THE PREDICTION THEORY OF MULTIVARIATE STOCHASTIC PROCESSES, III

UNBOUNDED SPECTRAL DENSITIES

BY

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Contents

1. Introduction

We shall show that the algorithm for determining the generating function and prediction error matrix of a q-variate, discrete parameter, weakly stationary, stochastic process (S.P.), as well as the unique, mean-convergent, autoregressive series for the linear predictor in the time-domain, which were obtained by Wiener and the writer in [8, Part II] [2] in case the eigenvalues of the spectral density matrix $\mathbf{F}'$ are bounded above and away from zero, are valid under a more general setting. The algorithm will be shown to hold under the weaker conditions that the quotient of the largest to the smallest eigenvalue of $\mathbf{F}'$ is in L_1, $\mathbf{F}'$ is invertible a.e. and $\mathbf{F}'^{-1} \in \mathbf{L}_1$. The series for the predictor will be shown to prevail under the hypothesis $\mathbf{F}' \in \mathbf{L}_\infty$, $\mathbf{F}'^{-1} \in \mathbf{L}_1$, which while more stringent than the last is again weaker than that assumed in II. Our method will rest on extending to the q-variate case a theorem of Kolmogorov [2, Thm. 24] on simple minimal processes, i.e., those for which the random function

(1) The writer wishes to thank Harvard University and the Massachusetts Institute of Technology for visiting appointments in 1957–58, during which a part of this research was completed.

(2) In the sequel all references which are prefixed by I or II are to parts I or II of the paper [8].

at any time is outside the closed subspace spanned by the past and future functions of the process. These results were announced by the writer in [3, 4] along with indications of their proofs. At about the same time Rosanov obtained a similar extension of Kolmogorov's theorem but from a different standpoint [6, Thm. 17].

In § 2, we shall define a q-variate, *full-rank minimal S.P.* $(\mathbf{f}_n)_{-\infty}^{\infty}$ and the associated *two-sided, normalised, innovation process*. Unlike the ordinary (one-sided) innovation process this is not orthonormal, but we shall show that it is biorthogonal to $(\mathbf{f}_n)_{-\infty}^{\infty}$, and that as a consequence the reciprocal of the generating function $\mathbf{\Phi}$ of $(\mathbf{f}_n)_{-\infty}^{\infty}$ is in $\mathbf{L}_2^{0+}$, i.e. the entries of the matrix $\mathbf{\Phi}^{-1}$ are in L_2 and their Fourier series have no negative frequencies. This crucial fact enables us to extend *Kolmogorov's Theorem* to the q-variate case: $(\mathbf{f}_n)_{-\infty}^{\infty}$ is full-rank minimal, if and only if it has a spectral distribution $\mathbf{F}$ such that $\mathbf{F}'$ is invertible a.e. and $\mathbf{F}'^{-1} \in \mathbf{L}_1$. We shall amplify the proof of this extension sketched in [3], even though Rosanov [6, Thm. 17] has given such an extension in the meantime, because the latter uses a new definition of q-variate S.P., which while equivalent to ours, results in an appreciably different conceptual framework. (Reasons for adhering to our definition, which is due to Zasuhin [9], as against Rosanov's have already been given in [5, § 5].)

In § 3 we shall show that the new *spectral condition*, mentioned in the opening paragraph, allows an *initial factorization of* $\mathbf{F}'$ into a complex-valued function F_1' such that F_1', $1/F_1' \in L_1$ and a matrix-valued function $\mathbf{I}+\mathbf{M} \in \mathbf{L}_\infty$ such that $(\mathbf{I}+\mathbf{M})^{-1} \in \mathbf{L}_1$ and $|\mathbf{M}|_B < 1$, a.e., the subscript B referring to the Banach-norm. This yields a corresponding factorization of the generating function, in terms of which the frequency-response function for the process is expressible. Since methods for finding the generating function of a simple S.P. are known, we are left with the problem of determining the generating function of a q-variate process with spectral density of the form $\mathbf{I}+\mathbf{M}$, where $|\mathbf{M}|_B < 1$ a.e. and $(\mathbf{I}+\mathbf{M})^{-1} \in \mathbf{L}_1$.

In § 4 we shall solve the last problem with the aid of the operator $\mathfrak{D}$ defined in terms of $\mathbf{M}$ as in II, 6.2. But whereas in II $|\mathfrak{D}|_B < 1$, we now have $|\mathfrak{D}|_B \leqslant 1$. We shall show, nevertheless, that $\mathfrak{D}$ is a strict contraction operator on $\mathbf{L}_2^{0+}$, that the geometric series $\sum_0^\infty (-1)^k \mathfrak{D}^k$ converges strongly on the range of $\mathfrak{J}+\mathfrak{D}$ (which need not be the whole of $\mathbf{L}_2$)(1) and that this yields the same *algorithm for the generating function* as was found in II, 6.5, 6.6. As the frequency-response function is expressible in terms of the generating function and its reciprocal, we will have solved the prediction problem in the frequency domain.

(1) $\mathbf{I}$ being the identity operator on $\mathbf{L}_2$.

Finally, in § 5 we shall show that with the stronger condition $\mathbf{F}' \in \mathbf{L}_\infty$ replacing $\mathbf{F}' \in \mathbf{L}_1$ above, we can exploit the isomorphism between the temporal and spectral domains to get the mean-convergent, autoregressive series for the *linear predictor in the time domain,* which was given in II, 5.7. To get this series in the univariate case Akutowicz [1, § 3] has assumed that the Fourier series of Φ and Φ^{-1} converge absolutely. Our result, and indeed the weaker one given in II, 5.7, shows that this assumption is unduly restrictive. We shall also express the (one-sided) innovation process as a one-sided moving average of the given process $(\mathbf{f}_n)_{-\infty}^{+\infty}$. Our criterion for this, viz. $\mathbf{F}' \in \mathbf{L}_\infty$, $\mathbf{F}'^{-1} \in \mathbf{L}$, is the same as that given by Wiener-Kallianpur in the univariate case in [7, ch. IV] (unpublished). As stated above the condition $\mathbf{F}'^{-1} \in \mathbf{L}_1$ is equivalent to $\mathbf{\Phi}^{-1} \in \mathbf{L}_2^{0+}$, which Akutowicz [1, Thm. 1] has shown to be necessary in the univariate case for the existence of such a moving average, under the assumption $F' \in L_\infty$. But as remarked in II, § 5 (end) the last assumption seems to be unduly strong, and it would be worth while to try to relax it.

We shall use extensively the theory developed in I, II, and adhere to the notation followed therein. A list of errata to I, II is given in § 6. We shall recall here some of this material for ready reference, and state a lemma which we will need.

NOTATION. *As in* [I, II] *bold face letters* $\mathbf{A}$, $\mathbf{B}$, *etc. will denote* $q \times q$ *matrices with complex entries* a_{ij}, b_{ij}, *etc. and bold face letters* $\mathbf{F}$, $\mathbf{\Phi}$, *etc. will denote functions whose values are such matrices. The symbols* τ, Δ, *, *will be reserved for the trace, determinant and adjoint of matrices.* $|\mathbf{A}|_B$, $|\mathbf{A}|_E$ *will denote the Banach and Euclidean-norms of* $\mathbf{A}$ [II, 1.1]. *The letters* C, D_+, D_- *will refer to the sets* $|z|=1$, $|z|<1$, $1<|z|\leqslant\infty$ *of the extended complex plane.*

We shall be concerned with the sets $\mathbf{L}_p$ of $q \times q$ matrix-valued functions $\mathbf{F}=[f_{ij}]$ on C such that each entry f_{ij} is in L_p in the usual sense, $0<p\leqslant\infty$. For $p\geqslant 1$, $\mathbf{L}_p^+$, $\mathbf{L}_p^{0+}$, $\mathbf{L}_p^-$, $\mathbf{L}_p^{0-}$ will denote the subsets of functions in $\mathbf{L}_p$ whose nth Fourier coefficients vanish for $n\leqslant 0$, $n<0$, $n\geqslant 0$, $n>0$, respectively. If $\mathbf{F}\in\mathbf{L}_p$, $p\geqslant 2$, and has Fourier coefficients $\mathbf{A}_k$, $-\infty<k<\infty$, then $\mathbf{F}_+$, $\mathbf{F}_{0+}$, $\mathbf{F}_-$, $\mathbf{F}_{0-}$ will denote the functions in $\mathbf{L}_p^+$, $\mathbf{L}_p^{0+}$, $\mathbf{L}_p^-$, $\mathbf{L}_p^{0-}$, whose nth Fourier coefficients are $\mathbf{A}_n$, for $n>0$, $n\geqslant 0$, $n<0$, $n\leqslant 0$ respectively (and zero for the remaining n). $\mathbf{F}_0$ will denote the constant function with value $\mathbf{A}_0$.

In $\mathbf{L}_2$ we introduce the Gramian, inner product and norm

$$\left.\begin{aligned}(\mathbf{\Phi}, \mathbf{\Psi}) &= \frac{1}{2\pi}\int_0^{2\pi} \mathbf{\Phi}(e^{i\theta})\,\mathbf{\Psi}^*(e^{i\theta})\,d\theta \\ ((\mathbf{\Phi}, \mathbf{\Psi})) &= \tau(\mathbf{\Phi}, \mathbf{\Psi}), \qquad \|\mathbf{\Phi}\| = \sqrt{\tau(\mathbf{\Phi}, \mathbf{\Phi})}.\end{aligned}\right\} \tag{1.1}$$

We readily infer the following relations, the second of which follows from the Parseval relations II, 1.13:

1.2. LEMMA. *For* $\mathbf{\Phi}$, $\mathbf{\Psi} \in \mathbf{L}_2$ *and* $\mathbf{X} \in \mathbf{L}_\infty$,

$$(\mathbf{\Phi X}, \mathbf{\Psi}) = (\mathbf{\Phi}, \mathbf{\Psi X}^*), \quad (\mathbf{\Phi}_+, \mathbf{\Psi}) = (\mathbf{\Phi}_+, \mathbf{\Psi}_+) = (\mathbf{\Phi}, \mathbf{\Psi}_+).$$

In I, II we defined a *q-variate, discrete parameter S.P.* as a sequence $(\mathbf{f}_n)_{-\infty}^{\infty}$ of q-dimensional vector-valued functions $\mathbf{f}_n = (f_n^i)_{i=1}^q$ such that each component f_n^i is L_2 on a probability space $(\Omega, \mathfrak{B}, P)$. This definition is germane to all stochastic applications of prediction theory, but is overspecific as far as much of the theory itself is concerned: it suffices, following Zasuhin [9], to treat each f_n^i as a vector in some (fixed) complex Hilbert space $\mathcal{H}$, so that $\mathbf{f}_n$ belongs to the Cartesian product $\mathcal{H}^q$. We call $(\mathbf{f}_n)_{-\infty}^{\infty}$ *weakly stationary* in case the Gram matrix

$$(\mathbf{f}_m, \mathbf{f}_n) = [(f_m^i, f_n^j)] = \mathbf{\Gamma}_{m-n} \tag{1.3}$$

depends only on $m-n$. In this case there exists a unitary operator U on $\mathcal{H}$ into itself such that $U^n f_0^i = f_n^i$. U is called the *shift operator* of the process. We let

$$\mathfrak{M}_n^i = \mathfrak{S}(f_k^i)_{k=-\infty}^{n}, \qquad \mathfrak{m}_n = \mathfrak{S}(\mathbf{f}_k)_{k=-\infty}^{n}, \qquad 1 \leqslant i \leqslant q, \qquad -\infty < n \leqslant \infty.$$

The first is the (closed) subspace of $\mathcal{H}$ generated by the elements f_k^i for $k \leqslant n$; the second is the corresponding subspace of $\mathcal{H}^q$ defined similarly except that linear combinations are taken with $q \times q$ matrix coefficients and the closure with respect to the induced topology in $\mathcal{H}^q$, cf. I, 5.6. The *orthogonal projection* $(\boldsymbol{\varphi} \mid \mathfrak{m}_n)$ of a vector $\boldsymbol{\varphi} = (\varphi^i)_{i=1}^q \in \mathcal{H}^q$ on the subspace $\mathfrak{m}_n$ is defined as the vector whose ith component is component is the projection $(\varphi_i \mid \text{clos.} \sum_{i=1}^{q} \mathfrak{M}_n^i)$, cf. I, 5.8 (a), 5.9.

We call the S.P. *non-deterministic* if $\mathfrak{m}_n \neq \mathfrak{m}_{n+1}$, and *regular* if $(\mathbf{f}_0 \mid \mathfrak{m}_{-n}) \to \mathbf{0}$ as $n \to \infty$. In the non-deterministic case

$$\mathbf{g}_n = \mathbf{f}_n - (\mathbf{f}_n \mid \mathfrak{m}_{n-1}) \neq \mathbf{0}. \tag{1.4}$$

We call $\mathbf{G} = (\mathbf{g}_0, \mathbf{g}_0)$ the *prediction error matrix* for lag 1, and say that the S.P. has *full rank* in case $\Delta\,\mathbf{G} > 0$.

On taking the spectral resolution of the shift operator U we get a bounded, non-decreasing, right continuous, $q \times q$ matrix-valued function $\mathbf{F}$ on $[0, 2\pi]$ such that $\mathbf{F}(0) = \mathbf{0}$ and

$$\mathbf{\Gamma}_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-ni\theta}\, d\,\mathbf{F}(\theta).$$

The corresponding function on C, also denoted by $\mathbf{F}$, is called the *spectral distribution* of the S.P. We know [I, 7.12] that $(\mathfrak{f}_n)_{-\infty}^{\infty}$ is regular and of full-rank, if and only if $\mathbf{F}$ is absolutely continuous and $\log \Delta \mathbf{F}' \in L_1$. For such processes, we define the space $\mathbf{L}_{2,F}$ as comprising all functions $\boldsymbol{\Phi}$ on C such that $\boldsymbol{\Phi}\mathbf{F}'\boldsymbol{\Phi}^* \in \mathbf{L}_1$. The structure of this space is governed by the Gramian, inner product and norm:

$$\left.\begin{aligned}(\boldsymbol{\Phi}, \boldsymbol{\Psi})_F &= \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{\Phi}(e^{i\theta})\,\mathbf{F}'(e^{i\theta})\,\boldsymbol{\Psi}^*(e^{i\theta})\,d\theta,\\ ((\boldsymbol{\Phi}, \boldsymbol{\Psi}))_F &= \tau(\boldsymbol{\Phi}, \boldsymbol{\Psi})_F, \qquad \|\boldsymbol{\Phi}\|_F = \sqrt{\tau(\boldsymbol{\Phi}, \boldsymbol{\Phi})_F}.\end{aligned}\right\} \tag{1.5}$$

In terms of this structure the space $\mathbf{L}_{2,F}$ is isomorphic to $\mathfrak{M}_{\infty}$, as shown in II, 4.10.

2. Full-rank minimal processes

In this section we shall extend to the q-variate case the spectral characterization of simple minimal sequences due to Kolmogorov [2, Thm. 24]. Our approach [3] will differ from Kolmogorov's in that we shall lean on the Wold-Zasuhin decomposition rather than on results on subordinate sequences.

2.1. Definition. *We shall call a q-ple stationary process* $(\mathfrak{f}_n)_{-\infty}^{\infty}$ minimal, *if and only if for some n, $\mathfrak{f}_n \notin \mathfrak{M}_n'$, where $\mathfrak{M}_n' = \mathfrak{S}(\mathfrak{f}_k)_{k \neq n}$.*

From the stationarity property it follows that the relation given in 2.1 holds for a single n only if it holds for all n. Hence for a minimal process

$$\boldsymbol{\varphi}_n = \mathfrak{f}_n - (\mathfrak{f}_n \mid \mathfrak{M}_n') \neq \mathbf{0}, \qquad -\infty < n < \infty. \tag{2.2}$$

We shall call the functions $\boldsymbol{\varphi}_n$ the *two-sided innovations* of the $\mathfrak{f}_n$-process. Unlike the ordinary (one-sided) innovation functions $\mathfrak{g}_n$, they do not form an orthogonal set, but we still have

$$\boldsymbol{\varphi}_n = U^n \boldsymbol{\varphi}_0, \qquad (\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_n) = (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0) \tag{2.3}$$

where U is the shift operator of $(\mathfrak{f}_n)_{-\infty}^{\infty}$.

2.4. Definition. *We shall call* $(\mathfrak{f}_n)_{-\infty}^{\infty}$ *a* full-rank minimal process, *if and only if the Gramian $(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0)$ is positive definite, i.e., $\Delta(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0) > 0$.*

Since $\mathfrak{M}_{n-1} \subseteq \mathfrak{M}_n'$ it follows readily that $(\mathfrak{g}_n, \mathfrak{g}_n) \succ (\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_n)$ (1), cf. I, 5.10 (b); whence we get:

(1) $\mathbf{A} \succ \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is non-negative hermitian.

2.5. LEMMA. *A minimal S.P. is non-deterministic; a full-rank minimal S.P. is non-deterministic and of full-rank.*

We shall "normalise" the two-sided innovations as follows (1):

2.6. DEFINITION. *Let* $(\mathbf{f}_n)_{-\infty}^{\infty}$ *be full-rank minimal, and let, cf.* (2.2),

$$\boldsymbol{\psi}_n = (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0)^{-1} \boldsymbol{\varphi}_n,$$

We shall call $(\boldsymbol{\psi}_n)_{-\infty}^{\infty}$ *the* normalised two-sided innovation process *of* $(\mathbf{f}_n)_{-\infty}^{\infty}$.

The following lemma gives the basic properties of this process:

2.7. LEMMMA. *Let* $(\mathbf{f}_n)_{-\infty}^{\infty}$ *be a minimal, full-rank process, and let* $(\boldsymbol{\psi}_n)_{-\infty}^{\infty}$ *be its normalised two-sided innovation process. Then*

(a) *the sequences* $(\boldsymbol{\psi}_n)_{-\infty}^{\infty}$, $(\mathbf{f}_n)_{-\infty}^{\infty}$ *are biorthogonal, i.e.,* $(\boldsymbol{\psi}_m, \mathbf{f}_n) = \delta_{mn}\mathbf{I}$;

(b) $\boldsymbol{\psi}_n = \sum_{k=0}^{\infty} \mathbf{D}_k^* \mathbf{h}_{n+k},$

where $(\mathbf{h}_n)_{-\infty}^{\infty}$ *is the normalised (one-sided) innovation process of* $(\mathbf{f}_n)_{-\infty}^{\infty}$ *cf.* [I, 6.12], *and* $\mathbf{D}_k$ *is the k-th Taylor coefficient of the reciprocal* $\boldsymbol{\Phi}_+^{-1}$ *of the inner holomorphic extension* $\boldsymbol{\Phi}_+$ *of its generating function* $\boldsymbol{\Phi}$, *cf.* [II, 2.6] (2);

(c) $\boldsymbol{\Phi}^{-1} \in \mathbf{L}_2^{0+}$.

Proof. (a) Write $\mathbf{A}$ for $(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0)^{-1}$ and P for the projection on $\mathfrak{M}_\infty$ onto $\mathfrak{M}_\infty \cap (\mathfrak{M}_0')^{\perp}$, so that $\boldsymbol{\varphi}_0 = P\mathbf{f}_0$. Then

$$(\boldsymbol{\psi}_0, \mathbf{f}_0) = (\mathbf{A}\boldsymbol{\varphi}_0, \mathbf{f}_0) = \mathbf{A}(P\mathbf{f}_0, \mathbf{f}_0) = \mathbf{A}(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0) = \mathbf{I}. \tag{1}$$

Also for $n \neq 0$, $\mathbf{f}_0 \in \mathfrak{M}_n' \perp \boldsymbol{\psi}_n$; hence

$$(\boldsymbol{\psi}_n, \mathbf{f}_0) = \mathbf{0}. \tag{2}$$

Applying the shift operator U^k in (1) and (2) we get (a).

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(1) This normalization, which differs from that adopted by Kolmogorov, is chosen so that the resulting process $(\boldsymbol{\psi}_n)_{-\infty}^{\infty}$ may have spectral density $\mathbf{F}'^{-1}$, where $\mathbf{F}$ is the spectral distribution of $(\mathbf{f}_n)_{-\infty}^{\infty}$, cf. Cor. 2.9 below.

(2) In view of the first and last equalities in II (2.5) it follows that $\boldsymbol{\Phi}_+^{-1}$ is holomorphic on the disk D_+.

(b) Since $\boldsymbol{\psi}_0 \perp \mathfrak{M}_0' \supseteq \mathfrak{M}_n$ for $n<0$, therefore

$$\boldsymbol{\psi}_0 \perp \bigcap_{n<0} \mathfrak{M}_n = \mathfrak{M}_{-\infty}. \tag{3}$$

It follows that $\boldsymbol{\psi}_0 \in \mathfrak{S}(\mathbf{h}_k)_{-\infty}^{\infty}$. for obviously $\boldsymbol{\psi}_0 \in \mathfrak{M}_\infty$, and by I, 6.10 (b)

$$\mathfrak{M}_\infty = \mathfrak{M}_{-\infty}^{\infty} + \mathfrak{S}(\mathbf{h}_k)_{-\infty}^{\infty}, \qquad \mathfrak{M}_{-\infty} \perp \mathfrak{S}(\mathbf{h}_k)_{-\infty}^{\infty}.$$

Actually $\boldsymbol{\psi}_0 \in \mathfrak{S}(\mathbf{h}_k)_{k=0}^{\infty}$, since for $k<0$, $\mathbf{h}_k \in \mathfrak{M}_{-1} \subseteq \mathfrak{M}_0' \perp \boldsymbol{\psi}_0$. Hence

$$\boldsymbol{\psi}_0 = \sum_{k=0}^{\infty} \mathbf{A}_k \mathbf{h}_k, \qquad \sum_0^\infty |\mathbf{A}_k|_E^2 < \infty. \tag{4}$$

Now by I, 6.11 (a), 6.12

$$\mathbf{f}_n = \sum_{k=0}^{\infty} \mathbf{C}_k \mathbf{h}_{n-k} + \mathbf{v}_n,$$

where $\mathbf{v}_n \in \mathfrak{M}_{-\infty}$. Hence by (3) and (4)

$$(\boldsymbol{\psi}_0, \mathbf{f}_n) = \left(\sum_{j=0}^{\infty} \mathbf{A}_j \mathbf{h}_j, \quad \sum_{k=0}^{\infty} \mathbf{C}_k \mathbf{h}_{n-k} \right) = \sum_{j=0}^{n} \mathbf{A}_j \mathbf{C}_{n-j}^*.$$

Hence by (a)

$$\sum_{j=0}^{n} \mathbf{A}_j \mathbf{C}_{n-j}^* = \delta_{n0} \mathbf{I}. \tag{5}$$

But by II, 2.6, $\boldsymbol{\Phi} = \sum_{k=0}^{\infty} \mathbf{C}_k e^{ki\theta}$; hence

$$\sum_{k=0}^{n} \mathbf{C}_{n-k} \mathbf{D}_k = \delta_{n0} \mathbf{I}.$$

Taking the adjoint, and comparing with (5) we see by a simple inductive argument that $\mathbf{A}_j = \mathbf{D}_j^*$. On applying U^n to both sides of (4) we now get (b).

(c) It follows from (b) and (4) that $\sum_0^\infty |\mathbf{D}_k|_E^2 < \infty$. Each entry of the matrix-function $\boldsymbol{\Phi}_+^{-1}$ is therefore in the Hardy class H_2 on D_+, and hence its radial limit is in L_2^{0+}. This means that each entry of $\boldsymbol{\Phi}^{-1}$ is in L_2^{0+}, i.e., $\boldsymbol{\Phi}^{-1} \in \mathbf{L}_2^{0+}$. (Q.E.D.)

We are now ready to give the spectral criterion for full-rank minimality:

2.8. Theorem. *Let $(\mathbf{f}_n)_{-\infty}^{\infty}$ be a q-ple stationary S.P. with spectral distribution* $\mathbf{F}$. *Then $(\mathbf{f}_n)_{-\infty}^{\infty}$ is full-rank minimal, if and only if $\mathbf{F}'$ is invertible a.e. on C, and $\mathbf{F}'^{-1} \in \mathbf{L}_1$.*

Proof. Let $(\mathbf{f}_n)_{-\infty}^{\infty}$ be full-rank minimal. Then by 2.5 it is non-deterministic and of full rank. Hence by I, 7.10 $\Delta \mathbf{F}' \neq 0$, a.e., and therefore $\mathbf{F}'^{-1}$ is defined a.e. Since by 2.7 (c), $\boldsymbol{\Phi}^{-1} \in \mathbf{L}_2$, it follows that $\mathbf{F}'^{-1} = (\boldsymbol{\Phi}^*)^{-1} \boldsymbol{\Phi}^{-1} \in \mathbf{L}_1$.

Next, suppose that $\mathbf{F}'^{-1} \in \mathbf{L}_1$. Then by I, 3.7 (e), $1/\Delta\,\mathbf{F}' \in L_{1/q}$. Likewise $\Delta\,\mathbf{F}' \in L_{1/q}$, since $\mathbf{F}' \in \mathbf{L}_1$. The inequality

$$|\log \Delta\,\mathbf{F}'| = q\,|\log \sqrt[q]{\Delta\,\mathbf{F}'}| \leqslant q \max\left(\sqrt[q]{\Delta\,\mathbf{F}'},\ \frac{1}{\sqrt[q]{\Delta\,\mathbf{F}'}}\right)$$

now shows that $\log \Delta\,\mathbf{F}' \in L_1$. Hence by I, 7.10, $(\mathbf{f}_n)_{-\infty}^{\infty}$ is non-deterministic and of full-rank. We therefore have the Wold-Zasuhin decomposition

$$\mathbf{f}_n = \mathbf{u}_n + \mathbf{v}_n = \sum_{k=0}^{\infty} \mathbf{C}_k\,\mathbf{h}_{n-k} + \mathbf{v}_n, \tag{1}$$

where $\mathbf{v}_n \in \mathfrak{M}_{-\infty}$ and $(\mathbf{u}_n)_{-\infty}^{\infty}$ is regular of full-rank with spectral density

$$\mathbf{F}'_u = \mathbf{F}' = \mathbf{\Phi}\,\mathbf{\Phi}^*, \tag{2}$$

cf. I, 6.12 and 7.11.

Now let $\mathfrak{N}_n = \mathfrak{S}\,(\mathbf{u}_k)_{k=-\infty}^{n}$, $\mathfrak{N}_\infty = \mathfrak{S}\,(\mathbf{u}_k)_{k=-\infty}^{\infty}$. By II, 4.10 there is an isomorphism on $\mathfrak{N}_\infty$ onto $\mathbf{L}_{2,F_u}$ such that to $\mathbf{u}_k$ corresponds $e^{-ki\theta}\,\mathbf{I}$, and if to $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2 \in \mathfrak{N}_\infty$ correspond $\mathbf{X}_1, \mathbf{X}_2 \in \mathbf{L}_{2,F_u}$, then (cf. (2))

$$(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{X}_1(e^{i\theta})\,\mathbf{F}'(e^{i\theta})\,\mathbf{X}_2^*(e^{i\theta})\,d\theta. \tag{3}$$

Now $\mathbf{F}'^{-1} \in \mathbf{L}_{2,F_u}$, since $\mathbf{F}'^{-1}\,\mathbf{F}'_u\,(\mathbf{F}'^{-1})^* = \mathbf{F}'^{-1} \in \mathbf{L}_1$. Hence if $\boldsymbol{\chi}$ is the function in $\mathfrak{N}_\infty$ corresponding to $\mathbf{F}'^{-1}$, then by (3)

$$(\boldsymbol{\chi}, \boldsymbol{\chi}) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{F}'^{-1}(e^{i\theta})\,\mathbf{F}'(e^{i\theta})\,\{\mathbf{F}'^{-1}(e^{i\theta})\}^*\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{F}'^{-1}(e^{i\theta})\,d\theta, \tag{4}$$

$$(\boldsymbol{\chi}, \mathbf{u}_n) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{F}'^{-1}(e^{i\theta})\,\mathbf{F}'(e^{i\theta})\,e^{ni\theta}\,d\theta = \delta_{n0}\,\mathbf{I}. \tag{5}$$

Since $\boldsymbol{\chi} \in \mathfrak{N}_\infty \subseteq \mathfrak{S}\,(\mathbf{h}_k)_{-\infty}^{\infty}$, $\mathbf{v}_n \in \mathfrak{M}_{-\infty}$, $\mathfrak{S}\,(\mathbf{h}_k)_{-\infty}^{\infty} \perp \mathfrak{M}_{-\infty}$, it follows from (1) and (5) that

$$(\boldsymbol{\chi}, \mathbf{f}_n) = (\boldsymbol{\chi}, \mathbf{u}_n) + (\boldsymbol{\chi}, \mathbf{v}_n) = \delta_{n0}\,\mathbf{I}.$$

Thus $\boldsymbol{\chi} \perp \mathbf{f}_n$, for $n \neq 0$; therefore $\boldsymbol{\chi} \perp \mathfrak{M}'_0 = \mathfrak{S}\,(\mathbf{f}_k)_{k\neq 0}$. But obviously $\boldsymbol{\chi} \in \mathfrak{M}_\infty$, since $\mathfrak{N}_\infty \subseteq \mathfrak{M}_\infty$. Also $\boldsymbol{\chi} \neq \mathbf{0}$, since by (4) $(\boldsymbol{\chi}, \boldsymbol{\chi})$ is positive definite. Thus $\mathfrak{M}'_0 \neq \mathfrak{M}_\infty$, which means that $\mathbf{f}_0 \notin \mathfrak{M}'_0$, i.e., $(\mathbf{f}_n)_{-\infty}^{\infty}$ is a minimal S.P.

That $(\mathbf{f}_n)_{-\infty}^{\infty}$ is full-rank minimal depends on the easily established relation

$$\mathfrak{M}_\infty = \mathfrak{M}'_0 + \mathfrak{S}\,(\boldsymbol{\varphi}_0), \qquad \boldsymbol{\varphi}_0 \perp \mathfrak{M}'_0. \tag{6}$$

For since $\boldsymbol{\chi} \in \mathfrak{M}_\infty$ and $\boldsymbol{\chi} \perp \mathfrak{M}_0'$, it follows from (6) that $\boldsymbol{\chi} \in \mathfrak{S}(\boldsymbol{\varphi}_0)$, i.e., $\boldsymbol{\chi} = \mathbf{A}\,\boldsymbol{\varphi}_0$. Hence

$$(\boldsymbol{\chi}, \boldsymbol{\chi}) = \mathbf{A}\,(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0)\,\mathbf{A}^*.$$

Since by (4) $\Delta(\boldsymbol{\chi}, \boldsymbol{\chi}) > 0$, it follows that $\Delta(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0) > 0$, i.e., $(\mathbf{f}_n)_{-\infty}^{\infty}$ is full-rank minimal. (Q.E.D.)

2.9. Corollary. *If $(\mathbf{f}_n)_{-\infty}^{\infty}$ is a full-rank minimal process with spectral distribution $\mathbf{F}$, then $\mathbf{F}'^{-1}$ is the spectral density of its normalised two-sided innovation process $(\boldsymbol{\psi}_n)_{-\infty}^{\infty}$* (cf. 2.2, 2.6), *which is a regular process of full-rank; moreover*

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} \mathbf{F}'^{-1}(e^{i\theta})\,d\theta\right\}^{-1} = (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0),$$

where $\boldsymbol{\varphi}_0$ is as in (2.2).

Proof. By 2.7 (b)

$$(\boldsymbol{\psi}_n, \boldsymbol{\psi}_0) = \left(\sum_{k=0}^{\infty} \mathbf{D}_k^*\,\mathbf{h}_{n+k},\ \sum_{k=0}^{\infty} \mathbf{D}_k^*\,\mathbf{h}_k\right) = \sum_{k=\max\{-n,0\}}^{\infty} \mathbf{D}_k^*\,\mathbf{D}_{k+n}.$$

In this $\mathbf{D}_k$ is the kth Fourier coefficient of the reciprocal $\boldsymbol{\Phi}^{-1}$ of the generating function of $(\mathbf{f}_n)_{-\infty}^{\infty}$. Since the kth coefficient of $(\boldsymbol{\Phi}^{-1})^*$ is $\mathbf{D}_{-k}^*$, it follows readily that the last sum is the nth Fourier coefficient of $(\boldsymbol{\Phi}^{-1})^*\,\boldsymbol{\Phi}^{-1}$, i.e., of $\mathbf{F}'^{-1}$. The $\boldsymbol{\psi}_n$-process thus has spectral density $\mathbf{F}'^{-1}$.

By 2.5 $(\mathbf{f}_n)_{-\infty}^{\infty}$ is non-deterministic and of full-rank. Therefore by I, 7.10 $\log \Delta\,\mathbf{F}' \in L_1$. Hence $\log \Delta\,\mathbf{F}'^{-1} \in L_1$. It follows by I, 7.12 that $(\boldsymbol{\psi}_n)_{-\infty}^{\infty}$ is a regular process of full-rank. Finally,

$$\frac{1}{2\pi}\int_0^{2\pi} \mathbf{F}'^{-1}(e^{i\theta})\,d\theta = (\boldsymbol{\psi}_0, \boldsymbol{\psi}_0) = (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0)^{-1},$$

in view of 2.6. (Q.E.D.)

We conclude this section with the following obvious consequence of 2.7 (c) and the Convolution Rule I, 3.9 (d):

2.10. Corollary. *If $(\mathbf{f}_n)_{-\infty}^{\infty}$ is a regular, full-rank minimal process with spectral density $\mathbf{F}'$ and generating function $\boldsymbol{\Phi}$, and for $\nu > 0$,*

$$\mathbf{Y}_\nu(e^{i\theta}) = [e^{-\nu i\theta}\,\boldsymbol{\Phi}(e^{i\theta})]_{0+}\,\boldsymbol{\Phi}^{-1}(e^{i\theta}),$$

then $\mathbf{Y}_\nu \in L_1^{0+}$ and its k-th Fourier coefficient is given by

$$\mathbf{E}_{\nu k} = \sum_{n=0}^{k} \mathbf{C}_{\nu+n}\, \mathbf{D}_{k-n},$$

where $\mathbf{C}_k$, $\mathbf{D}_k$ *are the k-th Fourier coefficients of* $\mathbf{\Phi}$, $\mathbf{\Phi}^{-1}$.

$\mathbf{Y}_\nu$ corresponds in the frequency domain to the linear predictor $\hat{\mathbf{f}}_\nu = (\mathbf{f}_\nu \mid \mathfrak{M}_0)$ in the time-domain under the isomorphism between $\mathfrak{M}_\infty$ and $\mathbf{L}_{2,F}$, cf. II, 4.11. It is therefore the *frequency-response* or *transfer-function* of the S.P. for lag ν (cf. II, 5.8 et seq.). The corollary shows that the Fourier series of this function can be derived from those of the generating function and its reciprocal. Two problems now arise:

(1) To obtain the generating function from the spectral density matrix $\mathbf{F}'$.

(2) To find out if the Fourier coefficients $\mathbf{E}_{\nu k}$ of $\mathbf{Y}_\nu$ can be used to get the predictor $\hat{\mathbf{f}}_\nu$ in the time-domain.

Both problems were solved in II under the Boundedness Condition II, 5.1. In § 4 we shall solve problem (1) under the weaker conditions 3.1. In § 5 we shall solve problem (2) under conditions, which though more stringent than 3.1 are weaker than II, 5.1.

3. The spectral assumption; initial factorization

To use the $\mathcal{D}$ operator mentioned in § 1, it is necessary to factor the spectral density in the form $\mathbf{F}' = F_1' \tilde{\mathbf{F}}'$, where F_1' is complex-valued and in L_1, $\tilde{\mathbf{F}}' \in \mathbf{L}_\infty$, and $1/F_1' \in L_1$, $\tilde{\mathbf{F}}'^{-1} \in \mathbf{L}_1$. The conditions on the reciprocals are required in order that we may utilize the preceding results on minimal processes (§ 2). To be able to carry out such a factorization we have to make the following assumption regarding $\mathbf{F}'$:

3.1. Assumption. *Our q-ple regular full-rank process* $(\mathbf{f}_n)_{-\infty}^{\infty}$ *has a spectral density* $\mathbf{F}'$ *such that*

(i) $\mathbf{F}'^{-1} \in \mathbf{L}_1$ *on* C,

(ii) *if* $\lambda(e^{i\theta})$, $\mu(e^{i\theta})$ *are the smallest and largest eigenvalues of* $\mathbf{F}'(e^{i\theta})$, *then* $\mu/\lambda \in L_1$ *on* C.

Now suppose that $\mathbf{F}'$, $\mathbf{F}'^{-1} \in \mathbf{L}_2$. Then obviously $\mathbf{F}'$, $\mathbf{F}'^{-1} \in \mathbf{L}_1$. Since for any θ, the trace of $\mathbf{F}'(e^{i\theta})$ is equal to the sum of its eigenvalues and similarly for $\mathbf{F}'^{-1}$, we will have, cf. II, 1.9 (c),

$$0 \leqslant \mu \leqslant \tau(\mathbf{F}') \in L_2, \qquad 0 \leqslant 1/\lambda \leqslant \tau(\mathbf{F}'^{-1}) \in L_2,$$

and therefore $\mu/\lambda \in L_1$. Thus

3.2. LEMMA. *The condition* 3.1 *will be satisfied if* $\mathbf{F}'$, $\mathbf{F}'^{-1} \in \mathbf{L}_2$.

We shall use the following notation:

$$\left.\begin{aligned} F_1'(e^{i\theta}) &= \tfrac{1}{2}\{\lambda(e^{i\theta}) + \mu(e^{i\theta})\} \\ \mathbf{M}(e^{i\theta}) &= \frac{1}{F_1'(e^{i\theta})}\mathbf{F}'(e^{i\theta}) - \mathbf{I}, \text{ a.e.} \end{aligned}\right\} \tag{3.3}$$

Since by 3.1 (i), $F_1'(e^{i\theta}) \geqslant \frac{1}{2}\lambda(e^{i\theta}) > 0$, a.e., the function $\mathbf{M}$ is well-defined a.e. The initial factorization alluded to above is given by the following:

3.4. LEMMA. *If* $(\mathfrak{f}_n)_{-\infty}^{\infty}$ *satisfies Assumption* 3.1, *then*

(a) $$\mathbf{F}' = F_1'(\mathbf{I} + \mathbf{M}), \ a.e.;$$

(b) $$|\mathbf{M}(e^{i\theta})|_B \leqslant 1, \ 0 \leqslant \theta \leqslant 2\pi; \ |\mathbf{M}(e^{i\theta})|_B < 1, \ a.e.;$$

(c) $\mathbf{I}+\mathbf{M}$, $(\mathbf{I}+\mathbf{M})^{-1} \in \mathbf{L}_1$; $\mathbf{I}+\mathbf{M}$ *is therefore the spectral density of a q-ple, regular, full-rank-minimal process* (cf. 2.8);

(d) F_1', $1/F_1' \in L_1$; F_1' *is therefore the spectral density of a simple regular, minimal process.*

Proof. (a) is obvious from (3.3).

(b) Since

$$\lambda(e^{i\theta})\mathbf{I} \prec \mathbf{F}'(e^{i\theta}) \prec \mu(e^{i\theta})\mathbf{I}, \tag{1}$$

we get, cf. II, 1.5 (c),

$$|\mathbf{M}(e^{i\theta})|_B = \left|\frac{2}{\lambda(e^{i\theta}) + \mu(e^{i\theta})}\mathbf{F}'(e^{i\theta}) - \mathbf{I}\right| \leqslant \frac{\mu(e^{i\theta}) - \lambda(e^{i\theta})}{\mu(e^{i\theta}) + \lambda(e^{i\theta})} \leqslant 1. \tag{2}$$

Also by (3.1) (ii), $\lambda > 0$, a.e. and therefore $|\mathbf{M}(e^{i\theta})|_B < 1$, a.e.

(c) By (b), $\mathbf{I}+\mathbf{M} \in \mathbf{L}_\infty \subseteq \mathbf{L}_1$. Next, from (2)

$$\frac{2\lambda}{\lambda+\mu}\mathbf{I} \prec \mathbf{I}+\mathbf{M} \prec \frac{2\mu}{\lambda+\mu}\mathbf{I}, \text{ a.e.},$$

whence since $\lambda > 0$, a.e. we get

$$\tfrac{1}{2}(1+\lambda/\mu)\mathbf{I} \prec (\mathbf{I}+\mathbf{M})^{-1} \prec \tfrac{1}{2}(1+\mu/\lambda)\mathbf{I}.$$

Since $\mu/\lambda \in L_1$, we conclude that $(\mathbf{I}+\mathbf{M})^{-1} \in \mathbf{L}_1$.

(d) Since for any θ, the trace of $\mathbf{F}'(e^{i\theta})$ is the sum of the eigenvalues we have

$$0 \leqslant F_1' = \tfrac{1}{2}\{\lambda+\mu\} \leqslant \tfrac{1}{2}\tau(\mathbf{F}') \in L_1,$$

so that $F_1' \in L_1$. Similarly,

$$0 \leqslant \frac{1}{F_1'} = \frac{2}{\lambda + \mu} \leqslant \frac{2}{\mu} \leqslant 2\,\tau(\mathbf{F}'^{-1}) \in L_1,$$

and hence $1/F_1' \in L_1$. (Q.E.D.)

The next theorem shows that our factorization of $\mathbf{F}'$ yields corresponding factorizations of the generating function and of the prediction-error matrix.

3.5. THEOREM. *If* (i) $(\mathbf{f}_n)_{-\infty}^{\infty}$ *satisfies the Assumption* 3.1

(ii) $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$, $\boldsymbol{\Phi}_1$ *are the generating functions of the processes with spectral densities* $\mathbf{F}'$, $\mathbf{I}+\mathbf{M}$, F_1',

(iii) $\mathbf{G}$, $\tilde{\mathbf{G}}$ *are the predictor-error matrices with* lag 1 *of the first two processes, and* g *the innovation function of the third,*

then (a) $\boldsymbol{\Phi}^{-1}$, $\tilde{\boldsymbol{\Phi}}^{-1} \in \mathbf{L}_2^{0+}$, $1/\boldsymbol{\Phi}_1 \in L_2^{0+}$

(b) $\boldsymbol{\Phi} = \boldsymbol{\Phi}_1 \tilde{\boldsymbol{\Phi}}$,

(c) $\mathbf{G} = |g|^2 \tilde{\mathbf{G}}$.

Proof. (a) follows from 2.7 (c), since by hypothesis and 3.4 (i) the processes referred to are full-rank-minimal.

(b) We have (cf. II, 2.5, 2.6) the factorizations

$$\mathbf{F}' = \boldsymbol{\Phi}\boldsymbol{\Phi}^*, \qquad F_1' = \Phi_1 \overline{\Phi}_1, \qquad \mathbf{I}+\mathbf{M} = \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^*.$$

Hence by 3.4 (a)

$$\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \Phi_1 \overline{\Phi}_1 \cdot \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* = (\Phi_1 \tilde{\boldsymbol{\Phi}})\,(\Phi_1 \tilde{\boldsymbol{\Phi}})^* = \boldsymbol{\Psi}\boldsymbol{\Psi}^*, \text{ say.} \tag{1}$$

Our problem is to identify $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. For this we appeal to our Uniqueness Thm. II, 8.12.

We first note, cf. (a), that

$$\boldsymbol{\Phi},\ \boldsymbol{\Phi}^{-1} \in \mathbf{L}_2^{0+}. \tag{2}$$

We assert next that

$$\boldsymbol{\Psi},\ \boldsymbol{\Psi}^{-1} \in \mathbf{L}_2^{0+}. \tag{3}$$

This follows, for on the other hand by (1) $\boldsymbol{\Psi}\boldsymbol{\Psi}^* = \mathbf{F}' \in \mathbf{L}_1$, $(\boldsymbol{\Psi}^{-1})^*\boldsymbol{\Psi}^{-1} = \mathbf{F}'^{-1} \in \mathbf{L}_1$ and so we have $\boldsymbol{\Psi}, \boldsymbol{\Psi}^{-1} \in \mathbf{L}_2$; and on the other hand we have $\boldsymbol{\Psi} = \Phi_1 \tilde{\boldsymbol{\Phi}} \in \mathbf{L}_2^{0+}$, $\boldsymbol{\Psi}^{-1} = (1/\Phi_1)\tilde{\boldsymbol{\Phi}}^{-1} \in \mathbf{L}_1^{0+}$ since $\Phi_1 \in L_2^{0+}$, $\tilde{\boldsymbol{\Phi}} \in \mathbf{L}_2^{0+}$ and (cf. (a)) $1/\Phi_1 \in L_2^{0+}$, $\tilde{\boldsymbol{\Phi}}^{-1} \in \mathbf{L}_2^{0+}$. Finally, since by II, (2.5), $\boldsymbol{\Phi}(0) = \sqrt{\mathbf{G}}$, $\tilde{\boldsymbol{\Phi}}_0(0) = \sqrt{\tilde{\mathbf{G}}}$ are positive definite and $\Phi_1(0) = |g| > 0$, we have

$$\mathbf{\Phi}(0),\ \mathbf{\Psi}(0) \textit{ are positive definite.} \tag{4}$$

From (1)–(4) and the Uniqueness Thm. II, 8.12 it follows that $\mathbf{\Phi} = \mathbf{\Psi}$, i.e., $\mathbf{\Phi} = \Phi_1 \tilde{\mathbf{\Phi}}$.

(c) clearly follows from (b). (Q.E.D.)

We know how to find the generating function of any simple regular S.P. by optimal factorization of its spectral density [II, p. 103]. In view of the last theorem all we have to do to get the generating function of the given S.P. $(\mathbf{f}_n)_{-\infty}^{\infty}$ is to determine that of the process whose spectral density is $\mathbf{I}+\mathbf{M}$. An algorithm for accomplishing this is given in the next section.

4. Determination of the generating and frequency-response functions

Let $(\mathbf{f}_n)_{-\infty}^{\infty}$ be a S.P. satisfying the Assymption 3.1 and let $\mathbf{M}$ be as in (3.3). Then by 3.4 (b), (c), $\mathbf{M}$ certainly satisfies the following:

4.1. Conditions. (i) $\mathbf{M}$ *is hermitian-valued on* C, *and* $\mathbf{M} \in \mathbf{L}_\infty$.

(ii) $|\mathbf{M}(e^{i\theta})|_B \leqslant 1$ *on* C.

(iii) $|\mathbf{M}(e^{i\theta})|_B < 1$ *on a subset of* C *of positive measure.* (1)

(iv) $(\mathbf{I}+\mathbf{M})^{-1} \in \mathbf{L}_1$.

We shall therefore assume that the function $\mathbf{M}$ to be dealt with in this section satisfies 4.1. We can then define our $\mathcal{D}$ operator exactly as in II, 6.2, the definition being meaningful, since for $\mathbf{\Psi} \in \mathbf{L}_2$ and $\mathbf{M} \in \mathbf{L}_\infty$ we have $\mathbf{\Psi}\mathbf{M} \in \mathbf{L}_2$:

4.2. Definition. *For any* $\mathbf{\Psi} \in \mathbf{L}_2$, $\mathcal{D}(\mathbf{\Psi}) = (\mathbf{\Psi}\mathbf{M})_+$.

The following properties of $\mathcal{D}$ are easily established using 4.1 (ii):

4.3. Lemma. (a) $\mathcal{D}$ *is a bounded linear operator on* $\mathbf{L}_2$ *into* $\mathbf{L}_2^+$, *and* $|\mathcal{D}|_B \leqslant 1$.

(b) $\mathcal{D}(\mathbf{I}) = \mathbf{M}_+$, $\mathcal{D}^2(\mathbf{I}) = (\mathbf{M}_+\mathbf{M})_+$, $\mathcal{D}^3(\mathbf{I}) = (\mathbf{M}_+\mathbf{M})_+\mathbf{M}_+$, *and so on.*

The relevance of the operator $\mathcal{D}$ to the problem of determining the generating function $\mathbf{\Phi}$ of a S.P. with spectral density $\mathbf{I}+\mathbf{M}$ is seen from the following lemma:

4.4. Main Lemma. *Let* $\tilde{\mathbf{\Phi}}$, $\tilde{\mathrm{G}}$ *be the generating function and prediction error matrix with* lag 1 *of a S.P. with spectral density* $\mathbf{I}+\mathbf{M}$. *Then*

$$(\mathcal{J}+\mathcal{D})(\sqrt{\tilde{\mathrm{G}}}\,\tilde{\mathbf{\Phi}}^{-1}) = \mathbf{I},$$

where $\mathcal{J}$ *is the identity operator on* $\mathbf{L}_2$.

---

(1) Actually $|\mathbf{M}(e^{i\theta})|_B < 1$ a.e., but the weaker condition (iii) is all we will need.

Proof. The proof rests on the crucial fact that $\tilde{\mathbf{\Phi}}^{-1} \in \mathbf{L}_2^{0+}$, which stems from 4.1 (iv), 2.8, 2.7 (c). Now let $\mathbf{\Psi} = \sqrt{\tilde{\mathbf{G}}}\,\tilde{\mathbf{\Phi}}^{-1}$. Then $\mathbf{\Psi} \in \mathbf{L}_2^{0+}$. Also, since $\tilde{\mathbf{\Phi}}(0) = \sqrt{\tilde{\mathbf{G}}}$, we have $\mathbf{\Psi}(0) = \mathbf{I}$. Hence

$$\mathbf{\Psi} = \mathbf{I} + \mathbf{\Psi}_+. \tag{1}$$

Next, since $\mathbf{I} + \mathbf{M} = \tilde{\mathbf{\Phi}}\,\tilde{\mathbf{\Phi}}^*$, we have

$$\mathbf{\Psi} + \mathbf{\Psi}\mathbf{M} = \sqrt{\tilde{\mathbf{G}}}\,\tilde{\mathbf{\Phi}}^{-1}(\mathbf{I} + \mathbf{M}) = \sqrt{\tilde{\mathbf{G}}}\,\tilde{\mathbf{\Phi}}^* \in \mathbf{L}_2^{0-}.$$

Hence
$$\mathbf{\Psi}_+ + (\mathbf{\Psi}\mathbf{M})_+ = \mathbf{0},$$

i.e., by (1),
$$\mathbf{\Psi} - \mathbf{I} + (\mathbf{\Psi}\mathbf{M})_+ = 0,$$

i.e.,
$$(\mathcal{J} + \mathcal{D})\mathbf{\Psi} = \mathbf{I}. \quad \text{(Q.E.D.)}$$

This lemma shows that we can get $\tilde{\mathbf{\Phi}}$, if we can invert the operator $\mathcal{J} + \mathcal{D}$. We proceed to show that this can be done in view of the condition 4.1 (iii).

4.5. LEMMA. (a) $\mathcal{D}$ *is a strict contraction operator on* $\mathbf{L}_2^{0+}$; *i.e.,*

$$\mathbf{0} \neq \mathbf{\Psi} \in \mathbf{L}_2^{0+} \textit{ implies } \quad \|\mathcal{D}(\mathbf{\Psi})\| < \|\mathbf{\Psi}\|.$$

(b) $\mathcal{J} + \mathcal{D}$ *is one-one on* $\mathbf{L}_2$ *into itself.*

(c) $\mathcal{D}$ *is hermitian on* $\mathbf{L}_2^+$, *i.e., for all* $\mathbf{\Psi}$, $\mathbf{X} \in \mathbf{L}_2^+$,

$$(\mathcal{D}(\mathbf{\Psi}), \mathbf{X}) = (\mathbf{\Psi}, \mathcal{D}(\mathbf{X})), \qquad ((\mathcal{D}(\mathbf{\Psi}), \mathbf{X})) = ((\mathbf{\Psi}, \mathcal{D}(\mathbf{X}))).$$

Proof. (a) By 4.1 (iii), there is a positive number ε and subset C_ε of C such that

$$|\mathbf{M}(e^{i\theta})|_B < \sqrt{(1-\varepsilon)} \quad \text{on } C_\varepsilon, \qquad \text{meas. } C_\varepsilon > 0.$$

Since $\|\mathcal{D}(\mathbf{\Psi})\| = \|(\mathbf{\Psi}\mathbf{M})_+\| \leqslant \|\mathbf{\Psi}\mathbf{M}\|$ we have, writing θ for brevity instead of $e^{i\theta}$,

$$\begin{aligned}
\|\mathcal{D}(\mathbf{\Psi})\|^2 &\leqslant \frac{1}{2\pi}\int_0^{2\pi} |\mathbf{\Psi}(\theta)\mathbf{M}(\theta)|_E^2\, d\theta \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} |\mathbf{\Psi}(\theta)|_E^2\,|\mathbf{M}(\theta)|_B^2\, d\theta \qquad \text{(by II, 1.4)} \\
&\leqslant (1-\varepsilon)\frac{1}{2\pi}\int_{C_\varepsilon} |\mathbf{\Psi}(\theta)|_E^2\, d\theta + \frac{1}{2\pi}\int_{C-C_\varepsilon} |\mathbf{\Psi}(\theta)|_E^2\, d\theta \\
&\leqslant \|\mathbf{\Psi}\|^2 - \frac{\varepsilon}{2\pi}\int_{C_\varepsilon} |\mathbf{\Psi}(\theta)|_E^2\, d\theta.
\end{aligned} \tag{1}$$

Now let $\mathbf{0} \neq \mathbf{\Psi} \in \mathbf{L}_2^{0+}$. Then its entries ψ_{ij} vanish almost nowhere on C. Hence

$$|\mathbf{\Psi}(\theta)|_E^2 = \sum_{i=1}^{q} \sum_{j=1}^{q} |\psi_{ij}(\theta)|^2 > 0 \quad \text{a.e.}$$

on C_ε. It follows that the last integral is positive, so that $\|\mathcal{D}(\mathbf{\Psi})\| < \|\mathbf{\Psi}\|$.

(b) Let $\mathbf{\Psi} \in \mathbf{L}_2$ and suppose that $(\mathcal{J} + \mathcal{D})(\mathbf{\Psi}) = \mathbf{\Psi} + \mathcal{D}(\mathbf{\Psi}) = \mathbf{0}$. Then

$$\mathbf{\Psi} = -\mathcal{D}(\mathbf{\Psi}) \in \mathbf{L}_2^+, \quad \text{and} \quad \|\mathbf{\Psi}\| = \|\mathcal{D}(\mathbf{\Psi})\|.$$

Hence by (a) $\mathbf{\Psi} = \mathbf{0}$. Thus $\mathcal{J} + \mathcal{D}$ is one-one on $\mathbf{L}_2$.

(c) For $\mathbf{\Psi}, \mathbf{X} \in \mathbf{L}_2^+$ we have the equalities

$$\begin{aligned}(\mathcal{D}(\mathbf{\Psi}), \mathbf{X}) &= ((\mathbf{\Psi}\mathbf{M})_+, \mathbf{X}) = (\mathbf{\Psi}\mathbf{M}, \mathbf{X}_+) = (\mathbf{\Psi}\mathbf{M}, \mathbf{X}) \\ &= (\mathbf{\Psi}, \mathbf{X}\mathbf{M}) = (\mathbf{\Psi}_+, \mathbf{X}\mathbf{M}) = (\mathbf{\Psi}, (\mathbf{X}\mathbf{M})_+) \\ &= (\mathbf{\Psi}, \mathcal{D}(\mathbf{X})),\end{aligned}$$

the second up to the sixth of which follow respectively from 1.2, the facts that $\mathbf{X} = \mathbf{X}_+$, $\mathbf{M} = \mathbf{M}^*$. $\mathbf{\Psi} = \mathbf{\Psi}_+$, and 1.2. On taking the trace we get the second equality in (c). (Q.E.D.)

By 4.5 (b) the operator $\mathcal{J} + \mathcal{D}$ (with domain $\mathbf{L}_2$) is invertible on its range $\mathbf{R}$, but $\mathbf{R}$ may not be the whole of $\mathbf{L}_2$. (From 4.5 (c) it follows that $\mathbf{R}$ has a subset everywhere dense in $\mathbf{L}_2^+$.) We know from 4.4, however, that $\mathbf{I} \in \mathbf{R}$. We shall now show that the usual geometric series for the (posssibly unbounded) operator $(\mathcal{J} + \mathcal{D})^{-1}$ converges strongly on $\mathbf{R}$. The application of this operator-series to $\mathbf{I}$ will yield our algorithm for the generating function.

4.6. Theorem. (a) *$\mathcal{D}^n \to 0$ strongly on $\mathbf{L}_2$, as $n \to \infty$; i.e., for all $\mathbf{\Psi} \in \mathbf{L}_2$,* $\lim_{n\to\infty} \|\mathcal{D}^n(\mathbf{\Psi})\| = 0$.

(b) *If $\mathbf{\Psi}$ is in the range of $\mathcal{J} + \mathcal{D}$, then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \mathcal{D}^k(\mathbf{\Psi}) = (\mathcal{J} + \mathcal{D})^{-1}(\mathbf{\Psi}),$$

the limit being taken with respect to the L_2-norm $\|\ \|$.

Proof. (a) Let $\mathbf{\Psi} \in \mathbf{L}_2$. Then, cf. 1.1,

$$\|\mathcal{D}^m(\mathbf{\Psi}) - \mathcal{D}^n(\mathbf{\Psi})\|^2 = \|\mathcal{D}^m(\mathbf{\Psi})\|^2 + \|\mathcal{D}^n(\mathbf{\Psi})\|^2 - 2 \text{ real } ((\mathcal{D}^m(\mathbf{\Psi}), \mathcal{D}^n(\mathbf{\Psi}))). \quad (1)$$

Consider first the case $m=n+2k$, $k\geqslant 0$. Since $\mathcal{D}^n(\boldsymbol{\Psi})$, $\mathcal{D}^{n+k}(\boldsymbol{\Psi})\in \mathbf{L}_2^+$, it follows from 4.5 (c) that

$$((\mathcal{D}^m(\boldsymbol{\Psi}),\ \mathcal{D}^n(\boldsymbol{\Psi})))=((\mathcal{D}^k\mathcal{D}^{n+k}(\boldsymbol{\Psi}),\ \mathcal{D}^n(\boldsymbol{\Psi})))=\|\mathcal{D}^{n+k}(\boldsymbol{\Psi})\|^2.$$

Hence letting $\mathbf{X}=\mathcal{D}(\boldsymbol{\Psi})\in\mathbf{L}_2^+$, (1) becomes

$$\|\mathcal{D}^{n+2k}(\boldsymbol{\Psi})-\mathcal{D}^n(\boldsymbol{\Psi})\|^2=\|\mathcal{D}^{n+2k-1}(\mathbf{X})\|^2+\|\mathcal{D}^{n-1}(\mathbf{X})\|^2-2\|\mathcal{D}^{n+k-1}(\mathbf{X})\|^2. \tag{2}$$

By 4.5 (a) the sequence $(\|\mathcal{D}^\nu(\mathbf{X})\|)_{\nu=1}^\infty$ is monotonic decreasing. Hence $\|\mathcal{D}^\nu(\mathbf{X})\|\to l\geqslant 0$, as $\nu\to\infty$. It follows from (2) that

$$\|\mathcal{D}^{n+2k}(\boldsymbol{\Psi})-\mathcal{D}^n(\boldsymbol{\Psi})\|\to 0,\qquad \text{as } n\to\infty. \tag{2$'$}$$

Next, take separately n even and n odd in (2′). We see that

$$\lim_{m,n\to\infty}\|\mathcal{D}^{2m}(\boldsymbol{\Psi})-\mathcal{D}^{2n}(\boldsymbol{\Psi})\|=0=\lim_{m,n\to\infty}\|\mathcal{D}^{2m+1}(\boldsymbol{\Psi})-\mathcal{D}^{2n+1}(\boldsymbol{\Psi})\|.$$

$\mathbf{L}_2$ being complete, this means that

$$\lim_{n\to\infty}\mathcal{D}^{2n}(\boldsymbol{\Psi})=\mathbf{X}_0,\quad \lim_{n\to\infty}\mathcal{D}^{2n+1}(\boldsymbol{\Psi})=\mathbf{X}_1,\quad \mathbf{X}_0,\mathbf{X}_1\in\mathbf{L}_2^+. \tag{3}$$

Now since $\mathcal{D}$ is bounded (and therefore continuous)

$$\mathbf{X}_1=\lim_{n\to\infty}\mathcal{D}(\mathcal{D}^{2n}(\boldsymbol{\Psi}))=\mathcal{D}(\mathbf{X}_0) \tag{4}$$

$$\mathbf{X}_0=\lim_{n\to\infty}\mathcal{D}(\mathcal{D}^{2n-1}(\boldsymbol{\Psi}))=\mathcal{D}(\mathbf{X}_1)=\mathcal{D}^2(\mathbf{X}_0). \tag{5}$$

Since (cf. 4.5 (a)) $\mathcal{D}$ and therefore $\mathcal{D}^2$ are contraction operators on $\mathbf{L}_2^+$, it follows from (5) that $\mathbf{X}_0=\mathbf{0}$, and from (4) that $\mathbf{X}_1=\mathbf{0}$. Thus from (3) $\lim_{n\to\infty}\mathcal{D}^n(\boldsymbol{\Psi})=\mathbf{0}$ in the L_2-norm.

(b) Let
$$S_n=\mathcal{J}-\mathcal{D}+\mathcal{D}^2-\cdots+(-1)^n\mathcal{D}^n.$$

Then
$$S_n(\mathcal{J}+\mathcal{D})=\mathcal{J}+(-1)^n\mathcal{D}^{n+1}. \tag{6}$$

Hence for any $\boldsymbol{\Psi}$ in the range of $\mathcal{J}+\mathcal{D}$, say $\boldsymbol{\Psi}=(\mathcal{J}+\mathcal{D})(\mathbf{X})$, $\mathbf{X}\in\mathbf{L}_2$, we have

$$S_n(\boldsymbol{\Psi})=S_n(\mathcal{J}+\mathcal{D})(\mathbf{X})=\mathbf{X}+(-1)^n\mathcal{D}^{n+1}(\mathbf{X})=(\mathcal{J}+\mathcal{D})^{-1}(\boldsymbol{\Psi})+(-1)^n\mathcal{D}^{n+1}(\mathbf{X}).$$

Thus
$$\|S_n(\boldsymbol{\Psi})-(\mathcal{J}+\mathcal{D})^{-1}(\boldsymbol{\Psi})\|=\|\mathcal{D}^{n+1}(\mathbf{X})\|.$$

Since by (a) the last term tends to 0 as $n\to\infty$, we get (b). (Q.E.D.)

Since by 4.4 $\mathbf{I}$ is in the range of $\mathcal{J}+\mathcal{D}$, it follows from 4.6 (b) that

$$\sqrt{\tilde{\mathbf{G}}}\,\tilde{\mathbf{\Phi}}^{-1}=(\mathcal{J}+\mathcal{D})^{-1}(\mathbf{I})=\lim_{n\to\infty}\sum_{k=0}^{\infty}(-1)^k\mathcal{D}^k(\mathbf{I})=\mathbf{\Psi},\ \text{say}.$$

By 4.3 (b) $$\mathbf{\Psi}=\mathbf{I}-\mathbf{M}_++(\mathbf{M}_+\mathbf{M})_+-\{(\mathbf{M}_+\mathbf{M})_+\mathbf{M}\}_++\cdots.$$

Also $$\mathbf{\Psi}(\mathbf{I}+\mathbf{M})\mathbf{\Psi}^*=\sqrt{\tilde{\mathbf{G}}}\,\tilde{\mathbf{\Phi}}^{-1}\,\tilde{\mathbf{\Phi}}\,\tilde{\mathbf{\Phi}}^*\,(\tilde{\mathbf{\Phi}}^{-1})^*\sqrt{\tilde{\mathbf{G}}}=\tilde{\mathbf{G}}.$$

Thus

4.7. THEOREM. *If* $\tilde{\mathbf{\Phi}}$, $\tilde{\mathbf{G}}$ *are the generating function and prediction-error matrix with* lag 1 *of a S.P. with spectral density* $\mathbf{I}+\mathbf{M}$ *subject to the Conditions* 4.1, *then*

(a) *the series* $\mathbf{I}-\mathbf{M}_++(\mathbf{M}_+\mathbf{M})_+-\{(\mathbf{M}_+\mathbf{M})_+\mathbf{M}\}_++\cdots$ *is mean-convergent,*

(b) $\mathbf{\Psi}$ *being its sum, we have*

$$\mathbf{\Psi}=\sqrt{\tilde{\mathbf{G}}}\,\tilde{\mathbf{\Phi}}^{-1}\in\mathbf{L}_2^{0+},\qquad \tilde{\mathbf{G}}=\mathbf{\Psi}(\mathbf{I}+\mathbf{M})\mathbf{\Psi}^*.$$

This theorem is an extension of II, 6.6. It follows that $\mathbf{\Psi}^{-1}=\tilde{\mathbf{\Phi}}(\sqrt{\tilde{\mathbf{G}}})^{-1}\in\mathbf{L}_\infty^{0+}$. Hence letting

$$\mathbf{\Psi}(e^{i\theta})\sim\sum_0^\infty\mathbf{A}_k e^{ki\theta},\qquad \mathbf{\Psi}^{-1}(e^{i\theta})\sim\sum_0^\infty\mathbf{B}_k e^{ki\theta}$$

we get, exactly as in II, 6.10, $\mathbf{A}_0=\mathbf{I}$ and for $m>0$

$$\mathbf{A}_m=-\mathbf{\Gamma}'_m+\sum_n\mathbf{\Gamma}'_n\mathbf{\Gamma}'_{m-n}-\sum_n\sum_p\mathbf{\Gamma}'_p\mathbf{\Gamma}'_{n-p}\mathbf{\Gamma}'_{m-n}+\cdots,$$

where $\mathbf{\Gamma}'_k$ is the kth Fourier coefficient of $\mathbf{M}$, and all subscripts run from 1 to ∞. The coefficients $\mathbf{B}_k$ can be found from the $\mathbf{A}_k$ by the usual recurrence relations (cf. II, 6.11). The kth Fourier coefficients of $\tilde{\mathbf{\Phi}}$, $\tilde{\mathbf{\Phi}}^{-1}$ can then be had from the relations

$$\tilde{\mathbf{C}}_k=\mathbf{B}_k\sqrt{\tilde{\mathbf{G}}},\qquad \tilde{\mathbf{D}}_k=\sqrt{\tilde{\mathbf{G}}}^{-1}\mathbf{A}_k.$$

As in II, 6.9 somewhat different expressions for $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{\Phi}}$ are also available. For since $\mathbf{M}=\mathbf{M}^*$, we get

$$\mathbf{\Psi}^*=\mathbf{I}-\mathbf{M}_-+(\mathbf{M}\mathbf{M}_-)_--\{\mathbf{M}(\mathbf{M}\mathbf{M}_-)_-\}_-+\cdots\in\mathbf{L}_2^{0-}. \tag{4.8}$$

Letting $$\mathbf{X}=(\mathbf{I}+\mathbf{M})\mathbf{\Psi}^* \tag{4.9}$$

is follows from 4.7 (b) that $\mathbf{X}=\mathbf{\Psi}^{-1}\tilde{\mathbf{G}}=\tilde{\mathbf{\Phi}}\sqrt{\tilde{\mathbf{G}}}\in\mathbf{L}_\infty^{0+}$, and since $\tilde{\mathbf{\Phi}}_0=\sqrt{\mathbf{G}}$ that $\mathbf{X}=\tilde{\mathbf{\Phi}}\sqrt{\mathbf{X}_0}$. Thus

4.10. Corollary. *Let* $\mathbf{X}$ *be defined by* (4.9) *and* (4.8). *Then*

$$\tilde{\mathbf{\Phi}} = \mathbf{X}(\sqrt{\mathbf{X}_0^{-1}}), \quad \tilde{\mathbf{G}} = \mathbf{X}_0.$$

The last theorem and corollary provide methods for determining the generating function $\tilde{\mathbf{\Phi}}$ and prediction-error matrix $\tilde{\mathbf{G}}$ from the spectral density $\mathbf{I}+\mathbf{M}$. As remarked at the end of § 3 we can get from these the generating function $\mathbf{\Phi}$ and prediction-error matrix $\mathbf{G}$ of any S.P. whose spectral density satisfies Assumption 3.1. For such a proccess the *frequency-response* or *transfer-function* can therefore be determined, cf. 2.10. Its linear prediction in the *frequency domain* is thus accomplished.

5. Determination of the predictor in the time-domain

We shall now show that the unique mean-convergent series for the linear predictor obtained in II, 5.7, 6.13 under the Boundedness Condition II, 5.1 is available under the weaker conditions $\mathbf{F}' \in \mathbf{L}_\infty$ and $\mathbf{F}'^{-1} \in \mathbf{L}_1$.

5.1. Lemma. *A S.P. for which* $\mathbf{F}' \in \mathbf{L}_\infty$ *and* $\mathbf{F}'^{-1} \in \mathbf{L}_1$ *satisfies Assumption* 3.1.

Proof. For letting $\mathbf{F}'(e^{i\theta}) \prec \beta \mathbf{I}$, it follows that $\mu(e^{i\theta})/\lambda(e^{i\theta}) \leqslant \beta/\lambda(e^{i\theta})$. But since $\mathbf{F}'^{-1} \in \mathbf{L}_1$, therefore $1/\lambda(e^{i\theta}) \leqslant \tau(\mathbf{F}'^{-1}) \in L_1$. Hence $\mu/\lambda \in L_1$. (Q.E.D.)

All results established in §§ 3, 4 will therefore apply to a process satisfying our new conditions.

5.2. Theorem. *If* $\mathbf{F}' \in \mathbf{L}_\infty$ *and* $\mathbf{F}'^{-1} \in \mathbf{L}_1$, *and* $\mathbf{Y}_\nu$, $\mathbf{E}_{\nu k}$ *are defined as in* 2.10, *then as* $N \to \infty$

(a) $$\sum_{k=0}^{\infty} \mathbf{E}_{\nu k}\, e^{ki\theta} \to \mathbf{Y}_\nu(e^{i\theta}) \text{ in the } \mathbf{L}_{2,F}\text{-norm}.$$

(b) $$\sum_{k=0}^{\infty} \mathbf{E}_{\nu k}\, \mathbf{f}_{-k} \to \hat{\mathbf{f}}_\nu \text{ in } \mathfrak{M}_\infty.$$

Proof. (a) We have, cf. 2.10,

$$[e^{-\nu i\theta}\mathbf{\Phi}(e^{i\theta})]_{0+} = \sum_{k=0}^{\infty} \mathbf{C}_{k+\nu}\, e^{ki\theta} = e^{-\nu i\theta}\{\mathbf{\Phi}(e^{i\theta}) - \sum_{k=0}^{\nu-1} \mathbf{C}_k\, e^{ki\theta}\}.$$

Now $\mathbf{\Phi} \in \mathbf{L}_\infty$, since $\mathbf{F}' \in \mathbf{L}_\infty$. Hence both the terms inside $\{\ \}$ are in $\mathbf{L}_\infty$, and so therefore is the term on the L.H.S. Since $\mathbf{\Phi}^{-1} \in \mathbf{L}_2$, as $\mathbf{F}'^{-1} \in \mathbf{L}_1$, it follows that

$$\mathbf{Y}_\nu(e^{i\theta}) = [e^{-\nu i\theta}\mathbf{\Phi}(e^{i\theta})]_{0+}\, \mathbf{\Phi}^{-1}(e^{i\theta}) \in \mathbf{L}_2.$$

Hence (cf. 2.10)

$$\sum_{k=0}^{N} \mathbf{E}_{\nu k}\, e^{ki\theta} \to \mathbf{Y}_\nu (e^{i\theta}) \qquad \text{in the } \mathbf{L}_2\text{-norm.} \tag{1}$$

But since $\mathbf{F}'(e^{i\theta}) \prec \beta \mathbf{I}$ a.e., $\beta < \infty$, it easily follows (cf. II, 5.2 (b)) that

$$\Big\| \sum_{k=0}^{N} \mathbf{E}_{\nu k}\, e^{ki\theta} - \mathbf{Y}_\nu \Big\|_F \leqslant \sqrt{\beta}\, \Big\| \sum_{k=0}^{N} \mathbf{E}_{\nu k}\, e^{ki\theta} - \mathbf{Y}_\nu \Big\|.$$

Hence (a) follows from (1).

(b) follows from (a) in view of the isomorphism between $\mathfrak{M}_\infty$ and $\mathbf{L}_{2,F}$, cf. II, 4.10, 4.11. (Q.E.D.)

The same conditions on the spectral density also enable us to express the ordinary (one-sided) normalised innovation process $(\mathbf{h}_k)_{-\infty}^{\infty}$ as a one-sided moving average of the given process $(\mathbf{f}_k)_{-\infty}^{\infty}$:

5.3. THEOREM. *If a regular S.P. $(\mathbf{f}_n)_{-\infty}^{\infty}$ has a spectral density $\mathbf{F}'$ such that $\mathbf{F}' \in \mathbf{L}_\infty$, $\mathbf{F}'^{-1} \in \mathbf{L}_1$, then its (ordinary) normalised innovation process $(\mathbf{h}_n)_{-\infty}^{\infty}$ is given by*

$$\mathbf{h}_n = \sum_{k=0}^{\infty} \mathbf{D}_k\, \mathbf{f}_{n-k},$$

where $\mathbf{D}_k$ is the k-th Fourier coefficient of the reciprocal of the generating function $\mathbf{\Phi}$.

Proof. As remarked just before 5.2, the conclusions of 2.8 and 2.7 (c) apply to our process. Thus

$$\mathbf{\Phi}^{-1} \sim \sum_{k=0}^{\infty} \mathbf{D}_k\, e^{ki\theta} \in \mathbf{L}_2^{0+},$$

and therefore, as $N \to \infty$,

$$\sum_{k=0}^{N} \mathbf{D}_k\, e^{ki\theta} \to \mathbf{\Phi}^{-1} (e^{i\theta}) \qquad \text{in the } \mathbf{L}_2\text{-norm.} \tag{1}$$

But since $\mathbf{F}'(e^{i\theta}) \prec \beta \mathbf{I}$, a.e., $\beta < \infty$, it follows that

$$\Big\| \sum_{k=0}^{N} \mathbf{D}_k\, e^{ki\theta} - \mathbf{\Phi}^{-1} \Big\|_F \leqslant \sqrt{\beta}\, \Big\| \sum_{k=0}^{N} \mathbf{D}_k\, e^{ki\theta} - \mathbf{\Phi}^{-1} \Big\|.$$

Hence (1) implies

$$\sum_{k=0}^{N} \mathbf{D}_k\, e^{ki\theta} \to \mathbf{\Phi}^{-1} (e^{i\theta}) \qquad \text{in the } \mathbf{L}_{2,F}\text{-norm.} \tag{2}$$

But by II, 4.9 (a) for any regular, full-rank process $\mathbf{\Phi}^{-1} \in \mathbf{L}_{2,F}$ and corresponds to $\mathbf{h}_0 \in \mathbf{M}_\infty$ under the isomorphism between $\mathbf{L}_{2,F}$ and $\mathfrak{M}_\infty$. Hence (2) implies

$$\sum_{k=0}^{N} \mathbf{D}_k \mathbf{f}_{-k} \to \mathbf{h}_0 \quad \text{in } \mathfrak{M}_\infty.$$

We get the desired result on applying the shift operator U^n to both terms in the last relation. (Q.E.D.)

6. Errata to Parts I and II

(1) The second equation in I (4.3), p. p. 124 should read:

$$\mathbf{F}^{(d)}(x) = \mathbf{F}(a+0) - \mathbf{F}(a) + \sum_{a<t<x} \{\mathbf{F}(t+0) - \mathbf{F}(t-0)\} + \mathbf{F}(x) - \mathbf{F}(x-0).$$

But when $\mathbf{F}$ is right continuous this reduces to the (generally erroneous) equation given in the paper. Since the spectral distribution of a S.P. is right continuous this error does not affect any result in I § 7.

(2) In I, 4.13 (a) (b), p. 128, the qualification "essentially", i.e., up to a set of zero Lebesgue measure, is required. This in turn shows that in the proof cf I, 7.8 (a), p. 143, the relation

$$\mathbf{F} = \mathbf{F}_u + \mathbf{F}_v + \mathbf{const.}$$

is valid only a.e. But in view of the right continuity of the functions involved and the fact that all vanish for $\theta = 0$, it still follows that $\mathbf{F} = \mathbf{F}_u + \mathbf{F}_v$ throughout $[0, 2\pi]$, as desired.

(3) In the proof of I, 5.11 (b) the equalities on the second, third and fourth lines from the bottom of p. 133 are not proven, for the term-by-term application of the limit as $k \to \infty$ to the infinite sum on the R.H.S. of (3) is unjustified. But the proof can still be completed as follows. From (3)

$$\sum_{|n|<N} |(\mathbf{A}_{j,n} - \mathbf{A}_{k,n}) \mathbf{K}^{\frac{1}{2}}|_E^2 \leqslant \|\mathbf{g}_j - \mathbf{g}_k\|^2.$$

Letting $k \to \infty$, we get

$$\sum_{|n|<N} |(\mathbf{A}_{j,n} - \mathbf{B}_n) \mathbf{K}^{\frac{1}{2}}|_E^2 \leqslant \|\mathbf{g}_j - \mathbf{g}\|^2,$$

whence

$$\sum_{n=-\infty}^{\infty} |(\mathbf{A}_{j,n} - \mathbf{B}_n) \mathbf{K}^{\frac{1}{2}}|_E^2 \leqslant \|\mathbf{g}_j - \mathbf{g}\|^2.$$

But L.H.S. $= \left\| \sum_{n=-\infty}^{\infty} (\mathbf{A}_{j,n} - \mathbf{B}_n) \boldsymbol{\varphi}_n \right\|^2 = \left\| \mathbf{g}_j - \sum_{n=-\infty}^{\infty} \mathbf{B}_n \boldsymbol{\varphi}_n \right\|.$

Thus

$$\left\| \mathbf{g}_j - \sum_{n=-\infty}^{\infty} \mathbf{B}_n \boldsymbol{\varphi}_n \right\|^2 \leqslant \|\mathbf{g}_j - \mathbf{g}\|^2.$$

Now let $j \to \infty$. We then get $\mathbf{g} = \sum_{n=-\infty}^{\infty} \mathbf{B}_n \boldsymbol{\varphi}_n \in \mathfrak{M}$; i.e., $\mathfrak{M}$ is closed, as desired.

(4) It should have been remarked that when the rank $\varrho < q$, the $\mathbf{A}_k$ occurring in I, 6.11 (b), p. 137, are not uniquely determined although the products $\mathbf{A}_k \mathbf{G}$ are. It follows easily that the products

$\mathbf{A}_k \mathbf{G}^{\frac{1}{2}}$ are also uniquely determined. Hence the expression for $\mathbf{u}_n$ in I, 6.11 (b), p. 137, is unequivocal, as is that for the function $\mathbf{\Phi}$ in I, 7.8 (b), p. 143.

(5) Most of the matrix equations on II, p. 101, are incorrect to the extent that the rows and columns of the block matrix have been transposed. With the following changes, however, the method given for solving the Prediction Problem becomes valid:

(i) In the matrices $[\Gamma_{k-j}]$ interchange N and $-N$ (3 places).

(ii) The last line of (3) p. 101 should read

$$= \sum_{j=0}^{N} \sum_{k=0}^{N} \mathbf{B}_j \mathbf{\Gamma}_{k-j} \mathbf{B}_k^* = (\sum_0^N \mathbf{B}_j \mathbf{f}_{-j}, \ \sum_0^N \mathbf{B}_k \mathbf{f}_{-k}),$$

(iii) The equation following (3) should read

$$\sum_0^N \mathbf{B}_j \mathbf{f}_{-j} = \mathbf{B}_0 (\mathbf{f}_0 + \sum_1^N \mathbf{B}_0^{-1} \mathbf{B}_j \mathbf{f}_{-j}) = \mathbf{B}_0 (\mathbf{f}_0 - \mathbf{g}).$$

(iv) In the subsequent discussion (p. 101 bottom, p. 102, 1st paragraph) replace the subscript N by 0.

(6) The following minor errata may be noted:

	Location	*For*	*Read*
		PART I	
1.	P. 116, inequality after (2)	$\frac{1}{2\pi}\int_0^{2\pi} \phi(e^{it}) P(r e^{i\theta}, e^{it})\, dt,$	$\frac{1}{2\pi}\int_0^{2\pi} \phi(e^{it})^{\delta} P(r e^{i\theta}, e^{it})\, dt$
2.	P. 123, formula before 3.13	$\overline{d\theta}$	$d\theta$
3.	P. 123, 3.13 (second line	$n>0$	$n<0$
4.	P. 126, 4.9 (c)	$\int_a^b \mathbf{G}(x)\, d\mathbf{F}(x)$	$\int_a^b d\mathbf{F}(x)\cdot \mathbf{G}(x)$
5.	P. 128, line 7	$d\mathbf{F}(\theta)$	$\mathbf{F}(\theta)\, d\theta$
6.	P. 128, 4.12	$\theta<\theta<2\pi$	$0<\theta<2\pi$
7.	P, 128, line 3* (1)	If	If $\mathbf{F}$ is of bounded variation and
8.	P. 130, line 9	produkt	product
9.	P. 130, 5.4, formula	$[(f^{(j)}, g^{(k)}]$	$[(f^{(j)}, g^{(k)})]$
10.	P. 131, 5.7 (g)	δ_{mm}	δ_{mn}
11.	P. 132, footnote 3	(4.3)	(5.3)
12.	P. 135, line 10*	f_n	$\mathbf{f}_n$
13.	P. 140, § 7, line 1	so in	so in the
14.	P. 148, line 10*	δ_{ij} I	$\delta_{ij}\mathbf{I}$
		PART II	
1.	P. 105, line 1	$\mathbf{G}_k$	$\mathbf{C}_k$
2.	P. 111, line 2*	L_∞	$\mathbf{L}_\infty$
3.	P. 111, line 1*	$L_{2,F}$	$\mathbf{L}_{2,F}$

(1) An asterisk indicates that the line is to be counted from the bottom of the page.

	Location	For	Read
4.	P. 114, line 5	$\mathbf{\Phi}$	$\boldsymbol{\varphi}$
5.	P. 115, line 9	*Proof.*	*Proof.* (a)
6.	P. 115. line 2*	$e^{ki\theta})$	$e^{ki\theta}$
7.	P. 116, line 1*	$\mid \mathbf{\Phi}(e^{i\theta}) \sqrt{\mathbf{F}'(e^{i\theta})} \mid_E^2$	$\mid \mathbf{\Phi}(e^{i\theta}) \sqrt{}\mathbf{F}'(e^{i\theta}) \mid_E^2$
8.	P. 119, line 1*	$\mathbf{g}_k$ (two places)	$\mathbf{g}_n$
9.	P. 119, line 1*	$\mathbf{f}_{-n}$	$\mathbf{f}_{-k}$
10.	P. 123, line 15	subtile	subtle
11.	P. 125, (6.7)	L_2^{0+}	$\mathbf{L}_2^{0+}$
12.	P. 128, line 8	choise	choice

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