JAKOB NIELSEN IN MEMORIAM

BY

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It is but natural that the Danish Mathematical Society should wish to commemorate Professor Jakob Nielsen by a meeting, for his death meant to our society the loss of one of its most active and prominent members. He joined it in 1921, sat on its committee for eight years, and in no less than 28 flawless and inspiring lectures given to it, he communicated to the mathematical circle the results of his own research, and often also that of others.

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Not without grave misgivings have I undertaken to speak about Jakob Nielsen, the man and the scientist, fearing that I should not be able to do justice to all essential aspects of his rich personality, but I have done so out of admiration for his work, and out of deep gratitude for his faithful friendship to me through 30 years, good and bad.

Jakob Nielsen was born on the 15th of October 1890 at the village of Mjels on the island of Als (then under Germany) as the youngest of four children. His father owned a small farm, where Jakob Nielsen spent his childhood, attending the village school until 1900. A relative of his, a teacher at Rendsburg, who took charge of him after the early death of his mother, must have become aware of his unusual gifts. At any rate, she took him with her to Rendsburg, where he went to the so-called Realgymnasium. Here, Latin was an item of considerable importance in the curriculum, and throughout his life he retained a deep love of Roman poetry. The relations between him and his aunt deteriorated, however, both being rather uncompromising of character, and at the age of 14 he left her home to take up a job as private tutor. For the remainder of his schooldays, and throughout his years of study, he earned his own living by coaching in a variety of subjects—even Norwegian, as he once said with a broad smile.

In December 1907 his secondary school career was cut short, he and some schoolfellows being expelled because they had founded a (quite harmless) pupil's club, which was against the rules. However, this did no great harm, for he continued his studies on his own, even matriculating at the University of Kiel in the spring of 1908, although he did not take his school-leaving certificate until the autumn of 1909 (privately at Flensborg). A little episode from this examination illustrates his early developed critical faculty: On one of the examiners asking him whether he had read the (as we know now, false) accounts of F. A. Cook's expedition to the North Pole, he said that he had, remarking that he could not understand how it was possible for Cook to fix his position so far to the north with a precision as great as that claimed by him.

Jakob Nielsen spent his years of study at Kiel, except for the summer term of 1910, spent at the University of Berlin. His interests and gifts were many-sided. At the start, he attended lectures in physics, chemistry, geology, biology, and literature. Only after some terms did mathematics take a prominent place, but philosophy was also strongly represented. I cannot tell what made him finally choose mathematics as the science to whose service he wanted to devote himself. Of course, we mathematicians cannot but feel that he chose rightly!

Among the teachers of mathematics of the time at Kiel, Georg Landsberg was undoubtedly the most prominent, and we note that in the short autobiography appended to Jakob Nielsen's thesis for the doctorate, he gives particular thanks to Landsberg, both for his

teaching and for his instigation to take up the problems which form the subject of the thesis. The list of lectures on mathematics attended by Jakob Nielsen shows that the instruction he received at Kiel scarcely exceeded what must be looked upon, at the time, as belonging to the general education of a mathematician. The only mathematician who had an appreciable influence on Jakob Nielsen's start as a scientist was Max Dehn, 12 years his senior, who was attached to the University of Kiel at the end of the year 1911, being already then a scientist of repute. Through Dehn he came into contact with the most recent advances and methods in topology and group theory, fields to which Dehn had made fundamental contributions. This contact between the two men developed into a life-long friendship.

In 1913 Jakob Nielsen took his doctor's degree with the thesis already mentioned: "Kurvennetze auf Flächen". Although he himself did not set great store by this work, for reasons to be mentioned presently, it is of importance for the understanding of his later research to have a look at it. The three first sections deal with the connection between the total curvature of a surface and the topological properties of families of (possibly degenerate) curves covering the surface simply and completely. The last section, added later, is but loosely connected with those preceding it. It deals with the topological properties of the closed surface of genus 1, the torus. The fundamental group F, the free abelian group with two generators, is introduced, and the homotopy class of a closed curve on the surface is characterized by the pair of exponents in the expression of the corresponding element of F as a product of powers of the generators. The minimal number of double points of a curve and of the points of intersection of two curves, the curves varying within their homotopy classes, are determined in terms of these exponents. Then topological mappings of the torus onto itself are studied. Each such mapping induces an automorphism of F which depends only on the homotopy class of the mapping and which is characterized by the two pairs of exponents belonging to the images of the generators. The homotopy classes of mappings form a group isomorphic with the group of automorphisms of F. For this group a system of generators and defining relations is found, and it is shown that it is closely related with the modular group, well known from the theory of elliptic functions. Finally the minimal number of fixed points of a topological mapping, varying within a homotopy class, is determined in terms of the above-mentioned exponents. After the composition of this second part, where both the approach, the tools and the methods reflect the then new views in topology, it must have been clear to Jakob Nielsen that cogent answers to the interesting questions dealt with in the first part would require considerably more refined and rigorous notions than he had had at his disposal.

In his little publication "A Mathematician's Apology", G. H. Hardy calls mathematics "a young man's game". Understood as a contention that the main work of a mathematician

is done in youth it is obviously wrong; it is easy to name many who have done excellent work at a ripe age. But there may be this much to the truth in it that the world of ideas in which the faculties develop is formed early in life. At any rate, this holds good of Jakob Nielsen. The problems dealt with in the second part of the thesis, generalized to arbitrary surfaces, came to be the governing idea of the greater part of his life-work. Although they were fairly simple in the case of the torus, and to some extent ended in things that were familiar from elsewhere, yet he must at an early stage have realized both the enormous difficulties presented by the general approach, and the fact that the necessary tools had to be created first. But with singular tenacity of purpose he did carry out these investigations after many years' work.

But there was to be a long and almost complete interruption: After taking his German M. A. degree in the summer of 1913, he was called up for service in the German navy. Owing to the outbreak of the 1914–18 war the prescribed year of military service became five years. Attached as he was to the coast defence artillery, he was sent first to Belgium and then, in April 1915, to Constantinople as one of the German officers functioning as advisers to the Turkish government on the defence of the Bosporus and the Dardanelles. He stayed there till the end of the war, finding a little time for scientific work. Besides a paper on a ballistic subject grown out of his work there, he wrote two short papers (published in Mathematische Annalen in 1917 and 1918). It is the problem occurring in the thesis for the (abelian) fundamental group of the torus which is studied here for the finitely generated free groups, namely to find a system of generators for the group of automorphisms of a given group. During his return journey from Turkey through Russia and Poland, in November 1918, at the time of the collapse of Germany, Jakob Nielsen had kept a diary, published in the newspaper Politiken on the tenth anniversary of Armistice Day. I shall quote a few lines, "I walked about here as a stranger the first two years, but in the wonderful summer of 1917 and its echo in 1918, I came to love this country under the all-powerful sway of the sun, in the company of people I liked. I know that I shall often languish for the sun. I want to go there again some day." It is probable that his experiences there contributed to his complete open-mindedness towards people of a background and with an attitude totally different from his own, that open-mindedness of his which qualified him eminently for his achievements in international co-operation, to which he devoted so great a part of his powers in later years.

Jakob Nielsen spent the summer term of 1919 at Göttingen. Among the many mathematicians he met in this centre of mathematical studies, he was specially attracted by Erich Hecke, three years his senior. When in that same year Hecke received nomination to the recently established University of Hamburg, Jakob Nielsen accompanied him as his

assistant and "Privatdozent". A close and life-long friendship developed between the two men, not so much on account of mathematical interests they had in common, for there are few and loose points of contact between Hecke's work in number theory and function theory and Nielsen's in topology and group theory, but on account of their spiritual affinity, the depth and seriousness of their attitude to science. We have two papers by Jakob Nielsen from that period, both dealing with the fixed point problem for surface mappings. In the first an investigation by L. E. J. Brouwer, concerning surfaces which admit of topological mappings onto themselves without fixed points, is completed. In the second the fixed point problem for the torus is taken up again, this time with the intuitive arguments occurring in the thesis replaced by rigorous proofs. At the same time the problem is also solved for the non-orientiable torus, the Klein bottle.

Already in 1920 Jakob Nielsen received nomination to a professorship at the Institute of Technology at Breslau. Here he had an opportunity to resume contact with Max Dehn, who had for some years been attached to the university. Of interest are mimeographed notes to some lectures he gave at Breslau in 1921. Here he formulated clearly the central problem he had set himself to solve: To determine and to investigate the group of homotopy classes of topological mappings of a given surface. In the general case, there corresponds to a homotopy class of a topological mapping of the surface not one automorphism, but a coset of the group of inner automorphisms in the group of all automorphisms, a "family of automorphisms". The group in question is thus the quotient group of the latter with respect to the group of inner automorphisms. In order to throw light on his co-operation with Dehn I may point out that one link of this investigation, namely the proof of the fact that every automorphism of the fundamental group of a closed surface is induced by some topological mapping, had been communicated to him by Dehn, who never published it, however. It is characteristic of Jakob Nielsen that whenever he needed this theorem, or merely an idea that resembled the proof, he would stress his debt to Dehn. Although the spheres of the two men's studies were closely related, no further interference proper ever came about. Jakob Nielsen went his own way.

The Breslau stay was to be as brief as the one in Hamburg, for at the plebiscite preceding the Reunion of North Schleswig with Denmark in 1920, Jakob Nielsen opted for Denmark, moving to Copenhagen in 1921 and taking over the vacant lecturership in mathematics at the Royal Veterinary and Agricultural College. There now followed a long succession of happy and fruitful years of work. Let me quote what Harald Bohr says about this in the lecture "Et Tilbageblik" he gave on his 60th birthday: "We all have a very living impression of what Jakob Nielsen means for mathematics in Denmark—for our science and teaching. At the return of North Schleswig to Denmark after the first world war, he came

to Copenhagen after having been professor in Breslau. Within the topology of surfaces, the field of mathematics to which he has so purposefully devoted his efforts in science, he is a recognized master who has been the example for many followers and whose influence will surely be of lasting importance for the future development of mathematics here at home. Bonnesen, Jakob Nielsen, and I followed each other's work with keen interest during those years, and many Tuesday evenings Bonnesen and I walked out to Hellerup to visit Jakob Nielsen, who was distinguished among us by being in possession of a blackboard, and in a cosy atmosphere we told each other what was on our minds." Let me here add a few words about the summers on Als, which have played such an important part in the mathematical world of Denmark. In 1919 Jakob Nielsen bought a little house near Fynshav on Als, and a few years later Harald Bohr followed his example. Year after year, in the summer vacation, a group of mathematicians, young and old, Danish and foreign, gathered about those two. Apart from the normal holiday activities, the study of mathematics was pursued. Not a few advances and discoveries were submitted in Bohr's little half-timbered house, in the study remarkable for its blackboard—unforgettable experiences which are remembered with gratitude by all who had the privilege of attending.

Let me return to Jakob Nielsen's scientific work. Starting in 1921 there appeared some purely group theoretic papers. In the first one, published in Matematisk Tidsskrift under the title "Om Regning med ikke-kommutative Faktorer og dens Anvendelse i Gruppeteorien", a major result is that every subgroup of a finitely generated free group is itself free. The proof is based on an ingeneous method of reduction of systems of generators. In 1927 the theorem was extended by Otto Schreier to arbitrary free groups, and under the name of the Nielsen-Schreier Theorem it contributes now one of the bases of the theory of infinite groups. The two other papers, from 1924, continue earlier investigations of the group of automorphisms of a given group. In the cases of the finitely generated free groups and the free abelian group with three generators not only a system of generators, but now also a system of defining relations is found. In the same year he once more resumed the subject of his thesis, the topology of the torus, in the little paper "Ringfladen og Planen". Here appears for the first time the important notion of a fixed point class. Let a continuous mapping t of the torus T into itself be given. The euclidean plane E being the universal covering surface of T, there exist infinitely many continuous mappings of E into itself which "lie over" the given mapping t. Two fixed points p_1 and p_2 of t are said to belong to the same class if at least one of these covering mappings has two fixed points which lie over p_1 and p_2. This is the case if and only if there exists a curve on T joining p_1 and p_2 such that the closed curve consisting of it and its image by t is homotopic to zero. Determining the number of classes of fixed points, which depends only on the homotopy class of the

mapping t, Jakob Nielsen obtains his former result concerning the minimal number of fixed points, now for not necessarily topological, continuous mappings, in a most satisfactory manner. Along with these investigations Jakob Nielsen took up the study of discontinuous groups of motions in the non-euclidean plane and devoted several papers (1923, 1925, 1927) to this subject. His interest in it arose from the fact that the fundamental groups of the surfaces of genus greater than 1 admit of representations by such groups. Among other results he obtained the following: On every closed surface with constant negative curvature there exist geodesics which are everywhere dense on the surface and approach every direction. This theorem is of importance in the theory of dynamical systems. But Jakob Nielsen's paper, published as it was in Danish, long remained unnoticed, and the theorem was found independently by Marston Morse. This case, like others, seems to show that in his modesty, Jakob Nielsen was apt to underrate the importance of the results he reached as bi-products on his way to the goal he had set himself.

These apparently somewhat desultory investigations turned out to be stones that went to the erection of an impressive building. Hints of this are to be found in some lectures given in Hamburg in 1924 and in Copenhagen in 1925, at the 6th Scandinavian Congress of Mathematicians. But in its final form it appeared in three long memoirs (300 pages in all) from the years 1927, 1929, and 1932 in Acta Mathematica under the common title "Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen". Here we find again the notions and methods he had previously used or developed: The universal covering surface interpreted as the non-euclidean plane, the latter represented by the conformal model in the interior of the unit circle, the fundamental group as a discontinuous group of motions in the non-euclidean plane, the mappings of the latter onto itself which lie over a given surface mapping, the automorphisms induced by them. As an essential new tool comes here the following theorem: Every mapping of the non-euclidean plane onto itself which lies over some surface mapping can be extended continuously to the points on the unit circle, representing the points at infinity of the non-euclidean plane, and the mapping of the circumference which arises in this way depends only on the homotopy class of the surface mapping. Hereby a two-dimensional topological problem is reduced to a one-dimensional one. A thorough analysis of this mapping leads to a wealth of information about homotopy classes of surface mappings. Further we find the notion of a class of fixed points generalized to the present case. To each class of fixed points an index is attached which gives the "algebraic number" of fixed points in the class. For a given homotopy class of surface mappings, estimates are found for the number of classes of fixed points and for their indices. Finally many profound results concerning the group of automorphisms and the group of homotopy classes of mappings are obtained. What we see here are results that for penetration surpass

what had otherwise been achieved in the topology of surfaces. That he could achieve so much was due to his unique geometrical intuition, his power of concentration, and his immense capacity for work, enabling him to work out examples complicated enough to show what general regularities might be expected. The special cases previously dealt with were far from sufficient for this purpose. With these memoirs Jakob Nielsen had definitely broken new ground, and they gave him great international reputation. Many scientists have in some way combined their investigations with his. It should be particularly emphasised that the concept of a class of fixed points has undergone a great and important generalisation. Jakob Nielsen's results are not easily accessible, and even today no substantially simpler way to them has been found. So his achievement still stands as a challenge to younger scientists.

In 1925 Jakob Nielsen became professor of theoretical mechanics at the Technical University after the retirement of C. Juel. During the first few years he continued to base his instruction on Juel's textbook; but gradually there arose weighty reasons for a revision. To be sure, mechanics of particles and rigid bodies is among the best worked-out parts of applied mathematics, and in the introductory teaching with its century-old tradition the contents are well-defined, determined by their application in physics, astronomy, and technology. Nor was it on this point that the new textbook, which Jakob Nielsen published in two volumes in 1933–34, came to differ considerably from others. Rather, the object was an adaptation to the development that had taken place in mathematics, and especially in the teaching of mathematical analysis. The point was to introduce and exploit recent mathematical tools, such as vectors and matrices. As an example may be mentioned his elucidation of the close connection between on one hand the theory of frameworks and the principle of virtual work, and on the other hand the theory of systems of linear equations, which has given this subject a conciseness hardly to be found in the numerous other textbooks the world over. The book is not easily read, and Jakob Nielsen's lectures demanded much of the students. He had an unusual power of expressing himself with great lucidity, but also with great terseness. It was incredible how much he could tell, and in great detail, in an hour's lecture.

The textbook was published in German in 1935. A second edition in Danish came out in 1943 and 1945 (revised owing to changes in the teaching of geometry), and a third in 1950 and 1952.

Jakob Nielsen was faced with a new task when the teaching of aerodynamics was to be established at the Technical University. His lectures were published in mimeographed form in the years 1940–42. His presentation of this subject, theoretically not easily accessible in parts, is remarkable for its clear distinction between the empirical foundation and the

mathematical theory. He published the more theoretical part of these lectures in 1952, as volume three of the textbook of theoretical mechanics.

Immediately after finishing the first edition of the textbook, Jakob Nielsen resumed his studies in topology and group theory. It is not possible here to mention the many papers, about 20, among them several comprehensive ones, which he published in the years after 1935, most of them carrying on his investigations on surface mappings. By means of the powerful tools developed in the previous papers, he succeeded in solving a series of related problems. In 1937 he gave a complete classification of the periodic mappings of a surface onto itself, and in 1942 a fourth great memoir, "Abbildungsklassen endlicher Ordnung", was published in Acta Mathematica. It deals with a problem to which he had been led in the third of the above-mentioned Acta-papers, and which he had solved there in some special cases: Does every homotopy class of surface mappings which is of finite order, in the sense that a certain power of it is the class of the identity mapping, contain a periodic mapping, that is, a mapping the same power of which is the identity? The proof that this is the case is extremely difficult and makes up all the 90 pages long paper. One cannot but admire the intellectual vigour with which this investigation is carried out. Finally I shall mention one more large paper: "Surface transformation classes of algebraically finite type" from 1944, in which more general classes of surface mappings are thoroughly investigated.

On several occasions Jakob Nielsen lectured at the Mathematical Institute of the university to a small circle of young mathematicians on subjects that occupied him in connection with his research. Of special interest is a series of lectures on discontinuous groups of motions in the non-euclidean plane, given in the year 1938–39. Such groups, whose theory goes back to H. Poincaré's fundamental work on automorphic functions, are, as noted,an important tool in Jakob Nielsen's investigations in surface topology. In the above-mentioned lectures he took up the theory for a certain class of these groups for its own sake. He thus arrived at interesting new notions and results. In connection with this, Svend Lauritzen has, in his thesis for the doctorate, carried out a corresponding investigation for another class of these groups. However, it proved desirable and expedient, in view of their many and important fields of application, to take up the theory of discontinuous groups of motions in the non-euclidean plane in its full generality and from the bottom. By degrees it became clear that this task, which Jakob Nielsen took up together with me, was considerable more extensive and timeconsuming than anticipated; many problems, great and small, which turned up on the way had to be elucidated before the monograph aimed at could take shape.

Although Jakob Nielsen's heart was in this task, he could only devote to it a moderate part of his great working power, for since the end of the 1939–45 war he was occupied on tasks of a totally different kind. It was almost a matter of course that his name should

come up when positions of trust and responsibility were to be filled, for he possessed to an eminent degree the qualities of integrity and open-mindedness, and he could make up his mind quickly on a problem and act resolutely when necessary. He required much of himself in attending to the duties he took upon himself. I shall not go into detail, but merely mention that his participation in international co-operation, especially the work of UNESCO, occupied a considerable part of his working power. Instead of trying to describe his activities as a delegate in 1945 to the conference which created UNESCO, as the leader of the Danish delegation to several sessions of the General Conference and as a member of the Executive Board of UNESCO from 1952 to 1958, I shall quote from the "Ad Honorem Jakob Nielsen" which the chairman of the Executive Board, Sir Ben Bowen Thomas, wrote after Jakob Nielsen's death:

"Those of us who worked with Professor Nielsen as members of the Board will recall his services to UNESCO with gratitude. We shall treasure his memory as an inspiration. He was punctual at our meetings, he was industrious in mastering our papers, he was able in our discussions, he was disinterested and wise in his approach to our problems, he was completely and utterly devoted to the service of humanity as a whole within the spheres of our responsibilities in Education, Science and Culture."

"I repeat those words: Devoted to the service of humanity as a whole. In a sense he was a keeper of the Board's conscience, unhappy when lack of resources in money, ability or techniques hampered our work, sad when factors other than the merits of the projects being examined, demanded and obtained overriding weight or authority, as it seemed to him, when we reached some decisions; but no matter how the issue was decided, he always marched breast forward to the end, leaving with those of us who knew him the abiding memory of his powerful, energetic frame, his bright blue eyes and ready smile, his high standards, his unselfish willingness to accept the tasks which we only too readily entrusted to him. An admirable representative of his people, a true son of Denmark, a distinguished scholar, a great public servant, a very humble, modest man."

In 1951, Jakob Nielsen was nominated Harald Bohr's successor at the University of Copenhagen. Here he lectured with delight and zeal to young mathematicians on subjects close to his heart. But the growing demands made upon him by his UNESCO work, and the frequent and at times long journeys abroad, which interrupted his lectures, caused him to feel after a few years that he could no longer give satisfaction as a university teacher. Add to this that during these years there was not much time for scientific work, especially on the above-mentioned monograph, although he took it up untiringly and zealously whenever his duties permitted. So already in 1955, he resigned his professorship, and after finishing his UNESCO work in 1958, he could devote himself whole-heartedly to research

work. There now followed a period of intense and successful work, especially on the monograph. He succeeded in surmounting a difficulty which had long prevented a satisfactory conclusion. But already in January 1959 he was stricken with the disease which carried him off on the 3rd of August. It is sincerely to be regretted that he did not live to see the publication of the monograph.

Jakob Nielsen was a sterling character and warm-hearted man. He showed life-long gratitude to those to whom he thought himself indebted, whether in the private or the scientific sphere. He was his friends' most faithful friend in times good and bad. He had always praise and encouragement for his collaborators, and if criticism was required, he always knew how to find a form that could not hurt even the most sensitive. Still, in the face of intolerance and narrow-mindedness he could find sharp words.

His memory as a man and as a scientist will be cherished with deep gratitude.

The publications of Jakob Nielsen

Memoirs and Articles

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