}} h\left(x_{2} \mathbf{v}\right) .
$$

Putting (4.2) in this last equation and rearranging we get
${ }^{(1)}$ The restriction of $f(\mathbf{x}-\mathbf{y})$ to any such line is given by $f\left(x_{1}-y_{1}, x_{2}-x_{2}\right)-f\left(x_{1}-y_{1}, 0\right)$. Any such line is to be taken as traversing the entire width of $Q$.

$$
\begin{aligned}
& \frac{1}{r_{2}} \int_{0}^{r_{2}} C_{2}^{*} g(\mathbf{x}+t \mathbf{v}) d t-\frac{e^{-x_{1}}}{r_{2}} \int_{0}^{r_{2}} C_{2}^{*} g\left(x_{2} \mathrm{v}+t \mathbf{v}\right) d t \\
&=-\frac{i}{r_{2}}\left[g\left(\mathbf{x}+r_{2} \mathbf{v}\right)-e^{-x_{1}} g\left(x_{2} \mathbf{v}+r_{2} \mathbf{v}\right)-g(\mathbf{x})+e^{-x_{1}} g\left(x_{2} \mathbf{v}\right)\right] .
\end{aligned}
$$

Letting $r_{2} \rightarrow 0$ we get

$$
\begin{equation*}
C_{2}^{*} g(\mathbf{x})-e^{-x_{1}} C_{2}^{*} g\left(x_{2} \mathbf{v}\right)=-i \frac{\partial\left[g(\mathbf{x})-e^{-x_{1}} g\left(x_{2} \mathbf{v}\right)\right]}{\partial x_{2}} \tag{4.3}
\end{equation*}
$$

From the facts that $e^{-\left(x_{1}+y_{i}\right)} \gg 0$ and $f\left(0, x_{2}-y_{2}\right) \gg 0$ we get [2; 357-361] $e^{-\left(x_{1}+y_{1}\right)} f\left(0, x_{2}-y_{2}\right) \gg 0$. Consequently, if $K(\mathbf{x}, \mathbf{y})=f(\mathbf{x}-\mathbf{y})+e^{-\left(x_{1}+y_{1}\right)} f\left(0, x_{2}-y_{2}\right)$, then $K(\mathbf{x}, \mathbf{y})>0$. Let $\mathcal{F}_{2}^{\neq}$be the Hilbert space corresponding to the kernel $e^{-\left(x_{1}+y_{1}\right)} f\left(0, x_{2}-y_{2}\right)$ and $\boldsymbol{F}_{3}$ the space corresponding to the kernel $K(\mathbf{x}, \mathbf{y})$. The elements of $\mathcal{F}_{2}^{*}$ are of the form $e^{-x_{1}} g\left(x_{2} \mathbf{v}\right)$, where $g(\mathbf{x}) \in \mathcal{F}$ and hence the elements of $\mathcal{F}_{3}$ are of the form $h(\mathbf{x})+e^{-x_{1}} g\left(x_{2} \mathbf{v}\right)$, where $g, h \in \mathcal{F}[2 ; 361]$. Since $D_{1}$ is self-adjoint $\mathcal{F}$ and $\mathcal{F}_{2}^{*}$ have only the zero element in common and hence $\mathfrak{F}_{3}=\mathfrak{F} \oplus \mathcal{F}_{2}^{*}[2 ; 352-353]$; i.e., $\mathcal{F}$ and $\mathcal{F}_{2}^{*}$ are orthogonal complements of each other in $\mathcal{F}_{3}$.

Let $A_{2}^{\prime}=-i \partial / \partial x_{2}$ be the differential operator acting in $\mathcal{F}_{2}^{\prime \prime}$ and $A_{2}^{\prime \prime}$ the differential operator acting in $\mathfrak{F}_{3}$. More precisely, the domain of $A_{2}^{\prime}$ consists of those elements in $\mathcal{F}_{2}^{*}$ whose partial derivative with respect to $x_{2}$ exists and again belongs
 adjoint. ( ${ }^{1}$ ) Hence, since $A_{2}=D_{2}$ is self-adjoint, the operator $A_{2} \oplus A_{2}^{\prime}$ is self-adjoint, and since the domain of $A_{2}^{\prime \prime}$ consists of those elements in $\mathfrak{F}_{3}$ whose partial derivative with respect to $x_{2}$ exists and again belongs to $\mathcal{F}_{3}$, we have

$$
A_{2}^{\prime \prime *} \subseteq A_{2} \oplus A_{2}^{\prime} \subseteq A_{2}^{\prime \prime}
$$

The operator $A_{2}^{\prime \prime}$ is closed, but we do not know, as yet, that it is symmetric. Symmetry, together with the above relation would imply that $A_{2}^{\prime \prime}$ is self-adjoint. We shall now show that the conditions put on $f$ in Theorem 1 imply that $A_{2}^{\prime \prime}$ is self-adjoint.

The operator $A_{2}^{\prime \prime *}$ is closed and symmetric and since $A_{2}^{\prime \prime}$ is closed, $A_{2}^{\prime \prime}=A_{2}^{\prime * * *}$. In order to show that $A_{2}^{\prime \prime}$ is self-adjoint we shall show that the deficiency spaces connected with the symmetric operator $A_{2}^{\prime \prime *}$ consist only of the zero element. Now, for any closed symmetric operator $T$, the deficiency spaces consist of those elements which satisfy the relations $T^{*} g=i g, T^{*} h=-i h$, respectively. Hence, in our situa-

[^1]tion let us consider, for example, those $g \in \mathcal{F}_{3}$ such that $A_{2}^{\prime \prime * *} g=A_{2}^{\prime \prime} g=i g$. Solving this partial differential equation yields $g(\mathbf{x})=e^{-x_{1}} g\left(x_{2} \nabla\right)$. For any fixed $x_{1}$, it follows by Theorem A that $g_{x_{1}}\left(x_{2}\right)=g\left(x_{1}, x_{2}\right)$ belongs to the space generated by the kernel $K_{x_{1}}\left(x_{2}, y_{2}\right)=K\left(\left(x_{1}, x_{2}\right),\left(x_{1}, y_{2}\right)\right)=\left(1+e^{-2 x_{2}}\right) f\left(0, x_{2}-y_{2}\right)$. Since $f\left(0, x_{2}\right)$ has a unique positive definitive extension it follows that the function $\left(1+e^{-2 x_{1}}\right) f\left(0, x_{2}\right)$ has a unique positive definite extension. Hence, by Lemmas 3.1 and 3.7, the only element $g$ in the space generated by $K_{x_{1}}$ which satisfies $-i d g / d x_{2}=i g$ must be the zero element. Consequently, $g_{x_{1}}\left(x_{2}\right)=g\left(x_{1}, x_{2}\right)=0$. By the same reasoning the only solution in $\mathfrak{F}_{3}$ to the equation $A_{2}^{\prime \prime} h=-i h$ is the zero element. This shows that $A_{2}^{\prime \prime}$ is selfadjoint and hence
$$
A_{2}^{\prime \prime}=A_{2} \oplus A_{2}^{\prime}
$$

This, in turn, implies $\mathcal{F}$ and $\mathcal{F}_{2}^{*}$ reduce $A_{2}^{\prime \prime}$.
Returning now to (4.3) we see that the right side belongs to $\mathcal{D}\left(A_{2}^{\prime \prime}\right)$ and hence $g \in \mathcal{D}\left(D_{2}\right)$ and $e^{-x_{1}} g\left(x_{2} v\right) \in \mathcal{D}\left(A_{2}^{\prime}\right)$. Hence,

$$
C_{2}^{*} g=D_{2} g
$$

This means $C_{2}^{*}=D_{2}$, which is what we set out to prove, namely, that the closure of the restriction of $D_{2}$ to $m$ is $D_{2}$.

Let $U_{1}$ be the Cayley transform of $D_{1}$. If $g \in T$, i.e., $g=\left(D_{1}+i I\right) h, h \in \mathcal{D}$, then $U_{1} D_{2} g=U_{1}\left(D_{1}+i I\right) D_{2} h=\left(D_{1}-i I\right) D_{2} h=D_{2} U_{1}\left(D_{1}+i I\right) h$. Hence,

$$
U_{1} D_{2} g=D_{2} U_{1} g
$$

Since $D_{2}$ is the closure of its restriction to $m$, it follows that

$$
U_{1} D_{2} \subseteq D_{2} U_{1}
$$

This says $D_{1}$ and $D_{2}$ permute; i.e., their canonical spectral measures permute.
Let $d E_{1}$ and $d E_{2}$ be the canonical spectral measures of $D_{1}$ and $D_{2}$ respectively and $d E=d E_{1} d E_{2}$. By Lemma 3.5 it follows that if

$$
U(\mathbf{x})=\int_{-\infty}^{\infty} e^{\mathbf{x} \cdot \mathbf{t}} d E(\mathbf{t})
$$

then

$$
F(\mathbf{x})=\left(U(\mathbf{x}) f_{0}, f_{0}\right)=\int_{-\infty}^{\infty} e^{\mathbf{x} \cdot \mathbf{t}} d\left(E(\mathbf{t}) f_{0}, f_{0}\right)
$$

is an extension of $f(\mathbf{x})$.
Finally, the unicity of the extension of $f(\mathbf{x})$ follows by a two-dimensional analogue of the argument in Lemma 3.7. This completes the proof of Theorem 1.

## 5. Proof of Theorem I'

We shall start with the proof of two lemmas. In the next two lemmas the symbols $j, k, n$ will be integers.

Lemma 5.1. If $\mu(j-k) \gg 0$ for $0 \leqslant j, k \leqslant n$, then $\mu(k)$ has a unique positive definite extension to all of the integers if and only if $\operatorname{det}\{\mu(j-k)\}^{n}, k=0=0$.

Proof. Let $\mathcal{F}$ be the Hilbert space corresponding to the kernel $\mu(j-k)$. Suppose $\mu(k)$ has a unique positive definite extension; then $\left\{\mu_{k}\right\}_{0}^{n-1}$ generates $\mathcal{F}$, where $\mu_{k}(\cdot)=\mu(\cdot-k)$. For, otherwise consider the operator $U \mu_{k}=\mu_{k+1}, 0 \leqslant k \leqslant n-1 . U$ is an isometric operator defined on a subspace of $\mathcal{F}$ and clearly this can be extended in many ways to be unitary on $\mathcal{F}$. Let $V$ be any such unitary extension. Then $V^{k} \mu_{0}=\mu_{k}, 0 \leqslant k \leqslant n$, and if $d E(t)$ is the canonical spectral measure of $V^{*}$ we have for $k \geqslant 0$,

$$
\mu(k)=\left(\mu_{0}, V^{k} \mu_{0}\right)=\int_{0}^{1} e^{2 \pi i k t} d\left(E(t) \mu_{0}, \mu_{0}\right),
$$

and since $\mu(-k)=\bar{\mu}(k)$, this formula gives a positive definite extension of $\mu(k)$. Hence every different unitary extension of $U$ will give a different extension which is impossible. Consequently, $\left\{\mu_{k}\right\}_{0}^{n-1}$ generates $\mathcal{F}$ which gives the necessity part of our lemma.

Suppose now that $\operatorname{det}\{\mu(j-k)\}_{j, k=0}^{n}=0$; then $\left\{\mu_{k}\right\}_{0}^{n-1}$ generates $\mathcal{F}$ and the translation operator $U \mu_{k}=\mu_{k+1}, 0 \leqslant k \leqslant n-1$, is unitary. Suppose $\nu(k)$ is a positive definite extension of $\mu(k)$ to all of the integers, $\mathcal{F}_{1}$ the space generated by the kernel $v(j-k),-\infty<j, k<\infty$, and $V$ the translation operator $V \nu_{k}=v_{k+1}$. Let $m$ be the space generated by $\left\{v_{k}\right\}_{0}^{n}$; then there exists a $(1-1)$ isometric map between $m$ and $\mathcal{F}$ and hence $\left\{\nu_{k}\right\}_{k=0}^{n-1}$ generates $m$ and $V m=m$. Since $V$ is unitary, $m$ reduces $V$.

Let $W$ be the isometric map from $\mathcal{F}$ to $\mathbb{T}$ defined by $W \mu_{k}=\nu_{k}$. If $d E(t)$ is the canonical spectral measure of $V$ we have

$$
\left(V v_{j}, v_{k}\right)=\left(U \mu_{j}, \mu_{k}\right)=\int_{0}^{1} e^{2 \pi t t} d\left(W^{-1} E(t) W \mu_{j}, \mu_{k}\right) .
$$

Hence $d F=d W^{-1} E W$ is the canonical resolution of the identity of $U$. Now, for $-n \leqslant k \leqslant n$,

$$
\begin{aligned}
\mu(k)=\left(\mu_{0}, U^{k} \mu_{0}\right)=\left(v_{0}, V^{k} v_{0}\right) & =\int_{0}^{1} e^{-2 \pi t k t} d\left(E(t) v_{0}, \nu_{0}\right) \\
& =\int_{0}^{1} e^{-2 \pi t k t} d\left(F(t) \mu_{0}, \mu_{0}\right) .
\end{aligned}
$$

Hence the measure which gives rise to any extension is the same, namely $d\left(F(t) \mu_{0}, \mu_{0}\right)$ which shows we can have only one extension. This completes the proof of Lemma 1.

In our next lemma the symbols $\mathbf{j}, \mathbf{k}, \mathbf{n}$ will be the lattice points $\left(j_{1}, j_{2}\right),\left(k_{1}, k_{2}\right)$, $\left(n_{1}, n_{2}\right)$ respectively.

Lemma 5.2. Let $\mu(\mathbf{j}-\mathbf{k})>0$ for $\mathbf{0} \leqslant \mathbf{j}, \mathbf{k} \leqslant \mathbf{n}$ and $\mu\left(k_{1} \mathbf{u}\right), \mu\left(k_{\mathbf{2}} \mathbf{v}\right)$ have unique positive definite extensions. If $\mathcal{F}$ is the space generated by the kernel $\mu(\mathbf{j}-\mathbf{k})$, then $\mathcal{I}$ is generated by $\left\{\mu_{\mathbf{k}} ; \mathbf{0} \leqslant \mathbf{k} \leqslant\left(n_{1}-1, n_{2}-1\right)\right\}$.

Proof. Suppose $g \in \mathcal{F}$ is perpendicular to the space generated by $\left\{\mu_{\mathbf{k}} ; \mathbf{0} \leqslant \mathbf{k} \leqslant\right.$ $\left.\leqslant\left(n_{1}-1, n_{2}-1\right)\right\}$; then $g(\mathbf{k})=0$ for $0 \leqslant k \leqslant\left(n_{1}-1, n_{2}-1\right)$. By Lemma $1, g\left(k_{1} \mathbf{u}\right)=0$ for $0 \leqslant k_{1} \leqslant n_{1}$, since $\mu\left(k_{1} \mathbf{1}\right)$ has a unique extension. Further, if $1 \leqslant j_{2} \leqslant n_{2}-1, g\left(k_{1}, j_{2}\right)$ as a function of $k_{1}$ belongs to the space generated by the kernel $\mu\left(j_{1}-k_{1}, 0\right)$ and hence again by Lemma l, $g\left(k_{1}, j_{2}\right)=0$ for $0 \leqslant k_{1} \leqslant n_{1}-1$ implies $g\left(n_{1}, j_{2}\right)=0$. If we now argue on the second variable in the same way we get $g(\mathbf{k}) \equiv 0$. This completes the proof of Lemma 2.

We are now in a position to prove Theorem 1'. The notation in the following is the same as for Lemma 2. By Lemma 2 the set $\left\{\mu_{\mathbf{k}} ; \mathbf{0} \leqslant \mathbf{k} \leqslant\left(n_{1}-1, n_{2}-1\right)\right\}$ generates $\mathcal{F}$ and hence the operators $U_{1} \mu_{\mathbf{k}}=\mu_{\mathbf{k}+\mathbf{u}}, U_{\mathbf{2}} \mu_{\mathbf{k}}=\mu_{\mathbf{k}+\mathbf{v}}, \mathbf{k} \neq \mathbf{n}$, are unitary and clearly permute. If $d E_{1}$ and $d E_{2}$ are the canonical spectral measures of $U_{1}^{*}$ and $U_{2}^{*}$ respectively and $d E(\mathbf{t})=d E_{1}\left(t_{1}\right) E_{2}\left(t_{2}\right)$, then

$$
\mu(\mathbf{k})=\int_{0}^{1} e^{2 \pi i \mathbf{k} \cdot \mathbf{t}} d\left(E(\mathbf{t}) \mu_{0}, \mu_{0}\right)
$$

and the right hand integral gives our extension.
The proof of the fact that the measure which appears in the above representation is unique proceeds along the same lines as the second half of the proof of Lemma 5.1. This completes the proof of Theorem $\mathbf{1}^{\prime}$.

## 6. Further theorems

Lemma 6.1. Let $f(x)$ be defined and continuous on the interval (-2a, 2a) and $f(x-y) \gg 0$ for $x, y \in(-a, a)$. Suppose there exists an $\varepsilon>0$ such that the restriction of $f$ to $(-2 a+2 \varepsilon, 2 a-2 \varepsilon)$ has a unique positive definite extension to the whole axis. If $\mathcal{F}$ is the space corresponding to $f(x-y)$, for $x, y \in(-a, a)$, then $\mathcal{F}$ is generated by the set of elements $\left\{f_{y}(x)=f(x-y) ;-a+\varepsilon<y<a-\varepsilon\right\}$.

Proof. Let $\mathcal{F}_{\varepsilon}$ be the Hilbert space generated by the kernel $f_{\varepsilon}(x-y)=f(x-y)$ for $x, y \in(-a+\varepsilon, a+\varepsilon)$. By the hypothesis of the lemma and by Lemma 3.7 the differential operator $D_{s}$ on $\xi_{\varepsilon}$ as set $u p$ in $\S 3$ is self-adjoint. Hence, we have

$$
f(x)=\int_{-\infty}^{\infty} e^{i x t} d\left(E(t) f_{\varepsilon 0}, f_{\varepsilon 0}\right)_{\varepsilon}, \quad x \in(-2 a, 2 a)
$$

where $f_{\varepsilon 0}(x)=f_{\varepsilon}(x-0),(,)_{\varepsilon}$ is the inner product in $\mathcal{F}_{\varepsilon}$ and $d E$ is the spectral measure of $D_{\mathrm{s}}$.

If $D$ is the differential operator set up in the space $\mathcal{F}$, we also have

$$
f(x)=\int_{-\infty}^{\infty} e^{i x t} d\left(F(t) f_{0}, f_{0}\right)
$$

where $d F$ is the spectral measure of $D$. Since $f_{\varepsilon}$ has a unique extension we must have $d\left(E(t) f_{\varepsilon 0}, f_{\varepsilon 0}\right)_{\varepsilon}=d\left(F(t) f_{0}, f_{0}\right)$.

By a general Theorem [4;61], the space $\mathcal{F}$ is the set of all functions

$$
g(x)=\int_{-\infty}^{\infty} e^{i x t} G(t) d\left(F(t) f_{0}, f_{0}\right), \quad x \in(-a, a)
$$

with $G \in \mathbb{R}^{2}\left(d\left(F(t) f_{0}, f_{0}\right)\right)$, and

$$
\|g\|^{2}=\int_{-\infty}^{\infty}|G(t)|^{2} d\left(F(t) f_{0}, f_{0}\right)
$$

By the same general theorem, $\mathfrak{Q}^{2}\left(d\left(F(t) f_{0}, f_{0}\right)\right)=\mathfrak{Q}^{2}\left(d\left(E(t) f_{\varepsilon 0}, f_{\varepsilon 0}\right)_{\varepsilon}\right)$ is generated by elements of the form

$$
\sum_{k=1}^{n} \xi_{k} e^{i y_{k} t}
$$

where $\xi_{k}$ is a complex number and $y_{k} \in(-a+\varepsilon, a-\varepsilon)$. Hence, the lemma is proved.
Theorem 2. Let $Q$ be the open interval in the plane $-\mathbf{a}<\mathbf{x}<\mathbf{a}$ and suppose $f(\mathbf{x})$ defined on $2 Q$ is continuous and $f(\mathbf{x}-\mathbf{y}) \gg 0$ for $\mathbf{x}, \mathbf{y} \in Q$. Suppose there exists an $\varepsilon>0$ such that the function $f\left(x_{1}, 0\right)$ restricted to $\left(-2 a_{1}+2 \varepsilon, 2 a_{1}-2 \varepsilon\right)$ has a unique extension to the whole axis; then $f$ may be extended to be positive definite over the whole plane.

Proof. Let $Q_{\varepsilon}$ be the two dimensional interval defined by the inequalities $\left|x_{1}\right|<a_{1}-\varepsilon,\left|x_{2}\right|<a_{2}$. Let $\mathcal{F}_{0} \subseteq \mathcal{F}$ be the class of elements which vanish on $Q_{\varepsilon}$. If
$g \in \mathcal{F}_{0}$, then for fixed $x_{2}, g\left(x_{1}, x_{2}\right)$, considered as a function of $x_{1}$, belongs, by Theorem $A$, to the space generated by $f\left(x_{1}-y_{1}, 0\right)$ and is orthogonal to the space generated by the elements $f_{y_{1}}\left(x_{1}\right)=f\left(x_{1}-y_{1}, 0\right)$ for $\left|y_{1}\right|<a_{1}-\varepsilon$. By Lemma 6.1, this orthogonal space $\mathcal{F}_{0}$ consists only of the zero element. Hence $g\left(x_{1}, x_{2}\right) \equiv 0$. This means $\mathcal{F}$ is generated by the set of elements $f_{\mathbf{y}}$, for which $\left|y_{1}\right|<a_{1}-\varepsilon,\left|y_{2}\right|<a_{2}$.

Let us set up the space $\mathcal{D}_{2}^{\prime}$ of elements of the form

$$
\begin{equation*}
g_{\dot{r}}=\frac{1}{r} \int_{0}^{r} g_{t} d t \tag{6.1}
\end{equation*}
$$

where $g_{t}(\mathbf{x})=\sum_{1}^{n} \xi_{k} f\left(\mathbf{x}-\mathbf{x}_{k}-t \mathbf{v}\right),\left|\mathbf{x}_{k} \cdot \mathbf{u}\right|<a_{1}-\varepsilon$ and $\mathbf{x}_{k}+r \mathbf{v} \in Q$. We shall show that if $D_{2}^{\prime}$ is the restriction of $D_{2}$ to $D_{2}^{\prime}$ then the closure of $D_{2}^{\prime}$ is $D_{2}$.

As in the Lemmas 3.1 and 3.2 we set up the space $\mathcal{D}_{2}$ of elements of form (6.1), where $\left|\mathbf{x}_{k} \cdot \mathbf{u}\right|<a_{1},\left|\mathbf{x}_{k} \cdot \mathbf{v}\right|<a_{2}$. The lemmas mentioned proved for us that $D_{2}$ is the closure of its restriction to $\mathcal{D}_{2}$. Let $h_{\tilde{\mathbf{r}}} \in \mathcal{D}_{2}$; i.e. there exists an $h_{i}(\mathbf{x})=\sum_{1}^{n} \xi_{k} f\left(\mathbf{x}-\mathbf{y}_{k}-i \mathbf{v}\right)$ such that

$$
h_{\tilde{r}}=\frac{1}{r} \int_{0}^{r} h_{t} d t
$$

where $\left|\mathbf{y}_{k} \cdot \mathbf{u}\right|<a_{1},\left|\mathbf{y}_{k} \cdot \mathbf{v}\right|<a_{2}, \mathbf{y}_{k}+r \mathbf{v} \in Q$. Let $m=\max \left|\mathbf{y}_{k} \cdot \mathbf{v}\right|$ and $\boldsymbol{\mathcal { F }}_{m}$ the subspace of $\mathcal{F}$ generated by the set $\left\{f_{\mathbf{y}} ;|\mathbf{y} \cdot \boldsymbol{v}| \leqslant m\right\}$. The space $\mathcal{F}_{m}$ is the orthogonal complement of the set of functions in $\mathcal{F}$ which vanish on the rectangle $|\mathbf{x} \cdot \mathbf{u}|<a_{1},|\mathbf{x} \cdot \mathbf{v}| \leqslant m$. By an argument similar to the argument which we have already made previously, $\mathcal{F}_{m}$ is generated by the set of elements $\left\{f_{y} ;|\mathbf{y} \cdot \mathbf{u}|<a_{1}-\varepsilon,|\mathbf{y}-\mathbf{v}|<m\right\}$. Hence, there exist elements $h_{n, t}(\mathbf{x})=\sum \xi_{k}^{(n)} f\left(\mathbf{x}-\mathbf{x}_{k}^{(n)}-t \mathbf{v}\right)$ with $\left|\mathbf{x}_{k}^{(n)} \cdot \mathbf{u}\right|<a_{1}-\varepsilon,\left|\mathbf{x}_{k}^{(n)} \cdot \mathbf{v}\right| \leqslant m$ such that $h_{n, 0} \rightarrow h_{0}$ as $n \rightarrow \infty$.

Let $U(x \mathbf{v})$ be the group of unitary operators as constructed in Lemma 3.5 with any self-adjoint extension of $D_{2}$ as its infinitesimal generator. As shown in that lemma, this is a group of translation operators on $\mathcal{F}$, wherever the translations are defined. Hence, if $|t|<|r|$, since $U^{*}(t \mathbf{v}) h_{n, 0}=h_{n, t}$ and $U^{*}(t \mathbf{v}) h_{0}=h_{t}$, we have $h_{n, t} \rightarrow h_{t}$ uniformly in $t$. This means that if we set

$$
h_{n, r}=\frac{1}{r} \int_{0}^{t} h_{n, t} d t
$$

then $h_{\tilde{n}, r} \rightarrow h_{\tilde{r}}$. Now, by Lemma 3.2

$$
\begin{aligned}
& D_{2}^{\prime} h_{n, r}=\frac{i}{r}\left[h_{n, r}-h_{n, 0}\right]=\frac{i}{r}\left[U^{*}(r \mathrm{v})-I\right] h_{n, 0} \\
& D_{2} h_{\dot{r}}=\frac{i}{r}\left[h_{r}-h_{0}\right]=\frac{i}{r}\left[U^{*}(r \mathrm{v})-I\right] h_{0} .
\end{aligned}
$$

Hence, $D_{2}^{\prime} h_{\tilde{n}, r} \rightarrow D_{2} h_{r}$; and since $D_{2}$ is the closure of its restriction to $\mathcal{D}_{2}$ it follows that $D_{2}$ is the closure of $D_{2}^{\prime}$.

The differential operator $D_{1}$ is self-adjoint since $f\left(x_{1}-y_{1}, 0\right)$ has a unique extension. Let $U(x \mathbf{u})$ be the group of unitary operators as set up in Lemma 3.5. Choose any $x$ such that $|x|<\varepsilon$ and $g_{\tilde{r}} \in \mathcal{D}_{2}^{\prime}$; i.e.

$$
g \tilde{r}=\frac{1}{r} \int_{0}^{r} g_{t} d t
$$

where $g_{t}(\mathbf{x})=\sum \xi_{k} f\left(\mathbf{x}-\mathbf{x}_{k}-\boldsymbol{t} \mathbf{v}\right),\left|\mathbf{x}_{k} \cdot \mathbf{u}\right|<a_{1}-\varepsilon$ and $\mathbf{x}_{k}+r \mathbf{v} \in Q$. Further let $h_{t}(\mathbf{x})=$ $=\Sigma \xi_{k} f\left(\mathbf{x}-\mathbf{x}_{k}-x \mathbf{u}-t \mathbf{v}\right)$. Then clearly

$$
U^{*}(x \mathbf{u}) D_{2}^{\prime} g_{\tilde{r}}=\frac{i}{r}\left[h_{r}-h_{0}\right]=D_{2} U^{*}(x \mathbf{u}) g_{\dot{r}}
$$

Since the closure of $D_{2}^{\prime}$ is $D_{2}$ and $U(x \mathbf{u})$ is bounded we have

$$
U^{*}(x \mathbf{u}) D_{2} g=D_{2} U^{*}(x \mathbf{u}) g
$$

for every $g \in \mathcal{D}\left(D_{2}\right)$. Since this is true for every $x$ such that $|x|<\varepsilon$, it follows that the whole group $U(x \mathbf{u}),-\infty<x<\infty$, permutes with $D_{2}$. Hence $D_{1}$ and $D_{2}$ permute in the sense that $D_{2}$ permutes with the canonical spectral measure of $D_{1}$. This means that there exists a sequence of subspaces $\left\{m_{n}\right\}$ each of which reduces $D_{2}$, and therefore $D_{2}^{*}$, and moreover reduces $D_{1}$ to a bounded self-adjoint operator.

As in $\S 3$ define the conjugation operator $J$ by

$$
J g(\mathbf{x})=\overline{\mathbf{g}}(-\mathbf{x}) .
$$

It is clear that $J$ permutes with both $D_{1}$ and $D_{2}$. Hence $m_{n}$ reduces $J$ and the restrictions of $D_{1}$ and $D_{2}$ to $m_{n}$ are real with respect to $J$.

Let $D_{1 n}$ and $D_{2 n}$ be the restriction to $m_{n}$ of $D_{1}$ and $D_{2}$ respectively. Since $D_{2 n}$ is real with respect to $J$ it has equal deficiency indices and therefore self-adjoint extensions. The deficiency spaces $\mathcal{E}_{n}^{+}$and $\mathcal{E}_{n}^{-}$of $D_{2 n}$ are given by $\mathcal{E}_{n}^{+}=\left[g \mid g \in \boldsymbol{M}_{n}\right.$ and $\left.D_{2}^{*} g=i g\right]$ and $\mathcal{E}_{n}^{-}=\left[g \mid g \in m_{n}\right.$ and $\left.D_{2}^{*} g=-i g\right]$ respectively. Hence $\mathcal{E}_{n}^{+}$and $\mathcal{E}_{n}^{-}$contain only elements of the form $g\left(x_{1}, 0\right) e^{-x_{2}}$ and $g\left(x_{1}, 0\right) e^{x_{2}}$ respectively. On the other

[^2]hand, any elements of this form which belong to $m_{n}$ belong to $\mathcal{E}_{n}^{+}$and $\mathcal{E}_{n}^{-}$respectively. Since $D_{1 n}$ is bounded, the last statement immediately implies that $\mathcal{E}_{n}^{+}$and $\mathcal{E}_{n}^{-}$both reduce $D_{1 n}$ and hence both of these manifolds reduce each element of the group $U(x \mathbf{u})$.

Let $K_{n}(\mathbf{x}, \mathbf{y})$ be the reproducing kernel corresponding of $\mathcal{E}_{n}^{+}$. If $P_{n}$ is the projection ( ${ }^{1}$ ) onto $\mathcal{E}_{n}^{+}$, then $K_{n}(\mathbf{x}, \mathbf{y})=P_{n} f(\mathbf{x}-\mathbf{y})$. Further, since every element of the group $U(x \mathbf{u})$ is reduced by $\mathcal{E}_{n}^{+}$, every element of this group permutes with $P_{n}$. Using the fact that $K_{n}(\mathbf{y}, \mathbf{x})=\bar{K}_{n}(\mathbf{x}, \mathbf{y})$ we get

$$
\begin{aligned}
K_{n}(\mathbf{x}, \mathbf{y}) & =e^{-\left(x_{2}+y_{2}\right)} K_{n}\left(x_{1} \mathbf{u}, y_{1} \mathbf{u}\right)=e^{-\left(x_{2}+y_{2}\right)} P_{n} U\left(x_{1} \mathbf{u}-y_{1} \mathbf{u}\right) f_{0}(0) \\
& =e^{-\left(x_{2}+y_{2}\right)} U\left(x_{1} \mathbf{u}-y_{1} \mathbf{u}\right) P_{n} f_{0}(0)=e^{-\left(x_{2}+y_{2}\right)} U\left(x_{1} \mathbf{u}-y_{1} \mathbf{u}\right) K_{n}(0,0) .
\end{aligned}
$$

If we let $K_{1 n}\left(x_{1}-y_{1}\right)=U\left(x_{1} \mathbf{u}-y_{1} \mathbf{u}\right) K_{n}(0,0)$. then we get

$$
K_{n}(\mathbf{x}, \mathbf{y})=K_{1 n}\left(x_{1}-y_{1}\right) e^{-\left(x_{2}+y_{3}\right)}
$$

That is to say, $K_{n}(\mathbf{x}, \mathrm{y})$ is the product of two kernels and hence according to theorem B of $\S 2, \mathcal{E}_{n}^{+}$is the tensor product of the spaces $\mathcal{F}_{1 n}$ and $\mathcal{F}_{2 n}$, where $\mathcal{F}_{1 n}$ corresponds to the kernel $K_{1 n}$ and $\mathcal{F}_{2 n}$ to the kernel $e^{-\left(x_{2}+\nu_{z}\right)}$. Hence, if $g\left(x_{1}, 0\right) e^{-x_{2}} \in \mathcal{E}_{n}^{-}$, then $g\left(x_{1}, 0\right) \in \mathcal{F}_{1 n}$ and hence $\bar{g}\left(-x_{1}, 0\right) \in \mathcal{F}_{1 n}$. This implies that $\bar{g}\left(-x_{1}, 0\right) e^{-x_{2}} \in \mathcal{E}_{n}^{+}$and

$$
\begin{aligned}
\| e^{-x_{2}} g\left(x_{1}, 0\|=\| e^{-x_{2}}\left\|_{2 n}\right\| g\left(x_{1}, 0\right) \|_{1 n}\right. & =\left\|e^{-x_{2}}\right\|_{2 n} \| \bar{g}\left(-x_{1}, 0 \|_{1 n}\right. \\
& =\left\|e^{-x_{2}} \bar{g}\left(-x_{1}, 0\right)\right\| .
\end{aligned}
$$

Hence $J \bar{g}\left(-x_{1}, 0\right) e^{-x_{2}}=g\left(x_{1}, 0\right) e^{x_{2}} \in \mathcal{E}_{n}^{-}$and $g\left(x_{1}, 0\right) e^{-x_{2}}$ and $g\left(x_{1}, 0\right) e^{x_{2}}$ have the same norm.

Let $V_{2 n}$ be the operator defined from $\mathcal{E}_{n}^{+}$onto $\mathcal{E}_{n}^{-}$by the equation.

$$
V_{2 n} e^{-x_{1}} g\left(x_{1}, 0\right)=e^{x_{s}} g\left(x_{1}, 0\right)
$$

and let $V_{2 n}^{\prime}$ be the Cayley transform of $D_{2 n} . V_{2 n}$ is an isometric operator which clearly permutes with $D_{1 n}$ on $\mathcal{E}_{n}^{+}$. On the other hand, $\mathcal{D}\left(V_{2 n}^{\prime}\right)$ is given by the set of all elements of the form $h=\left(D_{2 n}+i I\right) g$, where $g \in \mathcal{D}\left(D_{2 n}\right)$, and hence $V_{2 n}^{\prime} h=$ $=\left(D_{2 n}-i I\right) g$. Since $D_{1 n}$ is bounded and self-adjoint and permutes with $D_{2 n}$ it permutes with $V_{2 n}^{\prime}$ on the orthogonal complement of $\mathcal{E}_{n}^{+}$. Consequently, if we set $U_{2 n}=V_{2 n}^{\prime} \oplus V_{2 n}$, we have that $D_{1 n}$ permutes with $U_{2 n}$. If $H_{2 n}$ is the self-adjoint extension of $D_{2 n}$ whose Cayley transform is $U_{2 n}$, then $D_{1 n}$ and $H_{2 n}$ permute.
${ }^{(1)} P_{n}$ is the projection from $\mathcal{F}$ onto $\mathcal{E}_{n}^{+}$.

If we construct such a self-adjoint operator $H_{2 n}$ for every $n$ and let $H_{2}=\sum_{1}^{\infty} \oplus H_{2 n}$, then $H_{2}$ is self-adjoint and $D_{1}$ and $H_{2}$ permute in the sense that their canonical spectral measures permute. Let $d E_{2}$ be the canonical spectral measure of $H_{2}$ and set

$$
U(x v)=\int_{-\infty}^{\infty} e^{i x t} d E_{2}(t)
$$

Clearly, as $\mathbf{x}$ varies over the plane, the operators $U(\mathbf{x})=U\left(x_{1} \mathbf{u}\right) U\left(x_{2} \mathbf{v}\right)$ form a group of unitary operators. If $d E=d E_{1} E_{\mathbf{2}}$, then

$$
F(\mathbf{x})=\left(U(\mathbf{x}) f_{0}, f_{0}\right)=\int_{-\infty}^{\infty} e^{i \mathbf{x} \cdot \mathbf{t}} d\left(E(\mathbf{t}) f_{0}, f_{0}\right)
$$

is a positive definite extension of $f(\mathbf{x})$. This concludes the proof of Theorem 2.
Theorem 3. Let $Q$ be an open symmetric neighbourhood in the plane (i.e., $x \in Q$ implies $(-\mathbf{x}) \in Q$ ) and $f(\mathbf{x})$ a continuous function defined on $2 Q$. Necessary and sufficient conditions that there exists a bounded measure $d F \geqslant 0$ whose support is in the half-plane $\mathbf{t} \cdot \mathbf{u} \geqslant 0$ and such that

$$
f(\mathrm{x})=\int_{-\infty}^{\infty} e^{\mathrm{x} \cdot \mathrm{t}} d F(\mathrm{t})
$$

are:
(a)

$$
f(\mathbf{x}-\mathbf{y}) \gg 0 \text { for } \mathbf{x}, \mathbf{y} \in Q
$$

(b) There exists a sequence of real numbers $r_{n} \rightarrow 0$ such that for $n=1,2, \ldots$,

$$
i \int_{0}^{\tau_{n}}[f(\mathbf{x}-\mathbf{y}-t \mathbf{u})-f(\mathbf{x}-\mathbf{y}+t \mathbf{u}] d t \gg 0
$$

for all $\mathbf{x}$ and $\mathbf{y}$ in $Q$ for which the functions are defined.
Proof. That these conditions are necessary may be checked immediately by a simple computation.

To prove the sufficiency we set up the space $\mathcal{F}$ corresponding to $f(x-y)$. Condition (b) together with Lemma 3.4 tells us that the operator $D_{1}$ is non-negative. Let $H_{1}$ be any positive self-adjoint extension of $D_{1}, d E_{1}$ its canonical spectral meas. ure and

$$
U(x \mathbf{u})=\int_{-\infty}^{\infty} e^{i x t} d E_{1}(t)
$$

Further, let $H_{2}$ be any self-adjoint extension of $D_{2}$ and $U_{2}$ its Cayley transform.
Suppose $h^{\sim} \in D_{2}$; i.e.,

$$
h^{\sim}=\frac{1}{r} \int_{0}^{T} h_{t} d t
$$

where $h_{t}(\mathbf{x})=\sum \xi_{k} f\left(\mathbf{x}-\mathbf{x}_{k}-t \mathbf{v}\right), \quad \mathbf{x}_{k} \in Q, \quad \mathbf{x}_{k}+r \mathbf{v} \in Q$. If we let $h^{\prime}=\left(H_{\mathbf{2}}+i I\right) h^{\prime}$, it is clear that for all sufficiently small $x$, say $|x|<s$, where $\mathrm{x}_{k}+r \mathbf{v}-s \mathbf{u} \in Q$,

$$
\begin{equation*}
U(x \mathbf{u}) U_{2} h^{\prime}=U_{2} U \cdot(x \mathbf{u}) h^{\prime} . \tag{6.2}
\end{equation*}
$$

Suppose $g \in \mathcal{D}\left(H_{1}\right)$ such that

$$
H_{1} g=\int_{0}^{a} t d E_{1}(t) g
$$

where $a$ is a finite positive number. We have then

$$
\begin{align*}
\left(U(x \mathbf{u}) U_{2} h^{\prime}, g\right) & =\int_{0}^{a} e^{i x t} d\left(E_{1}(t) U_{2} h^{\prime}, g\right) \\
& =\int_{0}^{\infty} e^{i x t} d\left(U_{2} E_{1}(t) h^{\prime}, g\right)=\left(U_{2} U(x \mathbf{u}) h^{\prime}, g\right) \tag{6.3}
\end{align*}
$$

Now,

$$
F_{1}(z)=\int_{0}^{a} e^{i(x+i y) t} d\left(E_{1}(t) U_{2} h^{\prime}, g\right), \quad z=x+i y
$$

exists for $-\infty<y<\infty$ and is analytic. On the other hand,

$$
F_{2}(z)=\int_{0}^{\infty} e^{i(x+i z) t} d\left(U_{2} E_{1}(t) h^{\prime}, g\right)
$$

exists for $y \geqslant 0$ and is analytic in the half plane $y>0$. Further from (6.3)

$$
\lim _{y \uparrow 0} F_{1}(z)=F_{1}(x)=\lim _{y \downarrow 0} F_{2}(z) .
$$

Since $F_{1}(x)$ is continuous, $F_{1}(z)$ is an analytic extension of $F_{2}(z)$ and hence $F_{1}(z) \equiv F_{2}(z)$ for $y \geqslant 0$.

By the uniqueness theorem for Laplace-Stieltjes transforms we must have

$$
\left(E_{1}(t) U_{2} h^{\prime}, g\right)=\left(U_{2} E_{1}(t) h^{\prime}, g\right),
$$

since both functions are normalized in the same way. Since the class of $g$ for which this is true is dense in $F$ we have

$$
\begin{equation*}
E_{1}(t) U_{2} h^{\prime}=U_{2} E_{1}(t) h^{\prime} \tag{6.4}
\end{equation*}
$$

Let $V_{2}$ be the Cayley transform of $D_{2}$. Since $U_{2} h^{\prime}=V_{2} h^{\prime}$, it follows from (6.4) that if we consider $U_{2}$ as a generic symbol for the Cayley transform of any selfadjoint extension of $D_{2}$, the element $U_{2} E_{1}(t) h^{\prime}$ remains constant as $U_{2}$ varies. This implies $E_{1}(t) h^{\prime} \in \mathcal{D}\left(V_{2}\right)$. For, since $U_{2} E_{1}(t) h^{\prime}$ remains constant, $g=\left(I-U_{2}\right) E_{1}(t) h^{\prime}$ remains constant as $U_{2}$ varies and is in the domain of every self-adjoint extension of $D_{2}$. Hence $g \in \mathcal{D}\left(D_{2}\right)$ (see [7; 494] and [1; 279]). Hence there exists an $h \in \mathcal{D}\left(V_{2}\right)$ such that

$$
g=\left(I-U_{2}\right) E_{1}(t) h^{\prime}=\left(I-V_{2}\right) h=\left(I-U_{2}\right) h .
$$

We get therefore

$$
E_{1}(t) h^{\prime}-h=U_{2}\left[E_{1}(t) h^{\prime}-h\right] .
$$

Since $I-U_{2}$ has as inverse we must have $h=E_{1}(t) h^{\prime}$ which proves the fact that $E_{1}(t) h^{\prime} \in \mathcal{D}\left(V_{2}\right)$. Using this fact and (6.4) we get

$$
E_{1}(t) V_{2} \subseteq V_{2} E_{1}(t)
$$

since in (6.4) $h^{\prime}$ runs over a dense set in $\mathcal{D}\left(V_{2}\right)$. Hence

$$
\begin{equation*}
E_{1}(t) D_{2} \subseteq D_{2} E_{1}(t) \tag{6.5}
\end{equation*}
$$

Let $E_{n}=E_{1}(n)-E_{1}(0-)$ and let $m_{n}$ be the range of $E_{n}$. From (6.5) it follows that $m_{n}$ reduces $D_{2}$ and therefore $D_{2}^{*}$. Further $m_{n}$ reduces $H_{1}$ to a bounded selfadjoint. If we now follow through the argument of the latter part of the proof of Theorem 2 we will have completed the proof of Theorem 3.

## 7. The circle group case

We shall here give an example which shows that the general extension problem formulated in §l is not true for the circle group. Let $f(x)$ be an analytic positive definite function defined on the real axis which is not periodic. Let $f_{1}(x)$ be the restriction of $f(x)$ to the interval $-a<x<a$, where $0<a<\pi$. Let $F\left(e^{i x}\right)=f_{1}(x)$; then
$F^{\prime}\left(e^{i x}\right)$ is a positive definite function defined on a symmetric neighborhood of the identity of the circle group in the sense that $F\left(e^{i x} e^{-i v}\right)>0$ for $-\frac{1}{2} a<x, y<\frac{1}{2} a$. Now, if $F\left(e^{i x}\right)$ had a positive definite extension to the whole circle group, then $f_{1}(x)$ would have a periodic positive definite extension to the real axis. But this is impossible since $f_{1}(x)$ is analytic and its only positive definite extension is $f(x)$.

## References

[1]. N. Acheser \& I. M. Glasmann, Theory of linear operators in Hilbert space. USSR, 1950 (Russian). German translation, Akademie-Verlag, Berlin, 1954.
[2]. N. Aronszajn. The theory of reproducing kernels. Trans. Amer. Math. Soc., 68 (1950), 337-404.
[3]. S. Bochner, Fouriersche Integrale. Leipzig, 1932.
[4]. A. Devinatz, Integral representations of positive definite functions. Irans. Amer. Math. Soc., 74 (1953), 56-77. Errata, 536.
[5]. - -, Integral representations of positive definite functions. II. Trans. Amer. Math. Soc., 77 (1954), 455-480. Errata, 79 (1955) 556.
[6]. ---, The representation of functions as Laplace-Stieltjes integrals. Duke Math. J., 22 (1955), 185-192.
[7]. - -, Two parameter moment problems. Duke Math. J., 24 (1957), 481-498.
[8]. N. Dunford, and I. E. Segal, Semi-groups of operators and the Weierstrass theorem, Bull. Amer. Math. Soc., 52 (1946), 911-914.
[9]. C. G. Esseen, Fourier analysis of distribution functions. Acta Math., 77 (1945), 1-125.
[10]. H. L. Hamburger, Contributions to the theory of closed Hermitian transformations of deficiency index ( $m, m$ ). Ann. of Math., 45 (1944), 55-99.
[11]. M. G. Krein, Sur le problem du prolongement des fonctions Hermitiennes positives et continues. C. R. (Doklady) Acad. Sci. URSS, N. S., 26 (1940), 17-22.
[12]. - -, On a generalized problem of moments. C. R. (Doklady) Acad. Sci. URSS, N.S., 26 (1944), 219-222.
[13]. --, On Hermitian operators with directed functionals. Acad. Nauk Ukrain. RSR, Zbirnik Prac'. Inst. Math. 1948, no. 10 (1948), 83-106. (Ukrainian-Russian summary).
[14]. - -, The fundamental propositions of the theory of representations of Hermitian operators with deficiency ( $m, m$ ). Ukrain. Mat. Zurnal 1, no. 2 (1949), 3-66 (Russian).
[15]. -- -, Hermitian positive kernels on homogeneous Spaces I. Ukrain. Mat. Zurnal 1, no. 4 (1949), 64-98 (Russian).
[16]. M. Livshitz, On an application of the theory of Hermitian operators to the generalized problem of moments. C. R. (Doklady) Acad. Sci. URSS N.S., 44 (1944), 3-7.
[17]. D. Rarkov, Sur les functions positivement defines. C. R. (Doklady) Acad. Sci. URSS, N.S. 26 (1940), 860-865.


[^0]:    ${ }^{(1)}$ This research was partially supported by the United States Air Force Office of Scientific Research of the Air Research and Development Command, under contracts No. AF18 (600)-1223 and AF18(600)-568. Reproduction in whole or part is permitted for any purpose of the United States Government.
    ${ }^{(2)}$ In a recent seminar at Washington University, Professor M. Cotlar has indicated the importance of such a theory in order to unify certain aspects of the theory of singular integrals, multiplier transforms and more general types of integral transforms.

[^1]:    (1) $A_{2}^{\prime}$ is the tensor product $I \otimes A_{2}$, where $I$ is the identity operator in the one dimensional space generated by $e^{-x_{1}}$ and $A_{2}$ is the differential operator on $\mathcal{F}_{2}$, the space generated by $f\left(0, x_{2}-y\right)$. Hence $A_{2}^{\prime}$ is self-adjoint if and only if $A_{2}$ is self-adjoint.

[^2]:    9-593804. Acta mathematica. 102. Imprimé le 28 septembre 1959

