QUASICONFORMAL REFLECTIONS

BY

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Let L be a Jordan curve on the Riemann sphere, and denote its completmentary components by Ω, Ω^*. Suppose that there exists a sense-reversing quasiconformal mapping λ of the sphere onto itselfs which maps Ω on Ω^* and keeps every point on L fixed. Such mappings are called quasiconformal reflections. Our purpose is to study curves L which permit quasiconformal reflections.

Let U denote the upper and U^* the lower halfplane. Consider a conformal mapping f of U on Ω and a conformal mapping f^* of U^* on Ω^*. Evidently, $f^{*-1}\lambda f$ defines a quasiconformal mapping of U on U^* which induces a monotone mapping $h = f^{*-1}f$ of the real axis on itself. It is not quite unique, for we may replace f by fS and f^* by f^*S^* where S and S^* are linear transformations with real coefficients and possitive determinant. This replaces h by $S^{*-1}hS$ which we shall say is equivalent to h. Observe that h, or rather its equivalence class, does not depend on λ. It is also unchanged if we replace the triple (Ω, L, Ω^*) by a conformally equivalent triple $(T\Omega, TL, T\Omega^*)$ where T is a linear transformation.

The mapping f of U has a quasiconformal extension to the whole plane, namely by the mapping with values $\lambda f(\bar{z})$ for $z \in U^*$. It is known that quasiconformal mappings carry nullsets into nullsets. Therefore L has necessarily zero area.

From this we may deduce that h determines Ω uniquely up to conformal equivalence. In fact, let f_1, f_1^* be another pair of conformal mappings on complementary regions, and suppose that $f_1^{*-1}f_1 = f^{*-1}f$ on the real axis. For a moment, let us write F for the mapping given by $f(z)$ in U and by $\lambda f(\bar{z})$ in U^*, and let F_1 have the corresponding meaning. The mapping $H = F_1^{-1}f_1^* f^{*-1}F$ is defined in U^* and reduces to the identity on the real axis. We extend it to the whole plane by setting $H(z) = z$

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in U. Then $F_1 H F^{-1}$ is a quasiconformal mapping. It reduces to $f_1 f^{-1}$ in Ω and to $f_1^* f^{*-1}$ in Ω^*. It is thus conformal, except perhaps on L. But a quasiconformal mapping which is conformal almost everywhere is conformal. Hence $f_1 = Tf$ where T is a linear transformation.

What are the properties of h? A necessary condition is that h can be extended to a quasiconformal mapping of U on U^*, namely to $f^{*-1}\lambda f$. This condition is also sufficient. To prove it, let g be a quasiconformal mapping of U on U^* with boundary values h. The function $g^*(z) = g(\bar{z})$, defined in U^*, has weak derivatives which satisfy an equation

$$g^*_{\bar{z}} = \mu g^*_z$$

with $|\mu| \leqslant k < 1$ (k constant). Set $\mu = 0$ in U. Consider the equation

$$F_{\bar{z}} = \mu F_z$$

for the extended μ. An important theorem (see [1]), sometimes referred to as the generalized Riemann mapping theorem, asserts the existence of a solution F which is a homeomorphic mapping of the sphere. Because z is a solution in U and g^* a solution in U^* it is possible to write $F = f$ in U, $F = f^* g^*$ in U^*, where f and f^* are conformal mappings. Clearly, $\Omega = f(U)$ and $\Omega^* = f^*(U^*)$ are quasiconformal reflections of each other.

To sum up, we have established a correspondence between equivalence classes of boundary correspondences h, conformal mappings f, and curves L which permit a quasiconformal reflection. It is a natural program to try to characterize the possible h, f and L in a more direct way. For boundary correspondences h this problem has been solved; we shall have occasion to recall the solution.

In Part I we solve the corresponding problem for L. It turns out that the curves which permit a quasiconformal reflection can be characterized by a surprisingly simple geometric property. (Partial results in this direction have been obtained by M. Tienari whose paper [7] came to my attention only when this article was already written.)

We have been less successful with the mappings f, but in Part II we show, at any rate, that the mappings f form an open set. To understand the meaning of this, we observe that the mappings equivalent to f are of the form TfS. To account for T we replace f by its Schwarzian derivative $\varphi = \{f, z\}$. The Schwarzian of fS is $\varphi(S)S'^2$, and to eliminate S it is indicated to consider φdz^2 in its role of quadratic differential.

If f is schlicht in U, Nehari [6] has shown that $|\varphi| y^2 \leqslant \frac{3}{2}$. We take the least upper bound of $|\varphi| y^2$ to be a norm of φ. In the linear space of quadratic differentials with finite norm, let Δ be the set of all φ whose corresponding f is schlicht and has a quasiconformal extension. We are going to show that Δ is an open set. For the significance of this result in the theory of Teichmüller spaces we refer to the companion article of L. Bers [4] in the next issue of this journal.

Part I

1. In 1956 A. Beurling and the author derived a neccessary and sufficient condition for a boundary h to be the restriction of a quasiconformal mapping of U on itself (or on its reflection U^*). This work is an essential preliminary for what follows.

We recall the main result. Without loss of generality it may be assumed that $h(\infty)=\infty$. Then h admits a quasiconformal extension if and only if it satisfies a ϱ-condition, namely an inequality

$$\varrho^{-1} \leqslant \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqslant \varrho, \tag{1}$$

which is to be fulfilled for all real x, t and with a constant $\varrho \neq 0, \infty$. More precisely, if h has a K-quasiconformal extension, then (1) holds with a $\varrho(K)$ that depends only on K, and if (1) holds, then h has a $K(\varrho)$-quasiconformal extension.

The necessity follows from the simple observation that the quadruple $(x-t, x, x+t, \infty)$ with cross-ratio 1 must be mapped on a quadruple with bounded cross-ratio. The sufficiency requires an explicit construction. We set $w(z)=u+iv$ with

$$\left.\begin{aligned} u(z) &= \int_0^1 [h(x+ty)+h(x-ty)]\,dt, \\ v(z) &= \int_0^1 [h(x+ty)-h(x-ty)]\,dt. \end{aligned}\right\} \tag{2}$$

It is proved in [2] that $w(z)$ is $K(\varrho)$-quasiconformal.

I am indebted to Beurling for the very important observation that the mapping (2) is also quasi-isometric, in the sense that corresponding noneuclidean elements of length have a bounded ratio. This condition can be expressed by

$$\left.\begin{aligned} |w_z| &\leqslant C(\varrho)\frac{v}{y}, \\ |w_{\bar z}| &\geqslant C(\varrho)^{-1}\frac{v}{y}, \end{aligned}\right\} \tag{3}$$

where $C(\varrho)$ depends only on ϱ. The proof is an immediate verification based on the estimate given in Lemma 6.5 of the cited paper.

2. Let L be a Jordan curve through ∞ which admits a quasiconformal reflection. The complementary regions determined by L are denoted by Ω, Ω^*, and the reflection is written as $z \to z^*$. We assume that the reflection is K-quasiconformal.

Constants which depend only on K will be denoted by $C(K)$, with or without subscripts. In different connections $C(K)$ may have different values. We emphasize that $C(K)$ is not allowed to depend on L.

The shortest distance from a point z to L will be denoted by $\delta(z)$.

LEMMA 1. *The following estimates hold for all z in the plane and all z_0 on L:*

(a) $$C(K)^{-1} \leqslant \left|\frac{z^* - z_0}{z - z_0}\right| \leqslant C(K)$$

(b) $$\frac{|z - z^*|}{\delta(z)} \leqslant C(K)$$

(c) $$C(K)^{-1} \leqslant \frac{\delta(z^*)}{\delta(z)} \leqslant C(K).$$

Proof. If the cross-ratio of a quadruple has absolute value $\leqslant 1$, then the cross-ratio of the image points under a K-quasiconformal mapping has an absolute value $\leqslant C(K)$. This assertion is contained in [1], Lemma 16. It is a rather elementary result.

If $|z^* - z_0| \leqslant |z - z_0|$ we can apply the above remark to (z^*, z, z_0, ∞) and conclude that $|z - z_0| \leqslant C(K)|z^* - z_0|$. Symmetrically, $|z - z_0| \leqslant |z^* - z_0|$ implies $|z^* - z_0| \leqslant C(K)|z - z_0|$. In all circumstances (a) follows.

From (a) we obtain

$$|z - z^*| \leqslant (C(K) + 1)|z - z_0| = C_1(K)|z - z_0|$$

and (b) follows when $|z - z_0| = \delta(z)$. Since $\delta(z^*) \leqslant |z - z^*|$ the second inequality (c) follows from (b), and the first is true by symmetry.

2. We introduce now the noneuclidean metrics $ds = \varrho(z)|dz|$ in Ω and Ω^*. Explicitly, if $z = z(\zeta)$ is a conformal map of $|\zeta| < 1$ on Ω we set

$$\varrho(z)|dz| = (1 - |\zeta|^2)^{-1}|d\zeta|.$$

The classical estimates

$$\delta(z) \leqslant \varrho(z)^{-1} \leqslant 4\delta(z) \tag{4}$$

follow by use of Schwarz' lemma and Koebe's one-quarter theorem.

LEMMA 2. *If L passes through ∞ and permits a K-quasiconformal reflection, then it also permits a $C(K)$-quasiconformal reflection with the additional property that corresponding euclidean line elements satisfy*

$$C_1(K)^{-1}|dz| \leqslant |dz^*| \leqslant C_1(K)|dz|. \tag{5}$$

Proof. As shown in the introduction, the given K-quasiconformal reflection induces a K-quasiconformal mapping of U on U^* with a boundary correspondence h. This h must satisfy a $\varrho(K)$-condition of type (1). The Beurling-Ahlfors construction permits us to replace the mapping of U on U^* with a $C(K)$-quasiconformal mapping with the same boundary values, in such a way that it satisfies condition (3). It follows that the corresponding reflection about L is $C(K)$-quasiconformal and satisfies

$$C_1(K)^{-1}\varrho(z)|dz| \leqslant \varrho(z^*)|dz^*| \leqslant C_1(K)\varrho(z)|dz|.$$

Use of (4) and (c) leads to the desired inequality (5).

3. We are now ready to characterize the curves L in purely geometric form:

THEOREM 1. *A Jordan curve L through ∞ permits a quasiconformal reflection if and only if there exists a constant C such that*

$$\overline{P_1P_2} \leqslant C \cdot \overline{P_1P_3} \tag{6}$$

for any three points P_1, P_2, P_3 on L which follow each other in this order.

Again, there is a more precise statement to the effect that C depends only on the K of the reflection, and vice versa. If L does not pass through ∞ condition (6) must be replaced by

$$\overline{P_1P_2} : \overline{P_1P_3} \leqslant C(\overline{P_4P_2} : \overline{P_4P_3}),$$

where (P_1, P_3) separates (P_2, P_4).

4. *Proof of the necessity.* We follow the segment P_1P_3 from P_1 to its last intersection with the subarc $P_2P_1\infty$ of L, and from there to the first intersection P_3' with the arc $P_2P_3\infty$. If $P_1P_2 > P_1P_3$ it is geometrically evident that

$$\overline{P_1P_2} : \overline{P_1P_3} \leqslant \overline{P_1'P_2} : \overline{P_1'P_3'}.$$

Therefore we may assume from the beginning that $\overline{P_1P_3}$ has only its endpoints on L. For definiteness, we suppose that the inner points lie in Ω.

By Lemma 2 there exists a quasiconformal reflection which multiplies lengths at most by a factor $C(K)$. Hence P_1 and P_3 can be joined in Ω^* by an arc γ^* of length $\leqslant C(K)\cdot\overline{P_1P_3}$. The Jordan curve formed by $\overline{P_1P_3}$ and γ^* separates P_2 from ∞. Hence γ^* intersects the extension of $\overline{P_1P_2}$ over P_2 and we conclude that $\overline{P_1P_2}\leqslant$ length of $\gamma^*\leqslant C(K)\cdot\overline{P_1P_3}$.

5. *Proof of the sufficiency.* We shall use the notations

$$\alpha=\text{arc } P_2P_3, \quad \alpha'=\text{arc } P_1P_2,$$
$$\beta=\text{arc } P_1\infty. \quad \beta'=\text{arc } P_3\infty.$$

Denote by $d(\alpha,\beta)$ and $d^*(\alpha,\beta)$ the extremal distances of α and β with respect to Ω and Ω^* respectively. With similar notations for α',β' one has the relations

$$d(\alpha,\beta)\,d(\alpha',\beta')=d^*(\alpha,\beta)\,d^*(\alpha',\beta')=1.$$

In a conformal mapping of Ω on the halfplane U with ∞ corresponding to ∞, let P_1, P_2, P_3 be mapped on x_1, x_2, x_3. It is evident that $d(\alpha,\beta)=1$ if and only if $x_3-x_2=x_2-x_1$. Furthermore, the ratio $|x_3-x_2|:|x_2-x_1|$ is bounded away from 0 and ∞ if and only if this is true of $d(\alpha,\beta)$. Consequently, in order to prove that the boundary correspondence induced by L satisfies (1) it is sufficient to show that $d(\alpha,\beta)=1$ implies $K(C)^{-1}\leqslant d^*(\alpha,\beta)\leqslant K(C)$.

Two elementary estimates are needed. We show first that $d(\alpha,\beta)=1$ implies

$$\overline{P_1P_2}:\overline{P_2P_3}\leqslant C^2e^{2\pi}. \tag{7}$$

Indeed, it follows from (6) that the points of β are at distance $\geqslant C^{-1}\cdot\overline{P_1P_2}$ from P_2 while the points of α have distance $\leqslant C\cdot\overline{P_2P_3}$ from P_2. If (7) were not true, α and β would be separated by a circular annulus whose radii have the ratio $e^{2\pi}$. In such an annulus the extremal distance between the circles is 1, and the comparison principle for extremal lengths would yield $d(\alpha,\beta)>1$, contrary to hypothesis. Hence (7) must hold. If P_1 and P_3 are interchanged we have in the same way

$$\overline{P_2P_3}:\overline{P_1P_2}\leqslant C^2e^{2\pi}. \tag{8}$$

Consider points $Q_1\in\alpha$, $Q_2\in\beta$. By repeated application of (6)

$$\overline{Q_1Q_2}\geqslant C^{-1}\,\overline{Q_1P_1}\geqslant C^{-2}\,\overline{P_1P_2}$$

and with the help of (8) we conclude that the shortest distance between α and β is $\geqslant C^{-4}e^{-2\pi}\,\overline{P_2P_3}$. To simplify notations, write $d=\overline{P_2P_3}$, $M_1=Cd$, $M_2=C^{-4}e^{-2\pi}d$. Because of (6), all points on α are within distance M_1 from P_2.

We recall that the definition of extremal length implies

$$d^*(\alpha, \beta) \geqslant \frac{(\inf \int_\gamma \varrho \, |dz|)^2}{\iint_{\Omega^*} \varrho^2 \, dx \, dy},$$

where the infimum is with respect to all arcs γ that join α and β within Ω^*, and ϱ is any positive function for which the right-hand side has a meaning. We choose $\varrho = 1$ in a circular disk with center P_2 and radius $M_1 + M_2$, $\varrho = 0$ outside of that disk. Then $\int_\gamma \varrho \, |dz| \geqslant M_2$ for all curves γ. Indeed, this is so whether γ stays within the disk or contains a point on its circumference. We conclude that

$$d^*(\alpha, \beta) \geqslant \frac{1}{\pi}\left(\frac{M_2}{M_1 + M_2}\right)^2 = \pi^{-1}(1 + C^5 e^{2\pi})^{-2}.$$

The same inequality, applied to α', β', yields an upper bound for $d^*(\alpha, \beta)$, and our proof of Theorem 1 is complete.

Part II

1. In the introduction we saw that the boundary correspondences h give rise to conformal mappings f, and with these we associated their Schwarzian derivatives $\varphi = \{f, z\}$. The set of all such φ was denoted by Δ. We formulate a precise definition:

The set Δ consists of all functions φ, holomorphic in U, such that the equation $\{f, z\} = \varphi$ has a solution f which can be extended to a schlicht quasiconformal mapping of the whole plane.

Our purpose is to prove:

THEOREM 2. *Δ is an open subset of the Banach space of holomorphic functions with norm* $\|\varphi\| = \sup |\varphi(z)| y^2$.

We know already that all $\varphi \in \Delta$ have norm $\leqslant \frac{3}{2}$. It will follow that the norms are in fact strictly less than $\frac{3}{2}$.

2. It is a known result that Δ contains a neighborhood of the origin ([3], [5]). As an illustration of the method we shall follow it is nevertheless useful to include a proof.

LEMMA 3. *Δ contains all functions φ with* $\|\varphi\| < \frac{1}{2}$.

Proof. Let η_1 and η_2 be linearly independent solutions of the differential equation

$$\eta'' = -\tfrac{1}{2}\varphi\eta. \tag{9}$$

normalized by $\eta_1'\eta_2 - \eta_2'\eta_1 = 1$. It is well known that $f = \eta_1/\eta_2$ satisfies $\{f, z\} = \varphi$. Observe that f may be meromorphic with simple poles, and that $f' \neq 0$ at all other points!

It is to be shown that f is schlicht and has a quasiconformal extension. To construct the extension we form

$$F(z)=\frac{\eta_1(z)+(\bar{z}-z)\eta_1'(z)}{\eta_2(z)+(\bar{z}-z)\eta_2'(z)} \quad (z\in U). \tag{10}$$

Because $\eta_1'\eta_2-\eta_2'\eta_1=1$ the numerator and denominator cannot vanish simultaneously. If the denominator vanishes we set $F=\infty$, and local assertions about F will apply to $1/F$.

A simple computation which makes use of (9) gives

$$F_z/F_{\bar{z}}=\tfrac{1}{2}(z-\bar{z})^2\varphi(z).$$

Under the assumption $\|\varphi\|<\frac{1}{2}$ we conclude that F is quasiconformal and sense-reversing. The mapping $z\to F(\bar{z})$ is quasiconformal and sense-preserving in U^*.

Our intention is to show that

$$\hat{f}(z)=\begin{cases} f(z) \text{ in } U \\ F(\bar{z}) \text{ in } U^* \end{cases} \tag{11}$$

gives the desired extension. To see this it is sufficient to know that $\hat{f}$ can be extended to the real axis by continuity, that the extended function is locally schlicht at points of the real axis, and that it tends to a limit for $z\to\infty$. Indeed, $\hat{f}$ will then be locally schlicht everywhere, and by a familiar reasoning is must be globally schlicht.

The missing information is easy to supply under strong additional conditions. We suppose that φ is analytic on the real axis, including ∞, where φ shall have a zero of order $\geqslant 4$ (this means that the quadratic differential φdz^2 is regular at ∞). It is immediate that f and F agree on the real axis, and that they are real-analytic in the closed half-planes. It follows easily that $\hat{f}$ is locally schlicht. At ∞ the assumption implies that equation (9) has solutions whose power series expansions begin with 1 and z respectively. Hence

$$\eta_1=a_1z+b_1+O(|z|^{-1}),$$
$$\eta_2=a_2z+b_2+O(|z|^{-1})$$

with $a_1b_2-a_2b_1=1$. Substitution in (10) shows that

$$F(z)=\frac{a_1\bar{z}+b_1+O(|z|^{-1})}{a_2\bar{z}+b_2+O(|z|^{-1})}$$

and therefore f and F have the same limit a_1/a_2 as $z\to\infty$.

To prove the lemma without additional assumptions we use an approximation method. We can find a sequence of linear transformations S_n such that the closure of $S_n U$ is contained in U and $S_n z \to z$ for $n \to \infty$. Take $\varphi_n(z) = \varphi(S_n z)\ S_n'(z)^2$. It follows by Schwarz' lemma that $\|\varphi_n\| < \|\varphi\|$. Moreover, φ_n is analytic on the real axis and has at least a fourth order zero at ∞. Consequently, there exist quasiconformal mappings $\hat{f}_n$, holomorphic with $\{\hat{f}_n, z\} = \varphi_n$ in U, with uniformly bounded dilatation. A subsequence of the $\hat{f}_n$ converges to a limit function $\hat{f}$ which is itself schlicht and quasiconformal, and which satisfies $\{\hat{f}, z\} = \varphi$ in U. This completes the proof.

With suitable normalizations it is possible to arrange that $\hat{f}_n \to \hat{f}$, the mapping defined by (11).

3. The method of the preceding proof can be carried over to the general case, although with some significant modifications.

Suppose that $\varphi_0 \in \Delta$ and $\{f_0, z\} = \varphi_0$. We may assume that f_0 maps U on a region Ω whose boundary L passes through ∞, and we know that L admits a quasiconformal reflection $w \to w^* = \lambda(w)$. We choose λ in accordance with Lemma 2.

If $\|\varphi - \varphi_0\| < \varepsilon$ and $\{f, z\} = \varphi$ the identity

$$\{f, z\} = \{f, f_0\} f_0'^2 + \{f_0, z\}$$

yields

$$|\{f, f_0\}|\, |f_0'|^2 y^2 < \varepsilon.$$

The non-euclidean metric in Ω is given by

$$\varrho(w)\,|dw| = \frac{|dz|}{2y},$$

and if we write $\tilde{f} = f f_0^{-1}$ we obtain

$$|\{\tilde{f}, w\}| < 4\,\varepsilon\,\varrho(w)^2. \tag{12}$$

If ε is sufficiently small it is to be proved that $\tilde{f}$ is schlicht and has a quasiconformal extension.

We set $\tilde{\varphi} = \{\tilde{f}, w\}$ and $\tilde{f} = \eta_1/\eta_2$ where η_1, η_2 are normalized solutions of

$$\eta'' = -\tfrac{1}{2}\tilde{\varphi}\eta.$$

In close analogy with (10) we form

$$F(w) = \frac{\eta_1(w) + (w^* - w)\,\eta_1'(w)}{\eta_2(w) + (w^* - w)\,\eta_2'(w)},$$

where $w \in \Omega$ and $w^* = \lambda(w)$. Computation gives

$$\frac{F_w}{F_{\bar{w}}} = \frac{\lambda_w}{\lambda_{\bar{w}}} + \frac{\tilde{\varphi}(w-w^*)^2}{2\lambda_{\bar{w}}}. \tag{13}$$

Here $|\lambda_w/\lambda_{\bar{w}}| \leq k < 1$ because λ is quasiconformal. To estimate the second term we have first, by (12), Lemma 1 (b) and (4),

$$|\tilde{\varphi}|\,|w - w^*|^2 < 4\varepsilon C^2.$$

On the other hand, $|\lambda_{\bar{w}}|$ stays away from 0, for Lemma 2 gives

$$C^{-1}|dw| \leq |dw^*| \leq 2|\lambda_{\bar{w}}|\,|dw|.$$

We conclude that $|F_w/F_{\bar{w}}| \lessapprox k' < 1$ provided that ε is sufficiently small.

4. We wish to show that

$$\hat{f} = \begin{cases} \hat{f}(w) & \text{in } \Omega \\ F(w^*) & \text{in } \Omega^* \end{cases}$$

is schlicht and quasiconformal. Again, the proof is easy under strong assumptions. This time we assume that L is an analytic curve, that $\tilde{\varphi}$ is analytic on L and that it has a fourth order zero at ∞. It is clear that we can prove $\hat{f}$ to be a quasiconformal homeomorphism exactly as in the proof of Lemma 3.

To complete the proof, let $\zeta = \omega(w)$ be a conformal mapping of Ω on $|\zeta| < 1$. Let Ω_n be the part of Ω that corresponds to $|\zeta| < r_n$, L_n its boundary. Here $\{r_n\}$ is a sequence which converges to 1.

A quasiconformal reflection λ_n across L_n can be constructed as follows: If $r_n^2 < |\omega(w)| < r_n$ we define $\lambda_n(w)$ so that $\omega(w)$ and $\omega(\lambda_n(w))$ are mirror images with respect to $|\zeta| = r_n$. If $|\omega(w)| \leq r_n^2$ we find w_n so that $\omega(w_n) = r_n^{-2}\omega(w)$ and choose $\lambda_n(w) = \lambda(w_n)$. The definitions agree when $|\omega(w)| = r_n^2$, and L_n is kept fixed. The dilatation of λ_n is no greater than the maximum dilatation of λ.

After a harmless linear transformation which throws a point on L_n to ∞ the part of the theorem that has already been proved can be applied to Ω_n. It is to be observed that $\varrho_n \geq \varrho$ where $\varrho_n|dw|$ is the noneuclidean metric in Ω_n. Therefore $\tilde{\varphi}$ satisfies

$$|\tilde{\varphi}| < 4\,\varepsilon\varrho_n(w)^2$$

with the same ε as before. Hence there exists a quasiconformal mapping $\hat{f}_n$ of the whole plane which agrees with f on Ω_n and whose dilatation lies under a fixed bound. A subsequence of the $\hat{f}_n$ tends to a limit mapping $\hat{f}$ which is schlicht, quasiconformal, and equal to f in Ω. The theorem is proved.

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