COMMUTATORS, PERTURBATIONS, AND UNITARY SPECTRA

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1. Introduction. Let A and B denote linear operators, bounded or unbounded, on a Hilbert space H of elements x. As is customary, let $\|x\| = (x, x)^{\frac{1}{2}}$ and put $\|A\| = \sup\|Ax\|$, where $\|x\| = 1$. If A and B are bounded and if C denotes the commutator of A and B,

$$C = AB - BA, \tag{1.1}$$

then it is well known that

$$\|C\| \leqslant 2\|A\|\,\|B\|, \tag{1.2}$$

and that the inequality cannot be improved by replacing the 2 by $2 - \varepsilon$ with $\varepsilon > 0$. Indeed, simple examples with finite matrices $A \neq 0$, $B \neq 0$ and A, iB (hence also C) even self-adjoint show that the equality of (1.2) may hold.

Part I of this paper will be concerned with an improvement of (1.2) when B is bounded but otherwise arbitrary, A and C are bounded and self-adjoint, *and C is non-negative.* If the space H is finite-dimensional this last restriction forces C to be 0, since the trace of C, which equals the sum of its eigenvalues, is 0. On the other hand, in the infinite dimensional case, examples show that both conditions $C \geqslant 0$, $C \neq 0$ are compatible; see, e.g., [20], [23]. The principal result of Part I will be an inequality corresponding to (1.2) but where $\|A\|$ is replaced by $(\frac{1}{2})$ meas $\operatorname{sp}(A)$, where $\operatorname{sp}(A)$ denotes the spectrum of A.

In Part II there will be considered a related problem concerning perturbations of a self-adjoint operator A. It will be supposed first (Theorem 2) that A and B are unitarily equivalent bounded self-adjoint operators whose difference D is semi-definite, so that

$$D = A - B \geqslant 0 \text{ (or } \leqslant 0) \text{ and } B = UAU^* \quad (U \text{ unitary}). \tag{1.3}$$

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In Theorem 3 the boundedness restriction on A and B will be relaxed to half-boundedness. The results to be obtained will concern not the spectra of A or B but rather the spectrum of any unitary operator U effecting their equivalence. In fact, the operator U will, in the theorems involving (1.3), play a role similar to that of A in (1.1) of Part I. It will be shown that under certain hypotheses the relation (1.3) assures the existence of continuous, and even absolutely continuous, spectra for U, and in addition, sometimes implies that the entire unit circle must belong to $\mathrm{sp}(U)$.

In Part III there will be given applications of the results of Part II to semi-normal operators, Laurent matrices, measure-preserving transformations, and to what correspond to certain operators occurring in scattering theory in quantum mechanics.

Part I. The commutator $AB - BA$

2. There will be proved the following

THEOREM 1. *Let B be arbitrary, A and C be self-adjoint, C satisfy $C \geqslant 0$, and suppose that all operators are bounded. Then*

$$\|C\| \leqslant \|B\| \text{ meas sp}(A), \tag{2.1}$$

where "meas" *refers to ordinary Lebesgue measure on the real line.*

Since A is self-adjoint, the set $S = \mathrm{sp}(A)$ is contained in the interval $-\|A\| \leqslant \lambda \leqslant \|A\|$ and so meas $S \leqslant 2\|A\|$. Consequently, the inequality (2.1) is, under the assumptions made, an improvement of (1.2). The proof has, in essentials, been given elsewhere, see [20] and the remarks of [25, p. 107], but, for completeness, will be given below.

3. *Proof of Theorem 1.* If A has the spectral resolution $A = \int \lambda \, dE(\lambda)$ and if Δ denotes any λ-interval, then multiplications on the left and right of both sides of (1.1) by $E(\Delta)$ lead to

$$E(\Delta)\, C\, E(\Delta) = \int_\Delta \lambda \, dE\, B\, E(\Delta) - E(\Delta)\, B \int_\Delta \lambda \, dE, \tag{3.1}$$

an equality which continues to hold if each of the integrands λ is replaced by $\lambda - \alpha$, where α is any constant. If α is taken to be the midpoint of Δ, then $|\lambda - \alpha| \leqslant \frac{1}{2}d$, where d is the length of Δ, and one obtains $\|C^{\frac{1}{2}} E(\Delta)x\| = (E(\Delta)CE(\Delta)x, x)^{\frac{1}{2}} \leqslant [2\|B\|\,\|E(\Delta)x\|^2(\frac{1}{2}d)]^{\frac{1}{2}}$, where $C^{\frac{1}{2}}$ denotes the non-negative square root of C, and the factor 2 corresponds to the two terms on the right of the operator equation (3.1). If the intervals $\{\Delta\}$ are disjoint and cover S, an application of the Schwarz inequality readily leads to $\|C^{\frac{1}{2}}x\| \leqslant \|B\|^{\frac{1}{2}}\,(\text{meas } S)\|x\|$ and, since $\|C^{\frac{1}{2}}\|^2 = \|C\|$, hence to (2.1) This completes the proof.

4. *Remarks.* It is clear from the proof of Theorem 1 (see [20, p. 1028], also [23, p. 514]) that (2.1) can be refined to the inequality

$$\|C\| \leqslant \|B\| \operatorname{meas} T, \tag{4.1}$$

where $\int_T dE = I$ (thus, T is a set for which $\int_T d\|Ex\|^2 = \|x\|^2$ for all x in H). In particular, if $C \neq 0$, A cannot have a pure point spectrum. Moreover, if 0 is not in the point spectrum of C, then A must be absolutely continuous, that is, $\|E(\lambda)x\|^2$ must be absolutely continuous for all x.

If $f(\lambda)$ is measurable with respect to $E(\lambda)$ (see, e.g., [27, p. 227] and [28, pp. 41 ff.]) and belongs to $L^2(-\infty, \infty)$, a modification of the argument of section 3 leads to the generalization of (2.1),

$$\left\| \int_{-\infty}^{\infty} f(\lambda)\, dE\, C \int_{-\infty}^{\infty} f(\lambda)\, dE \right\| \leqslant \|B\| \int_{-\infty}^{\infty} |f(\lambda)|^2 d\lambda. \tag{4.2}$$

According as $f(\lambda)$ is the characteristic function of S or of T one obtains (2.1) or (4.1).

Under the assumptions of Theorem 1, it is seen that if the equality of (1.2) holds, and if $C \neq 0$, then necessarily the spectrum and, by (4.1), even the continuous spectrum, of A is the interval $-\|A\| \leqslant \lambda \leqslant \|A\|$. If, in addition, both A and iB are self-adjoint, then it is clear from Theorem 1 that also the spectrum, as well as the continuous spectrum, of iB is the interval $-\|B\| \leqslant \lambda \leqslant \|B\|$. However, in the absence of an example, it will remain undecided whether this situation can actually obtain, that is, whether the *equality* of (1.2) can hold, with $C \geqslant 0$ and $C \neq 0$, and the pair A and iB, or even just A, self-adjoint.

Part II. Perturbations and unitary equivalence

5. There will be proved the following

THEOREM 2. *Let A and B denote bounded unitarily equivalent self-adjoint operators satisfying* (1.3) *for some unitary operator U. Then*

$$\operatorname{meas} \operatorname{sp}(U) \geqslant 2\pi \|D\| \delta^{-1}, \tag{5.1}$$

where δ denotes the distance between the maximum and minimum points of $\operatorname{sp}(A)$.

It is seen that (5.1) is similar to (2.1), especially if it is noted that $\delta \leqslant 2\|A\|$, so that (5.1) implies

$$\operatorname{meas} \operatorname{sp}(U) \geqslant \pi \|D\| \|A\|^{-1}, \tag{5.2}$$

or, if $A \geqslant 0$ so that $\delta \leqslant \|A\|$, implies

$$\operatorname{meas} \operatorname{sp}(U) \geqslant 2\pi \|D\| \|A\|^{-1} \quad (A \geqslant 0). \tag{5.3}$$

If, in (5.2), $\|D\|$ assumes the largest value consistent with (1.3), namely $2\|A\|$, it is seen that meas sp$(U) \geqslant 2\pi$ (hence, is 2π) and so the entire unit circle $|z| = 1$ belongs to sp(U). Unlike the corresponding situation in Part I (cf. the last sentence of section 4), in the present case it is easy to give an example where $\|D\| = 2\|A\|$ and hence equality holds in (5.2) or (5.1). In fact, let $A = \text{diag}(1, 0, -1, 0, 1; 1, 0, \ldots)$ and $B = \text{diag}(-1, 0, -1, 0, 1; -1, 0, \ldots)$. Then $D = A - B = \text{diag}(2, 0, 0, 0, 0; 2, 0, \ldots) \geqslant 0$ and the spectrum of A as well as that of B consists of $1, -1, 0$ each of infinite multiplicity. Hence A and B are unitarily equivalent and so (1.3) holds. It is clear that $\|D\| = 2 = 2\|A\|$ and so the *equality* of (5.1) holds and meas sp$(U) = 2\pi$ for *any* unitary operator U for which $B = UAU^*$.

6. *Proof of Theorem 2.* Condition (1.3) can be written either as $A - UAU^* = D$ or as $A - U^*AU = -U^*DU$. Since $\|D\| = \|U^*DU\|$ and since the assertions of Theorem 2 regarding U hold if and only if the corresponding assertions hold for U^*, there is no loss of generality in supposing $D \geqslant 0$.

Let U have the spectral resolution

$$U = \int_0^{2\pi} e^{i\lambda}\, dE(\lambda), \tag{6.1}$$

and let S denote the set of values λ on $0 \leqslant \lambda \leqslant 2\pi$ for which $e^{i\lambda}$ belongs to the spectrum of U. Let S^* denote the complement of S (with respect to the interval $0 \leqslant \lambda \leqslant 2\pi$). If $f(\lambda)$ is any E-measurable function which is 0 on S, then $\int_0^{2\pi} f(\lambda)\, dE(\lambda) = 0$. If $D^{\frac{1}{2}}$ is applied to both sides of this last operator equation, one obtains

$$D^{\frac{1}{2}} \int_0^{2\pi} f(\lambda)\, dE(\lambda) = 0, \quad f(\lambda) = 0 \text{ on } S. \tag{6.2}$$

Since, if S is the entire interval $[0, 2\pi]$, relation (5.1) surely holds, it can be supposed, in the proof of Theorem 2, that the (open) set S^* is not empty. Next, let $f(\lambda)$ be a function on $[0, 2\pi]$ equal to 0 on S and possessing a continuous first derivative. Then $f(\lambda)$ *equals* its Fourier series, thus

$$f(\lambda) = \sum_{-\infty}^{\infty} c_k e^{ik\lambda}, \quad c_k = (2\pi)^{-1} \int_0^{2\pi} f(\lambda)\, e^{-ik\lambda}\, d\lambda. \tag{6.3}$$

The reason for wanting the equality sign in (6.3) rather than merely "$\sim$" is to avoid possible trouble with zero sets in case U is not absolutely continuous (cf. [25], p. 103).

Substitution of the series (6.3) for $f(\lambda)$ into (6.2) yields, by virtue of (6.1),

$$c_0 D^{\frac{1}{2}} + \Sigma' c_k D^{\frac{1}{2}} U^k = 0, \tag{6.4}$$

where the prime means that $k=0$ is to be omitted from the summation. If x is an arbitrary element of the Hilbert space H, it follows from (6.4) and the Schwarz inequality that

$$\|c_0 D^{\frac{1}{2}} x\|^2 \leqslant (\Sigma' |c_k|^2)(\Sigma' \|D^{\frac{1}{2}} U^k x\|^2). \tag{6.5}$$

Next, as a straightforward consequence of (1.3), there follows the pair of relations

$$\sum_{k=0}^{n} U^{*k} DU^k = U^{*n} AU^n - UAU^* \tag{6.6}$$

and

$$\sum_{k=1}^{n} U^k DU^{*k} = UAU^* - U^{n+1} AU^{*n+1}, \tag{6.7}$$

valid for $n=0, 1, 2, \dots$. On adding the equations of (6.6) and (6.7) one obtains

$$\sum_{k=-n}^{n} U^k DU^{*k} = U^{*n} AU^n - U^{n+1} A u^{*n+1}. \tag{6.8}$$

Consequently, if δ is defined as in Theorem 2,

$$\left(\sum_{k=-n}^{n} U^k DU^{*k} x,\, x\right) = \sum_{k=-n}^{n} \|D^{\frac{1}{2}} U^{*k} x\|^2 \leqslant \delta(x, x)$$

for $n=1, 2, \dots$, and hence

$$\Sigma' \|D^{\frac{1}{2}} U^{*k} x\|^2 \leqslant \delta(x, x) - (Dx, x). \tag{6.9}$$

Next, choose $x = x_n$ to be unit vectors satisfying $Dx_n - \|D\| x_n \to 0$ as $n \to \infty$. Relation (6.5) then implies, by virtue of (6.9) and the Parseval relation

$$\Sigma |c_k|^2 = (2\pi)^{-1} \int_0^{2\pi} |f(\lambda)|^2 d\lambda,$$

that

$$\left|(2\pi)^{-1}\int_0^{2\pi} f(\lambda)\, d\lambda\right|^2 \|D\| \leqslant \left[(2\pi)^{-1}\int_0^{2\pi} |f|^2 d\lambda - \left|(2\pi)^{-1}\int_0^{2\pi} f\, d\lambda\right|^2\right] [\delta - \|D\|]. \tag{6.10}$$

Then let $f(\lambda) = f_n(\lambda)$ where $\{f_n(\lambda)\}$ denotes a uniformly bounded sequence of smooth functions equal to 0 on S and tending (almost everywhere) to the characteristic function $c(\lambda)$ of the set S^*. Thus one obtains a relation similar to (6.10) but in which $f(\lambda)$ is replaced by $c(\lambda)$. It then follows that

$$(\text{meas } S^*/2\pi)^2 \|D\| \leqslant (\text{meas } S^*/2\pi)(1 - \text{meas } S^*/2\pi)(\delta - \|D\|),$$

which, by the equality $1 - \text{meas } S^*/2\pi = \text{meas } S/2\pi$, simplifies to $\|D\| \leqslant \delta \text{ meas } S/2\pi$, that is, to (5.1).

7. *Half-boundedness.* In this section the boundedness restriction on A and B will be relaxed to half-boundedness.

THEOREM 3. *Let A and B denote unitarily equivalent half-bounded (say, from below) self-adjoint operators with a bounded difference $D = A - B \geq 0$ (or ≤ 0), thus,*

$$D = A - B,\ D \text{ bounded},\ A \geq kI,\ B = UAU^*. \tag{7.1}$$

Let $|D| = D$ or $-D$ according as $D \geq 0$ or $D \leq 0$ and let x be any element for which $y = |D|^{\frac{1}{2}} \neq 0$ and y is in the domain of B. Then

$$\text{meas sp}(U) \geq 2\pi[1 + 2\|x\|^2((B - kI)y, y)/\|y\|^4]^{-1}. \tag{7.2}$$

If A and B are half-bounded but not bounded, their domains are not the entire Hilbert space H. The equation $D = A - B$ with D bounded then means that A and B have the same (dense) domain and that $Dx = Ax - Bx$ for all elements x in this domain. (The domain of D is, of course, H.) Since D is bounded, it is clear that each of the operators $U^k A U^{*k}$, for $k = 0, \pm 1, \pm 2, \ldots$, has the same domain (namely, that of A).

8. *Proof of Theorem 3.* As in the proof of Theorem 2, it can be supposed that $D \geq 0$; the proof will be a modification of that of Theorem 2. If it is noted that

$$\sum_{k=-1}^{-\infty} c_k D^{\frac{1}{2}} U^k = \sum_{k=1}^{\infty} c_{-k} D^{\frac{1}{2}} U^{*k},$$

then relation (6.4) is seen to imply (since f is real and hence $c_{-k} = \bar{c}_k$)

$$c_0 D^{\frac{1}{2}} + \sum_{k=1}^{\infty} c_k D^{\frac{1}{2}} U^k + \sum_{k=1}^{\infty} \bar{c}_k D^{\frac{1}{2}} U^{*k} = 0, \tag{8.1}$$

and hence, on forming inner products,

$$-(c_0 D^{\frac{1}{2}} y, x) = \left(\sum_{k=1}^{\infty} c_k D^{\frac{1}{2}} U^k y, x\right) + \left(\sum_{k=1}^{\infty} \bar{c}_k D^{\frac{1}{2}} U^{*k} y, x\right), \tag{8.2}$$

where x and y are defined as in Theorem 3. But the first expression on the right of the equation (8.2) is equal to $\left(y, \sum_{k=1}^{\infty} \bar{c}_k U^{*k} D^{\frac{1}{2}} x\right)$, which is $\left(D^{\frac{1}{2}} x, \sum_{k=1}^{\infty} \bar{c}_k U^{*k} y\right)$, or $\left(x, \sum_{k=1}^{\infty} \bar{c}_k D^{\frac{1}{2}} U^{*k} y\right)$. Consequently, relation (8.2) implies

$$|c_0|\,|(D^{\frac{1}{2}} y, x)| \leq 2\left|\text{Re}\left(\sum_{k=1}^{\infty} \bar{c}_k D^{\frac{1}{2}} U^{*k} y, x\right)\right|$$

and hence, by the Schwarz inequality $\left(\text{together with } \Sigma' |c_k|^2 = 2\sum_{k=1}^{\infty} |c_k|^2\right)$,

$$|c_0|^2 \|y\|^4 \leqslant \|x\|^2 (\Sigma' |c_k|^2) \left(2 \sum_{k=1}^{\infty} \|D^{\frac{1}{2}} U^{*k} y\|^2\right). \tag{8.3}$$

Since (6.7) can be written also as

$$\sum_{k=0}^{n-1} U^k D U^{*k} = A - U^n A U^{*n},$$

it follows from the assumption $A \geqslant kI$ that the second parenthetical expression on the right of (8.3) is majorized by $2((B - kI)y, y)$. Proceeding as in section 6 one is led to an equation similar to (6.10) but in which f is replaced by the characteristic function of S^*. Thus,

$$(\text{meas } S^*/2\pi)^2 \|y\|^4 \leqslant (\text{meas } S^*/2\pi)(1 - \text{meas } S^*/2\pi)\,[2((B - kI)y, y)\|x\|^2], \tag{8.4}$$

which, on simplification, becomes (7.2).

9. *Unrestricted case.* In case relation (1.3) is assumed for the pair of self-adjoint operators A and B without any restriction as to boundedness or half-boundedness, it will remain undecided whether there exists an estimate for meas $\text{sp}(U)$ corresponding to (7.2) of Theorem 3. It can be pointed out that, under proper assumptions on the domains of A and B, if 0 is not in the set $\text{sp}(A)\,(=\text{sp}(B))$ and if the (very severe) condition $AB = BA \geqslant 0$ is imposed, then (1.3) implies a similar relation for certain bounded operators. Thus, proceeding formally, one obtains $B^{-1} - A^{-1} = (A - B)(AB)^{-1} = (AB)^{-\frac{1}{2}}(A - B)(AB)^{-\frac{1}{2}} \geqslant 0$, that is $UA^{-1}U^* - A^{-1} \geqslant 0$, where now A^{-1} is bounded. This case will not be considered further however.

Part III. Applications

10. *Semi-normal operators.* Let A be a bounded operator for which

$$AA^* - A^*A = C \geqslant 0. \tag{10.1}$$

If A is non-singular, then $A = PU$, where P is positive definite and U is unitary; see Wintner [33], also [32, p. 282]. Relation (10.1) then yields

$$P^2 - U^*P^2U = C \geqslant 0, \tag{10.2}$$

so that Theorem 2 (as well as Theorem 3) is applicable to U. It was shown by Hartman [6, p. 233], using a generalization due to von Neumann [19, p. 307], of a result of Wintner (*loc. cit.*) that, even if A is singular, AA^* and A^*A are unitarily equivalent in case the multiplicities of $\lambda = 0$ in the point spectra of AA^* and A^*A are equal. Thus, if $A^*A = U^*(AA^*)U$, relation (10.1) holds, that is, (1.3) holds, and Theorems 2 and 3 can be applied to determine properties of the spectrum of U. Since Hartman's result is valid even if A is not bounded, provided it has a domain dense in H, and since AA^* and A^*A are non-negative, Theorem 3 is applicable in this case also.

11. *Laurent matrices.* Let $\{c_n\}$, $n = 0, \pm 1, \pm 2, \ldots$ denote a sequence of complex numbers satisfying

$$c_{-n} = \bar{c}_n \quad \text{and} \quad \sum_{-\infty}^{\infty} |c_n|^2 < \infty, \tag{11.1}$$

and let $L = (c_{j-k})$, where $j, k = 0, \pm 1, \pm 2, \ldots$ denote the associated Laurent matrix. It was shown by Toeplitz ([29], [30]; cf. also [4, p. 62]) that, if the Laurent series $\sum_{-\infty}^{\infty} c_n z^n$ is convergent for $r_1 < |z| < r_2$, where $0 < r_1 < 1 < r_2$, so that, in particular, the function

$$f(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{11.2}$$

is continuous, then the spectrum of L is the range of $f(\theta)$ on $0 \leqslant \theta \leqslant 2\pi$. It has been noted by Hartman and Wintner [7] that, even without the restrictive assumption on the Laurent series (involving convergence on an annulus containing $|z| = 1$) mentioned above, but supposing only (11.1), then L is bounded if and only if the function $f(\theta)$ of (11.2) is essentially bounded (i.e., $|f(\theta)| \leqslant$ const. almost everywhere on $0 \leqslant \theta \leqslant 2\pi$) and, furthermore, the spectrum of L is the set of values λ for which

$$\operatorname{meas}\{\theta; |f(\theta) - \lambda| < \varepsilon\} > 0, \text{ for all } \varepsilon > 0. \tag{11.3}$$

In this section, a proof of the above-mentioned theorem, and even more, concerning the location of the spectrum of a bounded Laurent matrix, using the results of [25] and the present paper, will be given. To this end, let $U = (u_{jk})$ denote the unitary operator on the Hilbert space of sequences $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$ satisfying $\|x\|^2 = \sum_{-\infty}^{\infty} |x_n|^2 < \infty$, defined by $u_{jk} = 1$ or 0 according as $k = j+1$ or $k \neq j+1$, so that U effects the shift $x_n \to x_{n+1}$ $(n = 0, \pm 1, \pm 2, \ldots)$. Then it is easily verified that U^k effects the shift $x_n \to x_{n+k}$ $(n, k = 0, \pm 1, \pm 2, \ldots)$ and hence that L is given by

$$L = \sum_{k=-\infty}^{\infty} c_k U^k. \tag{11.4}$$

Consequently, if U has the spectral resolution (6.1) then

$$L = \int_0^{2\pi} f(\lambda)\, dE(\lambda), \tag{11.5}$$

where $f(\lambda)$ is defined by (11.2). At least (11.5) holds as soon as it is shown that U, with the spectral resolution of (6.1), is absolutely continuous (see [25, p. 103], also the remark following formula (6.3) of the present paper). The reason for this proviso is the fact that $f(\theta)$ is defined by its Fourier series in (11.2) only to within a zero set and that, if U were not absolutely continuous, the operator on the right side of equation (11.5) could depend upon this set.

The assertion involving (11.3) concerning the spectrum of L will then follow if it is verified that (i) $\mathrm{sp}(U)$ is the entire circle $|z|=1$, and that (ii) U is absolutely continuous. The assertion (i) follows from Toeplitz's result with a Laurent series consisting of the single term z, but will be deduced below as a consequence of Theorem 2 of the present paper.

Let $A=(a_{jk})$ denote the (doubly infinite) diagonal matrix defined by $a_{jk}=\delta_{jk}\lambda_k$ and let $B=(b_{jk})$ be that defined by $b_{jk}=\delta_{jk}\lambda_{k-1}$, where $\{\lambda_n\}$, for $n=0, \pm 1, \pm 2, \ldots$, denotes any sequence of real numbers satisfying $|\lambda_n|<$ const. and $\lambda_n<\lambda_{n+1}$ for all n. It is easily verified that condition (1.3) holds with $\|D\|=\sup(\lambda_{n+1}-\lambda_n)$. Since 0 is not in the point spectrum of D, assertion (ii) is a consequence of [25, p. 105]. Moreover, by (5.1) of Theorem 2,

$$\text{meas } \mathrm{sp}(U) \geqslant 2\pi[\sup(\lambda_{n+1}-\lambda_n)](\lambda_\infty-\lambda_{-\infty})^{-1}, \tag{11.6}$$

where λ_∞ and $\lambda_{-\infty}$ denote the limits of λ_n as n tends to ∞ or $-\infty$ respectively. Let $\varepsilon>0$ and choose the sequence $\{\lambda_n\}$ so that $\lambda_\infty=1$, $\lambda_{-\infty}=-1$, $\lambda_\infty-\lambda_1<\varepsilon$ and $\lambda_0-\lambda_{-\infty}<\varepsilon$. Then relation (11.6) implies

$$\text{meas } \mathrm{sp}(U) \geqslant 2\pi(\lambda_1-\lambda_0)2^{-1}>2\pi(1-\varepsilon) \tag{11.7}$$

for every $\varepsilon>0$. Thus meas $\mathrm{sp}(U)=2\pi$ and assertion (i) follows.

12. *Some continuity considerations.* Let B and D denote a fixed pair of bounded self-adjoint operators, ε be a real parameter, and let A_ε be the perturbed self-adjoint operator defined by

$$A_\varepsilon=B+\varepsilon D. \tag{12.1}$$

Suppose that A_ε is unitarily equivalent to B for all sufficiently small values of ε, or at least for all small ε satisfying either $\varepsilon \geqslant 0$ or $\varepsilon \leqslant 0$, so that, for such ε,

$$A_\varepsilon=U_\varepsilon B U_\varepsilon^* \tag{12.2}$$

holds for some (perhaps more than one) unitary operator U_ε. (Of course, if (12.2) holds for some $U_\varepsilon=V$ it holds for $U_\varepsilon=zV$ where $|z|=1$. Possibly, though, (12.2) holds for other unitary operators which are not constant multiples of V.) Define the function $\Phi(\varepsilon)$ by

$$\Phi(\varepsilon)=\inf\,[\text{meas } \mathrm{sp}(U_\varepsilon)], \tag{12.3}$$

where "inf" is taken with reference to all unitary operators satisfying (12.2). Since, for $\varepsilon = 0$, $A = B$, it is seen that (12.2) holds for $U_0 = I$ and so $\Phi(0) = 0$. The problem to be considered in this section concerns the behavior of $\Phi(\varepsilon)$ near $\varepsilon = 0$ and, in particular, whether or not (for a fixed pair B and D) the function $\Phi(\varepsilon)$, which is supposed to be defined at least on some interval having 0 as an end-point, is continuous at $\varepsilon = 0$.

In [1], Friedrichs considered the perturbation equation (12.1) for small ε, where B was a certain operator with an absolutely continuous spectrum and where D was an integral operator with a kernel satisfying certain Lipschitz conditions and then showed that (12.2) was valid where U_ε was an analytic function of ε of the type $U_\varepsilon = I + \varepsilon U_1 + \dots$. In particular, $\|U_\varepsilon - I\| \to 0$ as $\varepsilon \to 0$, for the unitary operators he obtained. As a consequence, the function $\Phi(\varepsilon)$ as defined by (12.3) is, in this case, continuous at $\varepsilon = 0$.

It will be shown below that there exist pairs B and D, where in fact $D \leqslant 0$, for which (12.1) and (12.2) hold for $0 \leqslant \varepsilon < 1$ and for which $\Phi(\varepsilon)$ not only fails to be continuous at $\varepsilon = 0$ but even satisfies $\Phi(\varepsilon) \equiv 2\pi$ for $0 < \varepsilon < 1$.

13. *An example.* Let $q(t)$ be defined for $0 \leqslant t < \infty$ by

$$q(t) = (t/(t+1))^{\frac{1}{2}} \tag{13.1}$$

and define a function $Q_\varepsilon(t)$ by

$$Q_\varepsilon(t) = \varepsilon(1+t)q(t)/(t+1+(1-\varepsilon)q(t)) \tag{13.2}$$

for $0 < \varepsilon < 1$. On the space $L^2(0, \infty)$ let N denote the multiplication operator $N = t$ and then define the multiplication operator M_ε by

$$M_\varepsilon = N + Q_\varepsilon. \tag{13.3}$$

It is clear that $N \geqslant 0$ and that, for $0 < \varepsilon < 1$, $|Q_\varepsilon| \leqslant \varepsilon$, so that $(M_\varepsilon + I)^{-1}$ and $(N+I)^{-1}$ are bounded. A straightforward calculation shows that

$$(M_\varepsilon + I)^{-1} - (N+I)^{-1} = \varepsilon D, \tag{13.4}$$

where D is defined by

$$D = -q(t)/(t+1)(t+1+q(t)), \tag{13.5}$$

so that D, regarded as a multiplication operator on $L^2(0, \infty)$ satisfies $D \leqslant 0$.

Next put
$$B = (N+I)^{-1} \text{ and } A_\varepsilon = (M_\varepsilon + I)^{-1}, \tag{13.6}$$

so that (12.1) holds by virtue of (13.4).

Next, it will be shown that for $0 < \varepsilon < 1$, relation (12.2) holds for at least one unitary $U\varepsilon$. To this end, it can first be noted that (12.2) holds for some U_ε, that is, that $(M_\varepsilon + I)^{-1}$ and $(N + I)^{-1}$ are unitarily equivalent, by some U_ε, if and only if M_ε and N are unitarily equivalent, by the same U_ε. Thus it is sufficient to consider the problem of unitary equivalence of the operators $N = t$ and $M_\varepsilon = t + Q_\varepsilon(t)$ on $L^2(0, \infty)$, where $Q_\varepsilon(t)$ is defined by (13.2).

It will next be shown that t and $t + Q_\varepsilon(t)$ are unitarily equivalent. Let U be the operator defined on $L^2(0, \infty)$ by

$$U\colon x(t) \to x(T)(dT/dt)^{\frac{1}{2}}, \tag{13.7}$$

where $T(t) = t + Q_\varepsilon(t)$. Since $dT/dt = 1 + \varepsilon[q^2(1-\varepsilon) + (1+t)^2 q']/(t + 1 + (1-\varepsilon)q)^2$, it is seen that $dT/dt > 0$ for $0 < \varepsilon < 1$ and $0 < t < \infty$. Moreover, since $T(0) = 0$ and $T(\infty) = \infty$, it follows that

$$(Ux,\ Ux) = \int_0^\infty |x(T)|^2 (dT/dt)\, dt = \int_0^\infty |x(T)|^2\, dT = (x,\ x), \tag{13.8}$$

and hence U of (13.7) is isometric. Similarly, U^{-1} is isometric and hence U is unitary. In addition, it is seen that the sequence of transformations $x \to Ux \to M_\varepsilon Ux \to U^* M_\varepsilon Ux$ is given by

$$x(t) \to x(T(t))(dT/dt)^{\frac{1}{2}} \to T(t)x(T(t))(dT/dt)^{\frac{1}{2}} \to tx(t). \tag{13.9}$$

Thus

$$U^* M_\varepsilon U = N \text{ or } M_\varepsilon = UNU^*; \tag{13.10}$$

hence, as was noted earlier,

$$A_\varepsilon = UBU^*. \tag{13.11}$$

Thus far it has been shown that, for the pair of (13.6), $\Phi(\varepsilon)$ is actually defined (that is, (12.2) holds for some U_ε) for $0 \leqslant \varepsilon < 1$. It will next be shown that if U_ε is *any* unitary operator satisfying (12.2) for $0 < \varepsilon < 1$, then necessarily meas $\operatorname{sp}(U_\varepsilon) = 2\pi$. In order to show this, use will be made of Theorem 3 of Part II. It is sufficient to apply relation (7.2) if use is made of the fact noted earlier that (13.10) holds if and only if (13.11) holds. It is clear that $M_\varepsilon \geqslant 0$, $N \geqslant 0$, and that $M_\varepsilon - N = Q_\varepsilon \geqslant 0$ for $0 < \varepsilon < 1$, so that the k of Theorem 3 can now be chosen to be 0. In order to show that meas $\operatorname{sp}(U_\varepsilon) = 2\pi$, it is sufficient, by virtue of (7.2), to show that

$$\inf\,[\|x\|^2 (Ny, y)/\|y\|^4] = 0 \tag{13.12}$$

for functions $x(t)$, $y(t)=Q_\varepsilon^{\frac{1}{2}}(t)x(t)$, $ty(t)$ of class $L^2(0,\infty)$. Thus it is sufficient to show that the expression

$$\int_0^\infty y^2(t)\,Q_\varepsilon^{-1}(t)\,dt\cdot\int_0^\infty ty^2(t)\,dt\cdot\left(\int_0^\infty y^2(t)\,dt\right)^{-2} \tag{13.13}$$

can be made arbitrarily small for suitably chosen real-valued functions $y=y(t)$ on $0\leqslant t<\infty$.

Let h and δ denote positive constants and put $y=t^{(2\delta-1)/4}$ on $0<t\leqslant h$ and $y=0$ for $t>h$. Clearly y and ty belong to $L^2(0,\infty)$. Since, near $t=0$, $Q_\varepsilon(t)\sim q(t)\sim t^{\frac{1}{2}}$, it is clear that $x(t)$, defined by $y(t)=Q_\varepsilon^{\frac{1}{2}}(t)x(t)$, also belongs to $L^2(0,\infty)$. Moreover, (13.13) reduces to

$$\int_0^h t^{\frac{1}{2}(2\delta-1)}\,Q_\varepsilon^{-1}(t)\,dt\cdot h^{\frac{1}{2}(1-2\delta)}\,O(1) \tag{13.14}$$

($O(1)$ depending only on δ), which, by the estimate $Q_\varepsilon(t)\sim t^{\frac{1}{2}}$ near $t=0$, reduces to $h^{\frac{1}{2}}O(1)$. This last estimate, for δ fixed, tends to 0 as $h\to 0$. Hence (13.12) holds and it follows that meas $\operatorname{sp}(U_\varepsilon)=2\pi$, as was to be shown.

14. On $-\infty<t<\infty$, consider a function $T=T(t)$ of class C^1 which is strictly increasing and has the range $(-\infty,\infty)$. A calculation similar to that of section 13 shows that the U defined by (13.7) is a unitary operator on the space $L^2(-\infty,\infty)$. If, for instance, α is a positive constant and $T(t)=t+\alpha$, then U of (13.7) is the translation operator defined by

$$U\colon x(t)\to x(t+\alpha),\quad -\infty<t<\infty, \tag{14.1}$$

where $x(t)$ belongs to $L^2(-\infty,\infty)$. Let A and B denote the bounded multiplication operators

$$A=\operatorname{arc\,tan}(\beta^{-1}(t+\alpha)),\ B=\operatorname{arc\,tan}(\beta^{-1}t), \tag{14.2}$$

where β denotes a positive constant. Then it is seen that $A=UBU^*$, where U is defined by (14.1), and that $A-B=\operatorname{arc\,tan}(\beta^{-1}(t+\alpha))-\operatorname{arc\,tan}(\beta^{-1}t)\equiv d(t)>0$. Thus relation (1.3) holds and Theorems 2 and 3 (the latter with $k=-\frac{1}{2}\pi$) are applicable. The second term in the bracket of (7.2) is twice

$$\int_{-\infty}^\infty d^{-1}y^2\,dt\cdot\int_{-\infty}^\infty(\pi/2+\operatorname{arc\,tan}(\beta^{-1}t))\,y^2\,dt\cdot\left(\int_{-\infty}^\infty y^2\,dt\right)^{-2}. \tag{14.3}$$

Choose a constant δ satisfying $0<\delta<\frac{1}{4}\alpha$ and then let y be chosen so that $y\equiv 0$ outside the interval $-2\delta\leqslant t\leqslant-\delta$ and $0<\int_{-\infty}^\infty y^2\,dt<\infty$. For values t on this interval, $t+\alpha\geqslant\frac{1}{2}\alpha$

and it is clear that for any $\varepsilon > 0$ it is possible to choose $\beta > 0$ so small, that both inequalities $\frac{1}{2}\pi + \arctan(\beta^{-1}t) < \varepsilon$ and $d(t) > \pi - \varepsilon$ hold t in $-2\delta \leqslant t \leqslant -\delta$. It follows from the same type of argument as that used in section 13 that the expression (14.3) can be made arbitrarily small by choosing β sufficiently small. Hence, by (7.2), meas $\operatorname{sp}(U) = 2\pi$. In addition, since $d(t)(=D) > 0$, then 0 is not in the point spectrum of D and it follows from [25] that the spectrum of U is absolutely continuous.

The example discussed above suggests generalizations to a measure preserving transformation T on a space Ω of points P. The transformation $U: f(P) \to f(TP)$ is then unitary. In addition it will be supposed that Ω is a metric space with a distance $|PQ|$ defined for any two points P and Q.

The transformation T is said to be dissipative (cf. Hopf [9, p. 46], Halmos [5, p. 11]) if there exists a set A of positive measure for which the images $A_n = T^n(A)$ are disjoint and $\Omega = \sum_{-\infty}^{\infty} A_n$. Let such a set A be called a generating set of Ω.

There will be proved the following

THEOREM 4. (i) *If T is dissipative on the space Ω and if U is the associated unitary transformation, then U is absolutely continuous.* (ii) *If, in addition, there exists some point R belonging to the interior of a generating set A, so that there exists some sphere S_R with center at R satisfying*

$$S_R \text{ is contained in } A, \tag{14.4}$$

and if

$$T \text{ is continuous at } R, \tag{14.5}$$

then $\operatorname{sp}(U)$ *is the entire unit circle* $|z| = 1$.

15. *Proof of Theorem 4.* Since $\Omega = \sum_{-\infty}^{\infty} A_n$ where meas $A > 0$ and the A_n are disjoint, then Ω is obtained by taking all the images of the set $A = A_0$. If P is an arbitrary point of Ω then P is in a (unique) set A_n and so $P = Q_n = T^n Q$, where Q belongs to A. If $n = 1$, let $s(P) = 0$. If $n \geqslant 2$, let $s(P) = \sum_{k=2}^{n} |Q_{k-1}Q_k|$, and if $n \leqslant 0$, let $s(P) = -\sum_{k=0}^{n} |Q_k Q_{k+1}|$. Thus $s(P)$ is the (signed) distance to P from its image in A_1. Since the sets A_n are disjoint, it is clear that $s(TP) - s(P) > 0$ for all P in Ω and hence $k(P) = g(s(P))$, where g is the principal inverse tangent function of section 14 defined by

$$g(t) = g(t, \beta) = \arctan(\beta^{-1}t), \quad \beta > 0, \tag{15.1}$$

satisfies the condition

$$D = k(TP) - k(P) \equiv d(P) > 0. \tag{15.2}$$

As in section 14, it follows that if A and B are the multiplication operators $A = g(s(TP))$ and $B = g(s(P))$, then $A = UBU^*$ and $A - B = d > 0$, so that 0 is not in the point spectrum of D. The result of [25] then implies the absolute continuity of U and (i) is proved.

It will follow from (7.2) that meas $\mathrm{sp}(U) = 2\pi$ if it can be shown that the expression corresponding to (14.3), namely,

$$\int_\Omega d^{-1}(P)\, y^2(P)\, dV \cdot \int_\Omega \left(\frac{\pi}{2} + k(P)\right) y^2(P)\, dV \cdot \left(\int_\Omega y^2(P)\, dV\right)^{-2}, \tag{15.3}$$

can be made arbitrarily small by choosing functions $y(P) = d^{\frac{1}{2}}(P)x(P)$, where both $x(P)$ and $y(P)$ belong to $L^2(\Omega)$. By (14.4) and (14.5), $s(TP) \equiv 0$ for P near R and $s(P)$ is continuous at R. Hence $k(TP) \equiv 0$ for P near R, also $k(P)$ and $d(P)$ are continuous at R. Since $s(R) < 0$ (R being in A) it follows that for $\varepsilon > 0$, then $\frac{1}{2}\pi + k(P) < \varepsilon$ holds, if $\beta > 0$ is sufficiently small, for all points P in the set A sufficiently close to R. In addition, $d(P) = d(TP) - k(P) = 0 - k(P)$ for all such points, since TP is in A_1. Thus $d(P) > \frac{1}{2}\pi - \varepsilon$. If now one considers functions y equal to 0 outside a sufficiently small sphere S_R satisfying (14.4), it follows from the type of argument used in section 13 and 14 that the expression of (15.3) can be made arbitrarily small by choosing β sufficiently small. This completes the proof of (ii).

16. The following theorem is similar to Theorem 4.

THEOREM 5. *Let T be a measure-preserving transformation, with associated unitary transformation U, on the space Ω. Suppose that there exists a real-valued measurable function $f(P)$ on Ω for which*

$$f(TP) - f(P) \geqslant 0 \text{ and } \not\equiv 0. \tag{16.1}$$

If U has the spectral resolution (6.1), *then*

$$\int_z dE(\lambda) < I, \tag{16.2}$$

where Z is any zero set, so that U must have some continuous spectrum. If, instead of (16.1), *it is assumed that even*

$$f(TP) - f(P) > 0 \text{ almost everywhere}, \tag{16.3}$$

then U is absolutely continuous. Furthermore, if only (16.1) *holds, if $f(P)$ and $f(TP)$ are continuous at some point R, if also $f(TR) - f(R) > 0$, and finally, if there exists some sphere S_R with center R satisfying*

$$S_R \text{ is contained in } \Omega, \tag{16.4}$$

then $\mathrm{sp}(U)$ *is the entire circle* $|z| = 1$.

Proof of Theorem 5. Introduce the functions $k(P) = g(f(P))$, where $g(t)$ is defined by (15.1), and $d(P) = k(TP) - k(P)$. Let $A = k(TP)$ and $B = k(P)$ and note that again $A = UBU^*$ and $A - B = d(P)$. Since $d(P)$ corresponds to D and since (16.1) implies $D \geqslant 0$ and $D \neq 0$, then (16.2) follows from [25, p. 105]. Since (16.3) implies that 0 is not in the point spectrum of D, the assertion concerning the absolute continuity of U also follows from [25]. In order to prove the last part of the theorem note that $f(P)$ can be replaced by $f(P) +$ const., so that it can be supposed that $f(TR) = 0$. The remainder of the proof is then similar to that of Theorem 4 and can therefore be omitted.

17. *Examples.* In order to illustrate the results of the last section, consider a conservative, incompressible, n-component vector system of differential equations

$$x' = F(x), \ F \text{ of class } C^1 \text{ and } \operatorname{div} F = 0. \tag{17.1}$$

Suppose that (17.1) possesses unique solutions $x = x(t)$ for $-\infty < t < \infty$ on the space Ω of points x. In addition, suppose that there exists a function $f(x)$ of class C^1 satisfying, for instance,

$$df/dt \equiv \operatorname{grad} f \cdot F > 0 \ \text{ almost everywhere on } \Omega. \tag{17.2}$$

Since the incompressibility assumption $\operatorname{div} F = 0$ assures that the flow $x(0) \to x(t)$ determined by (17.1) is measure preserving, that portion of Theorem 5, in which assumption (16.3) occurs, corresponding now to condition (17.2), is applicable. It follows that if $U = U_t$ is the associated unitary transformation $U_t : g(x(0)) \to g(x(t))$, where g is of class $L^2(\Omega)$, then U is absolutely continuous and sp (U) is the entire circle $|z| = 1$.

In case (17.1) holds with $n = 1$, then one obtains the single equation $x' = a$, $a =$ const. $\neq 0$, with the solution $x = at + b\,(b =$ const.). If $f = ax$, then (17.2) holds. This example is, in essentials, that of the translation operator considered in section 14.

In case (17.1) holds with $n = 2$, it is known (cf. Wintner [34, p. 88]) that the system is Hamiltonian, that is, there exists an energy function $H = H(x, y)$ such that (17.1) can be written as

$$x' = \partial H/\partial y, \quad y' = -\partial H/\partial x. \tag{17.3}$$

The condition (17.2) now becomes

$$(\partial f/\partial x)(\partial H/\partial y) - (\partial f/\partial y)(\partial H/\partial x) > 0 \text{ almost everywhere on } \Omega. \tag{17.4}$$

If, for example, H of (17.3) is harmonic, let f denote its harmonic conjugate, so that $\partial f/\partial x = \partial H/\partial y$ and $\partial f/\partial y = -\partial H/\partial x$ and (17.4) reduces to the condition

$$|\operatorname{grad} H|^2 > 0 \ \text{ almost everywhere on } \Omega. \tag{17.5}$$

For a system of the type (17.3) in which H is harmonic and $H \not\equiv$ const., then in fact, (17.5) does hold. It follows that the unitary operator U of the Hamiltonian system (17.3) is, in this case, absolutely continuous and has a spectrum consisting of the entire circle $|z| = 1$.

18. *Scattering operators.* Investigations of the perturbation equation $B = A - D$ (in the notation of (1.3)) where A and B are self-adjoint, and the problem of unitary equivalence of A and B, by Friedrichs [1, 2] and subsequently by Rosenblum [26] and Kato [12, 13] have been mentioned earlier. In these papers it was shown, under appropriate assumptions on A and the perturbations D, that

$$e^{itB}e^{-itA} \to U_+ \quad \text{as } t \to \infty \tag{18.1}$$

and

$$e^{itB}e^{-itA} \to U_- \quad \text{as } t \to -\infty, \tag{18.2}$$

where U_+ and U_- denote unitary operators and the limits are meant in the sense of strong convergence. See also Kuroda [16, 17]. Moreover, each of the operators $U = U_+$ and $U = U_-$ satisfies relation (1.3). The operator $U_+^* U_-$ corresponds to the scattering operator of quantum mechanics (see, e.g., Friedrichs [3], Jauch [10], Moses [18]); the operators U_+ and U_- have been termed "half-scattering operators" by Friedrichs [3, p. 233], and "wave operators" by Jauch [10, p. 137].

If D satisfies the additional assumption

$$D \text{ bounded, } D \geqslant 0 \text{ (or } D \leqslant 0) \text{ and } D \neq 0, \tag{18.3}$$

then the theorems of Part II relating to the spectra of any unitary U satisfying (1.3), hence in the present case, in particular, to U_+ and U_-, are applicable, at least if A and B are half-bounded. In the quantum mechanical case, when A corresponds to the energy of the system, this latter assumption appears to be natural (cf. Kemble [14, p. 107], Kato [11, p. 205], Jauch [10, p. 134].).

19. *Differential operators.* Consider a limit point differential equation $L(u) + \lambda u = 0$ on, say, $0 \leqslant t < \infty$ ($L(u)$, a linear differential operator) with a boundary condition at $t = 0$; see Weyl [31], also Kodaira [15]. Let A_1 and A_2 denote self-adjoint extensions of the associated symmetric operator, corresponding to two distinct boundary conditions, and suppose that $\lambda = \mu$ is real and belongs to the resolvent set of both A_1 and A_2. It is known [31, p. 251] that the difference

$$(A_1 - \lambda I)^{-1} - (A_2 - \lambda I)^{-1} = D \tag{19.1}$$

is a constant multiple of a one-dimensional projection operator. It was shown in [21], using results of Rosenblum [26] that if, in addition, each of the two boundary value problems had a purely continuous spectrum with absolutely continuous basis functions, then $A = (A_1 - \lambda I)^{-1}$ and $B = (A_2 - \lambda I)^{-1}$ are unitarily equivalent, and thus satisfy (1.3). (Incidentally, the projection $F(\lambda)$ of [21, p. 994] should be given by $\int_{\mu^{-1} \leqq \lambda} dE(\mu)$.) Theorems 2 and 3 are then applicable to any such unitary operator, in particular, to the U_+ and U_- occurring in section 18 and which exist in the present instance.

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