GLOBAL BOUNDARY BEHAVIOR OF MEROMORPHIC FUNCTIONS

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1. Introduction

Let f be a function sending the open unit disk D into the Riemann sphere S. A point y on S is in the *global cluster set* of f, denoted by $C(f)$, if and only if there exists a sequence of points z_n in D such that $\lim |z_n| = 1$ and $\lim f(z_n) = y$. Thus, for example, if f is continuous on D, and can be extended to be continuous on $\bar{D}$, then $C(f)$ is the image of the bounding circle and hence a Peano space.

If f is continuous, then $C(f)$ is a continuum. Conversely, it is easy to prove that any continuum C on the sphere S is the global cluster set for some continuous function f. Collingwood ([3], p. 123) and Cartwright asked whether every continuum on S is the global cluster set of a function f meromorphic on the open disk D. D. B. Potyagailo [8] and W. Rudin [10] independently gave as counter-example the continuum consisting of the union of (a) a spiral, $r = \theta/(\pi + \theta)$, $\pi \leqslant \theta < \infty$, (b) the unit circumference, and (c) an interval, $1 \leqslant x \leqslant 2$, $y = 0$.

Because this example is not locally connected, and because, if f is continous on $\bar{D}$, then $C(f)$ is locally connected, one might conjecture that every locally connected continuum is the global cluster set for some function f meromorphic on D. In Section 2, we give a counter-example to this conjecture. The example also answers in the negative a question of Gerald MacLane [5]: Is every Peano space the image of the bounding circle of a function f meromorphic on D and continuous on $\bar{D}$?

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In Section 3 a topological sufficient condition for a continuum to be the global cluster set for a function f meromorphic on D is given. This condition is different from and simpler than a sufficient condition given by D. B. Potyagailo [8].

Other work relating to the boundary behavior of functions analytic or meromorphic on the open disk and continuous on the closed disk has been done by Salem and Zygmund [11], Piranian, Titus, and Young [7], Schaeffer [12] and Marston Morse [6]. Point cluster sets of meromorphic functions have been studied by Gross [4].

It is significant that the results in this paper are proved almost entirely by topological techniques. Throughout the sequel "map" means continuous function, $S(x, \varepsilon)$ is the open disk about x of radius ε, S is the Riemann sphere, and D is the open unit disk.

2. The Example

THEOREM 2.1. *There exists a Peano space P which is not the global cluster set of any function f meromorphic on the open unit disk.*

COROLLARY 2.2. *There exists a Peano space P which is not the image of the bounding circle for any function f meromorphic on the open unit disk and continuous on the closed unit disk.*

Construction of the example. The Peano space P_1 is the union of $\bar{D}$ and the following sets described using polar coordinates:

(1) the closed disks with centers $(2-2^{-n}, m\pi/2)$ and radii 2^{-n-3}, together with the line segments $1 \leqslant r \leqslant 2$, $\theta = m\pi/2$, which join these disks to the unit disk (n a positive integer and m odd);

(2) the closed disks with centers $(\frac{1}{4}(2-2^{-n}), \frac{1}{4}m\pi)$ and radii 2^{-n-5}, together with the line segments $1 \leqslant r \leqslant 1+\frac{1}{4}$, $\theta = \frac{1}{4}m\pi$, which join these disks to the unit disk (n a positive integer and m odd); at the kth stage, the closed disks with centers $((2-2^{-n})4^{1-k}, m\pi 2^{-k})$ and radii $2^{-n-2k-1}$, together with the line segments $1 \leqslant r \leqslant 1+4^{1-k}$, $\theta = m\pi 2^{-k}$ (n a positive integer and m odd). This completes the definition of P_1.

Given any disk $D' \neq D$ of P_1, map the plane onto itself using the natural similarity transformation followed by a rigid motion that maps D onto D', sending the vertical line through D onto the ray from D that passes through D'. Hence, to each disk in P_1 are added its own "satellites". The resulting set is called P_2.

Now, given any disk D' on P_2 but not on P_1, add, in the manner above, satellites corresponding to those of D *in* P_2; the result is P_3.

In general, given P_n, and any disk D' on P_n but not on P_{n-1}, add satellites corresponding to those of D *in* P_n, and call the result P_{n+1}.

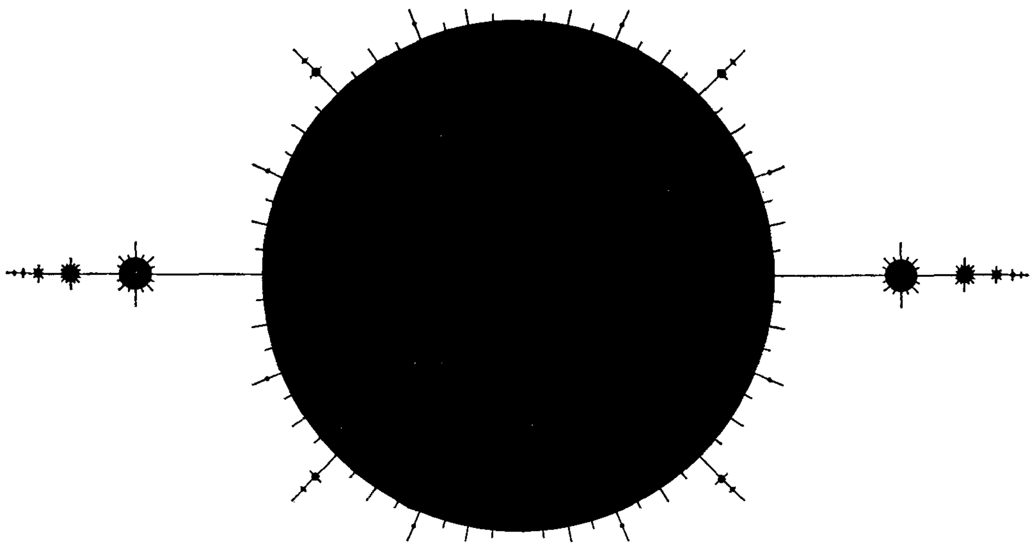

Fig. 1.

Let P be the closure of $\bigcup_{n=1}^{\infty} P_n$ (see the figure). From its definition P is clearly closed, bounded, and arcwise connected. To prove that P is locally connected, it is sufficient to show ([15], p. 20) that, for every $\varepsilon > 0$, P is the union of a finite number of connected sets each of diameter less than ε.

DEFINITIONS. Let f be a map of a topological space A into a topological space B. If, whenever U is open in A, $f(U)$ is open in B, then f is *interior*. If for every y in $f(A)$, $f^{-1}(y)$ is totally disconnected, then f is *light*.

A non-constant meromorphic function is light interior. Conversely, Stoilow ([14], p. 121) proved: If f is a light interior map of a plane region into the sphere S, then $f = gh$, where h is a homeomorphism, and g is meromorphic.

Proof of Theorem 2.1. We will prove the stronger result that there is no light interior map $f: D \to S$ having $C(f) = P$. The proof will use only the following properties (consequences of the Stoilow Theorem) of the maps:

(1) The set of points at which f is not one-to-one has no limit point in D.

(2) For each point q in the range of f, the set $f^{-1}(q)$ has no limit point in D.

Suppose there is such a light interior map f. We will show that this assumption leads to a contradiction. Since the range of f is open, it must meet $S - P$. But, since no point of $S - P$ is in $C(f)$, the range of f includes $S - P$. Let C be the set of points in D at which f is not one-to-one. Let B be $(S - P) - f(C)$, and let E be

$f^{-1}(B)$. Observe that, in any set at positive distance from P, the points of $f(C)$ are isolated.

Each q in B has only a finite number of inverse image points (otherwise, from property (2), q would be in $C(f)$). Let $q_1, q_2, \cdots, q_n$ be the points of $f^{-1}(q)$, and let N be a neighborhood of q in B at positive distance from P. Choose mutually disjoint neighborhoods N_i about q_i such that $N_i \subset f^{-1}(N)$ and the restriction of f to N_i is one-to-one $(i=1, 2, \ldots, n)$. Then $\bigcap_{i=1}^{n} f(N_i)$ is open, and its complete inverse image consists of n neighborhoods (of $q_1, q_2, \ldots, q_n$), each of which is mapped homeomorphically onto it. Thus E is a covering space of the base space B, with projection map f, and n, $1 \leqslant n < \infty$, the number of points in $f^{-1}(q)$, is independent of the choice of q in B ([13], p. 67).

For every ε, $0<\varepsilon<1$, let A_ε be the open annular region of points in D at distance less than ε from the boundary of D. Since the largest (open) disk of P, call it D_1, is in $C(f)$, the range of f must meet D_1. Therefore, there is an open set U_1 and an $\varepsilon_1>0$ such that U_1 does not meet A_{ε_1}, and $f(U_1)$ meets both D_1 and $S-P$ (and therefore B).

There is some $m\pi 2^{-k}$ such that the spike at $(1, m\pi 2^{-k})$, which has length 4^{1-k}, with all its attached disks lies entirely within $f(U_1)$. Let D_2 be the largest (open) disk on this spike. Since $D_2 \subset C(f)$, $f(A_{\varepsilon_1})$ meets D_2, and there is an open set $U_2 \subset A_{\varepsilon_1}$ and an $\varepsilon_2>0$, such that $A_{\varepsilon_2} \cap U_2 = 0$, $f(U_2) \subset f(U_1)$, and $f(U_2)$ meets both D_2 and $S-P$ (and therefore B). Since U_1 and U_2 are disjoint, $n \geqslant 2$.

In general, let open sets U_i and an $\varepsilon_k>0$ be given such that the U_i are mutually disjoint, $f(U_i) \subset f(U_{i-1})$ $(i=1, 2, \ldots, k)$, $\bigcup_{i=1}^{k} U_i$ does not meet A_{ε_k}, and $f(U_k)$ meets both P and B. There is some disk D_{k+1} of P contained in $f(U_k)$. Since $D_{k+1} \subset C(f)$, there is a neighborhood $U_{k+1} \subset A_{\varepsilon_k}$ and an $\varepsilon_{k+1}>0$, such that $A_{\varepsilon_{k+1}} \cap U_{k+1} = 0$, $f(U_{k+1}) \subset f(U_k)$, and $f(U_{k+1})$ meets both D_{k+1} and B. Since the U_i are pairwise disjoint, and $f(U_{i+1}) \subset f(U_i)$, $(i=1, 2, \ldots, k)$, we have $n \geqslant k+1$. Thus n is infinite, and $B \subset C(f)$, contradicting our assumption.

Remark. Let P' be the Peano space obtained from P by replacing each closed disk by its bounding circle, together with one of its diameters. The previous proof also shows that P' is a counter-example. We mention both of them so as to rule out conjectures that might otherwise be made. Observe that $S-P$ is simply connected, and that P' is one-dimensional.

Remark. Consider the following theorem of Rudin [9]. Suppose

(a) E is a closed subset of the boundary of D, E having Lebesgue measure zero;

(b) ϕ is a continuous function on E;

(c) T is a two-cell such that $\phi(E) \subset T$.

Then there exists a function f analytic on D and continuous on $\bar{D}$ such that

(i) $f(z) = \phi(z)$ for all z in E;

(ii) $f(\bar{D}) \subset T$.

Let E be a Cantor set (of zero measure) on bdy (D); there is a continuous map ϕ of E onto P ([15], p. 35). Let T_n be a sequence of simply connected regions containing P such that $T_{n+1} \subset T_n$ and $\bigcap_{n=1}^{\infty} T_n = P$, and let f_n be the functions given by the Rudin theorem. Our result implies that, if the (pointwise) limit function f exists, then it is not analytic. On the other hand, P is the intersection of the Peano spaces T_n (e.g., topological disks), each of which *is* the image of the bounding circle under a map f analytic on D and continuous on $\bar{D}$.

3. The Sufficient Condition

DEFINITION. A non-empty continuum C on the sphere S has *Property P* if there is a sequence (possibly finite) of simply connected regions $\{U_n\}$ $(n = 0, 1, 2, \ldots)$ such that:

(1) C is the closure of $\bigcup$ bdy (U_n);

(2) for every positive integer n, there is an integer $m < n$ such that bdy $(U_n) \cap$ bdy (U_m) contains a point p_n accessible from $U_n \cap U_m$;

(3) if the sequence U_n is infinite, the limit superior of $\{U_n\}$ is contained in C.

THEOREM 3.1. *If a continuum C on S has Property P, then C is the global cluster set of a function f meromorphic on D. Moreover, the range of f is $\bigcup U_n$.*

Let $d(U_n)$ denote the diameter of the largest circular open set in U_n. (Intuitively, $d(U_n)$ is the width of U_n.) If $d(U_n) \to 0$, then (3) is satisfied. In particular, if the measure of U_n converges to zero, then (3) is satisfied.

Property P is reasonably natural since *any* continuum C on S can be represented as the closure of $\bigcup$ bdy (U_n), where each U_n is a simply connected region. That is, $S - C$ has a finite, or countably infinite, number of components, $\{V_n\}$, each V_n simply connected. The boundary of each V_n is contained in C, and $C - \mathrm{Cl}\,[\bigcup \text{bdy}\,(V_n)]$ is open in S (Cl $[X]$ denotes the closure of X). If $C - \mathrm{Cl}\,[\bigcup \text{bdy}\,(V_n)] \neq 0$, then there is a countably infinity set of open disks, $\{D_n\}$, such that each D_n is contained in $C - \mathrm{Cl}\,[\bigcup \mathrm{bdy}\,(V_n)]$, and

$$C - \mathrm{Cl}\,[\bigcup \mathrm{bdy}\,(V_n)] \subset \mathrm{Cl}\,[\bigcup \mathrm{bdy}\,(D_n)] \subset C.$$

Condition (2) is necessary to eliminate the Rudin example [10] from our class, and (3) to eliminate the author's example of Section 2.

If R is any region, then $\bar{R}$ is a continuum with Property P. Any dendrite has Property P. If C has Property P, and h is a homeomorphism of S onto itself, then $h(C)$ has Property P.

On the other hand, from the author's example of Section 2, not every Peano space, boundary curve, or unicoherent continuum possesses Property P (for definitions, see [15]).

Definition. Two simply connected regions U and V on the sphere S are said to *meet properly* at p on $\mathrm{bdy}\,(U) \cap \mathrm{bdy}\,(V)$ if p is accessible by an arc in $U \cap V$.

Lemma 3.2. *Let U and V be simply connected regions on the sphere S such that* $\mathrm{bdy}\,(U)$ *and* $\mathrm{bdy}\,(V)$ *each contain more than one point,*

$$\mathrm{bdy}\,(U) \cap \mathrm{bdy}\,(V) \neq 0,\ U \cap V \neq 0,\ U \not\subset V,\ \textit{and}\ V \not\subset U.$$

Let p_i $(i = 1, 2, \ldots, n)$ be any finite set of points (of S). Then any component W of $U \cap V$ is a simply connected region such that U meets W properly at a point $q \neq p_i$ $(i = 1, 2, \ldots, n)$, W meets V properly, and $\mathrm{bdy}\,(W)$ *contains more than one point.*

Proof. There exist points r in $U \cap V$ and s in $V - U$, $r \neq p_i$, $s \neq p_i$ $(i = 1, 2, \ldots, n)$; hence, there is an infinite family of arcs in V, disjoint except that each has r and s as endpoints. Let γ be one of these arcs which does not meet any point p_i $(i = 1, 2, \ldots, n)$, and let W be the component of $U \cap V$ which contains r. Then γ has a subarc beginning at r which lies entirely in W except for its endpoint q on $\mathrm{bdy}\,(U) \cap \mathrm{bdy}\,(W)$. Thus U meets W properly at q, Similarly, W meets V properly.

Definition. A continuum C on the sphere S has *Property P'* if

(1) it has property P;

(2) there exists some function ϕ mapping the positive integers into the non-negative integers so that U_n meets $U_{\phi(n)}$ properly at p_n, $\phi(n) < n$; and

(3) if $n \neq n'$, $\phi(n) = \phi(n')$, and $p_n = p_{n'}$, then $U_n \cap U_{n'} = 0$.

Lemma 3.3. *If a continuum C on S has Property P, then it has Property P'. Moreover, if the simply connected regions given by Property P are denoted by U_n, and those of Property P' by V_n, then $\bigcup U_n = \bigcup V_n$.*

Proof. Given a sequence of simply connected regions $\{U_i\}$ satisfying P, we will construct a sequence of simply connected regions $\{V_j\}$ satisfying P'.

We may assume that the sequence $\{U_i\}$ has no repetitions. Let V_0 be U_0.

Now suppose that we have defined $V_0, V_1, \ldots, V_k$ to replace $U_0, U_1, \ldots, U_{n-1}$ such that

(1) For each i $(i=0, 1, \ldots, n-1)$ there is an integer $\varrho(i)$ $(\varrho(i)=0, 1, \ldots, k)$ such that $V_{\varrho(i)}=U_i$.

(2) If j $(j=0, 1, \ldots, k)$ is not in the range of ϱ, then $j+1$ is, and $V_j \subset V_{j+1}$.

(3) $\operatorname{bdy}(V_j) \subset C$.

(4) There is a function ϕ mapping the integers $1, 2, \ldots, k$ into the integers $0, 1, \ldots, k-1$, so that $\phi(j)<j$ and V_j meets $V_{\phi(j)}$ properly at a point q_j.

(5) Moreover, if $\phi(j)=\phi(h)$ and $q_j=q_h$, then $V_j \cap V_h = 0$.

Let us call conditions (1)–(5) Property P_{n-1} of $V_0, V_1, \ldots, V_k$. We will find at most two additional V_j's so that the enlarged family has Property P_n.

There is an integer $m<n$ such that U_n meets $U_m=V_{\varrho(m)}$ properly at a point p_n, by Property P. Let h $(h=0, 1, \ldots, k)$ be maximal such that either V_h and U_n meet properly, or they satisfy the hypothesis of Lemma 3.2. If V_h and U_n meet properly (at y), let U_n be V_{k+1}, h be $\phi(k+1)$, and y be q_{k+1}. Then the family $V_0, V_1, \ldots, V_{k+1}$ has Property P_n (condition (5) is satisfied because of the maximality of h).

If V_h and U_n do not meet properly, let the W given by Lemma 3.2 for $q \neq q_i$ $(i=1, 2, \ldots, k)$ be V_{k+1}, U_n be V_{k+2}, $k+1$ be $\phi(k+2)$ and h be $\phi(k+1)$. Then the set $V_0, V_1, \ldots, V_{k+2}$ has Property P_n.

The sequence $\{V_j\}$ (finite or infinite as the sequence $\{U_i\}$ is finite or infinite) thus constructed satisfies Property P'.

LEMMA 3.4. *Let U and V be simply connected (proper) subregions of the sphere S. Let F be a simply connected region bounded by a Jordan curve,* $\operatorname{bdy}(F)$ *containing a point p of* $\operatorname{bdy}(U) \cap \operatorname{bdy}(V)$, *$\bar{F}-\{p\}$ in $U \cap V$. Then there is a finite-to-one interior map g of U onto $U \cup V$ such that:*

(1) *The map g is the identity on $U-F$.*

(2) (a) *If* $\operatorname{bdy}(V)=\{p\}$ *and Γ is any arc ending at p, then there exists $\varepsilon>0$ and an open set F', $\bar{F}' \subset F \cup \{p\}$, such that g maps F' homeomorphically onto $S(p, \varepsilon)-\Gamma$.*

(b) *If* $\operatorname{bdy}(V)$ *is not a single point, then given any y on* $\operatorname{bdy} V$, *there exists $\varepsilon>0$ and an open set F', $\bar{F}' \subset F \cup \{p\}$, such that g maps F' homeomorphically onto $V \cap S(y, \varepsilon)$.*

(3) *If* $\{x_k\}$ *is a sequence in* F *converging to* p, *and if* $g(x_k)$ *converges to* y, *then* y *is on* bdy (V).

(4) *If* $\{y_k\}$ *is a sequence in* V *converging to a point* y *on* bdy (V), *then there is a sequence* $\{x_k\}$ *in* F *with* $g(x_k)=y_k$ *and* x_k *converging to* p. *Moreover, any such sequence* $\{x_k\}$ *in* F *converges to* p.

Proof. Suppose that S, viewed as the extended plane, has been assigned polar coordinates. There is no loss of generality in assuming that F is the hemisphere $0<r<\infty$, $0<\theta<\pi$, and p is infinity. Let g' be the finite-to-one interior map of U onto S which is the identity on $U-F$ and on F sends (r, θ) into $(r, 5\theta)$. If bdy (V) is a single point (i.e., $V=S-\{\infty\}$), let g be g'.

If bdy (V) is not a single point, let A and A' be the great circle arcs $r\geqslant 0$, $\theta=0$ and $\theta=\pi$, respectively. Let R, R', and R'' be the open sectors $0<\theta<2\pi/5$, $2\pi/5<\theta<4\pi/5$, and $4\pi/5<\theta<\pi$ in F, respectively. The map g' sends R and R' homeomorphically onto $S-A$, and R'' homeomorphically onto F. We will construct an orientation-preserving homeomorphism h of $S-\{\infty\}$ onto V such that h is the identity on $A-\{\infty\}$. Then g will be g' on $U-(R\cup R')$ and hg' on $\bar{R}\cup\bar{R}'$.

Let a_1 be the arc $r=1$, $0\leqslant\theta<\frac{1}{2}\pi$, b_1 the segment $\theta=0$, $-1\leqslant r\leqslant 1$, and c_1 the arc $r=1$, $\frac{1}{2}\pi\leqslant\theta\leqslant\pi$. Let D_1 denote the open upper half unit disk, and D_2 the lower one. There is a homeomorphism r of $\bar{D}_1$ onto the closed rectangle bounded by $x=0$, $x=-1$, $y=0$, and $y=1$, which maps b_1 onto the x-axis between $x=-1$ and $x=0$, and c_1 onto $x=-1$ between $y=1$ and $y=0$. Let s be the map of this closed rectangle onto the closed triangle bounded by the x- and y-axes, and by the line $x-y+1=0$, given by $s(x, y)=(x, y(1+x))$. There is a homeomorphism t of this closed triangle onto $\bar{D}_1$ which maps the x-axis between $x=-1$ and $x=0$ onto b_1.

Let u be the homeomorphism of $\bar{D}_1$ onto the quarter sphere $r\geqslant 0$, $\frac{1}{2}\pi\leqslant\theta\leqslant\pi$, given by $u(z)=(z-1)(z+1)^{-1}$. Let S_1 be the open hemisphere $r>0$, $0<\theta<\pi$, and let S_2 be $r>0$, $-\pi<\theta<0$. Let v be the homeomorphism of the quarter sphere onto $\bar{S}_1$ given by $v(r, \theta)=(r, 2\theta-\pi)$. Let w_1 be the composition $v\,u\,t\,s\,r$ of these functions in order, and let w_2 be the analogous map, defined by reflection, of $\bar{D}_2$ onto $\bar{S}_2$. The map w_i $(i=1, 2)$ is a homeomorphism of D_i onto S_i, of a_i onto A and of b_i onto A', which maps c_i into $\{\infty\}$.

Now, some great circle through the origin and infinity meets bdy (V) in a point other than infinity, since bdy (V) contains more than one point. There is an arc A'' on the great circle, A'' beginning at the origin, ending at a point $q\neq\infty$ of bdy (V), and lying entirely in V except for q. Then A and A'' meet only in the origin, and

$V-(A\cup A'')$ has two simply connected components. Let V_1 be the component for which $\theta>0$ and $r>0$, and let V_2 be the other component. Let x_i ($i=1, 2$) be a conformal map of the open unit disk D onto V_i. Let a_i', b_i', and c_i' be the arcs corresponding to $A-\{\infty\}$, $A''-\{q\}$, and bdy $(V_i)\cap$ bdy (V), respectively, under the Caratheodory [1] correspondence of prime ends. There is a homeomorphism y_i of $\bar{D}_i$ onto $\bar{D}$ mapping a_i onto a_i', b_i onto b_i', and c_i onto c_i' so that $y_i x_i w_i^{-1}$ (defined on $\bar{S}_i-\{\infty\}$) is the identity on $A-\{\infty\}$ and maps $A'-\{\infty\}$ onto $A''-\{q\}$ as a transformation of similitude. The homeomorphism h of $S-\{\infty\}$ onto V is $y_i x_i w_i^{-1}$ on $\bar{S}_i-\{\infty\}$ ($i=1, 2$).

The reader may verify that g is a finite-to-one interior map satisfying (1), (3), and (4). For (2), (in the case bdy $(V)\neq\{\infty\}$) suppose first that y is not infinity. Choose $\varepsilon>0$ so that $A\cap S(y, \varepsilon)=\mathbf{0}$. Since $g=hg'$ maps R homeomorphically onto $V-A$,

$$F'=g^{-1}(S(y, \varepsilon)\cap V)\cap R$$

will suffice. Suppose that y is infinity. Then g' maps the open sector R^*, $\pi/5<\theta<3\pi/5$ in $\bar{R}\cup\bar{R}'$, homeomorphically onto $S-A'$. But h maps $S-A'$ homeomorphically onto $V-A''$. Choose $\varepsilon>0$ so that $S(\infty, \varepsilon)\cap A''=\mathbf{0}$, and let

$$F'=g^{-1}[S(\infty, \varepsilon)\cap V]\cap R^*.$$

PROPOSITION 3.5. *If a continuum C possesses Property P, then there is a light interior map f of D into the sphere S such that C is the global cluster set $C(f)$. The range of f is* $\bigcup U_n$.

Proof. By Lemma 3.3, C thus has Property P'. Let $\{U_j\}$, $\{p_j\}$, and ϕ be the associated open sets, points, and function. For each $j>1$, p_j on bdy $(U_j)\cap$ bdy $(U_{\phi(j)})$ is accessible by an arc in $U_j\cap U_{\phi(j)}$; let Γ_j be one such arc. As before, A_ε will denote the annular region $1-\varepsilon<|z|<1$.

Let f_0 be a homeomorphism of D onto U_0. If the sequence consists of U_0 alone, then f_0 is f. Otherwise, about Γ_1 we can form a simply connected region F_1, bdy (F_1) a Jordan curve containing p_1, $\bar{F}_1-\{p_1\}$ in $U_0\cap U_1$. Let g_1 be the finite-to-one interior map of U_0 onto $U_0\cup U_1$ given by Lemma 3.4. Let $f_0^{-1}(F_1)=E_1$, and let $f_1=g_1 f_0$.

In general, suppose that we have constructed a set of functions f_0, f_1 $\dots$, f_{n-1} such that:

(1) Each f_j is a finite-to-one interior map of D onto $U_0\cup U_1\cup\dots\cup U_j$.

(2) There exist open sets E_j, E_j in $A_{1/j}$ (except for E_0 and E_1), such that $f_{j+1}=f_j$ on $D-E_j$; E_0 is D.

(3) The closure of E_j in D is contained in $E_{\phi(j)}$, and $f_{\phi(j)}$ (and f_{j-1}) maps E_j homeomorphically onto a simply connected region F_j, bdy (F_j) a Jordan curve containing p_j, $\bar{F}_j-\{p_j\}$ in $U_j \cap U_{\phi(j)}$.

(4) If for some positive integer h, $k=\phi^h(j)$, where ϕ^h is the hth iteration of ϕ, then $\bar{E}_j \cap D \subset E_k$; otherwise, $\bar{E}_j \cap \bar{E}_k \cap D = 0$. If $\phi(j)=\phi(m)=k$, then $\bar{F}_j \cap \bar{F}_m$ is 0 or $\{p_j\}$.

(5) (a) If bdy $(U_j)=\{p_j\}$ and Γ is any arc ending at p_j, then there exist $\varepsilon>0$ and an open set E, $\bar{E} \cap D \subset E_j$, such that f_j maps E homeomorphically onto $S(p_j, \varepsilon)-\Gamma$. (b) If bdy (U_j) is not a single point, then, given any point p on bdy (U_j), there exists $\varepsilon>0$ and an open set E, $\bar{E} \cap D \subset E_j$, such that f_j maps E homeomorphically onto $U_j \cap S(p_j, \varepsilon)$.

(6) The function f_j maps E_j onto U_j so that if $\{x_k\}$ is a sequence in E_j, $|x_k| \to 1$, and $f_j(x_k) \to y$, then $y \in$ bdy (U_j). Conversely, if $\{y_k\}$ is a sequence in U_j converging to a point y on bdy (U_j), then there exist x_k in E_j such that $f_j(x_k)=y_k$. Moreover, for any such x_k, $|x_k| \to .1$

Call properties (1)–(6) Property Q_{n-1} of $f_0, f_1, \ldots, f_{n-1}$. The function f_0 possesses Q_0. We will prove that, if $f_0, f_1, \ldots, f_{n-1}$ satisfy Q_{n-1}, then there exists f_n such that $f_0, f_1, \ldots, f_n$ satisfy Q_n. (The function f_1 was constructed separately for purposes of clarity, and we will not use the fact that f_0, f_1 satisfy Q_1 in the succeeding argument).

The set $\Gamma_n-\{p_n\}$ is contained in $U_n \cap U_{\phi(n)}$, and p_n is on bdy $(U_n) \cap$ bdy $(U_{\phi(n)})$. There exists $\delta>0$ such that $S(p_n, \delta) \cap U_{\phi(n)}$ is disjoint from each $\bar{F}_m$ having $m<n$, $\phi(m)=\phi(n)$, and $p_m \neq p_n$. By Property P', if $m<n$, $\phi(m)=\phi(n)$, and $p_m=p_n$, then $U_m \cap U_n = 0$. Thus $S(p_n, \delta) \cap U_{\phi(n)} \cap U_n$ is disjoint from each $\bar{F}_m-\{p_m\}$ having $m<n$ and $\phi(m)=\phi(n)$.

If bdy $(U_{\phi(n)})$ is a single point $\{p_n\}$, let Γ be any arc ending at p_n such that $\Gamma \cap \Gamma_n = 0$. There exists ε given by $Q_{\phi(n)}$ (5 (a)), such that $0<\varepsilon \leqslant \delta$, and

$$S(p_n, \varepsilon)-\Gamma \subset f_{\phi(n)}(E \cap A_{1/n}),$$

by $Q_{\phi(n)}$ (6). The arc Γ_n has a subarc Γ_n' containing p_n, Γ_n' in $S(p_n, \varepsilon)$. Thus, there is a simply connected region F_n such that bdy (F_n) is a Jordan curve containing p_n, and $\bar{F}_n-\{p_n\}$ is in $[U_n \cap S(p_n, \varepsilon)]-\Gamma$.

If bdy $(U_{\phi(n)})$ is not a single point, there exist ε and E given by $Q_{\phi(n)}$ (5 (b)), such that $0<\varepsilon \leqslant \delta$ and

$$S(p_n, \varepsilon) \cap U_{\phi(n)} \subset f_{\phi(n)}(E \cap A_{1/n}),$$

by $Q_{\phi(n)}$ (6). The arc Γ_n has a subarc Γ_n' in $S(p_n, \varepsilon)$. Thus, there is a simply connected region F_n such that bdy (F_n) is a Jordan curve containing p_n, and $\bar{F}_n - \{p_n\}$ is in

$$U_{\phi(n)} \cap U_n \cap S(p_n, \varepsilon).$$

Let g_n be the finite-to-one interior map of $U_{\phi(n)}$ onto $U_{\phi(n)} \cup U_n$ given by Lemma 3.4. Let

$$E_n = (f_{\phi(n)}^{-1}(F)) \cap E.$$

Since
$$\bar{E}_n \cap D \subset E_{\phi(n)} - \bigcup_{j=\phi(n)+1}^{n-1} (\bar{E}_j \cap D),$$

from the construction of E_n and from Q_{n-1} (4), we have $f_{n-1} = f_{\phi(n)}$ on $\bar{E}_n \cap D$. Let $f_n = g_n f_{n-1}$ on E_n, $f_n = f_{n-1}$ elsewhere. Then $f_1, f_2, \dots, f_n$ clearly satisfy Q_n ((2), (3), and (4)) by Q_{n-1} and the construction of f_n.

To prove that f_n is a finite-to-one interior map, observe that f_n is $g_n f_{n-1}$ on E, and is f_{n-1} on $D - \bar{E}_n$ (g_n is the identity on $U_{\phi(n)} - F_n$). Since f_{n-1} and g_n are finite-to-one interior, f_n is finite-to-one interior on E, and on $D - \bar{E}_n$. But E and $D - \bar{E}_n$ are open sets whose union is D, so f_n is finite-to-one interior on D, giving Q_n (1).

Given p on bdy (U_n) (or, if bdy $(U_n) = \{p_n\}$, given an arc Γ ending at p_n), let $F' \subset F_n$ be the set (given by Lemma 3.3) on which g_n is a homeomorphism. The function f_{n-1} maps E_n homeomorphically onto F_n. Since f_n is $g_n f_{n-1}$ on E_n, let E be

$$f_{n-1}^{-1}(F') \cap E_n,$$

giving Q_n (5).

For (6), suppose that $\{x_k\}$ is a sequence in E_n, with $|x_k| \to 1$, and $f_n(x_k) \to y$. Since $E_n \subset E_{\phi(n)}$, $f_{\phi(n)}(x_k)$ has all its limit points on bdy $(U_{\phi(n)})$ by Q_{n-1} (6). But $f_{\phi(n)}(E_n) = F_n$, and $\bar{F}_n \cap$ bdy $(U_{\phi(n)})$ is the point p_n. Also $f_{n-1} = f_{\phi(n)}$ on E_n. Thus, $f_{n-1}(x_k) \to p_n$. Now, applying Lemma 3.4, if $g_n(f_{n-1}(x_k)) \to y$, then y is on bdy (U_n). Since f_n is $g_n f_{n-1}$ on E_n, we have the desired result.

Conversely, suppose that $\{y_k\}$ is a sequence in U_n converging to y on bdy (U_n). From Lemma 3.4, there is a sequence $\{w_k\}$ in F_n such that $g(w_k) = y_k$ and $w_k \to p_n$. But F_n is contained in $U_{\phi(n)}$, and p_n is on bdy $(U_{\phi(n)})$. The function $f_{\phi(n)}$ maps E_n homeomorphically onto F_n, so there exist x_k in E_n such that $f_{\phi(n)}(x_k) = w_k$. By $Q_{\phi(n)}$ (6), since $E_n \subset E_{\phi(n)}$, $|w_k| \to 1$. But $f_{n-1} = f_{\phi(n)}$ on E_n, and $f_n = g_n f_{n-1}$ there, so that $f_n(w_k) \to y$.

Thus, there exists a sequence of functions $\{f_n\}$, corresponding with $\{U_n\}$ such that, for each n, $f_0, f_1, \dots, f_n$ satisfy Q_n.

If $\{U_n\}$ is a finite sequence of $m+1$ sets, then let f_m be f. The map f is light interior by Q_m (1), and

$$C(f) = \text{bdy}(U_0) \cup \text{bdy}(U_1) \cup \cdots \cup \text{bdy}(U_m) = C.$$

Thus, we may assume that the sequence $\{U_n\}$ is infinite. Let f be $\lim_{n\to\infty} f_n$. Given any z in D, choose a positive integer N so that z is in $D - \bar{A}_{1/N}$. Since $E_n \subset A_{1/N}$, for all $n \geq N$, by Q_n (2), $f_N = f$ on some neighborhood V of z. Since f_N is interior on V, and since z is arbitrary, f is interior. Let y be in the range of f, and let $z \in f^{-1}(y)$. Choose N and V as before. Since f_N is finite-to-one, $f^{-1}(y) \cap V$ is finite. Since z is arbitrary, $f^{-1}(y)$ consists of isolated points; thus f is light.

To prove that $C \subset C(f)$, it is sufficient to prove that, given any y in bdy (U_n), there is a sequence $\{z_k\}$ in D, $|z_k| \to 1$ such that $f(z_k) \to y$.

Let $\{U_{ni}\}$ denote those U_m's such that $\phi(m) = n$, and let $F_{ni} = f_n(E_{ni})$, as before. If $S(y, 1/k)$ is disjoint from all the F_{ni}, let y_k be any point of $S(y, 1/k) \cap U_n$. If $S(y, 1/k)$ meets some F_{ni}, and if $p_{ni} \neq y$, then there exists p in $S(y, 1/k) - F_{ni}$ and q in $S(y, 1/k) \cap F_{ni}$. There is some arc γ in $S(y, 1/k)$ which joins p and q and is disjoint from p_{ni}. Thus γ contains a point, call it y_k, of bdy $(F_{ni}) \cap U_n$. If $i \neq j$, then $\bar{F}_{ni} \cap \bar{F}_{nj}$ is 0 or the point p_{ni}, by Q_k (4) $(k = 1, 2, \ldots)$, so that y_k is not in any F_{ni}. Lastly, if $S(y, 1/k)$ meets some F_{ni} with $p_{ni} = y$, then the Jordan curve bdy (F_{ni}) meets $S(y, 1/k)$ in a point y_k not p_{ni}. Thus, again, y_k is in U_n, and not in any F_{ni}.

Thus, each y_k is in

$$f_n\Big(E_n - \bigcup_{i=n+1}^{\infty} E_i\Big),$$

so there exists x_k, $|x_k| \to 1$, with $f_n(x_k) = y_k$, by Q_n (6). But $f = f_n$ on

$$E_n - \bigcup_{i=n+1}^{\infty} E_i,$$

so y is in $C(f)$. Hence, $C \subset C(f)$.

Let y be any point of

$$C(f) - \bigcup_{n=0}^{\infty} \text{bdy}(U_n).$$

There is a sequence z_k in D, $|z_k| \to 1$, such that $f(z_k) \to y$. Since f agrees with f_n on

$$E_n - \bigcup_{i=n+1}^{\infty} E_i$$

$(n=0, 1, \ldots)$, each of these sets contains only a finite number of the z_k. (Otherwise, y would be in some bdy (U_n), by Q_n (6).) Since $\bigcup_{n=1}^{\infty} E_n = D$ and

$$f\,(E_n - \bigcup_{i=n+1}^{\infty} E_i) \subset U_n$$

$(n=0, 1, \ldots)$, $y \in \limsup U_n$. By condition (3) of Property P, $C(f) \subset C$; hence, $C(f) = C$.

LEMMA 3.6. *If a continuum C on the sphere S is the global cluster set $C(f)$ of a light interior map f of D into S, then there is a meromorphic function F on D such that $C(F)$ is C. The range of F is the range of f.*

Proof. In the special cases where C is a single point p or is S, we use $F(z) \equiv p$ or $F(z) = \exp[(z-1)^{-3}]$, respectively (see [13], p. 25). By the theorem of Stoilow ([14], p. 121), $f = gh$, where h is a homeomorphism, and g is meromorphic. The domain G of g is simply connected. If G is the plane, then either infinity is a removable singularity or a pole, and $C(f)$ is a single point; or infinity is an essential singularity, and $C(f)$ is S. Since we may assume that C is neither S nor a single point, there is a conformal map h' of D onto G. The desired map F is gh'.

Theorem 3.1 is an immediate consequence of Proposition 3.5 and Lemma 3.6.

COROLLARY 3.7. *If a continuum C on S possesses Property P, with no U_n containing infinity, then C is the global cluster set of an analytic map.*

Remarks. A slightly weaker sufficient condition results if (1) and (3) of Property P are replaced by: C is the closure of

$$[\bigcup \text{bdy}\,(U_n)] \cup [\limsup\,(U_n)].$$

In a later paper the author will discuss two natural questions:

(1) Is property P a necessary condition?

(2) What is a necessary and sufficient condition [2] for a continuum C on S to be the image of bdy (D), under a function f meromorphic on D and continuous on $\bar{D}$?

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