ON THE STRUCTURE OF MEASURE SPACES

BY

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1. Introduction

In two papers fundamental to the theory of measure, Halmos and von Neumann [3] and Maharam [4] have characterized the measure algebras associated with totally finite measure spaces by showing that each such algebra is isomorphic to the measure algebra of some canonical measure space. In this note, we shall show that the measurable sets of a given measure space are constructed of certain null subsets of that space in essentially the same manner as the measurable sets of the standard measure space, to which the given one is isomorphic, are constructed of points. The technique used depends on a theorem concerning the relation between the lattices of measurable functions modulo null functions defined on isomorphic measure spaces. We conclude the discussion with some applications to problems that arise in connection with the study of a theorem of Saks and Sierpinski [5] on the approximation of real functions by measurable functions.

2. Preliminary considerations

Let $(X, \mathbf{S}, \mu)$ be a measure space, and let $\mathbf{N}$ be the class of all measurable sets of measure zero. If E and F are elements of $\mathbf{S}$, and if $\triangle$ denotes the operation of symmetric difference, we write $E \sim F$ if and only if $E \triangle F$ belongs to $\mathbf{N}$. The relation $\sim$ thus defined on $\mathbf{S}$ is an equivalence relation. The quotient space $\mathbf{S}/\mathbf{N}$ to which $\sim$ gives rise, is denoted by $\mathbf{S}(\mu)$. If E is an element of $\mathbf{S}$, we denote by $[E]$ the equivalence class determined by E. The binary operations $+$, $\cdot$, are (well) defined on $\mathbf{S}(\mu)$ by means of the equations

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$$a+b=[E\triangle F],\quad a\cdot b=[E\cap F],$$

where E is an element of a and F is an element of b; it is easily verified that $\{S(\mu);\ +,\ \cdot\}$ is a Boolean ring. We write $a\leqslant b$ if and only if $ab=a$. It is clear that $\leqslant$ is a partial ordering and that $\{S(\mu);\ \leqslant\}$ is a σ-complete lattice. Moreover, if $[E_n]=a_n$, $n=1,2,\ldots$, then

$$\bigvee_{n=1}^{\infty} a_n=[\bigcup_{n=1}^{\infty} E_n]\quad\text{and}\quad \bigwedge_{n=1}^{\infty} a_n=[\bigcap_{n=1}^{\infty} E_n],$$

where $\bigvee$ and $\bigwedge$ denote least upper bound and greatest lower bound. The difference of two elements of $S(\mu)$ is defined by

$$a-b=a+ab.$$

If $[E]=a$ and $[F]=b$, then

$$a-b=[E-F];$$

thus, $a\leqslant b$ if and only if $E-F$ is an element of N. A measure, also denoted by μ, is (well) defined on $S(\mu)$ as follows: if $[E]=a$, then

$$\mu(a)=\mu(E).$$

The pair $(S(\mu),\ \mu)$ is called the measure ring associated with $(X,\ S,\ \mu)$.

The measure rings $(S(\mu),\ \mu)$ and $(T(\nu),\ \nu)$ associated with $(X,\ S,\ \mu)$ and $(Y,\ T,\ \nu)$ are isomorphic if there exists a one to one mapping θ of $S(\mu)$ onto $T(\nu)$ such that

$$(a\vee b)\theta=a\theta\vee b\theta,\ \ (a-b)\theta=a\theta-b\theta\quad\text{and}\quad \nu(a\theta)=\mu(a),$$

whenever a and b are elements of $S(\mu)$. Equivalently, $(S(\mu),\ \mu)$ and $(T(\nu),\ \nu)$ are isomorphic if there is a measure preserving ring isomorphism between $\{S(\mu);\ +,\ \cdot\}$ and $\{T(\nu);\ +,\ \cdot\}$. We remark that the conditions imposed on θ also imply that

$$(\bigvee_{n=1}^{\infty} a_n)\theta=\bigvee_{n=1}^{\infty} a_n\theta.$$

An element a of $S(\mu)$ is called an atom if $\mu(a)>0$ and if the conditions $b\leqslant a$, $b\neq a$ imply that $b=0$. A measure ring containing no atoms is termed nonatomic.

A family F of measurable sets is dense in S if to each measurable set E and positive number ε there corresponds an element F of F for which $\mu(E\triangle F)<\varepsilon$. The smallest cardinal number corresponding to a dense subset of S is called the character of the measure space. If its character is $\aleph_0$, then $(X,\ S,\ \mu)$ is called separable. The character of $(S(\mu),\ \mu)$ is defined analogously. If every principal ideal of a measure ring has the same character as the measure ring, then the measure ring and the measure space are homogeneous.

If f and g are extended-real-valued measurable (S) functions that assume different values only on a set of measure zero, then we write $f \sim g$. The relation $\sim$ is an equivalence relation on the set of all measurable functions. We denote by $[f]$ the equivalence class determined by the measurable function f and by $\mathcal{S}$ the collection of all such equivalence classes. If f and g are measurable functions and if $f(x) \leqslant g(x)$ a.e., then we write $[f] \leqslant [g]$. The relation $\leqslant$ is well defined, and $\{\mathcal{S}; \leqslant\}$ is a σ-complete lattice.

The following fundamental theorems occupy a central position in the rest of the discussion. Let $(I, \mathbf{L}, m)$ denote the ordinary Lebesgue measure space associated with $I = [0, 1]$.

THEOREM 1. (Halmos and von Neumann). *If $(X, \mathbf{S}, \mu)$ is totally finite, nonatomic and separable and if $\mu(X) = 1$, then $(\mathbf{S}(\mu), \mu)$ is isomorphic to $(\mathbf{L}(m), m)$.*

If γ is an ordinal number, we denote by $(I^\gamma, \mathbf{L}^\gamma, m^\gamma)$ the Cartesian product $\times_{\alpha<\gamma} (I, \mathbf{L}, m)_\alpha$, where each $(I, \mathbf{L}, m)_\alpha$ is a copy of $(I, \mathbf{L}, m)$.

THEOREM 2. (Maharam) *If $(X, \mathbf{S}, \mu)$ is totally finite, nonatomic and homogeneous and if $\mu(X) = 1$, then $(\mathbf{S}(\mu), \mu)$ is isomorphic to $(\mathbf{L}^\gamma(m^\gamma), m^\gamma)$, where γ is the least ordinal corresponding to the character of $(X, \mathbf{S}, \mu)$.*

THEOREM 3. (Maharam) *If $(X, \mathbf{S}, \mu)$ is totally finite, then there exists an at most denumerable family of homogeneous measure spaces $(X_n, \mathbf{S}_n, \mu_n)$ such that:*

(i) $X_n \| X_m$, *if* $n \neq m$,

(ii) $\bigcup_{n=1}^\infty X_n = X$,

(iii) $\{\bigcup_{n=1}^\infty E_n : E_n \in \mathbf{S}_n\} = \mathbf{S}$, *and*

(iv) $\mu(E) = \sum_{n=1}^\infty \mu_n(E \cap X_n)$, *for all E in* $\mathbf{S}$.

3. The main theorems

In [6] Segal has established a theorem connecting measure spaces with the algebras of bounded measurable functions defined on those spaces. We give now another theorem of this type, which, together with the isomorphism theorems of the last section, gives some insight into the point set structure of a wide class of measure spaces.

THEOREM 4. *Let $(X, \mathbf{S}, \mu)$ and $(Y, \mathbf{T}, \nu)$ be isomorphic totally finite measure spaces. To each isomorphic mapping θ of $\mathbf{S}(\mu)$ onto $\mathbf{T}(\nu)$ there corresponds an isomorphism φ between $\mathcal{S}$ and $\mathcal{T}$ satisfying the following condition:*

(*) if f is measurable (S), if f_0 is any element of $[f]\varphi$ and if B is any Borel subset of the two-point compactification of the real line, then $[f^{-1}(B)]\,\theta = [f_0^{-1}(B)]$.

The author is indebted to the referee for the elegant demonstration of this theorem that is given below. The argument is based on two lemmas, the second of which is known from one proof of the Radon–Nikodym theorem.

LEMMA 5. *Let* (X, S, μ) *be a measure space with* S *a* σ*-algebra, and let* f *and* g *be extended-real-valued measurable* (S) *functions. Then*

$$[f] \leqslant [g] \quad \text{if and only if} \quad [f^{-1}[-\infty, \alpha)] \geqslant [g^{-1}[-\infty, \alpha)],$$

for every extended real number α.

Proof. The necessity of the condition is trivial.

Suppose that $\{x : f(x) > g(x)\}$ were a set of positive measure. There would then exist a rational number r for which

$$\{x : f(x) \geqslant r\} \cap \{x : g(x) < r\}$$

would be a set of positive measure. But, in this case, $g^{-1}[-\infty, r) - f^{-1}[-\infty, r)$ would not be a null set, whence the inequality $[f^{-1}[-\infty, r)] \geqslant [g^{-1}[-\infty, r)]$ could not obtain. The sufficiency of the condition is thus established.

COROLLARY 6. *Subject to the general hypotheses of Lemma* 5,

$$[f] = [g] \quad \text{if and only if} \quad [f^{-1}[-\infty, \alpha)] = [g^{-1}[-\infty, \alpha)],$$

for every extended real number α.

LEMMA 7. *Let* (X, S, μ) *be a measure space with* S *a* σ*-algebra, and let* $\alpha \to e_\alpha$ *be a mapping of* $\bar{R}$*, the set of all extended real numbers, into* $\mathrm{S}(\mu)$*. In order that there should exist an extended-real-valued measurable* (S) *function* f *such that*

$$e_\alpha = [f^{-1}[-\infty, \alpha)],$$

for all α *in* $\bar{R}$*, it is both necessary and sufficient that the following conditions be satisfied:*

(i) *if* $\alpha < \beta$*, then* $e_\alpha \leqslant e_\beta$;

(ii) *if* $\sup_n \alpha_n = \alpha$*, then* $\bigvee_n e_{\alpha_n} = e_\alpha$;

(iii) $e_{-\infty} = 0$.

In the affirmative case, $[f]$ *is uniquely determined.*

Proof. The necessity of these conditions is obvious, since if f is such a function, then

(1) $f^{-1}[-\infty, \alpha) \subset f^{-1}[-\infty, \beta)$, if $\alpha < \beta$;

(2) $\bigcup_n f^{-1}[-\infty, \alpha_n) = f^{-1}[-\infty, \alpha)$, if $\sup_n \alpha_n = \alpha$; and

(3) $f^{-1}[-\infty, \alpha) = \emptyset$, if $\alpha = -\infty$;

from which immediately follow (i), (ii), (iii).

Now suppose that (i), (ii), (iii) are satisfied. For each rational number r let Fr be an element of e_r, and let

$$E_\alpha = \bigcup_{r<\alpha} F_r,$$

for each α in $\bar{R}$. Certainly the E_α are measurable sets satisfying the conditions:

(1) $E_\alpha \subset E_\beta$, if $\alpha < \beta$,

(2) $\bigcup_n E_{\alpha_n} = E_\alpha$, if $\sup_n \alpha_n = \alpha$,

(3) $E_{-\infty} = \emptyset$;

moreover, $$[E_\alpha] = [\bigcup_{r<\alpha} E_r] = \bigvee_{r<\alpha} [E_r] = \bigvee_{r<\alpha} e_r = e_\alpha.$$

Thus, if x is an arbitrary element of X, then

$$\{\alpha : x \in E_\alpha\} = (f(x), +\infty],$$

where $f(x)$ is a uniquely determined extended real number. The function f thus specified has the property that

$$f^{-1}[-\infty, \alpha) = E_\alpha,$$

for every α in $\bar{R}$. The uniqueness of $[f]$ follows at once from Corollary 6.

We now proceed to the proof of Theorem 4. Let f be an extended-real-valued measurable (S) function. For each extended real number α, let

$$e_\alpha = [f^{-1}[-\infty, \alpha)], \quad e_0^\alpha = e_\alpha \theta.$$

According to Lemma 7, the elements e_α possess properties (i), (ii), (iii). Since θ is an isomorphism, these conditions are also satisfied by the elements e_0^α of $\mathrm{T}(\nu)$. Hence, again by Lemma 7, there exists a unique element $[f_0]$ of $\mathcal{J}$ such that

$$e_0^\alpha = [f_0^{-1}[-\infty, \alpha)],$$

for each α in $\bar{R}$.

We shall show that the mapping φ of S into $\mathcal{J}$ defined by the equation

$$[f]\varphi = [f_0]$$

is the sought for isomorphism.

Suppose that $[f]$ and $[g]$ are two elements of S such that

$$[f]\varphi = [g]\varphi = [f_0].$$

Following the notation introduced above, we write, for each α,

$$e_\alpha = [f^{-1}[-\infty, \alpha)], \; d_\alpha = [g^{-1}[-\infty, \alpha)],$$
$$e_0^\alpha = e_\alpha \theta, \; d_0^\alpha = d_\alpha \theta.$$

According to the definition of $[f_0]$, the following equalities hold for all α in $\bar{R}$:

$$e_\alpha \theta = e_0^\alpha = [f_0^{-1}[-\infty, \alpha)] = d_0^\alpha = d_\alpha \theta.$$

Since θ is one to one, we must have $e_\alpha = d_\alpha$, for all α in $\bar{R}$. Hence, by Corollary 6, $[f] = [g]$, and it follows that φ is one to one.

Now let f_0 be an arbitrary function defined on Y and measurable (**T**). For each α in $\bar{R}$, let

$$e_0^\alpha = [f_0^{-1}[-\infty, \alpha)], \; e_\alpha = e_0^\alpha \theta^{-1}.$$

In exactly the same manner as before, Lemma 7 implies the existence of a function f, defined on X and measurable (S), such that

$$e_\alpha = [f^{-1}[-\infty, \alpha)],$$

for all α in $\bar{R}$. Since it is clear that $[f]\varphi = [f_0]$ and since $[f_0]$ was arbitrarily chosen from $\mathcal{T}$, it follows that φ is onto.

Hence, to prove that φ is an isomorphism, it will suffice to show that φ and φ^{-1} are order preserving, i.e.

$$[f] \geqslant [g] \quad \text{if and only if} \quad [f]\varphi \geqslant [g]\varphi.$$

Let f_0 be an element of $[f]\varphi$, and let g_0 be an element of $[g]\varphi$. By virtue of Lemma 5 and the fact that θ is an isomorphism, the following assertions are mutually equivalent:

$$[f] \leqslant [g];$$
$$[f^{-1}[-\infty, \alpha)] \geqslant [g^{-1}[-\infty, \alpha)], \quad \text{for all real } \alpha;$$
$$[f_0^{-1}[-\infty, \alpha)] \geqslant [g_0^{-1}[-\infty, \alpha)], \quad \text{for all real } \alpha;$$
$$[f]\varphi \leqslant [g]\varphi.$$

Finally, we note that the method of construction ensures that φ has the property (*), at least for intervals of the form $[-\infty, \alpha)$. Since it is clear that the class of all Borel subsets of $\bar{R}$ for which (*) subsists is a σ-algebra, the theorem is proved.

The following converse of this theorem is also true.

THEOREM 8. *Let* (X, S, μ) *and* (Y, T, ν) *be totally finite measure spaces, and let* θ *be a one to one mapping of* $\mathrm{S}(\mu)$ *onto* $\mathbf{T}(\nu)$ *such that* $\nu(a\theta) = \mu(a)$, *for all* a *in* $\mathrm{S}(\mu)$. *In order that* θ *be an isomorphism, it is sufficient that there should exist an isomorphism,* φ *of* $\mathcal{S}$ *onto* $\mathcal{T}$ *satisfying the following condition:*

(*) *if* f *is measurable* (S), *if* f_0 *is an element of* $[f]\varphi$ *and if* B *is any Borel subset of* $\bar{R}$, *then* $[f^{-1}(B)]\theta = [f_0^{-1}(B)]$.

Proof. Observe first that if E is an element of S and if g is an element of $[\chi_E]\varphi$, then

$$[g^{-1}\{0, 1\}] = [\chi_E^{-1}\{0, 1\}]\theta = [X]\theta = [Y];$$

thus, $g \sim \chi_{E_0}$, where $[E_0] = [g^{-1}\{1\}] = [\chi_E^{-1}\{1\}]\theta = [E]\theta$.

Suppose that a and b are elements of $\mathrm{S}(\mu)$ satisfying $a \leqslant b$. Let A, B be elements of a, b, and let A_0, B_0 be elements of $a\theta$, $b\theta$. It is clear that $[\chi_A] \leqslant [\chi_B]$ and, since φ is a lattice isomorphism, that

$$[\chi_{A_0}] = [\chi_A]\varphi \leqslant [\chi_B]\varphi = [\chi_{B_0}].$$

By Lemma 5, we have $[\chi_{A_0}^{-1}[-\infty, 1)] \geqslant [\chi_{B_0}^{-1}[-\infty, 1)]$; hence, $[\chi_{A_0}^{-1}[1, +\infty]] \leqslant [\chi_{B_0}^{-1}[1, +\infty]]$, since the latter elements are the complements in $\mathbf{T}(\nu)$ of the former. Therefore, summarizing the above remarks,

$$a\theta = [A_0] = [\chi_{A_0}^{-1}[1, +\infty]] \leqslant [\chi_{B_0}^{-1}[1, +\infty]] = [B_0] = b\theta.$$

Using the fact that φ^{-1} is also a lattice isomorphism, we can show, by an entirely parallel argument, that $a\theta \leqslant b\theta$ only if $a \leqslant b$.

Since a proof of the equivalence of $a \leqslant b$ and $a\theta \leqslant b\theta$ is sufficient, in this case, to show that θ is an isomorphism, the proof is complete.

By application of Theorem 4 and the fundamental results of Halmos, von Neumann and Maharam, we are now able to obtain some interesting theorems on the structure of the measurable sets of a totally finite measure space. We first consider the separable case.

THEOREM 9. *Let* (X, S, μ) *be a totally finite, nonatomic and separable measure space with* $\mu(X) = 1$, *let* $I = [0, 1]$ *and let* $(I, \mathbf{L}, m)$ *be the Lebesgue measure space. There exists a family* $\{X_r : r \in I\}$ *of measurable subsets of* X *for which the following conditions are satisfied:*

(i) $\mu(X_r)=0$, *for all* r *in* I,

(ii) $X_r \| X_s$, *if* $r \neq s$,

(iii) *to each element* E *of* S *there corresponds an* E_0 *in* L *such that* $E \sim \bigcup\{X_r : r \in E_0\}$, *and* $\mu(E)=m(E_0)$.

Proof. By virtue of the theorem of Halmos and von Neumann, there exists an isomorphic mapping θ of $\mathrm{S}(\mu)$ onto $\mathrm{L}(m)$. Thus, by Theorem 4, there exists an isomorphism φ between $\mathcal{S}$ and $\mathcal{L}$ satisfying the condition $[f^{-1}(B)]\theta=[f_0^{-1}(B)]$, whenever f is measurable (S), f_0 is an element of $[f]\varphi$ and B is a Borel subset of $\bar{R}$. Let i_0 be the identity function on I, and let i be a function, real valued and measurable (S), for which $[i]\varphi=[i_0]$. For each r in I, define

$$X_r = i^{-1}(\{r\}).$$

We see at once that $X_r \| X_s$, whenever $r \neq s$, and, since

$$\mu(X_r)=\mu(i^{-1}(\{r\}))=m(i_0^{-1}(\{r\}))=m(\{r\}),$$

the first condition is also satisfied. Now consider an arbitrary element E of S. If E_0 is a Borel set belonging to $[E]\theta$, then

$$[\bigcup\{X_r : r \in E_0\}]\theta=[i^{-1}(E_0)]\theta=[i_0^{-1}(E_0)]=[E_0]=[E]\theta,$$

whence

$$E \sim \bigcup\{X_r : r \in E_0\}.$$

Since $[E_0]=[E]\theta$, it is immediate from the definition of isomorphism that $\mu(E)=m(E_0)$; hence, the proof is complete.

We may assume that $\bigcup\{X_r : r \in I\}=X$, for if need be we could adjoin the exceptional null set to one of the sets X_r without affecting any of the conclusions of the theorem. Hence, the following proposition is an immediate consequence of Theorem 9.

COROLLARY 10. *Let* (X, S, μ) *be a totally finite, nonatomic and separable measure space with* $\mu(X)=1$, *let* $I=[0, 1]$ *and let* (I, L, m) *be the Lebesgue measure space. There exists a measure preserving transformation* T *of* (X, S, μ) *into* (I, L, m) *such that* $T^{-1}(\mathrm{L})$ *is equivalent to* S.

Thus, (I, L, m) is, in a sense, minimal for the class of normalized, totally finite, nonatomic and separable measure spaces.

In nonseparable spaces the situation is only slightly more complicated. By virtue of Theorem 3, we need to consider only the homogeneous case.

LEMMA 11. *Let* $(X, \mathbf{S}, \mu)$ *be a totally finite measure space, and let* $\mathbf{R}$ *be a subset of* $\mathbf{S}$. *If* E *belongs to* $\mathbf{S}$ *and if to each positive* ε *there corresponds an* R *in* $\mathbf{R}$ *satisfying* $\mu(E \triangle R) < \varepsilon$, *then* E *is equivalent to an element of the* σ*-ring generated by* $\mathbf{R}$.

Proof. Let $\delta > 0$ be specified. Choose R_n in $\mathbf{R}$ so that $\mu(E \triangle R_n) < \delta \cdot 2^{-n}$, let $S_n = \bigcup_{k=1}^{n} R_k$, for $n = 1, 2, \ldots$, and let $S = \bigcup_{n=1}^{\infty} S_n$. Then

$$\mu(E - S) = \lim_n \mu(E - S_n) = 0,$$

and since
$$\mu(S_n - E) \leq \sum_{k=1}^{\infty} \mu(R_k - E) < \delta, \quad n = 1, 2, \ldots,$$

we have
$$\mu(S - E) = \lim_n \mu(S_n - E) < \delta.$$

Proceeding in this manner, it is possible to choose, for each natural number n, a set T_n belonging to $\mathbf{R}_\sigma$ and satisfying the conditions

$$\mu(E - T_n) = 0, \quad \mu(T_n - E) < n^{-1}.$$

Let $U_n = \bigcap_{k=1}^{n} T_k$ for $n = 1, 2, \ldots,$ and let $U = \bigcap_{n=1}^{\infty} U_n$.

Then,
$$\mu(E - U_n) \leq \sum_{k=1}^{n} \mu(E - T_k) = 0,$$

whence
$$\mu(E - U) = \lim_n \mu(E - U_n) = 0.$$

On the other hand,

$$\mu(U_n - E) \leq \mu(T_n - E) < n^{-1}, \quad n = 1, 2, \ldots,$$

whence
$$\mu(U - E) = \lim_n \mu(U_n - E) = 0.$$

Thus, $\mu(E \triangle U) = 0$, where U is an element of $\mathbf{R}_{\sigma\delta}$, and the lemma is proved.

THEOREM 12. *Let* $(X, \mathbf{S}, \mu)$ *be a totally finite, nonatomic and homogeneous measure space, with* $\mu(X) = 1$. *There exist an ordinal number* γ *and a family* $\{X_{\alpha r} : \alpha \in [1, \gamma),\ r \in I\}$ *of measurable subsets of* X *such that the following conditions are satisfied:*

(i) $\mu(X_{\alpha r}) = 0$, *for all* $\alpha < \gamma$ *and for all* r *in* I,

(ii) $X_{\alpha r} \| X_{\alpha s}$, *if* $r \neq s$, *for all* $\alpha < \gamma$,

(iii) *the* σ*-ring generated by the class* $\{\bigcup \{X_{\alpha r} : r \in E_0\} : \alpha < \gamma,\ E_0 \in \mathbf{L}\}$ *is equivalent to* $\mathbf{S}$, *in the sense that every element of* $\mathbf{S}$ *differs from one of these sets by a set of measure zero, and conversely,*

(iv) $\mu(\bigcup \{X_{\alpha r} : r \in E_0,\ E_0 \in \mathbf{L}\}) = m(E_0)$, *for all* $\alpha < \gamma$.

Proof. By Maharam's theorem there exists an isomorphic mapping θ of $S(\mu)$ onto $L^{\gamma}(m^{\gamma})$,where γ is the least ordinal corresponding to the character of (X, S, μ). For each $\beta<\gamma$, let

$$L_{\beta}=\{E\times \mathop{\times}_{\substack{\alpha<\gamma\\ \alpha\neq\beta}} I_{\alpha}: E\in L\}$$

and let N_{β} denote the family of null sets in L_{β}. It is clear that $(L_{\beta}/N, m^{\gamma})$ is isomorphic to $(L_{\beta}/N_{\beta}, m^{\gamma})$ and that the latter is isomorphic to $(L(m), m)$. Thus, if

$$S_{\beta}=\{E: E\in S,\ [E]\theta\in L_{\beta}/N\},$$

then S_{β} is a σ-subalgebra of S, and $(S_{\beta}(\mu)\ \mu)$ is isomorphic to $(L(m), m)$. Hence, by Theorem 9, there exists a family $\{X_{\beta r}: r\in I\}$ of measurable subsets of X, such that:

(i) $\mu(X_{\beta r})=0$, for all r in I,

(ii) $X_{\beta r}\,\|\,X_{\beta s}$, if $r\neq s$,

(iii) to each E in S_{β} there corresponds an E_0 in L such that

$$E\sim\bigcup\{X_{\beta r}: r\in E_0\},\ \text{and}\ \mu(E)=m(E_0).$$

Let E be the class of all finite intersections of elements of $\bigcup_{\beta<\gamma} L_{\beta}$, and let R be the class of all finite disjoint unions of sets belonging to E. We note that R is an algebra of sets and that the σ-algebra generated by R is L^{γ}.

Consider an arbitrary element E of S. Let E_0 be a member of $[E]\theta$, and let ε be a specified positive number. By a well-known approximation theorem [2; 56], there exists a set G_0 in R such that $m^{\gamma}(E_0\,\triangle\, G_0)<\varepsilon$. Suppose that

$$G_0=\bigcup_{j=1}^{m}\bigcap_{k=1}^{n_j} F_{0jk},$$

where F_{0jk} belongs to $L_{\beta_{jk}}$. Then, there exist in S sets F_{jk} satisfying $[F_{jk}]\theta=[F_{0jk}]$ and

$$F_{jk}=\bigcup\{X_{\beta_{jk}r}: r\in L_{jk},\ L_{jk}\in L\},$$

for $j=1, 2, \dots, m$; $k=1, 2, \dots, n_j$. Certainly

$$G=\bigcup_{j=1}^{m}\bigcap_{k=1}^{n_j} F_{jk}$$

is an element of $[G_0]\theta^{-1}$; hence,

$$\mu(E\,\triangle\, G)=m^{\gamma}(E_0\,\triangle\, G_0)<\varepsilon.$$

By virtue of the lemma, the proof is complete.

Since we may assume without loss of generality that $\bigcup\{X_{\beta r}: r\in I\}=X$, for each $\beta<\gamma$, the preceding theorem also can be couched in the language of measurable transformations.

COROLLARY 13. *Let (X, S, μ) be a totally finite, nonatomic and homogeneous measure space, with $\mu(X)=1$, and let γ be the least ordinal corresponding to the character of this measure space. There exists a measure preserving transformation T of (X, S, μ) into $(I^\gamma, \mathrm{L}^\gamma, m^\gamma)$ such that $T^{-1}(\mathrm{L}^\gamma)$ is equivalent to* S.

Proof. We recall that the points of I^γ are precisely the functions defined on $[1, \gamma)$ and taking values in I. For each such point τ, define

$$Y_\tau=\bigcap\{X_{\alpha,\tau(\alpha)}:\alpha<\gamma\}$$

(where $X_{\alpha r}$ has the same meaning as above). It is clear from the foregoing remarks that each Y_τ is a null subset of X and that $\{Y_\tau:\tau\in I^\gamma\}$ is a partition of X. The transformation T is defined on X in a natural way as follows:

$$Tx=\tau \quad \text{if} \quad x\in Y_\tau.$$

It is not difficult to show that

$$X_{\alpha r}=T^{-1}\{\tau:\tau(\alpha)=r\},$$

for each $\alpha<\gamma$ and each r in I, and from this fact the desired conclusion follows rapidly from the theorem.

4. Applications

In [5], Saks and Sierpinski proved the following remarkable approximation theorem.

THEOREM 14. *If f is a real-valued function defined on $I=[0, 1]$, then there exists a Lebesgue measurable function φ such that, whatever be the positive number ε, the inequality*

$$|f(x)-\varphi(x)|<\varepsilon$$

holds for all x in I, save for a set of inner measure zero.

We now consider functions defined on the set X of an arbitrary totally finite measure space (X, S, μ) and taking values in a metric space (Y, ϱ). We shall show that a theorem of Saks-Sierpinski type holds when (Y, ϱ) is separable. Indeed, the proof given by Saks and Sierpinski, when modified only slightly, is sufficient to establish the more general result. However, if the given metric space is not separable, we find that the Saks–Sierpinski theorem fails to hold in every case, save perhaps the most trivial one, (X, S, μ) totally atomic.

DEFINITION 15. *Let* (X, S) *be a measurable space, and let* (Y, ϱ) *be a metric space. A function* f *defined on* X *and taking values in* Y *is measurable* (S) *if* $f^{-1}(U)$ *belongs to* S, *whenever* U *is an open subset of* Y.

In all that follows, we assume that the measures with which we deal are complete. Thus, if μ is a finite measure defined on S, a σ-algebra of subsets of X, and if μ^* is the outer measure generated by μ,

$$\mu^*(A) = \inf\{\mu(E) : A \subset E,\ E \in \mathrm{S}\}, \quad \text{for all } A \subset X,$$

then the class of μ^*-measurable sets coincides with S.

THEOREM 16. *Let* (X, S, μ) *be a totally finite measure space, and let* (Y, ϱ) *be a separable metric space. If* f *is an arbitrary function defined on* X *and taking values in* Y, *then there exists a function* g *defined on* X, *taking values in* Y *and measurable* (S) *such that, whatever be the positive number* ε, *the inequality*

$$\varrho(f(x),\ g(x)) < \varepsilon$$

holds on a set having outer measure equal to the measure of X.

Proof. We give a somewhat abbreviated argument. The omitted details can be supplied easily with the aid of [5]. Without loss of generality, we may assume that $\mu(X) = 1$.

LEMMA A. *If* E *is a subset of* X *and if* f *assumes at most a denumerable number of different values on* E, *then there exist a measurable* (S) *function* g *and a set* H *contained in* E *such that* $\mu^*(H) = \mu^*(E)$ *and* $f(x) = g(x)$, *for all* x *in* H.

LEMMA B. *If* E *is a subset of* X *and if* ε *is a positive number, then there exist a measurable* (S) *function* g *and a set* H *contained in* E *such that* $\mu^*(H) = \mu^*(E)$ *and* $\varrho(f(x), g(x)) < \varepsilon$, for all x in H.

Proof of Lemma B. Let $\{y_n : n = 1, 2, \ldots\}$, be a denumerable dense subset of Y. For each x in E, let $n(x)$ be the least integer n for which $\varrho(f(x), y_n) < \varepsilon$, and let $h(x) = y_{n(x)}$, for all x in E. By Lemma A, there exist a measurable (S) function g and a set H contained in E, such that $\mu^*(H) = \mu^*(E)$ and $h(x) = g(x)$, for all x in H. Evidently, the inequality $\varrho(f(x), g(x)) < \varepsilon$ is valid for all x in H.

We now proceed with the proof of the theorem. Let H_1 be a subset of X, and let g_1 be a measurable function such that $\mu^*(H_1) = \mu^*(X) = 1$ and $\varrho(f(x), g_1(x)) < 2^{-1}$, for all x in H_1. Now let n be a natural number greater than 1, and suppose that the set H_{n-1} and the function g_{n-1} have been defined. Let H_n be a subset of H_{n-1}, and let g_n be a measurable function such that $\mu^*(H_n)=1$ and $\varrho(f(x), g_n(x))<2^{-n}$, for all x in H_n. The sets H_n and the functions g_n are thus defined for all natural numbers n,

$$\mu^*(H_n)=1,\ H_n \supset H_{n+1}, \quad n=1, 2, \ldots,$$

and
$$\varrho(f(x), g_n(x))<2^{-n}, \text{ for all } x \text{ in } H_n, \quad n=1, 2, \ldots . \tag{*}$$

Since each g_n is measurable and since ϱ is a continuous mapping of $Y\times Y$ into the reals the sets

$$P_n=\{x:\varrho(g_{n+1}(x), g_n(x))<2^{-n+1}\}$$

are measurable. Evidently P_n contains H_{n+1}, whence $\bigcap_{k=1}^{n} P_k$ contains H_{n+1}, and thus $\mu(\bigcap_{k=1}^{n} P_k)=1$. Let $P=\bigcap_{n=1}^{\infty} P_n$. Then, P is measurable, $\mu(P)=1$ and $\varrho(g_{n+1}(x), g_n(x))<2^{-n+1}$, for all x in P. Hence, the sequence $\{g_n\}$ converges uniformly on P to a measurable function g. We extend the domain of definition of g to all of X by setting $g(x)=a$, for all x in $X-P$, where a is an arbitrary element of Y. The function thus defined is measurable, because P is a measurable set. It follows without difficulty that

$$\varrho(g(x), g_n(x))\leqslant 2^{-n+2}, \quad \text{for all } x \text{ in } P;$$

thus, in view of (*),

$$\varrho(f(x), g(x))<2^{-n+3}, \quad \text{for all } x \text{ in } P\bigcap H_n.$$

Now P is measurable; thus,

$$\mu^*(H_n)=\mu^*(H_n\cap P)+\mu^*(H_n-P).$$

Since $\mu(P)=1$, H_n-P is a null set, and it follows that $\mu^*(H_n\cap P)=1$.

Finally, if N is chosen so large that $2^{-N+3}<\varepsilon$ and if $H=H_N\cap P$, then $\mu^*(H)=1$ and $\varrho(f(x), g(x))<\varepsilon$, for all x in H.

To facilitate our further study of the approximation problem, we first recall the following lemma (see, for example, [7]).

LEMMA 17. *Let (Y, ϱ) be a nonseparable metric space. There exist a positive number δ and a nondenumerable subset D of Y, such that $\varrho(d_1, d_2)>\delta$, whenever d_1 and d_2 are distinct points of D.*

Definition 18. *We call a subset E of Y a δ-set if $\varrho(e_1, e_2) > \delta$, whenever e_1 and e_2 are distinct elements of E.*

Theorem 19. *Let (X, S, μ) be a totally finite, nonatomic and homogeneous measure space, with $\mu(X) > 0$, and let (Y, ϱ) be a nonseparable metric space. There exists a function defined on X and taking values in Y that cannot be approximated by a measurable function in the sense of Saks and Sierpinski.*

Proof. Without loss of generality we may assume that $\mu(X) = 1$. According to Lemma 17, there exists a positive number δ such that Y contains a nondenumerable 2δ-set. Thus, assuming the continuum hypothesis, there is contained in Y a 2δ-set D of potency c, say $D = \{y_r : r \in I = [0, 1]\}$. Let $\{X_r : r \in I\}$ be a family of null sets such that $X = \bigcup \{X_r : r \in I\}$ and $X_r \| X_s$, if $r \neq s$ (by virtue of Theorems 9 and 12, such a family always exists), and define the function f on X as follows:

$$f(x) = y_r, \text{ if } x \in X_r.$$

Suppose that g is a function on X to Y satisfying the inequality

$$\varrho(f(x), g(x)) < \delta$$

for all points x lying in a set P of outer measure one. We shall show that g is necessarily nonmeasurable.

Let $T = S \cap P$, and let ν be the measure defined on T in the following manner: if $F = E \cap P$ with E in S, then $\nu(F) = \mu(E)$. If ν^* is the outer measure generated by ν, then for every subset A of P,

$$\begin{aligned} \nu^*(A) &= \inf \{\nu(F) : A \subset F, F \in T\} \\ &= \inf \{\mu(E) : A \subset E, E \in S\} = \mu^*(A). \end{aligned}$$

Since ν is a complete measure on T, the class of all ν^*-measurable sets coincides with T. If ν_* is the inner measure engendered by ν, then for every subset A of P,

$$\begin{aligned} \nu_*(A) &= \sup \{\nu(F) : A \supset F, F \in T\} \\ &= 1 - \nu^*(P - A). \end{aligned}$$

Since ν is totally finite, the ν^*-measurable sets are precisely those sets A for which $\nu^*(A) = \nu_*(A)$.

We assert the existence of a set A_0 contained in I for which the following inequality holds:

$$\mu^*(P \cap \bigcup\{X_r : r \in A_0\}) + \mu^*(P \cap \bigcup\{X_r : r \in I - A_0\}) > 1. \tag{1}$$

Were there no such subset of I, we should have

$$\mu^*(P \cap \bigcup\{X_r : r \in B_0\}) + \mu^*(P \cap \bigcup\{X_r : r \in I - B_0\}) = 1,$$

for every B_0 contained in I. Then, for all such B_0, the following equality would obtain:

$$\nu^*(P \cap \bigcup\{X_r : r \in B_0\}) = \nu_*(P \cap \bigcup\{X_r : r \in B_0\}),$$

and, as a result, T would contain all sets of the form $P \cap \bigcup\{X_r : r \in B_0\}$. Consequently, the set function λ defined on Σ, the class of all subsets of $I_0 = \{r : P \cap X_r \neq \emptyset\}$, by means of the equation,

$$\lambda(B_0) = \nu(P \cap \bigcup\{X_r : r \in B_0\}), \quad \text{for all } B_0 \text{ in } \Sigma,$$

would be a nontrivial, countably additive, totally finite and nonatomic measure on Σ. According to the theorem of Banach and Kuratowski [1], this is impossible; hence, (1) most hold for at least one subset A_0 of I.

For such an A_0, let $U_1 = \bigcup\{S(y_r, \delta) : r \in A_0\}$, and let $U_2 = \bigcup\{S(y_r, \delta) : r \in I - A_0\}$, where $S(y, \eta) = \{z : \varrho(y, z) < \eta\}$. In view of the fact that D is a 2δ-set, the open sets U_1 and U_2 are disjoint; hence, $g^{-1}(U_1)$ and $g^{-1}(U_2)$ are also disjoint. Moreover, $P \cap \bigcup\{X_r : r \in A_0\}$ is a subset of $g^{-1}(U_1)$, and $P \cap \bigcup\{X_r : r \in I - A_0\}$ is a subset of $g^{-1}(U_2)$. Thus,

$$\mu^*(g^{-1}(U_1)) + \mu^*(g^{-1}(U_2)) > 1.$$

But this inequality shows that at least one of the sets $g^{-1}(U_1)$, $g^{-1}(U_2)$, is non-measurable; hence, g is a nonmeasurable function.

Since the above demonstration shows that

$$\mu^*(\{x : \varrho(f(x), g(x)) < \varepsilon\}) < 1,$$

for every ε not exceeding δ and every measurable function g, the theorem is proved.

Now let (X, S, μ) be a totally finite and not totally atomic measure space. Again we suppose that $\mu(X) = 1$. By Theorem 3, we are able to decompose (X, S, μ) into a countable number of homogeneous measure spaces $(X_n, \mathrm{S}_n, \mu_n)$ that are related to (X, S, μ) by these conditions:

(i) each X_n is a measurable subset of X,

(ii) $X_n \| X_m$, if $n \neq m$,

(iii) $\mathrm{S} = \{\bigcup_{n=1}^{\infty} E_n : E_n \in \mathrm{S}_n\}$,

(iv) $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E \cap X_n)$, for each E in S.

Since each of the component measure spaces is homogeneous, either $[X_n]$ is an atom or (X_n, S_n, μ_n) is atom free. We denote by Z the union of all those sets X_n for which $[X_n]$ is an atom, and we let $W = X - Z$. Then $\mu(W) = w > 0$, for (X, S, μ) is not totally atomic. Since W and Z are disjoint measurable sets and $W \cup Z = X$, it is clear that

$$\mu^*(A) = \mu^*(A \cap Z) + \mu^*(A \cap W),$$

for all subsets A of X; hence, if $\mu^*(P) = 1$, then $\mu^*(P \cap W) = w$, We note the existence of a continuum of nonintersecting μ null sets $\{W_r : 0 < r \leqslant 1\}$ such that $W = \bigcup\{W_r : 0 < r \leqslant 1\}$. This follows from the fact that W is the union of an at most denumerable family of sets each member of which has this property. Therefore, by the same argument as before, we see that it is possible to split $P \cap W$ into disjoint subsets P_1 and Q_1 such that $\mu^*(P_1) + \mu^*(Q_1) > w$. Writing $Q_2 = (P \cap Z) \cup P_1$, we have $P = Q_1 \cup Q_2$, $Q_1 \| Q_2$ and $\mu^*(Q_1) + \mu^*(Q_2) > 1$.

Let (Y, ϱ) be a nonseparable metric space, let $D = \{Y_r : r \in I\}$ be a 2δ-set $(\delta > 0)$ contained in Y, and let the function f be defined on X as follows:

$$f(x) = \begin{cases} y_r, & \text{if } x \in W_r; \\ y_0, & \text{if } x \in Z. \end{cases}$$

By means of an argument completely parallel to the one given in Theorem 19, it can be deduced easily that f has no Saks–Sierpinski approximant. Taking account of Theorem 16, we have proved the following general theorem.

THEOREM 20. *Let (X, S, μ) be a totally finite and not totally atomic measure space, and let (Y, ϱ) be a metric space. In order that every function defined on X and taking values in Y have a Saks–Sierpinski approximant, it is both necessary and sufficient that (Y, ϱ) be separable.*

The following elementary examples show that no general statement can be made in the totally atomic case.

Example 21. Let $X = I = [0, 1]$ let $S = \{\emptyset, X\}$ and let μ be the measure satisfying $\mu(\emptyset) = 0$, $\mu(X) = 1$. Then, every nonempty subset of X has outer measure one, and every function defined on X and taking values in an arbitrary metric space Y is trivially Saks–Sierpinski approximable.

Example 22. Let $X = I$, let S be the σ-algebra generated by the denumerable subsets of X, and let $\mu(E)$ be 0 or 1 according as E is countable or $X - E$ is countable. Let (Y, ϱ) be a nonseparable metric space, and let $D = \{Y_r : r \in I\}$ be a 2δ-set $(\delta > 0)$ contained in Y. The function f, where

$$f(x) = y_x, \quad \text{for all } x \text{ in } X,$$

is not Saks-Sierpinski approximable. This follows from the fact that every set of outer measure one can be written as the union of two disjoint sets each having outer measure one.

We conclude the discourse with a remark suggested by the proof of Theorem 19.

THEOREM 23. *Let (X, S, μ) be a totally finite, nonatomic and separable measure space. Suppose that μ is a complete measure, and let μ^* be the outer measure engendered by μ. If E is a set of positive outer measure, then there exist (nonmeasurable) subsets of E, E_1 and E_2, such that $E_1 \| E_2$, $E = E_1 \cup E_2$ and $\mu^*(E_1) = \mu^*(E_2) = \mu^*(E)$.*

Proof. Suppose first that E is an element of S. Without loss of generality, we may assume that $\mu(E) = 1$. The proof of Theorem 19 assures us of the existence of sets E_{11}^* and E_{21}^* satisfying the conditions:

$$E_{11}^* \| E_{21}^*, \; E_{11}^* \bigcup {}_{21}^* = E \text{ and } \mu^*(E_{11}^*) + \mu^*(E_{12}^*) > 1.$$

Let F_{11} and F_{21} be measurable kernels of E_{11}^* and E_{21}^*, and let $F_1 = F_{11} \cup F_{21}$. Since

$$\mu^*(E_{11}^*) + \mu^*(E_{21}^*) \leqslant \mu^*(E - F_{21}) + \mu^*(E - F_{11}) = 2 - \mu(F_1),$$

it follows that $\mu(F_1) = a_1 < 1$. Since

$$E_{11}^* - F_{11} \subset E - F_1,$$

it follows that $\mu^*(E_{11}^* - F_{11}) \leqslant 1 - a_1$; moreover, it is easy to see that the equality holds. For if $\mu^*(E_{11}^* - F_{11})$ were less than $1 - a_1$, there would exist a measurable set G containing $E_{11}^* - F_{11}$ and satisfying the inequality $\mu(G) < 1 - a_1$. But then $E - G$ would be a measurable subset of $E_{21}^* \cup F_{11}$ having measure greater than a_1, and, thus, $(E - G) - F_1$ would be a nonnull measurable subset of $E_{21}^* - F_{21}$, contrary to the fact that F_{21} is a measurable kernel of E_{21}^*. In similar fashion, we find that $\mu^*(E_{21}^* - F_{21}) = 1 - a_1$.

If $a_1 = 0$, then E_{11}^* and E_{21}^* are the desired sets. If $a_1 > 0$, then let

$$E_{11} = E_{11}^* - F_1, \; E_{21} = E_{21}^* - F_1.$$

Applying the foregoing technique to F_1, we obtain disjoint sets A_{12} and A_{22} such that

$$A_{12} \cup A_{22} = F_1 - F_2 \text{ and } \mu^*(A_{12}) = \mu^*(A_{22}) = a_1 - a_2,$$

where F_2 is measurable subset of F_1 with $\mu(F_2) = a_2 < a_1$. Let

$$E_{12} = E_{11} \cup A_{12}, \; E_{22} = E_{21} \cup A_{22}.$$

Then $$E_{12} \| E_{22},\ E_{12} \cup E_{22} = E - F_2 \text{ and } \mu^*(E_{12}) = \mu^*(E_{22}) = 1 - a_2.$$

If $a_2 = 0$, then $E_{12} \cup F_2$ and E_{22} have the desired properties. If $a_2 > 0$, then we repeat the process. We proceed by transfinite induction to construct for each ordinal $\alpha < \Omega$ (the first uncountable ordinal) a measurable set F_α and disjoint sets $E_{1\alpha}$, $E_{2\alpha}$ such that

$$E_{1\alpha} \supset E_{1\gamma},\ E_{2\alpha} \supset E_{2\gamma} \quad \text{and} \quad F_\alpha \subset F_\gamma \quad \text{if } \gamma < \alpha;$$
$$\mu(F_\alpha) = a_\alpha < a_\gamma = \mu(F_\gamma), \text{ if } \gamma < \alpha \quad \text{and} \quad a_\gamma > 0;$$
$$E_{1\alpha} \cup E_{2\alpha} = E - F_\alpha \quad \text{and} \quad \mu^*(E_{1\alpha}) = \mu^*(E_{2\alpha}) = 1 - a_\alpha.$$

Suppose that the sets $E_{1\alpha}$, $E_{2\alpha}$ and F_α have been constructed for each $\alpha < \beta$ $(< \Omega)$. If β is not a limit ordinal and if $a_{\beta-1} = 0$ we may take

$$F_\beta = F_{\beta-1},\ E_{1\beta} = E_{1,\beta-1},\ E_{2\beta} = E_{2,\beta-1},$$

while if $a_{\beta-1} > 0$, we find in the same manner as before, a measurable subset F_β of $F_{\beta-1}$, with $\mu(F_\beta) = a_\beta < a_{\beta-1}$, and disjoint sets $A_{1\beta}$, $A_{2\beta}$ such that

$$A_{1\beta} \cup A_{2\beta} = F_{\beta-1} - F_\beta \quad \text{and} \quad \mu^*(A_{1\beta}) = \mu^*(A_{2\beta}) = a_{\beta-1} - a_\beta.$$

In the latter situation, we define

$$E_{1\beta} = E_{1,\beta-1} \cup A_{1\beta},\ E_{2\beta} = E_{2,\beta-1} \cup A_{2\beta}.$$

Clearly, $$E_{1\beta} \| E_{2\beta},\ E_{1\beta} \cup E_{2\beta} = E - F_\beta \quad \text{and} \quad \mu^*(E_{1\beta}) = \mu^*(E_{2\beta}) = 1 - a_\beta.$$

If β is a limit ordinal, let $E^*_{1\beta} = \bigcup_{\alpha<\beta} E_{1\alpha}$, let $E^*_{2\beta} = \bigcup_{\alpha<\beta} E_{2\alpha}$ and let $F^*_\beta = \bigcap_{\alpha<\beta} F_\alpha$. Certainly F^*_β is measurable and

$$a^*_\beta = \mu(F^*_\beta) = \inf\{\mu(F_\alpha) : \alpha < \beta\}.$$

It is also clear that the inclusions

$$E^*_{1\beta} \supset E_{1\alpha} \quad \text{and} \quad E^*_{2\beta} \supset E_{2\alpha}$$

hold for all $\alpha < \beta$ and that $E^*_{1\beta} \cup E^*_{2\beta} = E - F^*_\beta$. Since $E - F^*_\beta \supset E^*_{1\beta} \supset E_{1\alpha}$, for all $\alpha < \beta$, we have

$$1 - a^*_\beta \geqslant \mu^*(E^*_{1\beta}) \geqslant 1 - a_\alpha \quad \text{for all } \alpha < \beta,$$

and thus $\mu^*(E^*_{1\beta}) = 1 - a^*_\beta$. By the same argument we find that $\mu^*(E^*_{2\beta}) = 1 - a^*_\beta$. If $a^*_\beta = 0$, we set

$$E_{1\beta} = E^*_{1\beta},\ E_{2\beta} = E^*_{2\beta},\ F_\beta = F^*_\beta.$$

If $a^*_\beta > 0$, we apply the basic technique once again in order to obtain a measurable subset F_β of F_β^*, satisfying $\mu(F_\beta)=a_\beta<a_\beta^*$, and disjoint sets $A_{1\beta}$, $A_{2\beta}$ satisfying the conditions

$$A_{1\beta}\,||\,A_{2\beta},\ A_{1\beta}\cup A_{2\beta}=F_\beta^*-F_\beta,\ \mu^*(A_{1\beta})=\mu^*(A_{2\beta})=a_\beta^*-a_\beta.$$

Letting

$$E_{1\beta}=E_{1\beta}^*\cup A_{1\beta},\ E_{2\beta}=E_{2\beta}^*\cup A_{2\beta},$$

we have $\quad E_{1\beta}\,||\,E_{2\beta},\ E_{1\beta}\cup E_{2\beta}=E-F_\beta\quad$ and $\quad\mu^*(E_{1\beta})=\mu^*(E_{2\beta})=1-a_\beta.$

By the principle of transfinite induction, the sets $E_{1\alpha}$, $E_{2\alpha}$, F_α, having the properties prescribed above, are thus defined for all ordinal numbers $\alpha<\Omega$.

Now it is clear that there must exist for each natural number n, an ordinal β_n such that $a_{\beta_n}<n^{-1}$; for otherwise we should have $a_\beta\geqslant n^{-1}$ for all $\beta<\Omega$, and this is impossible. If $\beta=\sup_n\beta_n$, then $a_\beta=0$; thus, $\{\alpha:a_\alpha=0\}$ is a nonempty set. Let $\gamma=\inf\{\alpha:a_\alpha=0\}$. Then $E_{1\gamma}$ and $E_{2\gamma}$ have outer measure 1 and are disjoint, and $N=E-(E_{1\gamma}\cup E_{2\gamma})$ is a null set. The proof of the theorem is completed in the measurable case by taking, for example,

$$E_1=E_{1\gamma}\cup N\ \text{ and }\ E_2=E_{2\gamma}.$$

In the general case, let F be a measurable cover of E, and let ν be the measure (well) defined on $\mathrm{S}\cap E$ as follows (cf. the proof of Theorem 19): if G is an element of S, then $\nu(G\cap E)=\mu(G\cap F)$. As the argument given above shows, when applied to ν, there exist sets E_1 and E_2 such that

$$E_1\,||\,E_2,\ E=E_1\cup E_2$$

and

$$\nu^*(E_1)=\nu^*(E_2)=\nu(E)=\mu^*(E).$$

Since μ^* and ν^* agree on the subsets of E, the theorem is proved.

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