THE COMBINATORIAL TOPOLOGY OF ANALYTIC FUNCTIONS ON THE BOUNDARY OF A DISK

BY

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PART I. THE GEOMETRIC PROBLEM

1. Preliminaries

A mapping $\zeta : M^1 \to E^2$, with M^1 an oriented 1-manifold (differentiable 1-manifold), is called a *representation* (*regular representation* if it possesses a continuous non-vanishing tangent ζ'); such a mapping will be described by complex valued function $\zeta(t) = \xi(t) + i\eta(t)$. An image point ζ_0 of a regular representation is a *simple crossing point* if there exist exactly two distinct points t_0' and t_0'' such that

$$\zeta(t_0') = \zeta(t_0'') = \zeta_0$$

and the tangents $\zeta'(t_0')$ and $\zeta'(t_0'')$ are linearly independent. A regular representation is *normal* (Whitney [12], p. 281) if it has a finite number of simple crossing points and for every other image point ζ has but one pre-image point t. A pair of representations (regular representations) ζ^1 and ζ^2 are *equivalent* if there exists a sense-preserving homeomorphism φ of M^1 onto M^1 such that $\zeta^2 = \zeta^1 \circ \varphi$ (and if $\varphi'(t)$ is continuous with $\varphi'(t) \neq 0$). With this equivalence relation one may define a *regular* (*normal*) *curve* as an oriented curve with a regular (normal) representation.

A mapping $F : M^2 \to E^2$, with M^2 a 2-manifold, is *open* if for every open set U in M^2 the set $F(U)$ is open in E^2; F is *light* if the pre-image of every point is totally disconnected; F is *interior* if F is light and open.

THEOREM (Stoilow [8], p. 121). *For every interior mapping F of a manifold M^2 into the complex plane there exists a homeomorphism H of M^2 onto a Riemann Surface R and an analytic function W of R into the complex plane such that $F = W \circ H$.*

In this paper the manifold M^2 will always appear as the oriented interior of a Jordan curve and will be denoted by D. In this context, with the use of the Riemann Mapping Theorem, one has

THEOREM 1. *Let F be a sense-preserving interior mapping of D into the complex plane such that $F(D)$ is bounded then there exists a sense-preserving homeomorphism H of D onto D and an analytic function on D such that $F = W \circ H$.*

THEOREM. (Caratheodory [2], p. 86). *A conformal mapping $W(D) = D^*$ (D and D^* each the interior of a Jordan curve) has a continuous univalent extension to the closure of D.*

A mapping F will be called *properly interior* if F is continuous on $\bar{D}$ ($=$closure of D), F is interior and sense-preserving on D; a mapping W will be called *properly analytic* if W is continuous on $\bar{D}$ and is analytic on D.

The Caratheodory Theorem and Theorem 1 combine to give:

THEOREM 2. *Let $F: D \to E^2$ be properly interior and $F|\text{bdy } D$ locally topological then there exists a sense-preserving homeomorphism H of $\bar{D}$ onto $\bar{D}$ and there exists a properly analytic mapping W such that $F = W \circ H$.*

A mapping $\zeta: \text{bdy } D \to E^2$ will be called an *interior boundary* (*analytic boundary*) if there exists a properly interior mapping F [properly analytic mapping W] such that $F|\text{bdy } D = \zeta$ [$W|\text{bdy } D = \zeta$]. A consequence of Theorem 2 is that every locally topological interior boundary is equivalent to a locally topological analytic boundary.

2. Statement of the Main Problem

The problem probably first arose in the study by Picard of the Schwarz-Christoffel mapping function for non-simple polygons and, in this context, was formulated essentially as follows:

Let $Z_0, Z_1, \ldots, Z_{n-1}$ be a sequence of complex numbers in general position. By connecting these points by directed line segments consecutively from Z_k to Z_{k+1}, mod n, a closed oriented polygon is formed. Let $\alpha_k \pi$ be the signed angle from $Z_k - Z_{k-1}$ to $Z_{k+1} - Z_k$ with $-1 < \alpha_k < 1$. For any set of n real numbers $a_0 < a_1 \ldots < a_{n-1}$ and any non-zero complex number A the function

$$\Phi = A(z - a_0)^{-\alpha_0}(z - a_1)^{-\alpha_1} \ldots (z - a_{n-1})^{-\alpha_{n-1}},$$

with $-\frac{1}{2}\pi < \arg(z - a_k) < \frac{1}{2}\pi$, is an analytic function in the upper half plane. Assume

$(\alpha_0+\alpha_1+\dots+\alpha_{n-1})\pi=2\pi$ and then, with B an arbitrary complex number, the function

$$W=\int_{z_0}^{z}\Phi\,dz+B,\quad \operatorname{Im} z_0>0,$$

besides being analytic in the upper half plane, maps the real axis onto a possibly different polygon with, say, $W(a_k)=Z'_k$, but with $Z'_k=Z'_{k-1}$ having the same direction as the given Z_k-Z_{k-1}.

PROBLEM A (Picard [7], p. 313). Find necessary and sufficient conditions on an oriented polygon so that there exists a mapping function W such that $W(a_k)=Z_k$; thus that the real axis is mapped onto the given polygon. (Picard and others near the time were also concerned with the problem of finding an effective method for computing the a_k.)

Some time ago, circa 1948, a clearly related problem was suggested by C. Loewner which will be stated in the following form.

PROBLEM B. Given a normal representation ζ of a closed oriented curve find necessary and sufficient conditions that ζ be an interior boundary.

Problem A is a corollary, by a simple limiting process, of a special case (with the tangent winding number of ζ equal to one) of Problem B. In this paper a complete answer, from the point of view of combinatorial topology, is given to Problem B.

3. Basic Concepts and Techniques

In what follows ζ will always be a representation of a closed curve and the simple crossing points will be called *vertices*. Let $\tau(\zeta)$ be the tangent winding number of ζ and let $\omega(\zeta,\pi)$ be the winding number (index) of ζ about a point π which is not on the point set $[\zeta]$.

LEMMA 1. *If ζ is an interior boundary then $\omega(\zeta,\pi)\geqslant 0$ for all $\pi\notin[\zeta]$; if ζ is also regular then $\tau(\zeta)\geqslant 1$.*

Proof. By Theorem 2 and the fact that both properties of are invariant under the equivalence of locally simple representations of the proof reduces to the case in which ζ is an analytic boundary. The proof then follows easily by standard Cauchy Integral type theorems.

The property of ζ that $\omega(\zeta, \pi) \geqslant 0$ for all $\pi \notin [\zeta]$ is called *non-negative circulation* (Loewner [5], p. 316). A corollary of the principal theorem of this paper will be, by the way, that there exist curves with any pre-assigned tangent winding number, which are of non-negative circulation and which are nevertheless not interior boundaries.

The *outer boundary* of ζ will be the subset of $[\zeta]$ which is contained in the closure of the unbounded component of the complement of $[\zeta]$; an outer point π is a point on the outer boundary such that $\zeta^{-1}(\pi)$ is a single point. Also, for normal ζ and such a point π one can define $\omega^{+}(\zeta, \pi)$ and $\omega^{-}(\zeta, \pi)$ as the larger and smaller winding numbers of point π' near π but not on $[\zeta]$; an outer point π is *positive* if $\omega^{+}(\zeta, \pi) = +1$; the outer boundary is called *positive* if every outer point is positive. The following Lemma follows easily from these notions and Lemma 1; the proof is omitted.

LEMMA 2. *If ζ is normal and is an interior boundary then, for any outer point π, $\omega^{+}(\zeta, \pi) = +1$.*

Let ζ be normal and let $\pi = \zeta(0)$ be an outer point with ζ given by the complex valued function $\zeta(t)$ and t the usual angle parameter, $0 \leqslant t < 2\pi$. Index the n vertices (it is clear there can only be a finite number) in the natural way by traversing the curve with increasing t and using consecutively the integers $0, 1, \ldots, n-1$; thus $\zeta_0, \zeta_1, \ldots, \zeta_{n-1}$ (see Fig. 1). Let the $2n$ preimages of the vertices be denoted by s_k and index so that $0 < s_0 < s_1 < \ldots < s_{2n-1} < 2\pi$. Denote s_j also by s_k^* if $\zeta(s_j) = \zeta(s_k^*)$, $j \neq k$; thus $\zeta(s_k^*) = \zeta(s_k)$ for all k. Define the function ν, with $\zeta(t) = \xi(t) + i\eta(t)$, by

$$\nu(s_k) = \nu_k = \operatorname{sgn} \begin{vmatrix} \xi'(s_k^*) & \eta'(s_k^*) \\ \xi'(s_k) & \eta'(s_k) \end{vmatrix}.$$

Let t_k' be yet another name for the smallest pre-image of ζ_k and let t_k'' be the other pre-image of ζ_k; thus given any k there is always a j such that

$$0 < t_k' = s_j < s_j^* = t_k'' < 2\pi.$$

Define the function

$$\lambda_k = \lambda(t_k') = \operatorname{sgn} \begin{vmatrix} \xi'(t_k'') & \eta'(t_k'') \\ \xi'(t_k') & \eta'(t_k') \end{vmatrix}.$$

Given a normal ζ with $\pi = \zeta(0)$ a positive outer point the *intersection sequence* of ζ with respect to π is defined by the sequence $\{s_k\}$, the values of s_k^* and the

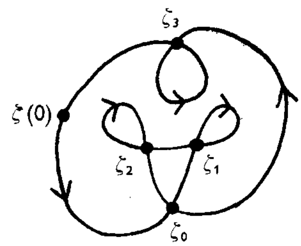

Fig. 1.

values of ν_k for each k. A pair of normal representations ζ and $\tilde{\zeta}$ have *isomorphic intersection sequences* if they have the same number of vertices, $s_j^* = s_k$ if $\tilde{s}_j^* = \tilde{s}_k$, and $\nu_k = \tilde{\nu}_k$.

THEOREM 3. *Let ζ and $\tilde{\zeta}$ be normal with $\zeta(0)$ and $\tilde{\zeta}(0)$ positive outer points. If ζ and $\tilde{\zeta}$ have isomorphic intersection sequences then there exists a sense preserving homeomorphism H of the plane onto itself such that the representation $H \circ \zeta$ is equivalent to the representation $\tilde{\zeta}$; thus the property that a normal representation be an interior boundary depends only on its intersection sequence.*

Proof. A proof can be constructed in a straightforward manner from work of Adkisson and MacLane [1]. However, partly for completeness and partly because of naturalness in this context, a different proof will be sketched.

The proof is by induction on the $\{s_m\}$ for each given ζ. If ζ has no vertices (but has a positive outer point) the Theorem is true.

If $\{s_m\}$ is not empty the following proposition will be proved (let ζ have n vertices): *For each m, $0 \leqslant m \leqslant 2n-1$, the partial intersection sequence $s_0, s_1, \ldots, s_m$ determines a normal representation $\zeta^m(t)$, $0 \leqslant t \leqslant s_m$, with $\zeta(0)$ on the boundary of the unbounded component of the complement of $[\zeta^m]$, up to a sense-preserving homeomorphism of the plane and an equivalence of the representation.* The Proposition is clearly true for $m=0$. If the Proposition is true for an $m>0$ then, because of the condition on $\zeta(0)$, it is true for $m+1$.

Now, to prove the Theorem, one has only to show that the representation is sufficiently well determined on the interval from s_{2n-1} to $2\pi\,(\zeta(2\pi)=\zeta(0))$. But this follows easily from the condition that $\zeta(0)$ is a positive outer point of ζ.

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4. Some Simple Necessary Conditions on the Intersection Sequence

The results in this section are logically independent of the rest of the paper and are presented here so that the reader can get some intermediate feeling for the kinds of conditions satisfied by an intersection sequences that belong to interior boundaries.

LEMMA (Whitney). *Let ζ be a normal representation with $\pi=\zeta(0)$ on the outer boundary and with $\omega^+(\zeta,\pi)=+1$ then*

$$\tau(\zeta)=\sum_{k=0}^{n}\lambda_k+1.$$

Proof. See Whitney [12], Theorem 2, p. 281 or [10], Lemma 3, p. 1086.

Let $C_k=\{\sigma\,|\,s_\sigma\leqslant s_k<s_k^*\leqslant s_\sigma^*\}$ and $L_k=\{\sigma\,|\,s_\sigma<s_k<s_\sigma^*<s_k^*\}$.

LEMMA ([10], Theorem 4, p. 1090). *Let ζ be a normal representation with $\pi=\zeta(0)$ on the outer boundary and with $\omega^+(\zeta,\pi)=+1$ then ζ is of non-negative circulation if and only if*

$$\sum_{C_k}\lambda_\sigma+\sum_{L_k}\lambda_\sigma\geqslant 0 \quad \textit{for all} \quad k.$$

These Lemmas together with Lemma 1 and 2 combine easily to give

THEOREM. *If $\tilde{\zeta}$ is normal and an interior boundary then there exists an equivalent representation ζ so that $\pi=\zeta(0)$ is an outer point, $\omega^+(\zeta,\pi)=+1$,*

$$\sum_{k=0}^{n}\lambda_k\geqslant 0 \quad \textit{and} \quad \sum_{C_k}\lambda_\sigma+\sum_{L_k}\lambda_\sigma\geqslant 0 \quad \textit{for all} \quad k.$$

5. Sketch of the Proof of the Main Theorem

By Lemma 1 every normal interior boundary has $\nu_0=+1$. Now choose the index k to be the smallest integer such that $\nu_k=-1$; such a k exists since $\nu_k=-\nu_k^*$, $\sum_{k=0}^{2n-1}\nu_k=0$. One has then the following list of all possible cases for a normal ζ with $\nu_0=+1$:

$$\textit{Case I} \quad s_k^*<s_k, \qquad \textit{Case II} \quad s_k<s_k^*;$$

in the later case one has for each j, $s_j<s_k$, the subcases:

Case II′ (j) $s_k < s_k^*$ and $s_j < s_k < s_k^* < s_j^*$,

Case II″ (j) $s_k < s_k^*$ and $s_j < s_k < s_j^* < s_k^*$.

For each k and j chosen as above (j is chosen only in Cases II) a "cut" will be defined. Each such "cut" will lead to a pair of normal representations ζ^1 and ζ^2 defined using the original normal ζ and the nature of the Case. It will then be shown that if a normal ζ is an interior boundary there exists a "cut" so that both ζ^1 and ζ^2 are interior boundaries; conversely if there exists a "cut" so that ζ^1 and ζ^2 are interior boundaries then ζ is an interior boundary. Also it will be shown that the ζ^1 and ζ^2 have less vertices than the original ζ. These results will form the basis of a complete algorithm for deciding whether or not a normal ζ is an interior boundary.

6. Definition of the Cuts

For a normal ζ defined on a circle of circumference c, define the cuts ζ^* and ζ^{**}, defined on oriented circles of circumference c^* and c^{**} respectively, as follows:

Cut of Type I

$$\zeta^*(t) = \zeta(t + s_k^*), \qquad 0 \leqslant t \leqslant s_k - s_k^* = c^*;$$

$$\zeta^{**}(t) = \begin{cases} \zeta(t), & 0 \leqslant t \leqslant s_k^*, \\ \zeta(s_k - s_k^* + t), & s_k^* \leqslant t \leqslant c - s_k + s_k^* = c^{**}. \end{cases}$$

Cut of Type II′ (j)

$$\zeta^*(t) = \begin{cases} \zeta(t + s_j), & 0 \leqslant t \leqslant s_k - s_j, \\ \zeta(s_j + s_k - s_k^* + t), & s_k - s_j \leqslant t \leqslant s_k - s_j + s_j^* - s_k^* = c^*; \end{cases}$$

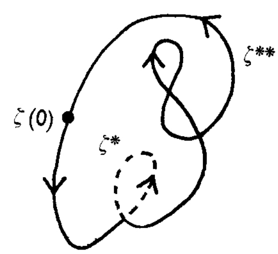

Fig. 2.

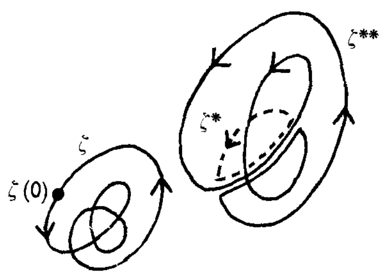

Fig. 3.

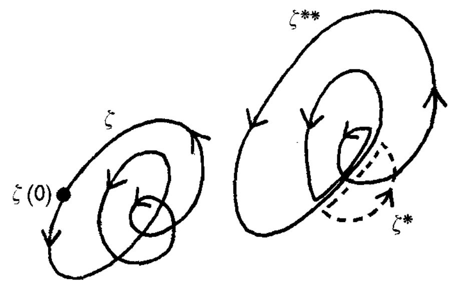

Fig. 4.

$$\zeta^{**}(t)=\begin{cases}\zeta(t), & 0\leqslant t\leqslant s_k^*,\\ \zeta(s_k-s_k^*-t), & s_k^*\leqslant t\leqslant s_k^*+s_k-s_j,\\ \zeta(s_j+s_j^*-s_k-s_k^*+t), & s_k^*+s_k-s_j\leqslant t\leqslant c+s_j+s_j^*-s_k-s_k^*=c^{**}.\end{cases}$$

Cut of Type II″ (j)

$$\zeta^*(t)=\begin{cases}\zeta(s_j^*+t), & 0\leqslant t\leqslant s_k^*-s_j^*,\\ \zeta(s_j^*+s_k^*+s_k-t) & s_k^*-s_j^*\leqslant t\leqslant s_k+s_j+s_k^*-s_j^*=c^*;\end{cases}$$

$$\zeta^{**}(t)=\begin{cases}\zeta(t), & 0\leqslant t\leqslant s_j^*,\\ \zeta(s_j-s_j^*+t), & s_j^*\leqslant t\leqslant s_j^*-s_j+s_k,\\ \zeta(s_j-s_j^*-s_k+s_k^*+t), & s_j^*-s_j+s_k\leqslant t\leqslant c+s_j^*-s_j+s_k-s_k^*=c^{**}.\end{cases}$$

These three Cases are illustrated, in order, in Figs. 2, 3 and 4 (the curves in 3 and 4 actually traverse on interval three times and this fact is indicated in a usual manner).

7. Properties of the Cuts of Type I

Some more lemmas will be needed.

LEMMA 3 ([11], Theorem 9). *Let U be a domain in the plane and X a subset that is closed in U and contains no open sets. If $F:U\rightarrow E^2$ is such that $F\,|\,U-X$ is sense-perserving and interior and if $F\,|\,X$ is light then $F:U\rightarrow E^2$ is interior.*

Define $p\subset q$ if $t_q'\leqslant t_p'<t_p''<t_q''$ and read ζ_p *contained in* $\zeta_q:\zeta_p$ *links* ζ_q *on the left if* $t_p'<t_q'<t_p''<t_q''$ *and write* $p\in L_q$; ζ_p *links* ζ_q *on the right if* $t_q'<t_p'<t_q''<t_p''$ *and write* $p\in R_q$.

LEMMA 4 (Corollary 1, [10], p. 1087). *Let T be the interval $t'_m \leqslant t \leqslant t''_m$: choose j so that $s_j = t'_m$; choose u_m so that $s_{j-1} < u_m < s_j < t'_m$ and assume that $\zeta(u_m)$ is in the unbounded component of the complement of $[\zeta \mid T_m]$ then*

$$\sum_{\sigma \in L_m} \lambda_\sigma = \sum_{\sigma \in R_m} \lambda_\sigma = 0.$$

LEMMA 5. *If a normal ζ determines a cut of Type I (k chosen as in 5) it follows that $s_k^* = s_{k-1}$; thus one has that $\zeta \mid T_k = \zeta^*$ represents a positively oriented Jordan curve and also that $[\zeta^*]$ intersects $[\zeta]$ only in the point $\zeta(s_k^*) = \zeta(s_k)$.*

Proof. Choose u_k as in Lemma 4, with then $s_{k-1} < u_m < s_k = t''_m$ $(t'_m = s_k^*)$. First, $\zeta(u_m)$ must be in the unbounded component of the complement of $[\zeta \mid T_m]$ for, if not, there would exist an s_p such that $s_p < s_k^* = t'_m < s_p^* < s_k = t''_m$ which, since $\nu(s_m) = -\nu(s_m^*)$, contradicts the choice of k. One can now apply Lemma 4 to $\zeta \mid T_m$.

If Lemma 5 is false then there must exist an s_p such that (i) $s_p < s_k^* < s_p^* < s_k$ or (ii) $s_k^* < s_p < s_p^* < s_k$ or (iii) $s_k^* < s_p < s_k < s_p^*$. Now (i) and (ii) are impossible by the choice of k and the fact that $\nu(s_p) = -\nu(s_p^*)$. Lemma 4 now gives

$$\sum_{s_k^* < s_\sigma < s_k < s_\sigma^*} \nu(s_\sigma) = 0;$$

and thus if there were any s_p of case (iii) there would have to be an s_q with $\nu(s_q) = -1$ contradicting the choice of s_k. The proof is complete.

Let ζ be a locally simple representation. When there exists a point $\pi \in [\zeta]$ such that $\zeta^{-1}(\pi)$ contains but one point one can make the following definition. An arc γ (homeomorph of a closed interval) is an *interior arc with endpoint at* π if π is one endpoint of γ, $\gamma - \pi$ is contained in a component of the complement of $[\zeta]$ and points π' on $\gamma - \pi$ are such that $\omega(\zeta, \pi') = \omega^+(\zeta, \pi)$; see Fig. 5.

Let ζ aug γ, read ζ *augmented by* γ, be a representation locally topological except at $\zeta^{-1}(\pi)$, which traverses $[\zeta] \cup \gamma$ in the following way: from $\zeta(0)$ to π in the same direction as ζ, from π to the other endpoint τ of γ along γ, from γ to π along γ and finally from π to $\zeta(0)$ along $[\zeta]$ in the direction of ζ; see Fig. 5.

LEMMA 6. *Let ζ be a locally simple representation and γ an interior arc with endpoint at π. A necessary and sufficient condition that ζ aug γ be an interior boundary is that ζ be an interior boundary.*

Sufficiency proof. Let F be the properly interior mapping of a disk that extends ζ (i.e., $F \mid \text{bdy}\, D = \zeta$). Select γ and note that, from the properties of γ, $F^{-1}(\gamma) = A$ is an interior arc of the circle that bounds D with one endpoint at say $p = \zeta^{-1}(\pi)$.

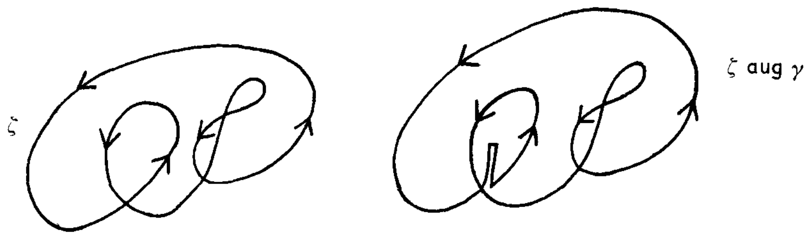

Fig. 5.

It is well known that there exists a conformal mapping H of the closure of the disk D such that $H(D) = D - A$; (let $h = H \mid \text{bdy } D$), $h(\text{bdy } D) = (\text{bdy } D) \cup A$; h is locally topological except at one point. (The proof is but a slight variation on the Caratheodory Theorem in Section 1.) One now sees that the mapping $F = F \circ H$ is properly interior and that $F \mid \text{bdy } D$ is equivalent to ζ aug γ.

Necessity Proof. The mapping H above will also be used here; let $H^{-1}(A) = B$ and note that B is an interval and that $H \mid B$ is at most 2 to 1. Let G be the properly interior mapping that extends ζ aug γ. It can be arranged that $B = G^{-1}(\gamma)$. Now form $G = G \circ H^{-1}$ and one sees by Lemma 3 that G is properly interior; that $G \mid \text{bdy } D$ is locally topological and is equivalent to ζ.

THEOREM 4. *If ζ is normal and has a cut of Type I and if ζ^* and ζ^{**} are interior boundaries then ζ is an interior boundary.*

Proof. Because of Lemma 5 there exists and arc γ which is an interior arc with respect to ζ, ζ^* and ζ^{**} (at the same time) with endpoint at $\pi = \zeta(s_k)$. It follows by Lemma 6 that both ζ^{**} aug γ and ζ^* aug γ are interior boundaries.

Let D^* and D^{**} be the right and left halves of a disk D. There exist properly interior mapping F^* and F^{**} on D^* and D^{**} which extend representations equivalent to ζ^* aug γ and ζ^{**} aug γ respectively and with the extra properties (with $d = \bar{D}^* \cap \bar{D}^{**}$) that $F^* \mid d = F^{**} \mid d$ and $F^*(d) = F^{**}(d) = \gamma$. Let F be defined by $F \mid D^* = F^*$ and $F \mid D^{**} = F^{**}$. Since F is at most 2 to 1 on d, Lemma 3 applies and F is seen to be a properly interior mapping. Also it is clear from the construction of F that it extends a representation equivalent to ζ and the proof is complete.

THEOREM 5. *If ζ is normal and has a cut of Type I and if ζ is an interior boundary then ζ^* and ζ^{**} are interior boundaries.*

Proof. By Lemma 5, ζ^* represents a positively oriented Jordan curve and thus ζ^* is an interior boundary. Let F be the properly interior mapping of the disk D

that extends ζ. By Lemma 5 $[\zeta^*]$ intersects $[\zeta]$ only in the point $\pi = \zeta(s_k) = \zeta(s_k^*)$. From these properties of ζ^* and from the fact that $s_k^* < s_k$ with $\nu(s_k^*) = +1$ it follows that $F^{-1}([\zeta^*]) \cap D = C$ is the interior of the closed arc $\bar{C}$ and that $\bar{C}$ connects s_k to s_k^*. Let R be the open 2-cell bounded by C and the interval from s_k^* to s_k. One mapping G is called *topologically equivalent* to another $\tilde{G}$ if there exists sense-preserving homeomorphisms H' and H'' (the compositions making sense) so that $\tilde{G} = H' \circ G \circ H''$. One sees that $F|R$ is topologically equivalent to $w = z^2$ on the disk (see, e.g., Whyburn [13], Theorem (4.3), p. 86); there exists therefore an open arc $B \subset R$ such that $F(B) = \gamma$ is an interior arc of ζ^* with endpoint at $\pi = \zeta(s_k)$. Let D^* be the open 2-cell bounded by the arc B and the interval $s_k^* \leqslant t \leqslant s_k$; let $D^{**} = D - \bar{D}^*$. By these constructions one sees that $F^* = F|\bar{D}^*$ and $F^{**} = F|\bar{D}^{**}$ are properly interior, that they extend ζ^* aug γ and ζ^{**} aug γ and that, by Lemma 6, both ζ^* and ζ^{**} are interior boundaries.

When ζ is normal and leads to a cut of Type I one can *modify* ζ^* and ζ^{**} by simply smoothing each near $\zeta(s_k)$ (denote these modified representations by mod ζ^* and mod ζ^{**}) so that each is normal; mod ζ^* is simple; mod ζ^{**}, with mod $\zeta^{**}(0) = \zeta(0)$, has the same intersection sequence as the intersection sequence of ζ with s_k and s_k^* deleted.

For reference purposes one can gather together these remarks together with Theorems 4 and 5 to obtain:

THEOREM 6. *Let ζ be a normal representation with $\zeta(0)$ a positive outer point. If ζ has a Cut of Type I, then ζ is an interior boundary if and only if the normal representations mod ζ^* and* mod *ζ^{**} are both interior boundaries. If ζ has n vertices then* mod *ζ^* has no vertices and* mod *ζ^{**} has $n-1$ vertices.*

8. Properties of Cuts of Type II

Recall, see Section 5, that with a normal ζ with $\zeta(0)$ a positive outer point, a Cut of Type II implies the existence of a k and j such that $s_k < s_k^*$ and

$$s_j < s_k < s_k^* < s_j^* \text{ or } s_j < s_k < s_j^* < s_k^*;$$

also one has that $\nu(s_m) = +1$ for $m < k$ and that $\nu(s_k) = -1$.

THEOREM 7'. *Let ζ* be a *normal representation, with $\zeta(0)$ a positive outer point, which has a Cut of Type II' (j). If ζ^* and ζ^{**} are both interior boundaries then is an interior boundary.*

Proof. Let D^* and D^{**} be the right and left open halves of a disk D and let d be the diameter $d = \bar{D}^* \cap \bar{D}^{**}$. Let F^* and F^{**} be the properly interior mappings that extend ζ^* and ζ^{**} respectively. Let T^* and T^{**} be the intervals $0 \leqslant t \leqslant s_k - s_j$ and $s_k^* \leqslant t \leqslant s_k^* + s_k - s_j$ on the positively oriented circles of circumferences C^* and C^{**} which in turn bound the disks $\tilde{D}^*$ and $\tilde{D}^{**}$ respectively. Let H^* and H^{**} be sense-preserving homeomorphisms of the closures of $\tilde{D}^*$ and $\tilde{D}^{**}$ onto the closures of D^* and D^{**} so that $H^* | T^* = H^{**} | T^{**}$, and thus that $H^*(T^*) = H^{**}(T^{**})$, respectively. Define the mapping F of $D = (\bar{D} \cup \bar{D}^{**})$ by $F | \bar{D}^* = F^* \circ H^*$ and $F | \bar{D}^{**} = F^{**} \circ H^{**}$. By use of Lemma 3, F is seen to be properly interior and by construction an extension of ζ.

THEOREM 7″. *Let ζ be a normal representation, with $\zeta(0)$ a positive outer point, which has a Cut of Type II″ (j). If ζ^* and ζ^{**} are both interior boundaries then ζ is an interior boundary.*

Proof. The proof is essentially the same as the proof of Theorem 7′ and is left to the reader (the main difference is in the choice of T^* and T^{**} which in this case are chosen as the intervals $s_k^* - s_j^* \leqslant t \leqslant s_k + s_j + s_k^* - s_j^*$ and $s_j^* \leqslant t \leqslant s_j^* - s_j + s_k$ respectively).

THEOREM 8. *Let ζ be a normal representation with $\zeta(0)$ a positive outer point; suppose also that ζ does not have a Cut of Type I. If ζ is an interior boundary then there exists a j, $j < k$, and the corresponding Cut of Type II′ (j) or II″ (j) such that ζ^* and ζ^{**} are both interior boundaries.*

Proof. Let F be the properly interior mapping that extends ζ. By the choice of k it follows that $s_0, s_1, \dots, s_k$ are equal to $t_0', t_1', \dots, t_k'$ (the smallest pre-images of $\zeta_0, \zeta_1, \dots, \zeta_k$), respectively; thus $\zeta(s_m) = \zeta_m$ for $m < k$. It is not difficult to show that if $\pi = \zeta(u_m)$, $s_{m-1} < u_m < s_m$, $0 \leqslant m < k$ (identify s_{-1} with $t = 0$), then $\zeta^{-1}(\pi)$ consists of precisely m points in $\bar{D}$ with one of these points on bdy (D) and the other m points in D. The proof of this fact is left to the reader (a formal proof can be constructed easily using the proof of Theorem 4 [10], p. 1090, and the fact that for an interior mapping F the number of points in a point inverse $\zeta^{-1}(\pi)$ is equal to the winding number of ζ about the point π). Let T_m be the interval $s_{m-1} < t < s_m$; then the pre-image of $T_m, m \leqslant k$, consists of m intervals in $\bar{D}$; exactly one of these intervals lies on the boundary of D and the other $m-1$ such intervals lie in D. Furthermore, all but one of these intervals in D have both endpoints in D and the other one has exactly one endpoint on bdy (D). By the choice of k, ζ is topological on the interval

$0 \leqslant t < s_{k+1}$. Let I_s be the interval $0 \leqslant t < s$. Consider I_t as t increases. A new pre-image interval is created each time t passes through an s_m, $m < k$ with one endpoint at s_m^*; but after t passes through s_k, since $\nu(s_k) = -1$, there are only $k-1$ points in the pre-image $\zeta^{-1}(t)$ and, therefore, some one of the previously created pre-image intervals must have an endpoint at s_k^*; choose j so that s_j^* is the other endpoint of this interval; call this interval A. Let D^* and D^{**} be components of $D - A$ with D^{**} the region with $t = 0$ on its boundary. By this construction $F^* = F \mid D^*$ and $F^{**} = F \mid D^{**}$ are properly interior and extend ζ^* and ζ^{**} respectively. The proof is complete.

Next, modifications of ζ^* and ζ^{**} (similar in purpose to those introduced at the end of Section 7) must be introduced; they must be normal and must have intersection sequences closely related to the original ζ, ζ^* and ζ^{**}. The situations for the ζ^* and for the ζ^{**} are quite different and will be treated separately. First, ζ^{**} will be modified.

Let I_{jk} be the interval $s_j \leqslant t \leqslant s_k$. By the choice of k, ζ is topological on I_{jk}; let ζ_{jk} be the oriented arc which is the image $\zeta(I_{jk})$. Now ζ^{**} traces over the arc ζ_{jk} exactly twice; if the cut is of Type II′ (j), ζ_{jk} is traced in opposite directions; if the cut is of Type II″ (j), ζ_{jk} is traced in the same direction. Define the *modifications of Type II* of ζ^{**}, written mod ζ^{**}, as follows (precise definition of ζ^{**} in Section 6):

Type II′ (j) (see Figures 3 and 6). For each $\varepsilon > 0$ let J'_ε be the interval

$$s_k^* - \varepsilon < t < s_k^* + s_k - s_j + \varepsilon.$$

Let mod $\zeta^{**} \mid \complement(J'_\varepsilon) = \zeta^{**}$ ($\complement(A)) =$ complement of A); on J' replace ζ^{**} by an arc which lies in a left-hand neighborhood of $\zeta^*(J'_\varepsilon)$; make sure that the resulting mod ζ^{**} is normal and that for each s_m such that $s_m \in I_{jk}$ and $s_m^* \in \complement(I_{k^*j^*})$ (note $m \neq j$ or k) there are exactly two vertices on mod ζ^{**}; these two vertices have opposite signs.

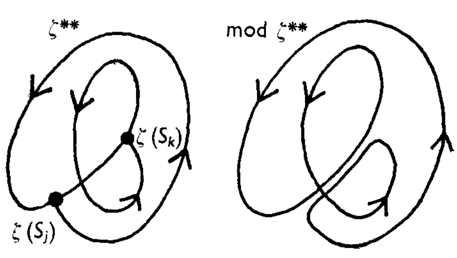

Fig. 6.

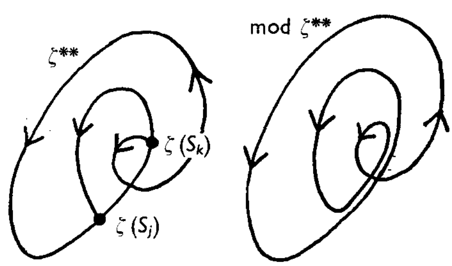

Fig. 7.

Type II'' (j) (see Figures 4 and 7). For each $\varepsilon > 0$ let J''_ε be the interval

$$s_j^* - \varepsilon < t < s_j^* - s_j + s_k + \varepsilon.$$

The construction is now the same as before except that for each s_m such that $s_m \in I_{jk}$ and $s_m^* \notin (I_{j^*k^*})$ the two associated vertices of mod ζ^{**} have the same sign.

To define the modifications of the ζ^* proceed as follows:

First smooth ζ^* at the only two points, $\zeta(s_j)$ and $\zeta(s_k)$, at which ζ^* fails to be regular; obtain a normal representation $\tilde{\zeta}^*$ arbitrarily close to ζ^* which has no new vertices. But for later purposes more must be done (to develop a well defined algorithm); one must find a way of selecting an outer point of $\tilde{\zeta}^*$. Let p be the smallest index such that $s_p \leqslant s_j$ and such that s_p^* is in the closed interval between s_j^* and s_k^*. It follows that $\zeta(s_p)$ is not a vertex of ζ^* and that it is an outer point of ζ^*. Thus, mod ζ^* is defined as a representation equivalent to $\tilde{\zeta}^*$ for which mod $\zeta^*(0) = \zeta(s_p)$.

This concludes the definitions of the modifications of the ζ^* and the ζ^{**}. From the nature of the construction of these modifications one can obtain the following Theorem; its proof is geometrically clear by previously developed techniques and is left to the reader.

THEOREM 9. *The representations ζ^* and ζ^{**} are interior boundaries if and only if* mod ζ^* *and* mod ζ^{**} *are interior boundaries.*

9. Further Properties of the Cuts of Type II

For later purposes (to obtain a finite algorithm) one must show that the number of vertices of mod ζ^* and of mod ζ^{**} are each strictly less than the number of vertices of the original ζ. Let, as before, I_{jk} be the interval, $s_j < t < s_k$; do not distinguish between I_{jk} and I_{kj}. Let R be the class of vertices $\zeta(s_m)$ such that $s_m \in I_{jk}$ and $s_m^* \notin I_{j^*k^*}$; these are the vertices of ζ that lead to pairs of vertices on mod ζ^{**}. Let P be the rest of the vertices of mod ζ^{**}; these are the vertices $\zeta(s_m)$ such that neither s_m nor s_m^* belongs to $I_{jk} \cup I_{j^*k^*}$. Let Q be the vertices $\zeta(s_m)$ such that $s_m \notin I_{jk} \cup I_{j^*k^*}$ and $s_m^* \in I_{j^*k^*}$; these are the vertices of ζ that do not give rise to vertices on either mod ζ^* or mod ζ^{**}. Let p, q, r be the number of vertices in P, Q, R respectively; let N, N^*, N^{**} be the number of vertices on ζ, mod ζ^*, mod ζ^{**} respectively. Recall that, by the choice of k, there does *not* exist an s_m such that $s_0 \leqslant s_m < s_m^* < s_k$ and obtain the equality: $N = p + q + r + N^* + 2$; obtain also that $N^{**} = p + 2r$.

LEMMA 7. *Let ζ be normal with $\zeta(0)$ a positive outer point. If ζ has a cut of Type II then $r \leqslant q$.*

Proof. Consider the oriented Jordan Curve B which is the outer boundary of ζ and the oriented (Jordan) arc A which is ζ restricted to the interval $s_0 \leqslant t \leqslant s_k$. By the choice of k, ζ can cross over A only to the right. A vertex in R is such a point $\zeta(s_m)$ but with $s_m \in I_{jk}$ and $s_m^* \notin I_{j^*k^*}$; there is therefore a pair of disjoint classes R_1 and R_2 that make up R. These are the following: R_1 contains $\zeta(s_m)$ when

$$s_j < s_m < s_k < s_m^* < s_k^*;$$

R_2 contains $\zeta(s_m)$ when

$$s_j < s_m < s_k \text{ and } s_m^* > \max(s_j^*, s_k^*).$$

Let C be the oriented curve represented by $\zeta | I_{j^*k^*}$ (oriented by increasing t). For each $\zeta(s_m)$ in R_1 let D_1' be the oriented curve given by $\zeta | I_{km^*}$ and for (s_m) in R_2 let D_2' be the oriented curve given by $\zeta | I_{j^*m^*}$; this construction is only valid for a Cut of Type II$'$ (j) but a similar construction can be made for a Cut of Type II$''$ (j). It is taken as geometrically clear, see Figures 8 and 9, that for each vertex in R there is an intersection of the curves C and D_1' or C and D_2' as the case may be. But such intersections are vertices in Q. Since, clearly, no pair of vertices in R can correspond to a single vertex in Q it follows that $r \leqslant q$.

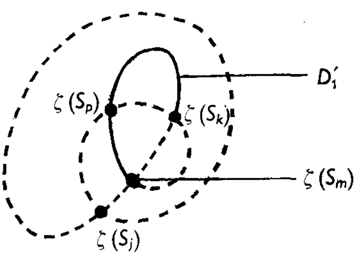

Fig. 8.

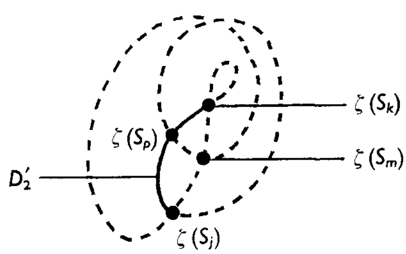

Fig. 9.

THEOREM 10. *Let ζ be a normal representation with $\zeta(0)$ a positive outer point. If ζ has a Cut of Type II then $N^* \leqslant N-2$ and $N^{**} \leqslant N-2$.*

Proof. From the equality $N = p+q+r+N^*+2$ one has immediately that

$$N^* = (N-2)-(p+q+r) \leqslant N-2.$$

Using Lemma 7 one has

$$N^{**} = p+2r \leqslant p+r+q = (N-2)-N^* \leqslant N-2.$$

10. The Principal Geometric Result

Theorems 5, 7′, 7″, 8, 9 and 10 can be put together to form

THEOREM 11. *Let ζ be a normal representation with $\zeta(0)$ a positive outer point. Then*

(i) *Either there exists a Cut of Type I or of Type II or ζ is not an interior boundary.*

(ii) *If the cut is of Type I, ζ is an interior boundary if and only if* mod ζ^* *and* mod ζ^{**} *are both interior boundaries. Furthermore, $N^*=0$ and $N^{**}=N-1$.*

(iii) *If the Cut is of Type II then ζ is an interior boundary if and only if there exists a j, $0 \leqslant j < k$, and the corresponding Cut of Type′ (j) or II″ (j) such that* mod ζ^* *and ζ^{**} are both interior boundaries. Furthermore, for any j, $0 \leqslant j < k$, the corresponding Cut of Type II′ (j) or II″ (j) satisfies the inequalities $N^* \leqslant N-2$ and $N^{**} \leqslant N-2$.*

Theorem 11 forms the basic reduction step of a finite algorithm for the reduction can be applied to the cuts themselves. Call the operation leading from a normal ζ to the mod ζ^* and mod ζ^{**} a *cut operation.* One has then a solution to Problem B:

THEOREM 12. *Let ζ be normal with $\zeta(0)$ a positive outer point. Then ζ is an interior boundary if and only if there a succession of cut operations that eventually leads to a collection of representations each of which describes a positively oriented Jordan curve.*

Remarks. (1) Since at any stage in a succession of cut operations all the cuts must satisfy any known necessary conditions that a normal representation be an interior boundary, the algorithm can be stated in considerably more efficient forms.

(2) If follows easily from the definitions that for cuts of Type II,

$$\tau(\zeta)-1=[\tau(\text{mod }\zeta^*)-1]+[\tau(\text{mod }\zeta^{**})-1];$$

for cuts of Type I,

$$\tau(\zeta)-1=[\tau(\text{mod }\zeta^{**})-1]+[\tau(\text{mod }\zeta^{**})-1]+1.$$

Thus, with ζ a normal interior boundary, $\tau(\zeta)=1$ if and only if there are no cuts of Type I. One has therefore that a normal ζ with $\zeta(0)$ a positive outer point has an extension by a sense-preserving *local homeomorphism* if and only if the complete cut sequence contains only cuts of Type II.

(3) Let a normal ζ be called an interior boundary of *degree n* if there exists a polynomial of degree n, $w=P_n(Z)$, such that $P_n(e^{it})$ is equivalent to ζ. It would be

interesting to classify the interior boundaries of given degree n. For example, as a beginning, it is probably true that the maximum number of vertices of a normal interior boundary of degree n is $(n-1)^2$.

(4) The analogous problem when the interior mappings are defined on manifolds, bounded by a Jordan curve, but other than a disk, is completely open.

(5) For certain sub-classes of normal curves one can obtain a complete solution to Problem B in terms of inequalities satisfied by the intersection sequence and thus no algorithm is necessary. See [9].

PART II. THE ALGEBRAIC PROBLEM

11. Preliminaries

The purpose of Part II is to show that the geometric algorithm, Theorem 12, can now be developed in a completely algebraic form. The main connection between the geometric objects (the normal representations with a positive outer point) and the algebraic objects (the intersection sequences) is given in Theorem 12. The development and notation will parallel the geometric treatment closely; for technical purposes many concepts will be reformulated.

Consider a set of p points on an oriented circle S. Begin with an *initial point* of the set; traverse S in the direction of its orientation; thus index the points $a_0, a_1, \dots, a_{p-1}$ with a_0 the initial point; a set of points so indexed will be called *ordered* and will be written $\alpha = (a_0, a_1, \dots, a_{p-1})$.

A collection $(\alpha, \phi_\alpha, \nu_\alpha) = S(\alpha)$ will be called an *intersection sequence* provided that α is an ordered set containing $2n$ points (a_0 the initial point); $a_m^* = \phi_\alpha(a_m)$ is a $1-1$ mapping of α onto α with no fixed points and such that $\phi_\alpha \circ \phi_\alpha$ is the identity mapping; $\nu_m = \nu_\alpha(a_m)$ is a mapping of α onto the set $(+1, -1)$ such that $\nu_\alpha \circ \phi_\alpha = -\nu_\alpha$.

An intersection sequence $S(\alpha)$ will be called an *interior boundary sequence* if there exists a normal interior boundary ζ, with $\zeta(0)$ a positive outer point, that possesses $S(\alpha)$ as its intersection sequence. An intersection sequence will be called *realizable* if there exists a normal representation ζ, with $\zeta(0)$ a positive outer point, that possesses $S(\alpha)$ as its intersection sequence. That not all intersection sequences are realizable was noted by Gauss; in fact he conjectured that if $S(\alpha)$ is realizable then the number of points between a_m and a_m^*, for any m, is always even ([3], pp. 272 and 282–286). In 1927 Nagy continued the study and showed that this was the case, ([6]). The condition is, however, far from sufficient for, among other things, it takes no account of the ν_α mapping (that signs the vertices). A stronger necessary condition, that is also not sufficient, was derived in [10], Theorem 1, p. 1087.

Problem B can be reformulated in this context as follows:

Problem B. Find necessary and sufficient conditions on a realizable intersection sequence $S(\alpha)$ that $S(\alpha)$ be an interior boundary sequence.

The following definitions will be useful.

With the ordered sets α and β disjoint define the *union* of $S(\alpha)$ and $S(\beta)$, written $S(\alpha \cup \beta) = S(\gamma)$ by defining: $\gamma = \alpha \cup \beta$; $\phi_\gamma | \alpha = \phi_\alpha$ and $\phi_\gamma | \beta = \phi_\beta$; $\nu_\gamma | \alpha = \nu_\alpha$ and $\nu_\gamma | \beta = \nu_\beta$. Let $S(0)$ be the empty intersection sequence. With $S(\alpha)$ and $S(\beta)$ intersection sequences and with $\beta \subset \alpha$ define $S(\alpha | \beta) = S(\gamma)$ by defining; $\gamma = \beta$; $\phi_\gamma = \phi_\alpha | \beta$; $\nu_\gamma = \nu_\alpha | \beta$. Let $p_m(\alpha) = \beta$ be the permutation defined by $b_i = a_{m+i}$ (mod p); β is then the ordered set containing the same points as α but with $a_m = b_0$ as its initial point. Let $P_m : S(\alpha) = S(\beta)$ be the intersection sequence defined by $\beta = p_m(\alpha)$; $\phi_\beta = p_m \circ \phi_\alpha \circ p_m^{-1}$; $\nu_\beta = \nu_\alpha \circ p_m^{-1}$.

12. Definition of the Cuts

Given a realizable $S(\alpha)$ let k be the smallest index such that $\nu_k = -1$. If $\nu_0 = -1$ then, by Lemma 1, $S(\alpha)$ is not an interior boundary sequence; if $\nu_0 = +1$ cuts of Type I and II are defined as follows:

Type I ($a_k^* < a_k$). In this case, by Lemma 5, it follows that $a_k^* = a_{k-1}$. Define the *cuts* $S^*(\alpha)$ and $S^{**}(\alpha)$ with β the ordered set (s_k^*, s_k), by

$$S^*(\alpha) = S(0),$$

$$S^{**}(\alpha) = S(\alpha | \alpha - \beta).$$

Type II ($a_k < a_k^*$). In this case there are for each j, $0 \leqslant j < k$, two subcases: Type II$'(j)$ and II$''(j)$. Here the definition of the cuts is considerably more complicated. In the first case let β be the ordered subset of α containing all a_m such that either $a_j < a_m < a_k < a_k^* < a_m^* < a_j^*$ or $a_k^* < a_m < a_m^* < a_j^*$ (these points correspond to the vertices of mod ζ^*). Let C be the set of points in α containing all a_m such that $a_m \leqslant a_j < a_k < a_k^* < a_m^* \leqslant a_j^*$; let p be the smallest index of a point in C; let q be the smallest index of the points in β such that $a_q > a_p^*$; select a permutation $p_r(\beta)$ so that a_q becomes the initial point of $p_r(\beta)$; define

$$S^*(\alpha) = P_r : S(\beta) \quad \text{(Cut of Type II}'(j)).$$

In case Type II$''(j)$ define β as above and define C as the set of a_m such that $a_m \leqslant a_j < a_k < a_j^* \leqslant a_m^* < a_k^*$; choose $p_r(\beta)$ as above; define

$$S^*(\alpha) = P_r(\beta) \quad \text{(Cut of Type II}''(j)).$$

To define the cut $S^{**}(\alpha)$ of Type II$'(j)$ let β be the ordered subset of α containing all a_m such that neither a_m nor a_m^* is between either a_j and a_k or a_j^* and a_k^* (these correspond to the vertices of that lead to single vertices on mod ζ^{**}); the initial point of $\beta\,(=b_0)$ is then the point in the subset with the smallest "α index". Let $S(\beta)=S(\alpha|\beta)$. Let γ be the ordered subset of α containing all a_m and a_m^* such that $a_j<a_m<a_k<a_m^*<a_k^*$; let δ be the ordered subset of α containing all a_m and a_m^* such that $a_j<a_m<a_k<a_k^*<a_j^*<a_m^*$; the initial points of $\gamma\,(=c_0)$ and of $\delta\,(=d_0)$ are chosen as the points in the respective subsets with the smallest "α index". Let $S(\gamma)=S(\alpha|\gamma)$ and $S(\delta)=S(\alpha|\delta)$. Let ε be the ordered subset of $\gamma\cup\delta$ containing all a_m such that $a_j<a_m<a_k$; the initial point of $\varepsilon\,(=e_0)$ is the point in the subset with the smallest "α index". Next, four sets of points will be defined by selecting the points according to certain inequalities. Let $(a_m)'=a_{m+1}$. Choose $\bar{C}$ to be the unordered (indexing implies nothing about order on S) set given by choosing points on S satisfying

$$c_m^*<\bar{c}_m^*<(c_m^*)' \quad \text{for } a_j<c_m<a_k<c_m^*<a_k^*;$$

choose $\bar{D}^*$ to be an unordered set given by

$$d_m^*<\bar{d}_m^*<(d_m^*)' \quad \text{for} \quad a_j<d_m<a_k<a_k^*<a_j^*<d_m^*;$$

this choice and indexing establishes a $1-1$ mapping f of ε onto $\bar{C}^*\cup\bar{D}^*$ such that $f(c_m)=\bar{c}_m^*$ and $f(d_m)=\bar{d}_m^*$. Define the unordered sets $\bar{C}$ and $\bar{D}$ by choosing $\bar{c}_m$ and $\bar{d}_m$ between a_k^* and a_j^*; define the $1-1$ mapping g of $\bar{C}\cup\bar{D}$ onto ε by defining $g(\bar{c}_m)=f^{-1}(\bar{c}_m^*)$ and $g(\bar{d}_m)=f^{-1}(\bar{c}_m^*)$; select the points in $\bar{C}$ and $\bar{D}$ so that this mapping g of $\bar{C}\cup\bar{D}$ onto ε inverts the natural order on S (i.e., with $\bar{\varepsilon}$ the ordered set composed of the points $\bar{C}\cup\bar{D}$ one has $g(\bar{e}_0)=e_{p-1}>g(\bar{e}_1)=e_{p-2}>\dots>g(\bar{e}_{p-1})=e_0)$. Let μ be the ordered set composed of $\bar{C}\cup\bar{D}\cup\bar{C}^*\cup\bar{D}^*$; define $S(\mu)$ by defining ϕ_μ to be the mapping which sends $\bar{c}_m\to\bar{c}_m^*$, $c_m^*\to\bar{c}_m$, $\bar{d}_m\to\bar{d}_m^*$, $\bar{d}_m^*\to\bar{d}_m$; $\nu_\mu(\bar{c}_m^*)=-\nu_\gamma(c_m^*)$, $\nu_\mu(\bar{d}_m^*)=-\nu_\delta(d_m^*)$ and then $\nu_\mu(\bar{c}_m)=-\nu_\mu(\bar{c}_m^*)$, $\nu_\mu(\bar{d}_m)=-\nu_\mu(\bar{d}_m^*)$. Finally, define

$$S^{**}(\alpha)=S(\beta\cup\gamma\cup\delta\cup\mu) \quad (\text{cut of Type II}'(j)).$$

The last case, the definition of $S^{**}(\alpha)$ for a cut of Type II$''(j)$, can be done in a similar fashion; the main difference being that the signing function ν_μ on $\bar{C}^*\cup\bar{D}^*$ has the same (instead of the opposite) sign as the ν_γ and ν_δ on the corresponding points in $\gamma\cup\delta$.

13. The Algebraic Algorithm

A realizable intersection sequence $S(\alpha)$ has a *complete cut sequence* provided that there exists an iteration of cuts of Types I and II such that the intersection sequences so generated ultimately all terminate in empty intersection sequences.

THEOREM (Algebraic Analogue of Theorem 12). *A realizable intersection sequence $S(\alpha)$ is an interior boundary sequence if and only if there exists a complete cut sequence.*

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