ON THE GEOMETRICAL MOMENTS OF SKEW-REGULAR SIMPLICES IN HYPERSPHERICAL SPACE, WITH SOME APPLICATIONS IN GEOMETRY AND MATHEMATICAL STATISTICS

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1. Introduction and summary

Consider a regular simplex constructed on the surface of a unit sphere immersed in N-space with each of its $N(N-1)/2$ primary bounding angles equal to θ. Denote the vertices of the simplex by $1, 2, \ldots, N$ and the simplex itself by $12 \ldots N$. Further, let i' denote the point antipodal to i. Then a simplex such as

$$1'\,2' \ldots \beta'\,(\beta+1) \ldots (\beta+\gamma) \quad (\beta=0, 1, \ldots, N,\ \beta+\gamma=N) \ \text{ has } \ [\beta(\beta-1)+\gamma(\gamma-1)]/2$$

angles equal to θ and the remaining $\beta\gamma$ angles to $\pi-\theta$. A simplex of this type will be referred to as a skew-regular simplex. In particular, for $\beta=0$ the simplex is regular, while for $\gamma=0$, we have the regular simplex antipodal to $12 \ldots N$. A regular simplex is then a particular case of a skew-regular simplex. The latter simplex is generated from the simplex $12 \ldots N$, hereafter called the base simplex, by the projection of an appropriate subset of vertices of the base simplex with respect to the centre of the sphere on to the surface of the sphere.

Let Π be the $(\beta+\gamma-1)$-flat through the centre of the sphere orthogonal to the line joining the latter point and the centroid of the base simplex. Let x be the distance of any point from Π; this distance will be regarded as positive if the point in question lies on the same side (half-space) of Π as does the base simplex, and

(1) This research was supported by the Office of Naval Research under Contract Number Nonr-266 (59), Project Number 042-205. Reproduction in whole or in part is permitted for any purpose of the United States Government.

negative otherwise. We shall investigate the values of the geometrical moments of $1'\,2'\,\ldots\,\beta'\,(\beta+1)\,\ldots\,(\beta+\gamma)$ relative to Π, represented by

$$p_{\beta,\beta+\gamma;\,s}(\theta)=\int x^s\,d\omega\Big/\int d\omega \qquad (s=0,1,2,\ldots), \tag{1}$$

where $d\omega$ denotes the content of an infinitesimal element on the surface of the sphere (or, equivalently, the solid angle subtended at the centre of the sphere by the element), the domains of integration in (1) being $1'\,2'\,\ldots\,\beta'\,(\beta+1)\,\ldots\,(\beta+\gamma)$. In the following section a generating function for the moments will be derived, which will enable each of the $(\beta+\gamma-1)$-fold integrals in (1) to be reduced to a univariate integral involving the error function and its integrals, provided θ is obtuse. An important consequence of this reduction is the rather striking result that the non-normalised moments, defined by

$$p'_{\beta,\beta+\gamma;\,s}(\theta)=\int x^s\,d\omega \qquad (s=0,1,2,\ldots), \tag{2}$$

may be expressed as linear combinations of the contents of the edges, of various dimensionality, of $1'\,2'\,\ldots\,\beta'\,(\beta+1)\,\ldots\,(\beta+\gamma)$, for all permissible θ (acute or obtuse). This result is derived in section 3.

We shall also consider the moments, both normalised and non-normalised, of the sector of the $(\beta+\gamma)$-sphere standing on $1'\,2'\,\ldots\,\beta'\,(\beta+1)\,\ldots\,(\beta+\gamma)$ as base and the centre of the sphere as pole, i.e.

$$q_{\beta,\beta+\gamma;\,s}(\theta)=\int x^s\,d\tau\Big/\int d\tau \qquad (s=0,1,2,\ldots), \tag{3}$$

and

$$q'_{\beta,\beta+\gamma;\,s}(\theta)=\int x^s\,d\tau \qquad (s=0,1,2,\ldots), \tag{4}$$

where $d\tau$ denotes the content of an infinitesimal element in $(\beta+\gamma)$-space, the domains of integration in (3) and (4) being the sector as described.

In order to relate the above discussion more directly to the investigation in the sections which follow, it will be convenient to refer to the class of skew-regular simplices in terms of a system of orthogonal coordinate axes rather than (as above) in purely geometrical terms. From this point of view the simplex $1'\,2'\,\ldots\,\beta'\,(\beta+1)\,\ldots$ $\ldots\,(\beta+\gamma)$ is represented formally as

$$\begin{aligned}1'\,2'\,\ldots\,\beta'\,(\beta+1)\,\ldots\,(\beta+\gamma)=\{\mathbf{x}\,|\,\mathbf{x}'\,\mathbf{x}=1,\ & L_i(\mathbf{x})\leqslant 0 \quad (i=1,2,\ldots,\beta),\\ & L_i(\mathbf{x})\geqslant 0 \quad (i=\beta+1,\beta+2,\ldots,\beta+\gamma)\},\end{aligned}$$

in which $\mathbf{x}$ denotes an arbitrary point in $(\beta+\gamma)$-space, $\mathbf{x}'\mathbf{x}=1$ the $(\beta+\gamma-1)$-dimensional surface of a unit sphere imbedded in $(\beta+\gamma)$-space, while $L_1(\mathbf{x})=0, \ldots, L_{\beta+\gamma}(\mathbf{x})=0$ represent a sheaf of $\beta+\gamma$ $(\beta+\gamma-1)$-flats, $\Lambda_1, \ldots, \Lambda_{\beta+\gamma}$, through the centre of the sphere, which are inclined equally to each other at an angle θ. Any vertex u, accented or otherwise, of the simplex $1'2' \ldots \beta'(\beta+1) \ldots (\beta+\gamma)$ is a join of the line of intersection of $\Lambda_1 \ldots, \Lambda_{u-1}, \Lambda_{u+1}, \ldots, \Lambda_{\beta+\gamma}$ with the surface of the sphere. The angle between Π and any of the flats Λ_i forming the faces ($\beta+\gamma-2$-edges) of the simplex, as measured by the angle between the oriented line joining the centre of the sphere and the centroid of the base simplex $1\,2 \ldots (\beta+\gamma)$, in that sense, and the normal to Λ_i, oriented towards the simplex $1'2' \ldots \beta'(\beta+1) \ldots (\beta+\gamma)$, is

$$\cos^{-1}\left(-[\{1-(\beta+\gamma-1)\cos\theta\}/(\beta+\gamma)]^{\frac{1}{2}}\right) \quad \text{if } i=1,2,\ldots,\beta,$$

and $$\pi-\cos^{-1}\left(-[\{1-(\beta+\gamma-1)\cos\theta\}/(\beta+\gamma)]^{\frac{1}{2}}\right) \quad \text{if } i=\beta+1, \beta+2, \ldots, \beta+\gamma.$$

Finally, in the concluding section some applications of the results of this paper to the contents of certain non-simplicial regions in hyperspherical space, to restricted multivariate normal distributions and to the distributional theory of order statistics in samples generated by a Gaussian population will be discussed briefly.

2. Determination of univariate integrals for the generalised centroids

Consider the integral defined by

$$I \equiv I_{\beta,\beta+\gamma}(z;\alpha;\lambda)=\int_{-\infty}^{\infty} e^{z\xi}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\alpha\xi^2}\prod_{1}^{\beta}F(\xi+\lambda_i)\prod_{\beta+1}^{\beta+\gamma}[1-F(\xi+\lambda_i)]\,d\xi, \tag{5}$$

in which the λ_i are real, α is real and positive and the *function* $F(.)$ is defined by

$$F(\xi)=\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\xi} e^{-\frac{1}{2}u^2}\,du. \tag{5.1}$$

It will appear subsequently that $I_{\beta,\beta+\gamma}(z;\alpha;\mathbf{0})$ is a generating function for the geometrical moments of skew-regular $(\beta+\gamma-1)$-dimensional spherical simplices with angles

$$\cos^{-1}\{-1/(1+\alpha)\} \quad \text{and} \quad \pi-\cos^{-1}\{-1/(1+\alpha)\}.$$

In order to evaluate this generating function it will be necessary first to study the more inclusive integral in (5) for arbitrary $\boldsymbol{\lambda}$.

On differentiation in (5) under the integral sign with respect to the λ_i (a stop which is easily justified), we find, after some reduction,

$$\frac{\partial^{\beta+\gamma} I}{\partial\lambda_1\partial\lambda_2\ldots\partial\lambda_{\beta+\gamma}} = (-)^\gamma \int_{-\infty}^{\infty} e^{z\xi}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\alpha\xi^2}\prod_1^{\beta+\gamma}\frac{1}{(2\pi)^{\frac{1}{2}}}e^{-\frac{1}{2}(\xi+\lambda_i)^2}\,d\xi$$

$$= (-)^\gamma\frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}}\exp\left\{-\tfrac{1}{2}\left[\sum_1^{\beta+\gamma}\lambda_i^2-\frac{\left(\sum_1^{\beta+\gamma}\lambda_i\right)^2}{\alpha+\beta+\gamma}\right]-\frac{z\sum_1^{\beta+\gamma}\lambda_i}{\alpha+\beta+\gamma}+\frac{z^2}{2(\alpha+\beta+\gamma)}\right\}$$

$$\cdot\int_{-\infty}^{\infty}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\exp\left\{-\tfrac{1}{2}(\alpha+\beta+\gamma)\left[\xi+\frac{\sum_1^{\beta+\gamma}\lambda_i-\theta}{\alpha+\beta+\gamma}\right]^2\right\}d\xi,$$

having "completed the square" with respect to ξ. Integrating out with respect to the latter variable,

$$\frac{\partial^{\beta+\gamma} I}{\partial\lambda_1\ldots\partial\lambda_{\beta+\gamma}} = (-)^\gamma\frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}}\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}\exp\left\{-\tfrac{1}{2}Q(\lambda)-\frac{z\sum\lambda_i}{\alpha+\beta+\gamma}+\frac{z^2}{2(\alpha+\beta+\gamma)}\right\},\tag{6}$$

where $Q(\lambda)$ is the definite positive function in the $\lambda_i\,(i=1,2,\ldots,\beta+\gamma)$ defined by

$$Q(\lambda)=\sum\lambda_i^2-\frac{(\sum\lambda_i)^2}{\alpha+\beta+\gamma}.\tag{6.1}$$

Hence,

$$I=\frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}}\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}\int_{R_{\beta,\beta+\gamma}(\lambda)}\exp\left\{-\tfrac{1}{2}Q(\mathbf{t})-\frac{z\sum t_i}{\alpha+\beta+\gamma}+\frac{z^2}{2(\alpha+\beta+\gamma)}\right\}d\mathbf{t}$$
$$+\sum C_{\beta,\beta+\gamma;i}(z;\,\alpha;\,\lambda^{(i)}),\tag{7}$$

where $R_{\beta,\beta+\gamma}(\lambda)$ is the infinitely extended orthotope in $(\beta+\gamma)$-dimensional $\mathbf{t}$-space defined by $t_i\leqslant\lambda_i$ $(i=1,2,\ldots,\beta)$, $t_i\geqslant\lambda_i\,(i=\beta+1,\beta+2,\ldots,\beta+\gamma)$, while $\lambda^{(i)}$ is the $(\beta+\gamma-1)$-dimensional vector with components $\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_{\beta+\gamma}$, so that $C_{\beta,\beta+\gamma;i}(.)$ represents a function independent of λ_i. Equation (7) incorporates the complete solution of the partial differential equation (6). This follows on noting that $\sum C_{\beta,\beta+\gamma;i}(z;\,\alpha;\,\lambda^{(i)})$ is the general solution of the differential equation $\partial^{\beta+\gamma}I/\partial\lambda_1\ldots\ldots\partial\lambda_{\beta+\gamma}=0$. We now establish that $\sum C_{\beta,\beta+\gamma;i}(z;\,\alpha;\,\lambda^{(i)})$, is identically zero. For this purpose, note that by (5)

$$\lim_{\lambda_i\to-\infty} I=0,\qquad\text{if } i\in\{1,2,\ldots,\beta\},$$

$$\lim_{\lambda_i\to+\infty} I=0,\qquad\text{if } i\in\{\beta+1,\beta+2,\ldots,\beta+\gamma\}.$$

For a fixed i then, let $\lambda_i \to \mp\infty$, according as to whether $i \in \{1, 2, \ldots, \beta\}$ or $i \in \{\beta+1, \beta+2, \ldots, \beta+\gamma\}$. This yields the relationship

$$C_{\beta,\beta+\gamma;i}(z;\ \alpha;\ \boldsymbol{\lambda}^{(i)}) + \lim_{\lambda_i \to \mp\infty} \sum_{j \neq i} C_{\beta,\beta+\gamma;j}(z;\ \alpha;\ \boldsymbol{\lambda}^{(j)}) = 0,$$

whence $C_{\beta,\beta+\gamma;i}(z,\ \alpha;\ \boldsymbol{\lambda}^{(i)})$ is expressible as a sum of functions each involving $\beta+\gamma-2$ arguments selected from the set $\{\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{\beta+\gamma}\}$. It follows that $\sum_i C_{\beta,\beta+\gamma;i}(z;\ \alpha;\ \boldsymbol{\lambda}^{(i)})$ may be expressed in the form

$$\sum_i C_{\beta,\beta+\gamma;i}(z;\ \alpha;\ \boldsymbol{\lambda}^{(i)}) = \sum_i \sum_j C_{\beta,\beta+\gamma;i,j}(z;\ \alpha;\ \boldsymbol{\lambda}^{(i,j)}),$$

where summation is extended over all distinct pairs i, j, with $i<j$, selected from the set $\{1, 2, \ldots, \beta+\gamma\}$, and $\boldsymbol{\lambda}^{(i,j)}$ denotes the $(\beta+\gamma-2)$-dimensional vector $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{\beta+\gamma})$, so that $C_{\beta,\beta+\gamma;i,j}(.)$ represents a function independent of λ_i and λ_j. Similarly,

$$\sum_i C_{\beta,\beta+\gamma;i}(z,\ \alpha;\ \boldsymbol{\lambda}^{(i)}) = \sum_i \sum_j \sum_k C_{\beta,\beta+\gamma;i,j,k}(z;\ \alpha;\ \boldsymbol{\lambda}^{(i,j,k)}),$$

with $i<j<k$, summation being extended over all such triples from $\{1, 2, \ldots, \beta+\gamma\}$, and $\boldsymbol{\lambda}^{(i,j,k)}$ is the $(\beta+\gamma-3)$-dimensional vector obtained from $\boldsymbol{\lambda}$ by deleting λ_i, λ_j and λ_k. The latter result is obtained by allowing fixed *pairs* of the λ_i to approach $\mp\infty$ simultaneously, according as to the sets in which they are included. Proceeding in this way, the result

$$\sum_i C_{\beta,\beta+\gamma;i}(z;\ \alpha;\ \boldsymbol{\lambda}^{(i)}) = C_{\beta,\beta+\gamma;2,3,\ldots,\beta+\gamma}(z;\ \alpha;\ \lambda_1) + \cdots + C_{\beta,\beta+\gamma;1,2,\ldots,\beta+\gamma-1}(z;\ \alpha;\ \lambda_{\beta+\gamma})$$

is deduced. On allowing $(\beta+\gamma-1)$-uples of the λ_i to approach their limiting values simultaneously it is established in the same manner as in the preceding stages that $\sum_i C_{\beta,\beta+\gamma;i}(z;\ \alpha;\ \boldsymbol{\lambda}^{(i)})$ is independent of all the λ_i. Finally, let the $\lambda_i\ (i=1, 2, \ldots, \beta+\gamma)$ approach their limiting values simultaneously. This yields the required result

$$\sum_i C_{\beta,\beta+\gamma;i}(z;\ \alpha;\ \boldsymbol{\lambda}^{(i)}) \equiv 0.$$

Set now $\boldsymbol{\lambda}=\mathbf{0}$ in equation (7) and use (5). Then

$$\begin{aligned} &\exp\left\{-\frac{z^2}{2(\alpha+\beta+\gamma)}\right\} \cdot I_{\beta,\beta+\gamma}(z;\ \alpha;\ \mathbf{0}) \\ &= \exp\left\{-\frac{z^2}{2(\alpha+\beta+\gamma)}\right\} \cdot \int_{-\infty}^{\infty} e^{z\xi} \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\alpha\xi^2} [F(\xi)]^{\beta} [1-F(\xi)]^{\gamma}\, d\xi \qquad (8) \\ &= \frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}} \left(\frac{\alpha}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}} \int_{R_{\beta,\beta+\gamma}} \exp\left\{-\tfrac{1}{2}Q(\mathbf{t}) - \frac{z\sum t_i}{\alpha+\beta+\gamma}\right\} d\mathbf{t}, \end{aligned}$$

where $R_{\beta,\beta+\gamma} \equiv R_{\beta,\beta+\gamma}(\mathbf{0})$ is the orthant

$$t_i \leqslant 0 \quad (i=1,2,\ldots,\beta), \qquad t_i \geqslant 0 \quad (i=\beta+1,\beta+2,\ldots,\beta+\gamma).$$

On differentiating the last two terms in (8) s times with respect to z at $z=0$ and recalling that

$$\frac{d^s}{dz^s}(e^{-\frac{1}{2}z^2+z\xi})\bigg|_{z=0} = H_s(\xi),$$

where $\{H_s(\xi)\}$ is the sequence of orthogonal polynomials, normalized so that the coefficient of ξ^s in $H_s(\xi)$ is 1, relative to the weight function $(2\pi)^{-\frac{1}{2}}\exp(-\xi^2/2)$ for real ξ, we obtain

$$\begin{aligned}\frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}}\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}\int_{R_{\beta,\beta+\gamma}} e^{-\frac{1}{2}Q(\mathbf{t})}(\Sigma t_i)^s\,d\mathbf{t}\\ =(-)^s(\alpha+\beta+\gamma)^{-\frac{1}{2}s}\int_{-\infty}^{\infty} H_s((\alpha+\beta+\gamma)^{\frac{1}{2}}\xi)\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\alpha\xi^2}[F(\xi)]^\beta[1-F(\xi)]^\gamma\,d\xi \quad (s=0,1,2,\ldots).\end{aligned} \tag{9}$$

Let $\mathbf{B}=(b_{ij})$ be an arbitrary orthogonal matrix of size $(\beta+\gamma)\times(\beta+\gamma)$ such that $b_{\beta+\gamma,j}=(\beta+\gamma)^{-\frac{1}{2}}$ $(j=1,2,\ldots,\beta+\gamma)$. This matrix achieves the diagonalisation of Q. On setting

$$\mathbf{y}=\mathbf{B}\mathbf{t}$$

we obtain

$$\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}\frac{(\beta+\gamma)^{\frac{1}{2}s}}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}}\int_{R_{\beta,\beta+\gamma}} e^{-\frac{1}{2}Q(\mathbf{t})}(\Sigma t_i)^s\,d\mathbf{t}=\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}\frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}}\int_{\bar{R}_{\beta,\beta+\gamma}} e^{-\frac{1}{2}\mathbf{y}'\mathbf{D}\mathbf{y}}\,y^s_{\beta+\gamma}\,d\mathbf{y},$$

where $\mathbf{D}$ is the diagonal matrix with diagonal elements $1,1,\ldots,1,\alpha/(\alpha+\beta+\gamma)$, and $\bar{R}_{\beta,\beta+\gamma}$ is the image of $R_{\beta,\beta+\gamma}$ under the mapping $\mathbf{B}$. ($\bar{R}_{\beta,\beta+\gamma}$ is an infinitely extended orthotope in $\mathbf{y}$-space.) On application of the scaling transformation

$$\mathbf{x}=\mathbf{D}^{\frac{1}{2}}\mathbf{y}$$

we obtain further

$$\begin{aligned}\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}\frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}}\int_{R_{\beta,\beta+\gamma}} e^{-\frac{1}{2}Q(\mathbf{t})}(\Sigma t_i)^s\,d\mathbf{t}\\ =\left[\frac{\alpha}{(\alpha+\beta+\gamma)(\beta+\gamma)}\right]^{-\frac{1}{2}s}\frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}}\int_{\bar{\bar{R}}_{\beta,\beta+\gamma}} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\,x^s_{\beta+\gamma}\,d\mathbf{x},\end{aligned} \tag{10}$$

where $\overline{\overline{R}}_{\beta,\beta+\gamma}$ is the polyhedral half-cone in **x**-space defined by

$$\overline{\overline{R}}_{\beta,\beta+\gamma}:\begin{cases} L_i(\mathbf{x}) = \sum_{k=1}^{\beta+\gamma-1} b_{ki}\, x_i + \left[\dfrac{\alpha(\beta+\gamma)}{\alpha+\beta+\gamma}\right]^{-\frac{1}{2}} x_{\beta+\gamma} \leqslant 0 \quad (i=1,2,\ldots,\beta), \\ L_i(\mathbf{x}) = \sum_{k=1}^{\beta+\gamma-1} b_{ki}\, x_i + \left[\dfrac{\alpha(\beta+\gamma)}{\alpha+\beta+\gamma}\right]^{-\frac{1}{2}} x_{\beta+\gamma} \geqslant 0 \quad (i=\beta+1,\beta+2,\ldots,\beta+\gamma). \end{cases}$$

Refer to the flat $L_i(\mathbf{x})=0$ as Λ_i $(i=1,2,\ldots,\beta+\gamma)$, and the angle between Λ_i and Λ_j interior to $\overline{\overline{R}}_{\beta,\beta+\gamma}$ as (ij) $(j\neq i)$. Then, on using the orthogonality properties of **B**,

$$\cos(ij) = \pm\frac{\sum_{k=1}^{\beta+\gamma-1} b_{ki}\,b_{kj} + \dfrac{\alpha+\beta+\gamma}{\alpha(\beta+\gamma)}}{\sum_{k=1}^{\beta+\gamma-1} b_{ki}^2 + \dfrac{\alpha+\beta+\gamma}{\alpha(\beta+\gamma)}} = \pm\frac{-\dfrac{1}{\beta+\gamma} + \dfrac{\alpha+\beta+\gamma}{\alpha(\beta+\gamma)}}{\left(1-\dfrac{1}{\beta+\gamma}\right) + \dfrac{\alpha+\beta+\gamma}{\alpha(\beta+\gamma)}} = \pm\frac{1}{1+\alpha}, \tag{11}$$

according as to whether i and j are members of different sets $\{1,2,\ldots,\beta\}$, $\{\beta+1,\beta+2,\ldots,\beta+\gamma\}$ or are both members of the same set. Finally, on transforming to polar coordinates such that $x_{\beta+\gamma}=r\cos\phi$, and integrating out with respect to r,

$$\begin{aligned} &\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}} \frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}} \int_{R_{\beta,\beta+\gamma}} e^{-\frac{1}{2}Q(\mathbf{t})} (\textstyle\sum t_i)^s\, d\mathbf{t} \\ &= \left[\frac{\alpha}{(\alpha+\beta+\gamma)(\beta+\gamma)}\right]^{-\frac{1}{2}s} \frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}} \int_{\overline{\overline{R}}_{\beta,\beta+\gamma}} e^{-\frac{1}{2}r^2} r^{\beta+\gamma+s-1} \cos^s\phi\, dr\, d\omega \\ &= \left[\frac{\alpha}{(\alpha+\beta+\gamma)(\beta+\gamma)}\right]^{-\frac{1}{2}s} \frac{1}{(2\pi)^{\frac{1}{2}(\beta+\gamma)}} 2^{\frac{1}{2}(\beta+\gamma+s-2)}\, \Gamma\left(\frac{\beta+\gamma+s}{2}\right) \int_{1'2'\ldots\beta'(\beta+1)(\beta+\gamma)} \cos^s\phi\, d\omega, \end{aligned} \tag{12}$$

where $d\omega$ denotes an infinitesimal element on the surface of the unit sphere $\mathbf{x}'\mathbf{x}=1$, and $1'2'\ldots\beta'(\beta+1)(\beta+2)\ldots(\beta+\gamma)$ is a skew-regular simplex of the type discussed in the introductory section, with $\theta=\cos^{-1}\{-1/(1+\alpha)\}$. Combining equations (9) and (12),

$$\begin{aligned} &\int_{1'\ldots\beta'(\beta+1)\ldots(\beta+\gamma)} \cos^s\phi\, d\omega = (-)^s \left[\frac{\alpha}{2(\beta+\gamma)}\right]^{\frac{1}{2}s} \frac{\Gamma\left(\dfrac{\beta+\gamma}{2}\right)}{\Gamma\left(\dfrac{\beta+\gamma+s}{2}\right)} \\ &\cdot \frac{2\pi^{\frac{1}{2}(\beta+\gamma)}}{\Gamma\left(\dfrac{\beta+\gamma}{2}\right)} \int_{-\infty}^{\infty} H_s\left((\alpha+\beta+\gamma)^{\frac{1}{2}}\xi\right) \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\alpha\xi^2} [F(\xi)]^\beta [1-F(\xi)]^\gamma\, d\xi \quad (s=0,1,2,\ldots). \end{aligned} \tag{13}$$

It should be noted that $\cos\phi$ is the distance of a point on the surface of a unit sphere from the flat Π which is equally inclined to the flats $\Lambda_1, \ldots, \Lambda_{\beta+\gamma}$. Therefore, in the notation of the introductory section,

$$\int\limits_{1'\ldots\beta'(\beta+1)\ldots(\beta+\gamma)} \cos^s\phi\, d\omega = p'_{\beta,\beta+\gamma;s}(\theta), \tag{13.1}$$

the non-normalized geometrical moments of the simplex $1' \ldots \beta' (\beta+1) \ldots (\beta+\gamma)$ with respect to Π. Equation (13) is then at the same time a formula for these moments in terms of a univariate integral involving simple functions. The normalised geometrical moments are given by

$$p_{\beta,\beta+\gamma;s}(\theta) = \frac{p'_{\beta,\beta+\gamma;s}(\theta)}{p'_{\beta,\beta+\gamma;0}(\theta)}$$

$$= (-)^s \left[\frac{\alpha}{2(\beta+\gamma)}\right]^{\frac{1}{2}s} \frac{\Gamma\left(\frac{\beta+\gamma}{2}\right)}{\Gamma\left(\frac{\beta+\gamma+s}{2}\right)} \frac{\int\limits_{-\infty}^{\infty} H_s((\alpha+\beta+\gamma)^{\frac{1}{2}}\xi)\, e^{-\frac{1}{2}\xi^2}[F(\xi)]^\beta[1-F(\xi)]^\gamma d\xi}{\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha\xi^2}[F(\xi)]^\beta[1-F(\xi)]^\gamma d\xi} \tag{14}$$

$$(s=0, 1, 2, \ldots).$$

We remark that equations (13), (13.1) and (14) give simultaneously simple integral forms for $q'_{\beta,\beta+\gamma;s}(\theta)$ and $q_{\beta,\beta+\gamma;s}(\theta)$, defined in section 1, since the latter functions are merely multiples of $p'_{\beta,\beta+\gamma;s}(\theta)$ and $p_{\beta,\beta+\gamma;s}(\theta)$, respectively. In fact,

$$q'_{\beta,\beta+\gamma;s}(\theta) = \int\limits_0^1 \int\limits_{1'\ldots\beta'(\beta+1)\ldots(\beta+\gamma)} r^s \cos^s\phi\, r^{\beta+\gamma-1}\, dr\, d\omega = \frac{1}{\beta+\gamma+s} p'_{\beta,\beta+\gamma;s}(\theta), \tag{15}$$

and

$$q_{\beta,\beta+\gamma;s}(\theta) = \frac{q'_{\beta,\beta+\gamma;s}(\theta)}{q'_{\beta,\beta+\gamma;0}(\theta)} = \frac{\beta+\gamma}{\beta+\gamma+s} \frac{p'_{\beta,\beta+\gamma;s}(\theta)}{p'_{\beta,\beta+\gamma;0}(\theta)} = \frac{\beta+\gamma}{\beta+\gamma+s} p_{\beta,\beta+\gamma;s}(\theta). \tag{16}$$

3. Relationship of geometrical moments of simplex to contents of edges

Define $\mathbf{A}$ as the shift operator which increases α by 1, and $\mathbf{B}$ and $\mathbf{\Gamma}$ as the operators which decrease β and γ, respectively, by 1. Further, let

$$\mathbf{P} \equiv \frac{1}{\alpha}\mathbf{A},$$

and

$$\mathbf{L} \equiv \beta\mathbf{B} - \gamma\mathbf{\Gamma}.$$

Then $$\mathbf{P}^m \equiv \frac{1}{\alpha^{-(m)}} \mathbf{A}^m \qquad (m = 0, 1, 2, \ldots),$$

and $$\mathbf{L}^m \equiv \sum_{j=0}^{m} (-)^j \binom{m}{j} \gamma^{(j)} \beta^{(m-j)} \mathbf{\Gamma}^j \mathbf{B}^{m-j} \qquad (m = 0, 1, 2, \ldots),$$

where $u^{-(m)}$ and $u^{(m)}$ denote the ascending and descending factorials, respectively, of degree m in u,

$$\left.\begin{aligned} &u^{(-0)} = 1, \\ &u^{-(m)} = u(u+1) \ldots (u+m-1) \qquad (m = 1, 2, \ldots), \end{aligned}\right\}$$

$$\left.\begin{aligned} &u^{(0)} = 1, \\ &u^{(m)} = u(u-1) \ldots (u-m+1) \qquad (m = 1, 2, \ldots). \end{aligned}\right\}$$

In passing, observe that $\mathbf{L}^m$ is an annihilation operator when $m > \beta + \gamma$.

Define for non-negative integral s, β and γ, and for real and positive α

$$\Psi(s; \alpha; \beta, \gamma) = \frac{1}{(2\pi)^{\frac{1}{2}\alpha}} \int_{-\infty}^{\infty} \xi^s e^{-\frac{1}{2}\alpha\xi^2} [F(\xi)]^\beta [1 - F(\xi)]^\gamma \, d\xi,$$

where $F(.)$ is defined in (5.1). Integration by parts gives

$$\Psi(s; \alpha; \beta, \gamma) = \frac{s-1}{\alpha} \Psi(s-2; \alpha; \beta, \gamma) + \mathbf{P}\,\mathbf{L}\,\Psi(s-1; \alpha; \beta, \gamma) \qquad (s = 1, 2, \ldots). \tag{17}$$

The latter relationship may be used to prove inductively the reduction formulae

$$\left.\begin{aligned} \Psi(2k+1; \alpha; \beta, \gamma) &= \sum_{i=0}^{k} a_{2k+1, 2i+1}(\alpha)\, (\mathbf{P}\,\mathbf{L})^{2i+1}\, \Psi(0; \alpha; \beta, \gamma), \\ \Psi(2k; \alpha; \beta, \gamma) &= \sum_{i=0}^{k} a_{2k, 2i}(\alpha)\, (\mathbf{P}\,\mathbf{L})^{2i}\, \Psi(0; \alpha; \beta, \gamma), \end{aligned}\right\} (k = 0, 1, 2, \ldots), \tag{18}$$

where the a's are rational functions satisfying the recursion relationships

$$a_{2k+2, 2i}(\alpha) = \frac{2k+1}{\alpha} a_{2k, 2i}(\alpha) + a_{2k+1, 2i-1}(\alpha+1) \qquad (i = 1, 2, \ldots k),$$

$$a_{2k+3, 2i+1}(\alpha) = \frac{2k+2}{\alpha} a_{2k+1, 2i+1}(\alpha) + a_{2k+2, 2i}(\alpha+1) \qquad (i = 0, 1, 2, \ldots k),$$

and $$a_{2k+2, 0}(\alpha) = 1.3.5 \ldots (2k+1)/\alpha^{k+1},$$

$$a_{2k+2, 2k+2}(\alpha) = 1 = a_{2k+3, 2k+3}(\alpha).$$

The above reduction formulae for the Ψ-functions have been derived elsewhere [6] together with explicit expressions for the coefficients $a_{p,q}(\alpha)$ as far as $a_{10,10}(\alpha)$. For convenience, and in order to render this paper completely self-contained, we state here the explicit forms of the coefficients as far as $a_{6,6}(\alpha)$:

$$a_{0,0}(\alpha)=1.$$

$$a_{1,1}(\alpha)=1.$$

$$a_{2,0}(\alpha)=\frac{1}{\alpha},\quad a_{2,2}(\alpha)=1.$$

$$a_{3,1}(\alpha)=\frac{2}{\alpha}+\frac{1}{\alpha+1},\quad a_{3,3}(\alpha)=1.$$

$$a_{4,0}(\alpha)=\frac{3}{\alpha^2},\quad a_{4,2}(\alpha)=\frac{3}{\alpha}+\frac{2}{\alpha+1}+\frac{1}{\alpha+2},\quad a_{4,4}(\alpha)=1.$$

$$a_{5,1}(\alpha)=\frac{4}{\alpha}\left(\frac{2}{\alpha}+\frac{1}{\alpha+1}\right)+\frac{3}{(\alpha+1)^2},\quad a_{5,3}(\alpha)=\frac{4}{\alpha}+\frac{3}{\alpha+1}+\frac{2}{\alpha+2}+\frac{1}{\alpha+3},\quad a_{5,5}(\alpha)=1.$$

$$a_{6,0}(\alpha)=\frac{15}{\alpha^3},\quad a_{6,2}(\alpha)=\frac{5}{\alpha}\left(\frac{3}{\alpha}+\frac{2}{\alpha+1}+\frac{1}{\alpha+2}\right)+\frac{4}{\alpha+1}\left(\frac{2}{\alpha+1}+\frac{1}{\alpha+2}\right)+\frac{3}{(\alpha+2)^2}.$$

$$a_{6,4}(\alpha)=\frac{5}{\alpha}+\frac{4}{\alpha+1}+\frac{3}{\alpha+2}+\frac{2}{\alpha+3}+\frac{1}{\alpha+4},\quad a_{6,6}(\alpha)=1.$$

Let $V_{\beta,\beta+\gamma}(\theta)$ denote the relative content of the simplex

$$1'2'\ldots\beta'(\beta+1)(\beta+2)\ldots(\beta+\gamma),\ \text{i.e.}$$

$$V_{\beta,\beta+\gamma}(\theta)=\int\limits_{1'\ldots\beta'(\beta+1)\ldots(\beta+\gamma)} d\omega\Bigg/\frac{2\pi^{\frac{1}{2}(\beta+\gamma)}}{\Gamma\{\frac{1}{2}(\beta+\gamma)\}}.$$

Then, by (13), on setting $s=0$,

$$V_{\beta,\beta+\gamma}(\theta)=\int_{-\infty}^{\infty}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}e^{-\frac{1}{2}\alpha\xi^2}[F(\xi)]^{\beta}[1-F(\xi)]^{\gamma}\,d\xi=\alpha^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(\alpha-1)}\Psi(0;\alpha;\beta,\gamma). \qquad (19)$$

On substitution for $\Psi(0;\alpha;\beta,\gamma)$ in (18) by means of (19),

$$\Psi(2k+1;\alpha;\beta,\gamma)$$
$$=\sum_{i=0}^{k}a_{2k+1,2i+1}(\alpha)\frac{1}{(2\pi)^{i+\frac{1}{2}\alpha}(\alpha+2i+1)^{\frac{1}{2}}\alpha^{-(2i+1)}}\mathrm{L}^{2i+1}V_{\beta,\beta+\gamma}\left(\cos^{-1}\left\{-\frac{1}{\alpha+2i+2}\right\}\right) \qquad (20)$$
$$(k=0,1,2,\ldots),$$

$$\Psi(2k;\,\alpha;\,\beta,\gamma)$$
$$=\sum_{i=0}^{k} a_{2k,2i}(\alpha)\,\frac{1}{(2\pi)^{i+\frac{1}{2}(\alpha-1)}(\alpha+2i)^{\frac{1}{2}}\,\alpha^{-2i}}\,\mathbf{L}^{2i}\,V_{\beta,\beta+\gamma}\left(\cos^{-1}\left\{-\frac{1}{\alpha+2i+1}\right\}\right) \tag{21}$$
$$(k=0,\,1,\,2,\,\ldots).$$

Equations (20) and (21) may now be used in (14) to derive the required relationship. For this purpose, let

$$H_{2m+1}(\xi)=\sum_{j=0}^{m} c_{2m+1,2j+1}\,\xi^{2j+1}\qquad (m=0,\,1,\,2,\,\ldots),$$

$$H_{2m}(\xi)=\sum_{j=0}^{m} c_{2m,2j}\,\xi^{2j}\qquad (m=0,\,1,\,2,\,\ldots),$$

$$c_{2m+1,2j+1}=(-)^{m-j}\frac{(2m+1)^{(2m-2j)}}{2^{m-j}(m-j)!},\quad c_{2m,2j}=(-)^{m-j}\frac{(2m)^{(2m-2j)}}{2^{m-j}(m-j)!}.$$

We then obtain

$$p_{\beta,\beta+\gamma;\,2m+1}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right)=-\left(\frac{\alpha}{2(\beta+\gamma)}\right)^{m+1}\left\{\frac{(\beta+\gamma)(\alpha+\beta+\gamma)}{\pi}\right\}^{\frac{1}{2}}\frac{\Gamma\left(\frac{\beta+\gamma}{2}\right)}{\Gamma\left(\frac{\beta+\gamma+1}{2}+m\right)}$$
$$\cdot\sum_{i=0}^{\min\{m,\,[\frac{1}{2}(\beta+\gamma-1)]\}}\frac{T_{2m+1,i}(\alpha;\,\alpha+\beta+\gamma)\,\mathbf{L}^{2i+1}\,V_{\beta,\beta+\gamma}\left(\cos^{-1}\left\{-\frac{1}{2i+2+\alpha}\right\}\right)}{(2\pi)^{i}(2i+1+\alpha)^{\frac{1}{2}}\,\alpha(\alpha+1)\ldots(\alpha+2i)\,V_{\beta,\beta+\gamma}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right)} \tag{22}$$
$$(m=0,\,1,\,2,\,\ldots),$$

and

$$p_{\beta,\beta+\gamma;\,2m}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right)=\left(\frac{\alpha}{2(\beta+\gamma)}\right)^{m}\sqrt{\alpha}\,\frac{\Gamma\left(\frac{\beta+\gamma}{2}\right)}{\Gamma\left(\frac{\beta+\gamma}{2}+m\right)}$$
$$\cdot\sum_{i=0}^{\min\{m,\,[\frac{1}{2}(\beta+\gamma)]\}}\frac{T_{2m,i}(\alpha;\,\alpha+\beta+\gamma)\,\mathbf{L}^{2i}\,V_{\beta,\beta+\gamma}\left(\cos^{-1}\left\{-\frac{1}{2i+1+\alpha}\right\}\right)}{(2\pi)^{i}(2i+\alpha)^{\frac{1}{2}}\,\alpha(\alpha+1)\ldots(\alpha+2i-1)\,V_{\beta,\beta+\gamma}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right)} \tag{23}$$

where the T-functions are polynomials defined by

$$T_{2m+1,i}(\alpha;\,x)=\sum_{j=i}^{m} c_{2m+1,2j+1}\,a_{2j+1,2i+1}(\alpha)\,x^{j}\qquad (m=0,\,1,\,2,\,\ldots),$$

$$T_{2m,i}(\alpha;\,x)=\sum_{j=i}^{m} c_{2m,2j}\,a_{2j,2i}(\alpha)\,x^{j}\qquad (m=0,\,1,\,2,\,\ldots).$$

(The upper limits of summation in (22) and (23) follow from the fact that $\mathbf{L}^r = 0$, for all positive integral $r > \beta + \gamma$.) Equations (22) and (23) provide the fundamental relationships sought after.

In particular, for $\gamma = 0$, the moments of a regular $(\beta - 1)$-dimensional spherical simplex are given by

$$p_{\beta,\beta;\,2m+1}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right) = -\left(\frac{\alpha}{2\beta}\right)^{m+1}\left\{\frac{\beta(\alpha+\beta)}{\pi}\right\}^{\frac{1}{2}} \frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta+1}{2}+m\right)}$$

$$\sum_{i=0}^{\min\{m,\,[\frac{1}{2}(\beta-1)]\}} \frac{T_{2m+1,\,i}(\alpha;\,\alpha+\beta)\,\beta(\beta-1)\ldots(\beta-2i)\,V_{\beta-2i-1,\,\beta-2i-1}\left(\cos^{-1}\left\{-\frac{1}{2i+2+\alpha}\right\}\right)}{(2\pi)^i(2i+1+\alpha)^{\frac{1}{2}}\,\alpha(\alpha+1)\ldots(\alpha+2i)\,V_{\beta,\,\beta}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right)} \tag{24}$$

$$p_{\beta,\beta;\,2m}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right) = \left(\frac{\alpha}{2\beta}\right)^{m}\sqrt{\alpha}\,\frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}+m\right)}$$

$$\cdot\sum_{i=0}^{\min\{m,\,[\frac{1}{2}\beta]\}} \frac{T_{2m,\,i}(\alpha;\,\alpha+\beta)\,\beta(\beta-1)\ldots(\beta-2i+1)\,V_{\beta-2i,\,\beta-2i}\left(\cos^{-1}\left\{-\frac{1}{2i+1+\alpha}\right\}\right)}{(2\pi)^i(2i+\alpha)^{\frac{1}{2}}\,\alpha(\alpha+1)\ldots(\alpha+2i-1)\,V_{\beta,\,\beta}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right)} \tag{25}$$

$$(m = 0,\ 1,\ 2,\ \ldots).$$

The general spherical simplices of dimensionality 0, 1, 2, 3, ... are respectively a point, an arc of a circle, a spherical triangle, a spherical tetrahedron, etc. The content of the general spherical simplex of arbitrary dimensionality was first investigated in a series of classic and remarkable papers by Schläfli [7] who derived a fundamental differential recursion relationship for the content. (For further work on this problem see Hoppe [4], Richmond [5], Coxeter [1], [2], Sommerville [8], Ruben [6], and Van der Vaart [9], [10].) The content of a spherical triangle on the surface of a unit sphere is given by its spherical excess, but for simplices of dimensionalities greater than 2 the contents cannot be expressed in terms of elementary functions. However, for the special case of skew-regular simplices the following simple recursion relationships may be derived using Schläfli's relationship (Ruben, [6])

$$\frac{d}{d\theta}V_{\beta,\,\beta+\gamma}(\theta) = \frac{1}{4\pi}\mathbf{L}^2\,V_{\beta,\,\beta+\gamma}\left(\cos^{-1}\left\{\frac{\cos\theta}{1-2\cos\theta}\right\}\right). \tag{26}$$

Tables of the contents of regular simplices for dimensionalities up to and including 48 and for various angles θ are available in [6].

As a simple application of our formulae, the distance of the centroid of a regular simplex from the centre of the corresponding sphere is

$$p_{\beta,\beta;1}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right)=-\frac{1}{2}\left\{\frac{\beta(\alpha+\beta)}{(1+\alpha)\pi}\right\}^{\frac{1}{2}}\frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta+1}{2}\right)}\frac{V_{\beta-1,\beta-1}\left(\cos^{-1}\left\{-\frac{1}{2+\alpha}\right\}\right)}{V_{\beta,\beta}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right)} \tag{27}$$

$$(\beta=1,\,2,\,\ldots).$$

Thus, trivially, (1)

$$p_{3,3;1}\left(\cos^{-1}\left\{-\frac{1}{1+\alpha}\right\}\right)=-\frac{1}{2}\left\{\frac{3(\alpha+3)}{1+\alpha}\right\}^{\frac{1}{2}}\frac{\cos^{-1}\left(-\frac{1}{2+\alpha}\right)}{3\cos^{-1}\left(-\frac{1}{1+\alpha}\right)-\pi}.$$

Since, for arbitrary θ,

$$\begin{aligned}\mathbf{L}^k V_{\beta,\beta+\gamma}(\theta)&=\sum_{j=0}^{k}(-)^j\binom{m}{j}\mathbf{B}^{m-j}\mathbf{\Gamma}^j V_{\beta,\beta+\gamma}(\theta)\\&=\sum_{j=0}^{k}(-)^j\binom{m}{j}\beta^{(m-j)}\gamma^{(j)}V_{\beta-(m-j),\beta+\gamma-m}(\theta),\end{aligned}$$

the numerators in equations (22) and (23) are linear functions of contents of skew-regular simplices of various dimensionalities. However, in order to provide a more significant geometrical interpretation of (22) and (23) we shall now demonstrate that these simplices are, in fact, the edges of various dimensionalities of the simplex $1'\,2'\ldots\beta'\,(\beta+1)\,(\beta+2)\ldots(\beta+\gamma)$. For this purpose, the following subsidiary two theorems must first be proved:

THEOREM I. *Let* $1'\,2'\ldots\beta'\,(\beta+1)\,(\beta+2)\ldots(\beta+\gamma)$ *be a skew-regular simplex in* $(\beta+\gamma-1)$*-dimensional spherical space, demarcated by a sheaf of* $\beta+\gamma$ $(\beta+\gamma-1)$*-flats,* $\Lambda_1,\ldots,\Lambda_{\beta+\gamma}$*, with its* $(\beta+\gamma)(\beta+\gamma-1)/2$ *angles given by*

$$\left.\begin{aligned}(\sigma\tau)&=\theta\ ,\ \sigma,\tau\text{ both accented or both unaccented vertices }(\sigma\neq\tau),\\&=\pi-\theta,\text{ one vertex accented and the other unaccented.}\end{aligned}\right\}$$

(1) "Trivially" from the point of view of our formulae, but even this formula is by no means easy to derive by direct integration.

Let $\{r_1', r_2', \ldots, r_{k-j}'\}$ *and* $\{s_1, s_2, \ldots, s_j\}$ *be arbitrary subsets of* $\{1', 2', \ldots, \beta'\}$ *and* $\{\beta+1, \beta+2, \ldots, \beta+\gamma\}$, *respectively, and let the complements of the latter subsets relative to the sets from which they are selected be* $\{t_1', t_2', \ldots, t_{\beta-(k-j)}'\}$ *and* $\{u_1, u_2, \ldots, u_{\gamma-j}\}$. *Then the* $(\beta+\gamma-1-k)$*-dimensional edge*

$$t_1' \ldots t_{\beta-(k-j)}' u_1 \ldots u_{\gamma-j} \equiv \overline{r_1' \ldots r_{k-j}' s_1 \ldots s_j},$$

formed jointly by the k *flats* $\Lambda_{r_1}, \ldots, \Lambda_{r_{k-j}}, \Lambda_{s_1}, \ldots, \Lambda_{s_j}$, *is itself a skew-regular simplex with its* $(\beta+\gamma-k)(\beta+\gamma-k-1)/2$ *angles given by*

$$\left.\begin{aligned}\overline{(r_1' \ldots r_{k-j}' s_1 \ldots s_j\, p\, q)} &= \phi_k(\theta), \ p, q \text{ both accented or both unaccented } (p \neq q),\\ &= \pi - \phi_k(\theta), \text{ one vertex accented and the other unaccented,}\end{aligned}\right\} \quad (28)$$

where p, q, *are to be chosen from the set* $\{t_1', \ldots, t_{\beta-(k-j)}', u_1, \ldots, u_{\gamma-j}\}$, *and* $\phi_k(\theta) = \cos^{-1}\{\cos\theta/(1-k\cos\theta)\}$ $(k=0, 1, 2, \ldots, \beta+\gamma)$.

Proof. Schläfli has shown in [7] that the angles of the edges of any spherical simplex may be expressed in terms of the determinants of certain bordered matrices with the angle-cosines of the simplex as elements. In fact, quite generally,

$$\cos \overline{(r_1' \ldots s_j\, p\, q)} = -\frac{\Delta\begin{pmatrix} p & r_1' \ldots s_j \\ q & \end{pmatrix}}{\{\Delta(p\, r_1' \ldots s_j)\, \Delta(q\, r_1' \ldots s_j)\}^{\frac{1}{2}}}, \tag{29}$$

with

$$\Delta\begin{pmatrix} p & r_1' \ldots s_j \\ q & \end{pmatrix} = \begin{vmatrix} -\cos(pq) & -\cos(p r_1') & \cdots & -\cos(p r_{k-j}') & -\cos(p s_1) & \cdots & -\cos(p s_j) \\ -\cos(r_1' q) & 1 & \cdots & -\cos(r_1' r_{k-j}') & -\cos(r_1' s_1) & \cdots & -\cos(r_1' s_j) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\cos(r_{k-j}' q) & -\cos(r_{k-j}' r_1') & \cdots & 1 & -\cos(r_{k-j}' s_1) & \cdots & -\cos(r_{k-j}' s_j) \\ -\cos(s_1 q) & -\cos(s_1 r_1') & \cdots & -\cos(s_1 r_{k-j}') & 1 & \cdots & -\cos(s_1 s_j) \\ -\cos(s_2 q) & -\cos(s_2 r_1') & \cdots & -\cos(s_2 r_{k-j}') & -\cos(s_2 s_1) & 1 & -\cos(s_2 s_j) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\cos(s_j q) & -\cos(s_j r_1') & \cdots & -\cos(s_j r_{k-j}') & -\cos(s_j s_1) & \cdots & 1 \end{vmatrix}, \tag{29.1}$$

$$\Delta\,(p\;r_1'\;\ldots\;s_j)=\begin{vmatrix} 1 & -\cos(p\,r_1') & \cdots & -\cos(p\,r_{k-j}') & -\cos(p\,s_1) & \cdots & -\cos(p\,s_j) \\ -\cos(r_1'\,p) & 1 & \cdots & -\cos(r_1'\,r_{k-j}') & -\cos(r_1'\,s_1) & \cdots & -\cos(r_1'\,s_j) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\cos(r_{k-j}'\,p) & -\cos(r_{k-j}'\,r_1') & \cdots & 1 & -\cos(r_{k-j}'\,s_1) & \cdots & -\cos(r_{k-j}'\,s_j) \\ -\cos(s_1\,p) & -\cos(s_1\,r_1') & \cdots & -\cos(s_1\,r_{k-j}') & 1 & \cdots & -\cos(s_1\,s_j) \\ -\cos(s_2\,p) & -\cos(s_2\,r_1') & \cdots & -\cos(s_2\,r_{k-j}') & -\cos(s_2\,s_1) & 1 & -\cos(s_2\,s_j) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\cos(s_j\,p) & -\cos(s_j\,r_1') & \cdots & -\cos(s_j\,r_{k-j}') & -\cos(s_j\,s_1) & \cdots & 1 \end{vmatrix}. \tag{29.2}$$

Hence, in our case,

$$\Delta\begin{pmatrix} p & r_1'\cdots s_j \\ q & \end{pmatrix}=\begin{vmatrix} -\cos(pq) & \mu & \mu & \cdots & \mu & -\mu & -\mu & \cdots & -\mu \\ \nu & 1 & -\cos\theta & \cdots & -\cos\theta & \cos\theta & \cos\theta & \cdots & \cos\theta \\ \nu & -\cos\theta & 1 & \cdots & -\cos\theta & \cos\theta & \cos\theta & \cdots & \cos\theta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \nu & -\cos\theta & -\cos\theta & \cdots & 1 & \cos\theta & \cos\theta & \cdots & \cos\theta \\ -\nu & \cos\theta & \cos\theta & \cdots & \cos\theta & 1 & -\cos\theta & \cdots & -\cos\theta \\ -\nu & \cos\theta & \cos\theta & \cdots & \cos\theta & -\cos\theta & 1 & \cdots & -\cos\theta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\nu & \cos\theta & \cos\theta & \cdots & \cos\theta & -\cos\theta & -\cos\theta & \cdots & 1 \end{vmatrix}, \tag{30}$$

$$\Delta\,(p\;r_1'\cdots s_j)=\begin{vmatrix} 1 & \mu & \mu & \cdots & \mu & -\mu & -\mu & \cdots & -\mu \\ \mu & 1 & -\cos\theta & \cdots & -\cos\theta & \cos\theta & \cos\theta & \cdots & \cos\theta \\ \mu & -\cos\theta & 1 & \cdots & -\cos\theta & \cos\theta & \cos\theta & \cdots & \cos\theta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu & -\cos\theta & -\cos\theta & \cdots & 1 & \cos\theta & \cos\theta & \cdots & \cos\theta \\ -\mu & \cos\theta & \cos\theta & \cdots & \cos\theta & 1 & -\cos\theta & \cdots & -\cos\theta \\ -\mu & \cos\theta & \cos\theta & \cdots & \cos\theta & -\cos\theta & 1 & \cdots & -\cos\theta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\mu & \cos\theta & \cos\theta & \cdots & \cos\theta & -\cos\theta & -\cos\theta & \cdots & 1 \end{vmatrix}, \tag{30.1}$$

where
$$\begin{aligned}\mu &= -\cos\theta,\ p\in\{1', 2', \ldots, \beta'\},\\ &= \cos\theta,\ p\in\{\beta+1, \beta+2, \ldots, \beta+\gamma\},\\ \nu &= -\cos\theta,\ q\in\{1', 2', \ldots, \beta'\},\\ &= \cos\theta,\ q\in\{\beta+1, \beta+2, \ldots, \beta+\gamma\}.\end{aligned}$$

It may be readily shown that the inverse of the $k\times k$ submatrix is (30) which is bordered by the first row and first column is

$$\begin{pmatrix}\mathbf{U} & \mathbf{W}\\ \mathbf{W}' & \mathbf{V}\end{pmatrix},$$

where $\mathbf{U}$ and $\mathbf{V}$ are square matrices of size $(k-j)\times(k-j)$ and $j\times j$, respectively, having all their diagonal elements equal to $\{1-(k-2)\cos\theta\}/(1+\cos\theta)\{1-(k-1)\cos\theta\}$ and all their off-diagonal elements equal to $\cos\theta/(1+\cos\theta)\{1-(k-1)\cos\theta\}$, while $\mathbf{W}$ is a matrix of size $(k-j)\times j$ having all its elements equal to $-\cos\theta/(1+\cos\theta)\{1-(k-1)\cos\theta\}$. ($\mathbf{W}'$ is the transpose of $\mathbf{W}$.) The corresponding cofactors in the submatrix are the products of these values and Δ_0, where Δ_0 is the value of the determinant of the submatrix. On using Cauchy's expansion for bordered determinants in terms of the elements of the bordering row and column and the corresponding cofactors,

$$\begin{aligned}\Delta\begin{pmatrix}p & r_1' \ldots s_j\\ q & \end{pmatrix} = &-\cos(pq)\,\Delta_0 - \Big[(k-j)\frac{1-(k-2)\cos\theta}{(1+\cos\theta)\{1-(k-1)\cos\theta\}}\\ &+(k-j)(k-j-1)\frac{\cos\theta}{(1+\cos\theta)\{1-(k-1)\cos\theta\}} + j\frac{1-(k-2)\cos\theta}{(1+\cos\theta)\{1-(k-1)\cos\theta\}}\\ &+j(j-1)\frac{\cos\theta}{(1+\cos\theta)\{1-(k-1)\cos\theta\}}\\ &+2(k-j)\frac{\cos\theta}{(1+\cos\theta)\{1-(k-1)\cos\theta\}}\Big]\mu\nu\Delta_0\\ = &-\left\{\cos(pq)+\frac{k\mu\nu}{1-(k-1)\cos\theta}\right\}\Delta_0.\end{aligned} \tag{31}$$

$\Delta(p\, r_1', \ldots, s_j)$ is obtained from $\Delta\begin{pmatrix}p & r_1' \ldots s_j\\ q & \end{pmatrix}$ by replacing $\cos(pq)$ by -1 and setting $\nu=\mu$. Similarly, $\Delta(q\, r_1' \ldots s_j)$ is obtained from $\Delta\begin{pmatrix}p & r_1' \ldots s_j\\ q & \end{pmatrix}$ by replacing $\cos(pq)$ by -1 and setting $\mu=\nu$. Hence, by (29) and (31),

$$\cos(\overline{r_1' \ldots s_j\, p\, q}) = \frac{\{1-(k-1)\cos\theta\}\cos(pq)+k\mu\nu}{[\{1-(k-1)\cos\theta-k\mu^2\}\{1-(k-1)\cos\theta-k\nu^2\}]^{\frac{1}{2}}}. \tag{32}$$

There are four possible cases, corresponding to the nature of the sets (accented or unaccented) in which p and q are contained. Examination of each of these cases yields on substitution in (32),

$$\left.\begin{aligned}\cos\overline{(r'_1 \dots s_j\, p\, q)} &= \frac{\cos\theta}{1-k\cos\theta}, \quad p, q \text{ belong to the same set of the} \\ &\qquad\qquad \text{sets } \{t'_1, \dots, t'_{\beta-(k-j)}\},\ \{u_1, \dots, u_{\gamma-j}\}, \\ &= -\frac{\cos\theta}{1-k\cos\theta}, \quad p, q \text{ belong to different sets.}\end{aligned}\right\} \tag{33}$$

Thus, $t'_1 \dots t'_{\beta-(k-j)}\, u_1 \dots u_{\gamma-j}$ is a $(\beta+\gamma-1-k)$-dimensional simplex with angles $\phi_k(\theta)$ and $\pi-\phi_k(\theta)$, as given in (28) of Theorem I. This completes the proof.

In view of Theorem I, the relative content of the edge $t'_1 \dots t'_{\beta-(k-j)}\, u_1 \dots u_{\gamma-j}$ is $V_{\beta-(k-j),\, \beta+\gamma-k}\,(\phi_k(\theta))$.

THEOREM II. *Let each edge $t'_1 \dots t'_{\beta-(k-j)}\, u_1 \dots u_{\gamma-j}$ be assigned a weight $+1$ or -1 according as to whether j is even or odd. Then the sum of the weighted relative contents of all $(\beta+\gamma-1-k)$-edges of the skew-regular spherical simplex $1'\,2' \dots \beta'\,(\beta+1)\,(\beta+2) \dots (\beta+\gamma)$ of Theorem I is equal to*

$$\mathbf{L}^k\, V_{\beta,\, \beta+\gamma}\,(\phi_k(\theta))/k!\,.$$

Proof. The weighted relative content of the edge $t'_1 \dots t'_{\beta-(k-j)}\, u_1 \dots u_{\gamma-j}$ is $(-)^j\, V_{\beta-(k-j),\, \beta+\gamma-k}\,(\phi_k(\theta))$, by Theorem I. Now

$$\frac{\mathbf{L}^k\, V_{\beta,\, \beta+\gamma}\,(\phi_k(\theta))}{k!} = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\beta^{(k-j)}\gamma^{(j)}\cdot(-)^j\, V_{\beta-(k-j),\, \beta+\gamma-k}\,(\phi_k(\theta)).$$

Again,

$$\frac{1}{k!}\binom{k}{j}\beta^{(k-j)}\gamma^{(j)} = \binom{\beta}{k-j}\binom{\gamma}{j}$$

is the number of distinct ways in which k flats may be selected from the $\beta+\gamma$ flats $\Lambda_1, \dots, \Lambda_{\beta+\gamma}$, such that $k-j$ flats are from the set $\{\Lambda_1, \dots, \Lambda_\beta\}$ and j from the set $\{\Lambda_{\beta+1}, \dots, \Lambda_{\beta+\gamma}\}$. Since each such selection corresponds to a $(\beta+\gamma-1-k)$-edge of signed (weighted) relative content $(-)^j\, V_{\beta-(k-j),\, \beta+\gamma-k}\,(\phi_k(\theta))$, the total signed relative content of all $(\beta+\gamma-1-k)$-edges is $\mathbf{L}^k\, V_{\beta,\, \beta+\gamma}\,(\phi_k(\theta))/k!$, as required.

Reverting now to equations (22) and (23), it appears through the last two theorems that *the r-th geometrical moment of a skew-regular spherical simplex is a linear function of the contents of all edges having dimensionalities $0, 2, \dots, r$ or $1, 3, \dots, r$*

less than that of the simplex, according as to whether r is even or odd. ($V_{h,k}(\theta)$ is here interpreted as zero when $k<h$).

We conclude this section by discussing briefly the range of validity of the formulae (22) and (23). These formulae have been proved for $\alpha>0$. If then α is replaced by $-(1+\cos\theta)/\cos\theta$ (recall that $\theta=\cos^{-1}\{-1/(1+\alpha)\}$), one obtains expressions for $p_{\beta,\beta+\gamma;s}(\theta)$ which have been proved only for $\frac{1}{2}\pi<\theta<\pi$. However, in view of the geometrical significance discussed above of the formulae it is clear that these must hold for all permissible θ, since there is no essential qualitative difference between obtuse or acute θ, so far as the intrinsic geometrical situation is concerned. This may be verified formally by analytic continuation. We shall not here proceed with this verification in detail, but remark merely that in (22) and (23)

$$\frac{1}{\alpha(\alpha+1)\dots(\alpha+k-1)}=\frac{(-)^k\cos^k\theta}{\prod_{p=0}^{k-1}\{1-(p-1)\cos\theta\}}$$

has infinities at $\theta=\pi,\ 0,\ \cos^{-1}1/2, \cos^{-1}1/3,\ \dots,\ \cos^{-1}1/(k-1)$, while $a_{2j+1,2i+1}(\alpha)$ and $a_{2j,2i}(\alpha)$, regarded as functions of θ, may be shown by induction from the recursion relationships given immediately after equation (18) to have infinities at

$$\theta=\pi,\ 0,\ \cos^{-1}1/2,\ \cos^{-1}1/3,\ \dots,\ \cos^{-1}1/2i$$

and at

$$\theta=\pi,\ 0,\ \cos^{-1}1/2,\ \cos^{-1}1/3,\ \dots,\ \cos^{-1}1/(2i-1),$$

respectively. Furthermore, by (26), it follows that the term $V_{\beta,\beta+\gamma}(\phi_k(\theta))$ in (22) and (23) is continuous and has finite derivatives of all order in the open interval $(\cos^{-1}1/(1+k),\pi)$. These facts may be exploited to show that the right-hand members of (22) and (23) are well-behaved over the entire domain of $p_{\beta,\beta+\gamma;s}(\theta)$, inasmuch as the infinities of the right-hand members are exterior to this domain. The domain itself may be established from a result of Schläfli [7]. Schläfli has shown that if all the angles of a $(N-1)$-dimensional spherical simplex are acute, then the simplex is null if, and only if,

$$\begin{vmatrix} 1 & -\cos(12) & -\cos(13) & \cdots & -\cos(1N) \\ -\cos(21) & 1 & -\cos(23) & \cdots & -\cos(2N) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\cos(N1) & -\cos(N2) & -\cos(N3) & \cdots & 1 \end{vmatrix}=0.$$

Thus, the regular spherical simplex $12 \ldots (\beta+\gamma)$ is degenerate, when the common primary angle θ is acute, if, and only if, $(1+\cos\theta)\{1-(N-1)\cos\theta\}=0$, that is, $\theta=\cos^{-1}\{1/(N-1)\}$. Since the skew-regular simplex $1'\,2' \ldots \beta'\,(\beta+1) \ldots (\beta+\gamma)$ is generated from $12 \ldots (\beta+\gamma)$ as base simplex, this means that the domain of $p_{\beta,\beta+\gamma;s}(\theta)$ is the open interval $(\cos^{-1}\{1/(\beta+\gamma-1)\}, \pi)$.

4. Some applications

(A) *Surface and volume-contents of varieties of revolution generated by the rotation of skew-regular hyperspherical simplices.* The $(\beta+\gamma-1)$-dimensional spherical simplex $1'\,2' \ldots \beta'\,(\beta+1)\,(\beta+2) \ldots (\beta+\gamma)$ may be regarded as imbedded in a $(\beta+\gamma)$-flat. Rotate the simplex round the $(\beta+\gamma-1)$-flat Λ_0, lying in the latter $(\beta+\gamma)$-flat, as axis. Assume that this axis is defined by $\beta+\gamma$ fixed points in a $(\beta+\gamma-1+r)$-flat, imbedded in its $(\beta+\gamma+r)$-space, which has therefore r degrees of freedom and can rotate round Λ_0 in such a way that each point of the simplex generates a r-dimensional hyperspherical surface. The simplex itself then generates a variety of revolution of dimensionality $\beta+\gamma-1+r$ and of species r (see e.g. Sommerville [8]) lying on the surface of a unit sphere in $(\beta+\gamma+r)$-space. According to the multi-dimensional generalization of Pappus's theorem (see for instance [8] again), the surface-content of the (non-simplicial) variety of revolution is the product of the content of the generating simplex and the surface-content of the hypersphere traced by the rth centroid. The distance of the latter point from Λ_0 is $\{p_{\beta,\beta+\gamma;r}(\theta)\}^{1/r}$. Hence, the surface-content of the generated surface is

$$\frac{2\,\pi^{\frac{1}{2}(r+1)}}{\Gamma(\frac{1}{2}(r+1))}\, p_{\beta,\beta+\gamma;r}(\theta) \cdot \frac{2\,\pi^{\frac{1}{2}(\beta+\gamma)}}{\Gamma(\frac{1}{2}(\beta+\gamma))}\, V_{\beta,\beta+\gamma}(\theta). \tag{34}$$

Similarly, consider the sector of the unit sphere in $(\beta+\gamma)$-space constructed by joining the vertices of $1'\,2' \ldots \beta'\,(\beta+1)\,(\beta+2) \ldots (\beta+\gamma)$ to the centre of the sphere. Then according to the extension of Pappus's theorem, the volume-content of that portion of a unit sphere in $(\beta+\gamma+r)$-space generated by rotation of the above sector round Λ_0 is equal to the product of the volume-content of the sector and the surface-content of the hypersphere traced by the rth centroid of the sector. The distance oft he latter point from Λ_0 is $\{q_{\beta,\beta+\gamma;r}(\theta)\}^{1/r}$. Hence, the volume-content of the generated figure is

$$\frac{2\,\pi^{\frac{1}{2}(r+1)}}{\Gamma(\frac{1}{2}(r+1))}\, q_{\beta,\beta+\gamma;r}(\theta) \cdot \frac{\pi^{\frac{1}{2}(\beta+\gamma)}}{\Gamma(\frac{1}{2}(\beta+\gamma)+1)}\, V_{\beta,\beta+\gamma}(\theta). \tag{35}$$

(B) *The moments of the sum of components of an equicorrelated random normal vector restricted to an orthant.* Consider a random normal N-dimensional vector $\mathbf{y}(N=\beta+\gamma)$ with zero expectation vector and with variance-covariance matrix having all its off-diagonal elements equal to ϱ, the common correlation between the components of the vector. The elementary probability law of $\mathbf{y}$ is then

$$f_N(\mathbf{y};\varrho)=\frac{1}{(2\pi)^{\frac{1}{2}N}\{(1-\varrho)^{N-1}(1+(N-1)\varrho)\}^{\frac{1}{2}}}\exp\left\{-\frac{1}{2(1-\varrho)(1+(N-1)\varrho)}\cdot\right.$$
$$\left.\cdot[(1+(N-2)\varrho)\sum y_i^2-\varrho\sum_{j\neq i}\sum y_i y_j]\right\}$$
$$=\frac{1}{(2\pi)^{\frac{1}{2}N}\{(1-\varrho)^{N-1}(1+(N-1)\varrho)\}^{\frac{1}{2}}}\exp\left\{-\frac{1}{2(1-\varrho)}\cdot\right.$$
$$\left.\cdot\left[\sum y_i^2-\frac{\varrho}{1+(N-1)\varrho}(\sum y_i)^2\right]\right\},$$

defined over the whole N-space. Suppose now $\mathbf{y}$ is conditioned to lie in the orthant $R_{\beta,\beta+\gamma}$, $y_i\leqslant 0$ $(i=1,2,\ldots,\beta)$, $y_i\geqslant 0$ $(i=\beta+1,\beta+2,\ldots,\beta+\gamma)$. Then the probability density function of the conditioned vector is

$$g_N(\mathbf{y};\varrho)=\frac{1}{K}f_N(\mathbf{y};\varrho),$$

where K is a normalising constant chosen so that the relation

$$\int_{R_{\beta,\beta+\gamma}} g_N(\mathbf{y};\varrho)\,d\mathbf{y}=1$$

is satisfied. The moments of the sum of the components are

$$\int_{R_{\beta,\beta+\gamma}}\left(\sum_1^N y_i\right)^s g_N(\mathbf{y};\varrho)\,d\mathbf{y}\qquad(s=0,1,2,\ldots),$$

and on replacing y_i by $y_i\sqrt{1-\varrho}$, this reduces to

$$(1-\varrho)^{\frac{1}{2}s}\frac{\{(1-\varrho)/(1+(N-1)\varrho)\}^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}N}K}\int_{R_{\beta,\beta+\gamma}}(\sum y_i)^s\cdot$$
$$\cdot\exp\left\{-\frac{1}{2}\left[\sum y_i^2-\frac{\varrho}{1+(N-1)\varrho}(\sum y_i)^2\right]\right\}dy_1\ldots dy_{\beta+\gamma}.\qquad(36)$$

(36) is to be compared with the left-hand number of (10). As in (10), the quadratic form in (36) defines a set of homothetic ellipsoids having one principal axis along the line of symmetry $y_1 = y_2 = \cdots = y_{\beta+\gamma}$, and with the remaining $\beta + \gamma - 1$ axes of equal magnitude and arbitrary, but mutually orthogonal directions in the flat $\sum_1^N y_i = 0$. A rotation of the coordinate axes chosen so as to orient one of the transformed axes along the first principal axis referred to above followed by a scaling transformation reduces

$$\sum_1^N y_i^2 - \varrho \left(\sum_1^N y_i\right)^2 / (1 + (N-1)\varrho) \quad \text{to} \quad \sum_1^N x_i^2.$$

Transformation to polar coordinates then gives

$$\frac{1}{K}\{2\,(1 + (\beta + \gamma - 1)\,\varrho)\,(\beta + \gamma)\}^{\frac{1}{2}s} \frac{\Gamma(\frac{1}{2}(\beta + \gamma + s))}{\Gamma(\frac{1}{2}(\beta + \gamma))} p_{\beta,\beta+\gamma;s} (\cos^{-1}(-\varrho))\, V_{\beta,\beta+\gamma}(\cos^{-1}(-\varrho))$$
$$(s = 0, 1, 2, \ldots)$$

as the moments of the sum, and since the zero-*th* moment is necessarily 1, $K = V_{\beta,\beta+\gamma}(\cos^{-1}(-\varrho))$, and the moments are

$$\{2\,(1 + (\beta + \gamma - 1)\,\varrho)\,(\beta + \gamma)\}^{\frac{1}{2}s} \frac{\Gamma(\frac{1}{2}(\beta + \gamma + s))}{\Gamma(\frac{1}{2}(\beta + \gamma))} p_{\beta,\beta+\gamma;s} (\cos^{-1}(-\varrho)) \quad (s = 0, 1, 2, \ldots). \tag{37}$$

(C) *The moments of order statistics in normal samples.*

The probability density function of the r*th* largest value in a sequence of n independent observations of a Gaussian stochastic variable with zero mean and unit variance is

$$n\binom{n-1}{r-1}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\xi^2} [F(\xi)]^{r-1} [1 - F(\xi)]^{n-r} \; (-\infty < \xi < \infty), \tag{38}$$

where $F(\xi)$ is as defined in (5.1) (see e.g. [3]).

The characteristic function of ξ, $E(\exp i\tau\xi)$, τ real, is then

$$n\binom{n-1}{r-1} I_{r-1,\,n-1}(i\tau;\, 1;\, \mathbf{0}), \tag{39}$$

in the notation of equation (5). From equation (8), this is equivalent to

$$e^{-\tau^2/2n} \frac{n^{\frac{1}{2}}\binom{n-1}{r-1}}{(2\pi)^{\frac{1}{2}(n-1)}} \int\limits_{R_{r-1,\,n-1}} \exp\left\{-\tfrac{1}{2}Q(\mathbf{t}) - i\tau \frac{\sum_1^{n-1} t_i}{n}\right\} d\mathbf{t}, \tag{40}$$

where Q is defined in (6.1).

The first factor in (40), $\exp(-\tau^2/2n)$, is the characteristic function of a Gaussian stochastic variable with zero mean and variance $1/n$ and is therefore the characteristic function of the arithmetic mean of the n observations. The second factor represents the characteristic function of $-1/n$ times the sum of the components of a $(n-1)$-dimensional random vector distributed initially as in a multivariate normal distribution with zero expectation vector and with a correlation matrix having all its off-diagonal elements equal to $1/2$ but constrained in the sense of (B) above to lie in the orthant $R_{r-1,\,n-1}$. It follows that the distribution of ξ is the convolution of two distributions, one of which is that of the arithmetic mean of the n observations. In particular, the distribution of ξ as $n\to\infty$ behaves as that of the negative arithmetic mean of the components of the $(n-1)$-dimensional random vector referred to previously.

Further, in view of the discussion in (B), it appears from (40) that *the moments of ξ may be expressed as linear fnnctions of the geometrical moments of a $(n-2)$-dimensional skew-regular spherical simplex $S=1'\,2'\,\ldots\,(r-1)'\,r\,(r+1)\,\ldots\,(n-1)$ with the angle* $\theta=2\pi/3$. The base regular simplex $12\,\ldots\,(n-1)$ may be obtained as follows: Inscribe a regular (linear) simplex in a unit $(n-1)$-sphere with the angle between any two flats at a vertex $\cos^{-1}\{1/(n-1)\}$. Then n regular $(n-2)$-dimensional regular simplices with common angle $2\pi/3$ result by joining the vertices of the linear simplex to the centre of the sphere. These n simplices are mutually non-overlapping and cover the entire surface of the sphere. Finally, we note from the preceding discussion that the limiting moments of ξ as $n\to\infty$ are simple multiples of the geometrical moments of the simplex S relative to the flat which is orthogonal to the line of symmetry of $12\,\ldots\,(n-1)$. Specifically,

$$E(\xi^s)\sim(-)^s\left(\frac{2(n-1)}{n}\right)^{\frac{1}{2}s}\frac{\Gamma(\frac{1}{2}(n-1+s))}{\Gamma(\frac{1}{2}(n-1))}\,p_{r-1,\,n-1;\,s}\left(\frac{2\pi}{3}\right). \tag{41}$$

These are the *exact* moments of the second component of ξ.

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(1) Translated in part by Q. J. Cayley from the author's Theorie der Vielfachen Kontinuität. *Neue Denkschr. Schweiz Ges. Naturwiss.*, 38 (1901), published posthumously.

See also Schläfli's. *Gesammelte Mathematische Abhandlungen*, I. Verlag Birkhäuser, Basel, 1950. (Lehre von den sphärischen Kontinuen, 227–298.)

Received May 20, 1959