A PRIORI INEQUALITIES CONNECTED WITH SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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Introduction

In the more recent development of the theory of general (linear) partial differential equations, the so-called a priori inequalities play a prominent role. For instance, the important comparison of two partial differential operators $P(D)$ and $Q(D)$ with constant coefficients, studied by L. Hörmander [1], depends on the existence of a constant C such that

$$\| Q(D) u \| \leqslant C \| P(D) u \|$$

for all functions $u = u(x)$ of class $\mathcal{D}(\Omega)$. (The norms are L^2-norms with respect to Lebesgue measure in a given region Ω in a Euclidean space R^n. The class $\mathcal{D}(\Omega)$ consists of all infinitely differentiable functions of compact support in Ω.) One of Hörmander's basic results [1, Theorem 2.2] asserts that, if Ω is bounded, such a constant C exists if and only if the ratio $\tilde{Q}(\xi)/\tilde{P}(\xi)$ remains bounded as a function of $\xi \in R^n$. Here $\tilde{P}(\xi)$ denotes a certain "norm function" associated with the polynomial $P(\xi)$ in terms of which the operator $P(D)$ is defined (cf. § 1 below).

The present paper is concerned with similar problems for *systems* of differential operators. Such a system is conveniently described as a matrix $\mathbf{P}(D)$ whose elements are partial differential operators $P_{ij}(D)$. If $\mathbf{Q}(D)$ denotes another such matrix, we shall find a necessary and sufficient condition for the existence of a constant C such that

$$\| \mathbf{Q}(D) \mathbf{u} \| \leqslant C \| \mathbf{P}(D) \mathbf{u} \|$$

for all column vectors $\mathbf{u} = \mathbf{u}(x)$ whose elements $u_j(x)$ are of class $\mathcal{D}(\Omega)$. (Theorems 3.1 and 4.) In § 5 we treat a more general problem of the same nature, viz., to decide under which conditions an inequality of the above type holds for all solutions $\mathbf{u} \in \mathcal{D}(\Omega)$ of a given homogeneous linear system of differential equations, $\mathbf{H}(D)\mathbf{u} = 0$.

If the L^2-norm of $\mathbf{Q}(D)\mathbf{u}$ is replaced by the L^∞-norm or the L^2-norm of the restriction of $\mathbf{Q}(D)\mathbf{u}$ to some affine subspace of R^n, the condition for the validity of an estimate of the above kind will be changed in the manner described in Hörmander [1, Theorems 2.6 and 2.8] for the case of single operators. This modification is discussed in the last section (§ 6).

It is a pleasure for me to acknowledge the valuable interest taken by professor Lars Hörmander in the present investigation, which is so intimately associated with some of his important contributions to the theory of general partial differential operators.

1. The norm function associated with a polynomial

For any given number n of dimensions we denote by R^n the Euclidean n-dimensional space. The points of R^n are denoted (in the present context) by $\xi = (\xi_1, \ldots, \xi_n)$, and we write $|\xi| = (\xi_1^2 + \ldots + \xi_n^2)^{\frac{1}{2}}$. An arbitrary partial derivative of order k may be denoted by

$$\partial_\alpha = \partial_{\alpha_1} \partial_{\alpha_2} \ldots \partial_{\alpha_k}, \quad \text{where } \partial_\nu = \partial/\partial \xi_\nu.$$

Here the multiindex $\alpha = (\alpha_1, \ldots, \alpha_k)$ ranges over all subsets of the set $(1, 2, \ldots, n)$. The length, or *order*, k, of such a multiindex α will be denoted by $|\alpha|$; it equals the order of ∂_α.

Throughout the paper we denote by $\mathcal{A}$ the integral domain $C[\xi_1, \ldots, \xi_n]$ of all polynomials $P(\xi)$ in n real variables $\xi = (\xi_1, \ldots, \xi_n)$ with complex coefficients. Equations and inequalities between polynomials are always understood to hold in $\mathcal{A}$, that is, identically with respect to $\xi \in R^n$. For any multiindex α, we write

$$P^{(\alpha)}(\xi) = \partial_\alpha P(\xi)$$

for the corresponding derivative of the polynomial $P(\xi)$.

The following "norm function" $\tilde{P}(\xi)$, associated with any given polynomial $P(\xi)$, plays an important role in Hörmander's investigations of general partial differential operators:

$$\tilde{P}(\xi) = (\textstyle\sum_\alpha |P^{(\alpha)}(\xi)|^2)^{\frac{1}{2}}. \tag{1}$$

The summation should be extended over all multiindices α, but the terms corresponding to orders $|\alpha|$ exceeding the degree of $P(\xi)$ vanish. The derivatives $P^{(\alpha)}(\xi)$

of order $|\alpha|$ = the degree of $P(\xi)$ are constants, not all zero (unless $P(\xi)\equiv 0$). Hence $\tilde{P}(\xi)$ is bounded away from 0, that is,

$$\tilde{P}(\xi)\geqslant c \tag{2}$$

for some constant $c>0$, provided $P(\xi)\not\equiv 0$.

It is solely the order of magnitude of the norm function which matters in the applications to differential operators. Accordingly, $\tilde{P}(\xi)$ could be replaced by any function $F(\xi)$ *equivalent* to $P(\xi)$ in the sense that

$$C^{-1}\tilde{P}(\xi)\leqslant F(\xi)\leqslant C\tilde{P}(\xi) \tag{3}$$

for some constant C. For instance, $\sum_\alpha |P^{(\alpha)}(\xi)|$ and $\max_\alpha |P^{(\alpha)})\xi)|$ are equivalent to $\tilde{P}(\xi)$. The following lemma contains further examples of such functions. We denote by C^n the complex n-dimensional space and write $|\zeta|^2=|\zeta_1|^2+\ldots+|\zeta_n|^2$ when $\zeta=(\zeta_1,\ldots,\zeta_n)\in C^n$.

LEMMA 1.1. *Each of the following functions is equivalent to $\tilde{P}(\xi)$:*

$$F_1(\xi)=\int\limits_{\{\eta\in R^n,\,|\eta|\leqslant 1\}}|P(\xi+\eta)|\,d\eta,\quad d\eta=d\eta_1\ldots d\eta_n,$$

$$F_2(\xi)=\max_{\eta\in R^n,\,|\eta|\leqslant 1}|P(\xi+\eta)|,$$

$$F_3(\xi)=\max_{\zeta\in C^n,\,|\zeta|\leqslant 1}|P(\xi+\zeta)|.$$

It will follow from the proof that the constant C in (3) may be chosen so as to depend only on the degree of $P(\xi)$ and the dimension n in each of the three cases $F=F_1$, $F=F_2$, and $F=F_3$. Note that

$$F_1(\xi)\leqslant V_n F_2(\xi)\leqslant V_n F_3(\xi)$$

if V_n denotes the volume of the unit ball in R^n. The inequality $F_3(\xi)\leqslant C\tilde{P}(\xi)$ follows from Taylors formula

$$P(\xi+\zeta)=\textstyle\sum_\alpha(|\alpha|!)^{-1}P^{(\alpha)}(\xi)\,\zeta_\alpha,\quad \zeta_\alpha=\zeta_{\alpha_1}\zeta_{\alpha_2}\ldots. \tag{4}$$

In order to prove the remaining inequality

$$\tilde{P}(\xi)\leqslant C F_1(\xi), \tag{5}$$

we choose an infinitely differentiable, non-zero function $k = k(t)$ of a single real variable t. This function k should vanish for $|t| > n^{-\frac{1}{2}}$ and satisfy the following $m+1$ conditions:

$$\int k(t)\, t^{\mu}\, dt = \delta_{\mu 0}, \quad \mu = 0, 1, \ldots, m.$$

Here m denotes a given integer. It follows that

$$\int k(t)\,(s-t)^{\mu}\, dt = s^{\mu}, \quad \mu = 0, 1, \ldots, m,$$

and hence $k * p = p$ for every polynomial $p = p(t)$ of degree $\leqslant m$. Writing

$$K(\xi) = k(\xi_1)\, k(\xi_2) \ldots k(\xi_n),$$

we infer that $K * P = P$ for every polynomial $P = P(\xi)$ of degree $\leqslant m$ in each of the n variables. Performing a differentiation ∂_{α}, we obtain $P^{(\alpha)} = K^{(\alpha)} * P$, that is,

$$P^{(\alpha)}(\xi) = \int K^{(\alpha)}(\eta)\, P(\xi - \eta)\, d\eta.$$

Since $k(t) = 0$ for $|t| > n^{-\frac{1}{2}}$, $K(\xi)$ and $K^{(\alpha)}(\xi)$ vanish for $|\xi| > 1$. Consequently

$$|P^{(\alpha)}(\xi)| \leqslant \max_{\eta \in R^n} |K^{(\alpha)}(\eta)| \cdot \int_{|\eta| \leqslant 1} |P(\xi - \eta)|\, d\eta.$$

Substituting $-\eta$ for η under the integral sign and summing over α, we arrive at (5).

We proceed to establish some properties of the norm function $\tilde{P}(\xi)$. First we note that, in view of Taylors formula (4),

$$C^{-1}\tilde{P}(\xi) \leqslant \tilde{P}(\xi + \eta) \leqslant C\,\tilde{P}(\xi), \tag{6}$$

uniformly with respect to η in bounded subsets of the η-space. Next we observe that $\tilde{P}(\xi)$ is positive homogeneous and subadditive in its dependence on the polynomial $P = P(\xi)$:

$$(a\,P(\xi))^{\sim} = |a|\,\tilde{P}(\xi); \quad (P_1(\xi) + P_2(\xi))^{\sim} \leqslant \tilde{P}_1(\xi) + \tilde{P}_2(\xi). \tag{7}$$

Less obvious is the following property of approximate multiplicativity:

LEMMA 1.2. *There is a constant C, depending only on the two polynomials $P_1(\xi)$ and $P_2(\xi)$, such that*

$$C^{-1}\tilde{P}_1(\xi)\,\tilde{P}_2(\xi) \leqslant (P_1(\xi)\,P_2(\xi))^{\sim} \leqslant C\,\tilde{P}_1(\xi)\,\tilde{P}_2(\xi).$$

While the latter inequality is quite elementary, the former is a consequence of an extension of Malgrange's Lemma due to Hörmander [2, Lemma 1.3], in view of which

$$|P_1(\xi)\, P_2^{(\alpha)}(\xi)| \leqslant C_\alpha F_3(\xi)$$

for every α. Here F_3 denotes the function defined in Lemma 1.1 above, now corresponding to the polynomial $P(\xi) = P_1(\xi)\, P_2(\xi)$. Using Taylor's formula (4), or Lemma 1.1, we obtain

$$|P_1(\xi)|\, \tilde{P}_2(\xi) \leqslant C\, \tilde{P}(\xi),$$

and hence, for $|\eta| \leqslant 1$,

$$|P_1(\xi+\eta)|\, \tilde{P}_2(\xi) \leqslant C'\, |P_1(\xi+\eta)|\, \tilde{P}_2(\xi+\eta) \leqslant C'\, C\, \tilde{P}(\xi+\eta) \leqslant C'\, C\, C''\, \tilde{P}(\xi).$$

The constants C' and C'' enter as a result of two applications of (6), first to $\tilde{P}_2(\xi)$ and next to $\tilde{P}(\xi)$. Maximizing over all $\eta \in R^n$ with $|\eta| \leqslant 1$, we obtain the desired estimate of $\tilde{P}_1(\xi)\, \tilde{P}_2(\xi)$ on account of Lemma 1.1.

Lemma 1.3. *If $P(\xi) = \sum_{\mu=1}^{m} \bar{P}_\mu(\xi)\, P_\mu(\xi)$, where $P_\mu(\xi)$, $\mu = 1, \ldots, m$, denote arbitrary polynomials, then there is a constant C such that*

$$C^{-1} \sum_{\mu=1}^{m} \tilde{P}_\mu(\xi)^2 \leqslant \tilde{P}(\xi) \leqslant C \sum_{\mu=1}^{m} \tilde{P}_\mu(\xi)^2.$$

Again, the latter inequality is elementary (and follows from (7) and Lemma 1.2). It remains to be proved that there are constants C_μ such that

$$\tilde{P}_\mu(\xi) \leqslant C_\mu \tilde{P}(\xi)^{\frac{1}{2}}, \tag{8}$$

and this follows from the trivial inequalities $|P_\mu(\xi)| \leqslant P(\xi)^{\frac{1}{2}} \leqslant \tilde{P}(\xi)^{\frac{1}{2}}$. Using (6), we obtain, in fact,

$$|P_\mu(\xi+\eta)| \leqslant \tilde{P}(\xi+\eta)^{\frac{1}{2}} \leqslant C\, \tilde{P}(\xi)^{\frac{1}{2}},$$

uniformly for $|\eta| \leqslant 1$. Hence (8) follows from Lemma 1.1.

A polynomial $P(\xi)$ is called *stronger* than another polynomial $Q(\xi)$, and $Q(\xi)$ is called *weaker* than $P(\xi)$, if there is a constant C such that,

$$\tilde{Q}(\xi) \leqslant C\, \tilde{P}(\xi)$$

for all $\xi \in R^n$. We write $P(\xi) \succ Q(\xi)$, or $Q(\xi) \prec P(\xi)$, to designate this transitive relation. It follows from Lemma 1.2 that, for any polynomial $R(\xi) \not\equiv 0$, *the relations $P(\xi) \succ Q(\xi)$ and $P(\xi)\, R(\xi) \succ Q(\xi)\, R(\xi)$ are equivalent.*

More generally, a polynomial $Q(\xi)$ is called weaker than a *family* of polynomials $P_\mu(\xi)$, $\mu = 1, \ldots, m$, if there is a constant C such that

$$\tilde{Q}(\xi)^2 \leqslant C \sum_{\mu=1}^{m} \tilde{P}_\mu(\xi)^2 \tag{9}$$

for all $\xi \in R^n$. According to Lemma 1.3, this amounts to the requirement that $\bar{Q}(\xi) Q(\xi)$ be weaker than $\sum_\mu \bar{P}_\mu(\xi) P_\mu(\xi)$.

Remark. In order that $Q(\xi)$ be weaker than the family $P_1(\xi), \ldots, P_m(\xi)$, it is (necessary and) sufficient that there be a constant C' such that

$$|Q(\xi)|^2 \leqslant C' \sum_{\mu=1}^{m} \tilde{P}_\mu(\xi)^2$$

for all $\xi \in R^n$. This follows from (6) and Lemma 1.1, applied to each $P_\mu(\xi)$ and to $Q(\xi)$, respectively.

2. A priori inequalities for certain special systems of differential operators

To every polynomial $P(\xi)$, $\xi = (\xi_1, \ldots, \xi_n)$, corresponds a (formal) partial differential operator $P(D)$ (with constant coefficients), obtained by the substitution

$$\xi_\nu \to D_\nu = i^{-1} \partial / \partial x_\nu.$$

Here $x = (x_1, \ldots, x_n)$ ist the set of Cartesian coordinates of the generic point $x \in R^n$. For any non-void open subset $\Omega \subset R^n$ we denote by $\mathcal{D}(\Omega)$ the linear space of all infinitely differentiable functions $u = u(x)$ whose supports are compact and contained in Ω. The above formal differential operator gives rise to a linear transformation $P(D)$ of $\mathcal{D}(\Omega)$ into itself, and the mapping $P(\xi) \to P(D)$ is an isomorphism of the ring $\mathcal{A}$ of all polynomials $P(\xi)$ and the ring of all partial differential operators (with constant coefficients) acting on $\mathcal{D}(\Omega)$.(1) A differential operator $Q(D)$ is called weaker than another such operator, $P(D)$, if the polynomial $Q(\xi)$ is weaker than $P(\xi)$ in the sense described at the end of the preceding section. If this is the case, we write also $Q(D) \prec P(D)$.

We shall now study two cases of a priori inequalities for certain simple systems of differential operators acting on $\mathcal{D}(\Omega)$, where the open set $\Omega \subset R^n$ is supposed to be *bounded*. The norms in question are L^2-norms: $\|f\|^2 = \int |f(x)|^2 dx$. The first system

(1) The only point in need of comment is the known fact that $P(\xi) \equiv 0$ if $P(D)\varphi = 0$ for all functions $\varphi \in \mathcal{D}(\Omega)$. This may be shown by use of Laplace transforms.

to be considered consists of operators denoted by

$$P_1(D),\ P_2(D),\ \ldots,\ P_m(D) \quad \text{and } Q(D).$$

THEOREM 2.1. *In order that there be a constant C such that the inequality*

$$\| Q(D)u \|^2 \leqslant C \sum_{\mu=1}^{m} \| P_\mu(D)u \|^2 \tag{10}$$

holds for all functions $u \in \mathcal{D}(\Omega)$, it is necessary and sufficient that $Q(\xi)$ be weaker than the family $P_1(\xi), \ldots, P_m(\xi)$; that is, for some constant C',

$$\tilde{Q}(\xi)^2 \leqslant C' \sum_{\mu=1}^{m} \tilde{P}_\mu(\xi)^2. \tag{11}$$

For $m=1$, this theorem is due to Hörmander [1, Theorem 2.2], whose proof of the necessity part may be carried over to the case $m>1$. The sufficiency part may be reduced to that of Hörmander's theorem, or rather to the crucial case thereof [1, Lemma 2.8], asserting that

$$\| P^{(\alpha)}(D)u \| \leqslant k \| P(D)u \| \tag{12}$$

for some k independent of $u \in \mathcal{D}(\Omega)$. The reduction to this case is simple and proceeds as in [1, p. 185] by application of Parseval's formula to both sides of (10). Denoting the Fourier transform of $u=u(x)$ by $\hat{u}=\hat{u}(\xi)$, we obtain under the hypothesis (11)

$$\begin{aligned} \| Q(D)u \|^2 &= \int | Q(\xi)|^2 |\hat{u}(\xi)|^2 d\xi \leqslant C' \sum_{\mu=1}^{m} \sum_{\alpha} \int | P_\mu^{(\alpha)}(\xi)|^2 |\hat{u}(\xi)|^2 d\xi \\ &= C' \sum_{\mu=1}^{m} \sum_{\alpha} \| P_\mu^{(\alpha)}(D)u \|^2 \leqslant C \sum_{\mu=1}^{m} \| P_\mu(D)u \|^2. \end{aligned}$$

The last step follows from (12) applied to $P=P_\mu$.

Before describing the second case of a priori inequalities to be considered in the present section, we recall that a *fundamental solution* of a differential operator $P(D)$ (with constant coefficients) is defined as a distribution in the sense of L. Schwartz [3] such that

$$P(D)E = \delta_0,$$

the Dirac measure at the origin. This definition may also be expressed as follows:

$$E * P(D)u = u \qquad \text{for every } u \in \mathcal{D}(R^n).$$

It was shown by Hörmander [2, Theorem 1.2] that every non-zero differential operator $P(D)$ has a fundamental solution which is *proper* in the sense that there corresponds to any pair of non-void bounded open subsets Ω and Ω' of R^n and to any differential operator $Q(D)$ weaker than $P(D)$ a constant C such that

$$\|E * Q(D)u\|_{\Omega'} \leqslant C\|u\| \qquad \text{for every } u \in \mathcal{D}(\Omega). \tag{13}$$

(We write $\|f\|_A$ for $\left(\int_A |f(x)|^2\,dx\right)^{\frac{1}{2}}$.) In the sequel we shall always take $\Omega' = \Omega$.

Consider now a system of operators $P(D)$ and $Q_1(D), \ldots, Q_m(D)$.

THEOREM 2.2. *Let $P(\xi) \not\equiv 0$. In order that there be a constant C such that the relation*

$$P(D)u = \sum_{\mu=1}^{m} Q_\mu(D)v_\mu, \qquad u,\, v_\mu \in \mathcal{D}(\Omega), \tag{14}$$

implies

$$\|u\|^2 \leqslant C\sum_{\mu=1}^{m}\|v_\mu\|^2, \tag{15}$$

it is necessary and sufficient that each $Q_\mu(\xi)$ be weaker than $P(\xi)$.

The *necessity* follows immediately from Hörmander's theorem (i.e. the above Theorem 2.1 for $m=1$) applied to the two operators $P(D)$ and $Q_\lambda(D)$, $\lambda = 1, \ldots, m$. In fact, for any $w \in \mathcal{D}(\Omega)$ the functions $u = Q_\lambda(D)w$ and $v_\mu = \delta_{\lambda\mu}P(D)w$ satisfy (14). In the proof of the *sufficiency* we apply a proper fundamental solution E of $P(D)$. Under the hypothesis (14) we obtain

$$u = E * P(D)u = \sum_{\mu=1}^{m} E * Q_\mu(D)v_\mu,$$

from which (15) follows on account of (13).

3. The regular case

Let $\mathbf{P}(\xi)$ and $\mathbf{Q}(\xi)$ denote two matrices over the ring $\mathcal{A}$ of all polynomials in $\xi = (\xi_1, \ldots, \xi_n)$, and let $\mathbf{P}(D)$ and $\mathbf{Q}(D)$ denote the corresponding matrices over the ring of all partial differential operators (with constant coefficients) acting on functions of class $\mathcal{D}(\Omega)$, where Ω is a given non-void bounded open subset of R^n. We propose to decide under which conditions the inequality

$$\|\mathbf{Q}(D)\mathbf{u}\| \leqslant C\|\mathbf{P}(D)\mathbf{u}\|$$

holds (with a suitable constant C) for all column vectors $\mathbf{u}$ of class $\mathcal{D}(\Omega)$ (i.e. with elements of class $\mathcal{D}(\Omega)$). Here $\|\mathbf{f}\|$ denotes the L^2-norm of a column vector $\mathbf{f}$, defined in terms of the elements f_j of $\mathbf{f}$ by $\|\mathbf{f}\|^2 = \int \sum_j |f_j(x)|^2\,dx$. Theorem 3.1 below is concerned with the *regular* case, i.e. the case where $\mathbf{P}(\xi)$ is a non-singular, square matrix. The general case is studied in the subsequent section (Theorem 4). Theorem 3.2 below deals, in the regular case, with a kind of dual problem to that of Theorem 3.1. The extension of Theorem 3.2 to the general case is contained in a result (Theorem 5) concerning "conditional" a priori inequalities, cf. § 5.

In the regular case, where $\mathbf{P}(\xi)$ is a square matrix whose determinant $P(\xi)$ does not vanish identically, we denote by $\check{\mathbf{P}}(\xi)$ the square matrix over $\mathcal{A}$ whose elements are the cofactors of the corresponding entries in the transpose of $\mathbf{P}(\xi)$. Then

$$\mathbf{P}(\xi)\,\check{\mathbf{P}}(\xi) = \check{\mathbf{P}}(\xi)\,\mathbf{P}(\xi) = P(\xi)\,\mathbf{I},$$

where $\mathbf{I}$ denotes the identity matrix. It follows that

$$\mathbf{P}(D)\,\check{\mathbf{P}}(D)\,\mathbf{u} = \check{\mathbf{P}}(D)\,\mathbf{P}(D)\,\mathbf{u} = P(D)\,\mathbf{u}$$

for any column vector $\mathbf{u}$ of class $\mathcal{D}(\Omega)$.

THEOREM 3.1. *Let $\mathbf{P}(\xi)$ denote a non-singular square matrix over $\mathcal{A}$ with r rows and columns, and let $\mathbf{Q}(\xi)$ denote any matrix over $\mathcal{A}$ with r columns. In order that there be a constant C such that the inequality*

$$\|\mathbf{Q}(D)\,\mathbf{u}\| \leqslant C\,\|\mathbf{P}(D)\,\mathbf{u}\| \tag{16}$$

holds for all r-dimensional column vectors $\mathbf{u} = \mathbf{u}(x)$ of class $\mathcal{D}(\Omega)$, it is necessary and sufficient that

$$\mathbf{Q}(\xi)\,\check{\mathbf{P}}(\xi) \prec P(\xi), \tag{17}$$

i.e., each element of $\mathbf{Q}(\xi)\,\check{\mathbf{P}}(\xi)$ should be weaker than the determinant $P(\xi)$ of $\mathbf{P}(\xi)$.

Proof of the necessity. Applying (16) to $\mathbf{u} = \check{\mathbf{P}}(D)\,\mathbf{w}$, where the column vector $\mathbf{w}$ is of class $\mathcal{D}(\Omega)$, we obtain

$$\|\mathbf{Q}(D)\,\check{\mathbf{P}}(D)\,\mathbf{w}\| \leqslant C\,\|\mathbf{P}(D)\,\check{\mathbf{P}}(D)\,\mathbf{w}\| = C\,\|P(D)\,\mathbf{w}\|.$$

Writing, for brevity, $\mathbf{Q}(\xi)\,\mathbf{P}(\xi) = \mathbf{R}(\xi)$, and putting $w_k = \delta_{jk}\varphi$, where $\varphi \in \mathcal{D}(\Omega)$ and $k = 1, \ldots, r$, we conclude that, for arbitrary indices i and j,

$$\|R_{ij}(D)\,\varphi\| \leqslant C\,\|P(D)\,\varphi\|,$$

and hence from Hörmander's Theorem ($=$Theorem 2.1 for $m=1$) that $R_{ij}(\xi) \prec P(\xi)$.

Proof of the sufficiency. For any column vector $\mathbf{u}$ of class $\mathcal{D}(\Omega)$, write $\mathbf{v} = \mathbf{P}(D)\,\mathbf{u}$, $\mathbf{w} = \mathbf{Q}(D)\,\mathbf{u}$. Then

$$P(D)\,\mathbf{w} = P(D)\,\mathbf{Q}(D)\,\mathbf{u} = \mathbf{Q}(D)\,P(D)\,\mathbf{u} = \mathbf{Q}(D)\,\check{\mathbf{P}}(D)\,\mathbf{P}(D)\,\mathbf{u} = \mathbf{R}(D)\,\mathbf{v}.$$

Explicitly, we obtain for every i,

$$P(D)\,w_i = \sum_j R_{ij}(D)\,v_j,$$

and hence it follows from Theorem 2.2 under the hypothesis (17) that

$$\|w_i\|^2 \leqslant C_i \sum_j \|v_j\|^2.$$

Summing over i, we arrive at (16) with $C^2 = \sum_i C_i$.

THEOREM 3.2. *Let $\mathbf{P}(\xi)$ denote a non-singular square matrix over $\mathcal{A}$ with r rows and columns, and let $\mathbf{Q}(\xi)$ denote any matrix over $\mathcal{A}$ with r rows. In order that there be a constant C such that*

$$\mathbf{P}(D)\,\mathbf{u} = \mathbf{Q}(D)\,\mathbf{v} \quad \textit{implies} \quad \|\mathbf{u}\| \leqslant C\,\|\mathbf{v}\| \tag{18}$$

when $\mathbf{u}$ and $\mathbf{v}$ denote column vectors of class $\mathcal{D}(\Omega)$, it is necessary and sufficient that

$$\check{\mathbf{P}}(\xi)\,\mathbf{Q}(\xi) \prec P(\xi).$$

Proof of the necessity. For brevity, write $\check{\mathbf{P}}(\xi)\,\mathbf{Q}(\xi) = \mathbf{S}(\xi)$. For any column vector $\mathbf{w}$ of class $\mathcal{D}(\Omega)$, put $\mathbf{u} = \mathbf{S}(D)\,\mathbf{w}$, $\mathbf{v} = P(D)\,\mathbf{w}$. Then

$$\mathbf{P}(D)\,\mathbf{u} = \mathbf{P}(D)\,\check{\mathbf{P}}(D)\,\mathbf{Q}(D)\,\mathbf{w} = P(D)\,\mathbf{Q}(D)\,\mathbf{w} = \mathbf{Q}(D)\,P(D)\,\mathbf{w} = \mathbf{Q}(D)\,\mathbf{v},$$

and hence the hypothesis (18) implies $\|\mathbf{S}(D)\,\mathbf{w}\| \leqslant C\,\|P(D)\,\mathbf{w}\|$. As in the necessity part of Theorem 3.1, we conclude that each $S_{ij}(\xi)$ is weaker than $P(\xi)$.

Proof of the sufficiency. From $\mathbf{P}(D)\,\mathbf{u} = \mathbf{Q}(D)\,\mathbf{v}$ follows

$$P(D)\,\mathbf{u} = \check{\mathbf{P}}(D)\,\mathbf{P}(D)\,\mathbf{u} = \check{\mathbf{P}}(D)\,\mathbf{Q}(D)\,\mathbf{v} = \mathbf{S}(D)\,\mathbf{v}.$$

As in the sufficiency part of Theorem 3.1, we conclude that $\|\mathbf{u}\| \leqslant C\,\|\mathbf{v}\|$ under the hypothesis $\mathbf{S}(\xi) \prec P(\xi)$.

Remark 1. The restriction $P(\xi) \not\equiv 0$ in Theorem 3.2 is necessary for the validity of the implication (18), even in the case $\mathbf{v} = \mathbf{0}$. In fact, if $\mathbf{X}(\xi)$ denotes any column vector over $\mathcal{A}$ such that $\mathbf{P}(\xi)\,\mathbf{X}(\xi) = \mathbf{0}$, and if $\varphi \in \mathcal{D}(\Omega)$, then the relation $\mathbf{P}(D)\,\mathbf{u} = \mathbf{0}$

is fulfilled by $\mathbf{u}=\mathbf{X}(D)\varphi$. If (18) subsists, we infer that $\mathbf{X}(D)\varphi=\mathbf{u}=\mathbf{0}$, and hence $\mathbf{X}(\xi)=\mathbf{0}$. This shows that $\mathbf{P}(\xi)$ is non-singular.

Remark 2. When the square matrix $\mathbf{P}(\xi)$ is non-singular, the differential operator $\mathbf{P}(D)=\{P_{ij}(D)\}$ has a fundamental solution with properties similar to those of the proper fundamental solutions discussed in Hörmander [2]. In fact, if E denotes a proper fundamental solution (as described in [2]) for the differential operator $P(D)$, where $P(\xi)=\det \mathbf{P}(\xi)$, then the matrix

$$\mathbf{E}=\check{\mathbf{P}}(D)\,E=\{\check{P}_{ij}(D)\,E\}$$

is a fundamental solution of $\mathbf{P}(D)$ because

$$\mathbf{P}(D)\,\mathbf{E}=\mathbf{P}(D)\,\check{\mathbf{P}}(D)\,E=P(D)\,E\,\mathbf{I}=\delta_0\mathbf{I}.$$

Note also that, for any column vector $\mathbf{u}$ of class $\mathcal{D}(R^n)$,

$$\mathbf{Q}(D)\,\mathbf{E}*\mathbf{u}=\mathbf{Q}(D)\,\check{\mathbf{P}}(D)\,E*\mathbf{u}=\mathbf{R}(D)\,E*\mathbf{u}$$

and

$$\mathbf{E}*\mathbf{Q}(D)\,\mathbf{u}=E*\check{\mathbf{P}}(D)\,\mathbf{Q}(D)\,\mathbf{u}=E*\mathbf{S}(D)\,\mathbf{u}$$

with the notations $\mathbf{R}(D)$ and $\mathbf{S}(D)$ from Theorem 3.1 and Theorem 3.2, respectively. Hence $\mathbf{E}$ is *proper* in two senses (left and right) corresponding to these two theorems.

4. Extension of Theorems 2.1 and 3.1 to the general case

We shall need the following lemma, which may be viewed as a generalization of the sufficiency part of Theorem 2.2. As usual, Ω denotes any given non-void bounded open subset of R^n.

LEMMA 4. *Let $P_\lambda(\xi)$, not all zero, and $Q_{\lambda\mu}(\xi)$ denote given polynomials ($\lambda=1, \ldots, l$; $\mu=1, \ldots, m$), and suppose each $Q_{\lambda\mu}(\xi)$ is weaker than the family $P_1(\xi), \ldots, P_l(\xi)$. Then there is a constant C such that the relations*

$$P_\lambda(D)\,u=\sum_{\mu=1}^{m} Q_{\lambda\mu}(D)\,v_\mu, \qquad \lambda=1, \ldots, l, \tag{19}$$

(with u, $v_\mu\in\mathcal{D}(\Omega)$) imply

$$\|u\|^2\leqslant C\sum_{\mu=1}^{m}\|v_\mu\|^2.$$

Proof. Writing $P(\xi)=\sum_\lambda \overline{P}_\lambda(\xi)\,P_\lambda(\xi)$ and $Q_\mu(\xi)=\sum_\lambda \overline{P}_\lambda(\xi)\,Q_{\lambda\mu}(\xi)$, we obtain from (19)

$$P(D)\,u=\sum_{\mu=1}^{m} Q_\mu(D)\,v_\mu. \tag{20}$$

The desired inequality now follows from (20) by application of Theorem 2.2 because each $Q_\mu(\xi)$ is weaker than $P(\xi)$. In fact,

$$\begin{aligned}\tilde{Q}_\mu(\xi) &\leqslant C_1\,(\textstyle\sum_\lambda \tilde{P}_\lambda(\xi)\,\tilde{Q}_{\lambda\mu}(\xi)) \leqslant C_1\,(\sum_\lambda \tilde{P}_\lambda(\xi)^2)^{\frac{1}{2}}\,(\sum_\lambda \tilde{Q}_{\lambda\mu}(\xi)^2)^{\frac{1}{2}}\\ &\leqslant C_2\textstyle\sum_\lambda \tilde{P}_\lambda(\xi)^2 \leqslant C_3\,\tilde{P}(\xi)\end{aligned}$$

by virtue of (7), § 1, and of Lemmas 1.2 and 1.3.

The extension of Theorems 2.1 and 3.1 is concerned with two rectangular matrices $\mathbf{P}(\xi)$ and $\mathbf{Q}(\xi)$ over $\mathcal{A}$ which are arbitrary, save for the obvious requirement that they have equally many, say s, columns. Let r denote the number of rows in $\mathbf{P}(\xi)$ and r' in $\mathbf{Q}(\xi)$, and let ϱ denote the rank of $\mathbf{P}(\xi)$ over $\mathcal{A}$. We shall decide under which conditions the inequality

$$\|\mathbf{Q}(D)\,\mathbf{u}\| \leqslant C\,\|\mathbf{P}(D)\,\mathbf{u}\| \tag{21}$$

holds (with a suitable constant C) for all column vectors $\mathbf{u}=\mathbf{u}(x)$ of class $\mathcal{D}(\Omega)$ (and of dimension s). In view of a later application (§ 5) we prefer to study the following slightly more general problem, in which $A(\xi)$ denotes a (single) non-zero polynomial: Under which conditions do the relations

$$A(D)\,\mathbf{v}=\mathbf{P}(D)\,\mathbf{u}, \qquad A(D)\,\mathbf{w}=\mathbf{Q}(D)\,\mathbf{u}, \tag{22}$$

in which $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are of class $\mathcal{D}(\Omega)$, imply $\|\mathbf{w}\| \leqslant C\,\|\mathbf{v}\|$ with a suitable constant C? It turns out that, in the affirmative case, the constant C may be chosen so as to be independent of $A(\xi)$ (as long as $A(\xi)\not\equiv 0$). The preceding problem (21) corresponds to $A(\xi)=1$.

In order to formulate the solution of these two problems, we shall consider certain minors of $\mathbf{P}(\xi)$ and of the matrix $\begin{Bmatrix}\mathbf{P}(\xi)\\ \mathbf{Q}(\xi)\end{Bmatrix}$. A minor of $\mathbf{P}(\xi)$ has the form

$$P_{\alpha,\beta}(\xi)=\begin{vmatrix} P_{\alpha_1\beta_1}(\xi) & \dots & P_{\alpha_1\beta_k}(\xi)\\ \cdot & \cdot\;\cdot\;\cdot & \cdot\\ P_{\alpha_k\beta_1}(\xi) & \dots & P_{\alpha_k\beta_k}(\xi)\end{vmatrix},$$

where $k \leqslant \min(r, s)$. Here $\alpha=(\alpha_1, \dots, \alpha_k)$ and $\beta=(\beta_1, \dots, \beta_k)$ are ordered sets consisting of $|\alpha|=|\beta|=k$ distinct numbers selected among the integers $1, 2, \dots, r$ and

1, 2, ..., s, respectively. In addition to these "*pure*" minors $P_{\alpha\beta}(\xi)$, we shall need "*mixed*" minors, in which the last row is taken from $\mathbf{Q}(\xi)$ instead of $\mathbf{P}(\xi)$:

$$R_{\gamma q,\beta}(\xi)=\begin{vmatrix} P_{\gamma_1\beta_1}(\xi) & \dots & P_{\gamma_1\beta_k}(\xi) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ P_{\gamma_{k-1}\beta_1}(\xi) & \dots & P_{\gamma_{k-1}\beta_k}(\xi) \\ Q_{q\beta_1}(\xi) & \dots & Q_{q\beta_k}(\xi) \end{vmatrix}=\sum_{j=1}^{k}(-1)^{k-j}P_{\gamma,\beta^{(j)}}(\xi)\,Q_{q\beta_j}(\xi). \tag{23}$$

Here $\gamma=(\gamma_1, \dots, \gamma_{k-1})$ denotes any ordered set of $k-1$ distinct numbers selected among the integers 1, 2, ..., r, whereas the number q ranges over the integers 1, ..., r'. Moreover, we obtain $\beta^{(j)}$ from $\beta=(\beta_1, \dots, \beta_k)$ by deleting β_j. We shall mainly consider the case $k=\varrho$, the rank of $\mathbf{P}(\xi)$.

THEOREM 4. *In order that there be a constant C such that $\|\mathbf{Q}(D)\mathbf{u}\|\leqslant C\|\mathbf{P}(D)\mathbf{u}\|$ for all column vectors $\mathbf{u}$ of class $\mathcal{D}(\Omega)$, it is necessary and sufficient that the following two conditions be fulfilled:*

1°) *The rank of the matrix $\begin{Bmatrix}\mathbf{P}(\xi)\\ \mathbf{Q}(\xi)\end{Bmatrix}$ over $\mathcal{A}$ equals the rank ϱ of $\mathbf{P}(\xi)$ over $\mathcal{A}$.*

2°) *Every mixed minor $R_{\gamma q,\beta}(\xi)$ of order ϱ is weaker than the family of corresponding pure minors $P_{\gamma p,\beta}(\xi)$, $p=1, \dots, r$. Explicitly: There is a constant C' such that*

$$\sum_{q=1}^{r'}\tilde{R}_{\gamma q,\beta}(\xi)^2\leqslant C'\sum_{p=1}^{r}\tilde{P}_{\gamma p,\beta}(\xi)^2. \tag{24}$$

More generally, the joint conditions 1° *and* 2° *are necessary and sufficient for* (22) *to imply $\|\mathbf{w}\|\leqslant C\|\mathbf{v}\|$ with a suitable constant C.*

Remarks. Ad 1°) This condition states that every row in $\mathbf{Q}(\xi)$ after multiplication by a suitable polynomial may be expressed as a linear combination (over $\mathcal{A}$) of the rows in $\mathbf{P}(\xi)$. In other words, if a column vector $\mathbf{X}(\xi)$ formed by s polynomials $X_k(\xi)$, $k=1, \dots, s$, fulfills the relation $\mathbf{P}(\xi)\mathbf{X}(\xi)=\mathbf{0}$ (identically), then likewise the relation $\mathbf{Q}(\xi)\mathbf{X}(\xi)=\mathbf{0}$. Ad 2°) The following weaker condition remains sufficient (when combined with 1°): Every mixed minor $R_{\gamma q,\beta}(\xi)$ is weaker than the system of *all* pure minors $P_{\alpha,\beta}(\xi)$ formed by means of the same column numbers $\beta_1, \dots, \beta_\varrho$. It suffices, moreover, to check this condition for one single set β of column numbers provided the corresponding columns in $\mathbf{P}(\xi)$ are linearly independent over $\mathcal{A}$.

Proof of the necessity of 1° *and* 2°. Suppose (22) implies $\|\mathbf{w}\|\leqslant C\|\mathbf{v}\|$ in case of some given non-zero polynomial $A(\xi)$. Then $\|\mathbf{Q}(D)t\|\leqslant C\|\mathbf{P}(D)\mathbf{t}\|$ for all column vectors $\mathbf{t}=\mathbf{t}(x)$ of class $\mathcal{D}(\Omega)$ because (22) is satisfied by $\mathbf{u}=A(D)\mathbf{t}$, $\mathbf{v}=\mathbf{P}(D)\mathbf{t}$, $\mathbf{w}=\mathbf{Q}(D)\mathbf{t}$.

It suffices, therefore, to prove that 1° and 2° are fulfilled under the hypothesis that (21) holds for all $\mathbf{u}$ of class $\mathcal{D}(\Omega)$. Ad 1°) Let $\mathbf{X}(\xi)$ denote any column vector over $\mathcal{A}$ such that $\mathbf{P}(\xi)\,\mathbf{X}(\xi)=\mathbf{0}$. For any function $\varphi\in\mathcal{D}(\Omega)$, write $\mathbf{u}=\mathbf{X}(D)\varphi$. Since $\mathbf{P}(D)\,\mathbf{u}=\mathbf{P}(D)\,\mathbf{X}(D)\,\varphi=\mathbf{0}$, we infer from (21) that $\mathbf{Q}(D)\,\mathbf{X}(D)\,\varphi=\mathbf{Q}(D)\,\mathbf{u}=\mathbf{0}$. Consequently, $\mathbf{Q}(\xi)\,\mathbf{X}(\xi)=\mathbf{0}$. Ad 2°) Again, let $\varphi\in\mathcal{D}(\Omega)$, and put

$$u_k=(-1)^{\varrho-h}\,P_{\gamma,\beta^{(h)}}(D)\,\varphi \qquad \text{if } k=\beta_h,$$

and $u_k=0$ if $k\notin\beta$. Then it follows from (23) and the analogue thereof for pure minors that

$$(\mathbf{P}(D)\,\mathbf{u})_p=\sum_{h=1}^{\varrho}P_{p\beta_h}(D)\,u_{\beta_h}=P_{\gamma p,\beta}(D)\,\varphi,$$

$$(\mathbf{Q}(D)\,\mathbf{u})_q=\sum_{h=1}^{\varrho}Q_{q\beta_h}(D)\,u_{\beta_h}=R_{\gamma q,\beta}(D)\,\varphi.$$

Hence we infer from (21) that

$$\|R_{\gamma q,\beta}(D)\,\varphi\|^2\leqslant C\sum_{p=1}^{r}\|P_{\gamma p,\beta}(D)\,\varphi\|^2, \qquad \varphi\in\mathcal{D}(\Omega).$$

It follows now from Theorem 2.1 that

$$\tilde{R}_{\gamma q,\beta}(\xi)^2\leqslant C_q\sum_{p=1}^{r}\tilde{P}_{\gamma p,\beta}(\xi)^2,$$

where C_q is independent of ξ. This implies (24) with $C'=\sum_{q=1}^{r'}C_q$.

Proof of the sufficiency of 1° *and* 2°. According to 1°,

$$\begin{vmatrix} P_{\alpha_1\beta_1}(\xi), & \dots, & P_{\alpha_1\beta_\varrho}(\xi), & P_{\alpha_1 k}(\xi)\\ \cdot & \cdot & \cdot & \cdot\\ P_{\alpha_\varrho\beta_1}(\xi), & \dots, & P_{\alpha_\varrho\beta_\varrho}(\xi), & P_{\alpha_\varrho k}(\xi)\\ Q_{q\beta_1}(\xi), & \dots, & Q_{q\beta_\varrho}(\xi), & Q_{qk}(\xi)\end{vmatrix}=0$$

for any set $\alpha=(\alpha_1,\dots,\alpha_\varrho)$ of ϱ distinct row numbers for $\mathbf{P}(\xi)$, any row number $q=1,2,\dots,r'$ for $\mathbf{Q}(\xi)$, any set $\beta=(\beta_1,\dots,\beta_\varrho)$ of distinct column numbers (for $\mathbf{P}(\xi)$ and $\mathbf{Q}(\xi)$), and any additional column number $k=1,2,\dots,s$. In fact, if $k\notin\beta$, the above determinant equals the mixed minor $R_{\alpha q,\beta k}(\xi)$ of order $\varrho+1$, and hence vanishes on account of 1°. And if $k\in\beta$, two of the columns are equal. Developing the determinant according to its last column, we obtain

$$P_{\alpha,\beta}(\xi)\,Q_{qk}(\xi)=\sum_{h=1}^{\varrho}(-1)^{h-\varrho}\,R_{\alpha^{(h)}q,\beta}(\xi)\,P_{\alpha_h k}(\xi).$$

Applying the associated differential operators to the kth component u_k of a column vector $\mathbf{u}=\mathbf{u}(x)$ of class $\mathcal{D}(\Omega)$, and summing over $k=1, 2, \ldots, s$, we obtain

$$P_{\alpha,\beta}(D)(\mathbf{Q}(D)\mathbf{u}))_q=\sum_{h=1}^{\varrho}(-1)^{h-\varrho}R_{\alpha(h)q,\beta}(D)(\mathbf{P}(D)\mathbf{u})_{\alpha_h}.$$

Suppose now (22) is fulfilled. Since $A(D)\varphi=0$ implies $\varphi=0$ when $\varphi\in\mathcal{D}(\Omega)$, we get the result

$$P_{\alpha,\beta}(D)w_q=\sum_{h=1}^{\varrho}(-1)^{h-\varrho}R_{\alpha(h)q,\beta}(D)v_{\alpha_h}.$$

We shall now keep $\beta=(\beta_1, \ldots, \beta_\varrho)$ fixed in such a way that the corresponding columns in $\mathbf{P}(\xi)$ are linearly independent over $\mathcal{A}$. Moreover, we fix q temporarily. Introducing the abbreviations $P_\alpha(\xi)=P_{\alpha,\beta}(\xi)$ and

$$Q_{\alpha,p}(\xi)=(-1)^{h-\varrho}R_{\alpha(h)q,\beta}(\xi) \qquad \text{if } p=\alpha_h \text{ for some } h,$$

whereas $Q_{\alpha,p}(\xi)=0$ if $p\notin\alpha$, we obtain the following relations

$$P_\alpha(D)w_q=\sum_{p=1}^{r}Q_{\alpha,p}(D)v_p.$$

Now Lemma 4 is applicable (with u replaced by w_q, μ by p, and λ by the multi-index $\alpha=(\alpha_1, \ldots, \alpha_\varrho)$. Our choice of β guarantees, in fact, that the polynomials $P_\alpha(\xi)$ are not all $\equiv 0$, and Condition 2° implies, in its weak form, that each $Q_{\alpha,p}(\xi)$ is weaker than the family of all $P_\alpha(\xi)$. We conclude that there is a constant C_q such that

$$\|w_q\|^2\leqslant C_q\sum_{p=1}^{r}\|v_p\|^2.$$

Summing over $q=1, \ldots, r'$, we obtain the desired inequality $\|\mathbf{w}\|\leqslant C\|\mathbf{v}\|$ with $C^2=\sum_q C_q$. This completes the proof of Theorem 4 and the remarks to it, as well as the fact that C may be chosen to as to be independent of the polynomial $A(\xi)\neq 0$.

Remark. Despite the close analogy between Theorems 3.1 and 3.2, there is a considerable difference between the extensions of these two theorems to the general case. Though expressible in terms of certain minors in the compound matrix $\{\mathbf{P}(\xi), \mathbf{Q}(\xi)\}$ (of order equal to the rank of this matrix), the condition in order that the relation $\mathbf{P}(D)\mathbf{u}=\mathbf{Q}(D)\mathbf{v}$ imply $\|\mathbf{u}\|\leqslant C\|\mathbf{v}\|$ for a suitable constant C is not in general a matter simply of comparing each of these minors with a family of other such minors. It turns out that the extension of Theorem 3.2 to the general case is best understood as a special case of *conditional* a priori inequalities, the subject of the following section.

5. Conditional a priori inequalities

In the present section we study a case of a priori inequalities containing all the preceding cases. Let $\mathbf{F}(\xi)$, $\mathbf{G}(\xi)$, and $\mathbf{H}(\xi)$ denote three given matrices over the ring $\mathcal{A}$ of all polynomials in $\xi = (\xi_1, \ldots, \xi_n)$. The only restriction imposed on these matrices is that they should have equally many, say s, columns. As before, Ω denotes a given non-void bounded open subset of R^n. We propose to answer the following question, in which $\mathbf{u} = \mathbf{u}(x)$ ranges over all s-dimensional column vectors of class $\mathcal{D}(\Omega)$: *Under which conditions is there a constant C such that*

$$\mathbf{H}(D)\mathbf{u} = \mathbf{0} \quad \textit{implies} \quad \|\mathbf{G}(D)\mathbf{u}\| \leqslant C\,\|\mathbf{F}(D)\mathbf{u}\|\,?$$

We shall reduce this problem to the "free" case $\mathbf{H}(\xi) = 0$ studied, in its general form, in Theorem 4. It is convenient to rephrase the question as follows: When is there a constant C such that the system of simultaneous differential equations

$$\mathbf{F}(D)\mathbf{u} = \mathbf{v}, \qquad \mathbf{G}(D)\mathbf{u} = \mathbf{w}, \qquad \mathbf{H}(D)\mathbf{u} = \mathbf{0}, \tag{25}$$

imply $\|\mathbf{w}\| \leqslant C\,\|\mathbf{v}\|$? In order to answer the question in this latter form, we begin by reducing the system (25) to a simpler system to which Theorem 4 is applicable. We may assume without loss of generality that the rows in the *"relation matrix"* $\mathbf{H}(\xi)$ are *linearly independent* over $\mathcal{A}$. If not, choose a base for these rows, and cancel the remaining rows. This does not affect the manifold of solutions of $\mathbf{H}(D)\mathbf{u} = 0$. In fact, each of the rows to be cancelled is expressible (after multiplication by a suitable non-zero polynomial) as a linear combination (over $\mathcal{A}$) of the basic rows, and hence the relations $H_{j1}(D)u_1 + \ldots + H_{js}(D)u_s = 0$ corresponding to the basic rows imply the remaining relations. (Recall that, for any polynomial $P(\xi) \not\equiv 0$, $P(D)\varphi = 0$ implies $\varphi = 0$ when $\varphi \in \mathcal{D}(\Omega)$.)

Having thus achieved that the rank s' of $\mathbf{H}(\xi)$ equals the number of rows in $\mathbf{H}(\xi)$, we select a base for the *columns* in $\mathbf{H}(\xi)$. In this way we obtain a non-singular square matrix $\mathbf{H}'(\xi)$ of order s'. The remaining columns in $\mathbf{H}(\xi)$ form a matrix $\mathbf{H}''(\xi)$ with s' rows and $s'' = s - s'$ columns. There is a corresponding decomposition of the s-dimensional column vector $\mathbf{u} = \mathbf{u}(x)$ into two column vectors $\mathbf{u}'$ and $\mathbf{u}''$ of dimensions s' and s'', respectively. Likewise, $\mathbf{F}(\xi)$ splits into two matrices $\mathbf{F}'(\xi)$ and $\mathbf{F}''(\xi)$, and $\mathbf{G}(\xi)$ into $\mathbf{G}'(\xi)$ and $\mathbf{G}''(\xi)$. The system (25) is now equivalent to the following system:

$$\mathbf{F}'(D)\,\mathbf{u}' + \mathbf{F}''(D)\,\mathbf{u}'' = \mathbf{v}, \tag{26}$$

$$\mathbf{G}'(D)\,\mathbf{u}' + \mathbf{G}''(D)\,\mathbf{u}'' = \mathbf{w}, \tag{27}$$

$$\mathbf{H}'(D)\,\mathbf{u}' + \mathbf{H}''(D)\,\mathbf{u}'' = \mathbf{0}. \tag{28}$$

Denoting by $A(\xi)$ the (non-zero) determinant of $\mathbf{H}'(\xi)$ and by $\check{\mathbf{H}}'(\xi)$ the matrix formed by the cofactors of the transpose of $\mathbf{H}'(\xi)$, we have, as in § 3, when $\mathbf{I}$ denotes the unit matrix of order s',

$$\check{\mathbf{H}}'(\xi)\,\mathbf{H}'(\xi) = \mathbf{H}'(\xi)\,\check{\mathbf{H}}'(\xi) = A(\xi)\,\mathbf{I}. \tag{29}$$

Writing $\mathbf{B}(\xi) = \check{\mathbf{H}}'(\xi)\,\mathbf{H}''(\xi)$, we obtain

$$\mathbf{H}'(\xi)\,\mathbf{B}(\xi) = A(\xi)\,\mathbf{H}''(\xi) = \mathbf{H}''(\xi)\,A(\xi). \tag{30}$$

Applying the operator $\check{\mathbf{H}}'(D)$ to both sides of (28), we obtain in view of (29)

$$A(D)\,\mathbf{u}' + \mathbf{B}(D)\,\mathbf{u}'' = \mathbf{0}. \tag{31}$$

Conversely, (28) follows from (31) by left application of $\mathbf{H}'(D)$ under observation of (30) and the fact that $A(D)\varphi = 0$, $\varphi \in \mathcal{D}(\Omega)$, implies $\varphi = 0$ because $A(\xi) \not\equiv 0$.

Applying $A(D)$ to both sides of (26) and (27), and eliminating $A(D)\,\mathbf{u}'$ by means of (31), we get the following new system

$$\mathbf{P}(D)\,\mathbf{u}'' = A(D)\,\mathbf{v}, \tag{32}$$

$$\mathbf{Q}(D)\,\mathbf{u}'' = A(D)\,\mathbf{w}, \tag{33}$$

$$-\mathbf{B}(D)\,\mathbf{u}'' = A(D)\,\mathbf{u}', \tag{34}$$

where the matrices $\mathbf{P}(\xi)$ and $\mathbf{Q}(\xi)$ over $\mathcal{A}$ are defined by

$$\mathbf{P}(\xi) = -\mathbf{F}'(\xi)\,\mathbf{B}(\xi) + \mathbf{F}''(\xi)\,A(\xi), \tag{35}$$

$$\mathbf{Q}(\xi) = -\mathbf{G}'(\xi)\,\mathbf{B}(\xi) + \mathbf{G}''(\xi)\,A(\xi). \tag{36}$$

This new system (32), (33), (34) is our *reduced system*. Since $A(\xi) \not\equiv 0$, the reduced system is equivalent to the preceding system (26), (27), (28), and hence to the original system (25).

The problem is now to decide under which conditions the reduced system (32), (33), (34) implies $\|\mathbf{w}\| \leqslant C\,\|\mathbf{v}\|$ for a suitable constant C. In order to obtain a *necessary* condition, we observe that the reduced system is satisfied by $\mathbf{v} = \mathbf{P}(D)\,\mathbf{t}$, $\mathbf{w} = \mathbf{Q}(D)\,\mathbf{t}$, $\mathbf{u}' = -\mathbf{B}(D)\,\mathbf{t}$, $\mathbf{u}'' = A(D)\,\mathbf{t}$, when $\mathbf{t} = \mathbf{t}(x)$ denotes an arbitrary column vector of class

$\mathcal{D}(\Omega)$ (and of dimension s''). Hence, necessarily,

$$\|\mathbf{Q}(D)\mathbf{t}\| \leqslant C\|\mathbf{P}(D)\mathbf{t}\| \quad \text{for all } \mathbf{t} \text{ of class } \mathcal{D}(\Omega).$$

The "algebraic" content of this condition was described in Theorem 4. Actually, this necessary condition is likewise *sufficient* to ensure that the reduced system (and hence also the given system (25)) implies $\|\mathbf{w}\| \leqslant C\|\mathbf{v}\|$ for a suitable constant, which we denote again by C. It follows, in fact, from the final assertion of Theorem 4 that, if conditions 1° and 2° of that theorem are fulfilled, then (32) and (33) *alone* imply $\|\mathbf{w}\| \leqslant C\|\mathbf{v}\|$ for a suitable C. — The content of the present section may be summarized in the following theorem, in which $\mathbf{P}(D)$ and $\mathbf{Q}(D)$, as above, denote the differential operators (in matrix form) associated with the matrices $\mathbf{P}(\xi)$ and $\mathbf{Q}(\xi)$ over $\mathcal{A}$ defined by (35) and (36):

THEOREM 5. *In order that there exist a constant C such that, for* $\mathbf{u}$ *of class* $\mathcal{D}(\Omega)$,

$$\mathbf{H}(D)\mathbf{u} = 0 \quad \textit{implies} \quad \|\mathbf{G}(D)\mathbf{u}\| \leqslant C\|\mathbf{F}(D)\mathbf{u}\|,$$

it is necessary and sufficient that there be a constant C' such that $\|\mathbf{Q}(D)\mathbf{t}\| \leqslant C'\|\mathbf{P}(D)\mathbf{t}\|$ *for all column vectors* $\mathbf{t}$ *of class* $\mathcal{D}(\Omega)$. (Cf. Theorem 4 for the "algebraic" content of this condition.)

6. Extensions to certain other norms

In this final section we indicate briefly how Theorems 2.6 and 2.8 in Hörmander [1] may be extended to systems. Like [1, Theorem 2.1], these theorems give conditions in order that $\|Q(D)u\| \leqslant C\|P(D)u\|$ for all functions $u \in \mathcal{D}(\Omega)$, but the norm of $Q(D)u$ should now be understood either (i) as the L^∞-norm $\sup_x |Q(D)u(x)|$ or (ii) as the L^2-norm of the restriction of $Q(D)u$ to some given affine subspace Σ of R^n. The norm of $P(D)u$ is the L^2-norm over R^n in both cases, as before. Hörmander's conditions are necessary and sufficient. In each of the two cases the condition in question involves the ratio $\tilde{Q}(\xi)/\tilde{P}(\xi)$, or equivalently $Q(\xi)/\tilde{P}(\xi)$; and the condition states that this ratio should be (i) of class $L^2(R^n)$, (ii) of class $L^\infty(\Sigma) \otimes L^2(\Sigma^\perp)$, that is, uniformly square integrable over all affine subspaces perpendicular to Σ and of dimension $n - \dim \Sigma$.

All the results of the present paper can be carried over to these two cases. For instance, the only change to be performed in Theorem 3.1 is that the stated condition (24) should be replaced by the requirement that each element of the matrix

$\tilde{P}(\xi)^{-1}\,\mathbf{Q}(\xi)\,\check{\mathbf{P}}(\xi)$ be (i) of class $L^2(R^n)$, (ii) of class $L^\infty(\Sigma)\otimes L^2(\Sigma^\perp)$. The proofs of this and related results proceed as before under observation of the following comments:

Property (13), p. 184, of the proper fundamental solution E of $P(D)$ constructed by Hörmander [2] remains in force in each of the two cases (i), (ii) with the new norm of $E * Q(D)u$ under the new hypothesis concerning the ratio $\tilde{Q}(\xi)/\tilde{P}(\xi)$. This follows from Theorem 1.3 in Hörmander [2] by application of the method described on p. 30 of [2]. We omit the details.

Our Theorem 2.2, with one of the new norms of u, is proved in the same way as Theorem 2.6 and Theorem 2.8, respectively, in Hörmander [1]. There is no difficulty in modifying the proof of Lemma 4. The rest of § 4 and all of § 5 can now be taken over to the case of the new norms without any further changes.

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