SOME GEOMETRIC AND ANALYTIC PROPERTIES OF HOMOGENEOUS COMPLEX MANIFOLDS

PART II: DEFORMATION AND BUNDLE THEORY

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9. Deformation Theory; Part I

(i) The Infinitesimal Theory

Let Y be a compact complex manifold and suppose that on the differentiable manifold Y_d we are given a 1-parameter family of complex structures $Y_t(Y_0 = Y)$. If $U = \{U_i\}$ is a covering of Y by complex coordinate neighborhoods, with coordinates $(z_i^1, \ldots, z_i^n)$ in U_i, the structure on Y_t is given by transition functions

$$z_j^r(z_i, t) = f_{ij}^r(z_i^1, \ldots, z_i^n; t);$$

letting

$$\theta_{ij}^r = \left[\frac{df_{ij}^r}{dt}\right]_{t=0}$$

and

$$\theta_{ij} = (\theta_{ij}^1, \ldots, \theta_{ij}^n),$$

the deformation Y_t of Y is represented infinitesimally (or linearly) by the 1-cocycle $\theta_{ij} \in H^1(N(U), \Theta)$ ($N(U)$ = nerve of U). Further details concerning the relation between the variation of structure of Y and its parametrization by $H^1(Y, \Theta)$ are given in [19] and [20]; we shall be concerned with the special cases when Y = non-Kähler C-space or $Y = \hat{X} \times T^{2a}$ where $\hat{X}$ is a Kähler C-space. We remark that by corollary 1 to Theorem 2, the structure of $\hat{X}$ is infinitesimally rigid. By way of notation, we let $X = G/U$ be a non-Kähler C-space with fundamental fibering $T^{2a} \to X \to \hat{X}$, $\hat{X} = G/\hat{U}$ a Kähler C-space, and we set $\hat{X}^b = \hat{X} \times T^{2b}$; the manifolds $\hat{X}^b$ are the most general compact homogeneous Kähler manifolds. If $Y = X$ or $\hat{X}^b$ (where b may be zero), the group $H^1(Y, \Theta_Y)$ is a representation space and for us this interpretation will be crucial.

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DEFINITION 9.1. Let $Y=\hat{X}^b$ or X and let Y_t be a 1-parameter deformation of $Y(Y_0=Y)$. We say that Y_t is a *homogeneous deformation* if all the Y_t are homogeneous complex manifolds.

PROPOSITION 9.1. *Let $Y=X$ or $\hat{X}^b$ and let $\theta\in H^1(Y,\Theta)$ be the infinitesimal element representing a 1-parameter deformation Y_t. Then Y_t is a homogeneous deformation $\Leftrightarrow\theta$ is invariant under the compact automorphism group of Y.*

The proof will be given later.

THEOREM 9. *Let $Y=X$ or $\hat{X}^a$. Then*

$$H^q(Y,\Theta)\cong A^q\oplus B^q \tag{9.1}$$

(as M-modules) where

$$A^q=\{\mathfrak{p}\otimes \mathbf{C}^{\binom{a}{q}}\} \tag{9.2}$$

with induced representation $(1\otimes 1)$, *and*

$$B^q=\{\mathfrak{g}\otimes \mathbf{C}^{\binom{a}{q}}\} \tag{9.3}$$

with induced representation $(\mathrm{Ad}\otimes 1)$.

The following are corollaries of Theorem 9 and its proof.

COROLLARY 1. *X is a homogeneous principal bundle $T^{2a}\to X\xrightarrow{\pi}\hat{X}$ (Proposition 5.2) and the connected automorphism group $A^0(X)$ is isomorphic to $G\times T^{2a}$ where G acts by lifting the action of G on $\hat{X}$ to X and T^{2a} acts as structure group in the principal fibering.*

COROLLARY 2. *Let $Y=X$, $g\in H^0(\hat{X},\Theta)$, and $\bar{\omega}_j$ be a component of the conjugated connexion form of the canonical complex connexion in $T^{2a}\to X\xrightarrow{\pi}\hat{X}$. In (9.2), the elements $p\in\mathfrak{p}$ may be interpreted as vertical holomorphic vector fields in the fibering $X\xrightarrow{\pi}\hat{X}$; a generic ($=$indecomposable) element in A^1 is of the form $p\otimes\bar{\omega}_j$ and a generic element in B^1 is of the form $\pi^*(g)\otimes\bar{\omega}_j$.*

Proof of Theorem 9. The proof is done in three steps:

(i) In the notations of § 1, we may write $L(X)=\mathfrak{n}^*_{\mathrm{Ad}}$ and $L(\hat{X})=\hat{\mathfrak{n}}^*_{\hat{\mathrm{Ad}}}$; we know that $H^q(\hat{X},\tilde{\mathfrak{n}}^*_{\hat{\mathrm{Ad}}})=0$ $(q>0)$ and $H^0(\hat{X},\tilde{\mathfrak{n}}^*_{\hat{\mathrm{Ad}}})\cong\mathfrak{g}$. It is almost obvious that the induced representation of $\mathfrak{g}$ on $H^0(\hat{X},\tilde{\mathfrak{n}}^*_{\hat{\mathrm{Ad}}})$ is "Ad"; for us, the geometric construction given now will be useful. The space $H^0(\hat{X},\tilde{\mathfrak{n}}^*_{\hat{\mathrm{Ad}}})$ is given by the analytic functions $f:G\to\hat{\mathfrak{n}}^*$

such that $f(gu) = \mathrm{Ad}\, u^{-1} f(g)$ $(u \in \hat{U})$; for such an f and $g, g' \in \mathfrak{g}$, $(g \circ f)(g') = f(g^{-1} g')$. In the fibration $\hat{U} \to G \xrightarrow{\hat{\pi}} G/\hat{U}$, we may identify $T_0(G/\hat{U})$ with $\hat{\mathfrak{n}}^*$ under $\hat{\pi}_*$. Letting $X_1, \ldots, X_s$ be a basis for $\mathfrak{g}$, we define analytic functions $X_i : G \to \hat{\mathfrak{n}}^*$ by $X_i(g) = \pi_*(\mathrm{Ad}\, g^{-1} X_i)$ (the geometric motivation being clear). Since, for $u \in \hat{U}$, $X_i(gu) = \pi_*(\mathrm{Ad}\, u^{-1}\, \mathrm{Ad}\, g^{-1} X_i) = u^{-1} \circ \hat{\pi}_*(\mathrm{Ad}\, g^{-1} X_i)$, we have a linear mapping $j : g \to H^0(\hat{X}, \tilde{\hat{\mathfrak{n}}}^*_{\hat{\mathrm{Ad}}})$ and it will suffice to show that $\ker j = 0$. If $\tilde{X} \in \mathfrak{g}$ and $j(\tilde{X}) = 0$, then $\hat{\pi}_*(\mathrm{Ad}\, g \tilde{X}) = 0$ for all $g \in G$ and thus $\exp \tilde{X}$ acts trivially on $\hat{X}$. However, this is impossible unless $\tilde{X} = 0$, for then we would have a representation of G into $A^0(\hat{X})$ with a non-discrete kernel which contradicts the semi-simplicity of G. Thus we may in this way identify $\mathfrak{g}$ with $H^0(\hat{X}, \tilde{\hat{\mathfrak{n}}}^*_{\hat{\mathrm{Ad}}})$; the action of $\mathfrak{g}$ on $\mathfrak{g}$ is given by

$$g \circ \tilde{X}(g') = \tilde{X}(g^{-1} g') = \hat{\pi}_*(\mathrm{Ad}\, g'^{-1}\, \mathrm{Ad}\, g \tilde{X}) = (\mathrm{Ad}\, g \tilde{X})(g'); \text{ i.e.,}$$

$$g \circ \tilde{X} = (\mathrm{Ad}\, g X). \tag{9.4}$$

(ii) If $Y = \hat{X}^b$, formula (9.1) is just the Künneth relation. Indeed, with obvious notation,

$$H^1(\hat{X}^a, \Theta) \cong H^0(\hat{X}, \Theta_{\hat{X}}) \otimes H^{0,1}(T^{2a}, \mathbf{C}) \oplus H^1(T^{2a}, \Theta_{T^{2a}})$$

and the induced representation is $(\mathrm{Ad} \otimes 1) \oplus 1$.

(iii) To derive (9.1) for X, we use Proposition 5.3. The exact sequence of vector spaces

$$0 \to \hat{\mathfrak{u}}/\mathfrak{u} \to \mathfrak{g}/\mathfrak{u} \to \mathfrak{g}/\hat{\mathfrak{u}} \to 0 \quad \text{or} \quad 0 \to \mathfrak{p} \to \mathfrak{n}^* \to \hat{\mathfrak{n}}^* \to 0$$

is an exact sequence of $\hat{U}$-modules (since $[\hat{\mathfrak{u}}, \mathfrak{u}] \subseteq \mathfrak{u}$) giving rise to the exact sequence of vector bundles over $\hat{X}$ $0 \to \hat{\mathfrak{p}} \to \mathfrak{n}^*_{\hat{\mathrm{Ad}}} \to \hat{\mathfrak{n}}^*_{\hat{\mathrm{Ad}}} \to 0$ where $\hat{\mathfrak{p}}$ is the trivial bundle $\hat{X} \times \mathfrak{p}$. From Theorem 3 and the exact cohomology sequence, $H^q(\hat{X}, \tilde{\mathfrak{n}}^*_{\hat{\mathrm{Ad}}}) = 0$ $(q > 0)$ and $H^0(\hat{X}, \tilde{\mathfrak{n}}^*_{\hat{\mathrm{Ad}}}) \cong \mathfrak{p} \oplus \mathfrak{g}$ with induced representation $1 \oplus \mathrm{Ad}$. Now apply Proposition 5.3. Q.E.D.

For later use, we record here the calculations of some more groups. Let $X = G/U$ be non-Kähler and consider the Atiyah sequence

$$0 \to \mathbf{L} \to \mathbf{Q} \to T(X) \to 0; \tag{9.5}$$

this sequence is constructed from the principal fibering $U \to G \to G/U$.

PROPOSITION 9.2.

(i) $H^0(X, \mathbf{\mathfrak{L}}) = 0$ *and* $H^q(X, \mathbf{\mathfrak{L}}) \cong \mathfrak{p} \otimes \mathbf{C}^{\binom{a}{q-1}}$ $(q > 0)$.

(ii) $H^q(X, \mathbf{Q}) \cong \mathfrak{g} \otimes \mathbf{C}^{\binom{a}{q}}$ $(q \geqslant 0)$.

Proof. Letting $T^{2a} \to X \to \hat{X}$ $(X = G/U,\ \hat{X} = G/\hat{U})$ be the fundamental fibering, U is normal in $\hat{U}$ and the exact sequence of vector spaces

$$0 \to \mathfrak{u} \to \hat{\mathfrak{u}} \to \hat{\mathfrak{u}}/\mathfrak{u} \to 0$$

is an exact sequence of either U- or $\hat{U}$-modules. Letting $\mathfrak{p} = \hat{\mathfrak{u}}/\mathfrak{u}$, we may apply Theorem 3 and have that $H^q(\hat{X}, (\tilde{\hat{\mathfrak{u}}}_{\hat{\mathrm{Ad}}})) = 0$ for all q which gives

$$H^{q+1}(\hat{X}, (\tilde{\mathfrak{u}}_{\hat{\mathrm{Ad}}})) \cong H^q(\hat{X}, (\tilde{\mathfrak{p}}_{\hat{\mathrm{Ad}}})) \cong H^q(\hat{X}, \Omega) \otimes \mathfrak{p}.$$

Thus $H^{q+1}(\hat{X}, (\tilde{\mathfrak{u}}_{\hat{\mathrm{Ad}}})) = 0$ $(q \neq 0)$ and $H^1(\hat{X}, (\tilde{\mathfrak{u}}_{\hat{\mathrm{Ad}}})) \cong \mathfrak{p}$, the induced M-action being trivial. Since $\mathbf{L} = \mathfrak{u}_{\mathrm{Ad}}$, an application of Proposition 5.3 completes the proof.

Remark. The exact cohomology sequence of (9.5) is

$$\ldots \to \mathfrak{p} \otimes \mathbf{C}^{\binom{a}{q-1}} \xrightarrow{j_q} \mathfrak{g} \otimes \mathbf{C}^{\binom{a}{q}} \to \{\mathfrak{g} \otimes \mathbf{C}^{\binom{a}{q}}\} \oplus \{\mathfrak{p} \otimes \mathbf{C}^{\binom{a}{q}}\} \xrightarrow{\delta_q} \mathfrak{p} \otimes \mathbf{C}^{\binom{a}{q}} \to \ldots; \tag{9.6}$$

to find the maps δ_q, we simply observe that the j_q are all zero.

(ii) Obstructions to Deformation

The second aspect of our general theory of deformations of C-spaces is concerned with the notion of "obstructions" to deformations as discussed in [19], § 6. Letting $U = \{U_i\}$ be a covering of Y as above, if $\theta, \lambda \in H^1(Y, \Theta)$ are represented by cocycles $\{\theta_{ij}\}$, $\{\lambda_{ij}\} \in H^1(N(U), \Theta)$, we may define a new element $\{\theta, \lambda\} \in H^2(Y, \Theta)$ by

$$\{\theta, \lambda\}_{ijk} = \tfrac{1}{2}([\theta_{ij}, \lambda_{jk}] + [\lambda_{ij}, \theta_{jk}]). \tag{9.7}$$

It is known (and easily checked) that if θ is an infinitesimal deformation element, $\{\theta, \theta\} = 0$ in $H^2(Y, \Theta)$; if θ, λ are deformation elements, then $\theta + \lambda$ may not be and the obstruction here is just $2\{\theta, \lambda\}$ (since $\{\theta, \lambda\} = \{\lambda, \theta\}$). We shall now calculate this bracket $\{\ ,\ \}$ in case $Y = \hat{X}^a$ or X; a maximal Abelian subspace $D \subseteq H^1(Y, \Theta)$ will be a "maximal" possibility for a deformation space, and in § 10 we shall explicitly construct a *local* family which is infinitesimally represented by D.

Letting Y be arbitrary for a moment and $\Xi \subseteq H^0(Y, \Theta)$ a sub-algebra, there is a natural mapping $j: \Xi \otimes H^1(Y, \Omega) \to H^1(Y, \Theta)$. If $\theta \otimes \bar{\omega}$, $\theta' \otimes \bar{\omega}' \in \Xi \otimes H^1(Y, \Omega)$ then we shall prove in § 10 below (see also [19], § 4) that

$$\{\theta \otimes \bar{\omega}, \theta' \otimes \bar{\omega}'\} = \theta \otimes \bar{\omega}' \wedge L_{\theta'} \bar{\omega} + \theta' \otimes \bar{\omega} \wedge L_\theta \bar{\omega}' + [\theta, \theta'] \otimes \bar{\omega} \wedge \bar{\omega}'; \tag{9.8}$$

here $L_\theta(\bar{\omega}')$ = Lie derivative of $\bar{\omega}'$ along the vector field θ.

THEOREM 10. *In the notation of Theorem 9, the* $\{\ ,\ \}$ *on* $H^1(Y,\Theta)$ *is given as follows:*

$$\left.\begin{array}{l}\text{(i) } A^1 \text{ is always Abelian and } \{A^1, B^1\}=0,\\ \text{(ii) if } a=1,\ B^1 \text{ is Abelian,}\\ \text{(iii) if } a>1, \text{ there exist maximal Abelian subspaces of } B^1 \text{ of the form}\\ \quad D_{\mathfrak{h}}=\mathfrak{h}\otimes \mathbf{C}^a \text{ where } \mathfrak{h}\subseteq \mathfrak{g} \text{ is a Cartan sub-algebra, and of the form}\\ \quad D_{\bar\omega}=\mathfrak{g}\otimes \mathbf{C}\bar\omega \text{ where } \bar\omega\in H^{0,1}(Y,\mathbf{C}).\end{array}\right\} \quad (9.9)$$

Proof. If $Y=\hat{X}^a=\hat{X}\times T^{2a}$, then $A^1\cong H^1(T^{2a},\Theta_T)$ and $B^1\cong H^0(\hat{X},\Theta_{\hat{X}})\otimes H^1(T^{2a},\Omega_T)$. Since, for $\theta\in H^0(\hat{X},\Theta_{\hat{X}})$, $\bar\omega\in H^1(T^{2a},\Omega_T)$, $L_\theta\bar\omega=0$, the statement (i) follows easily from (9.8).

In case $Y=X$ (where $T^{2a}\to X\to\hat{X}$), $A^1\cong\mathfrak{p}\otimes\mathbf{C}^a$ and an indecomposable element $p\otimes\bar\omega\in A^1$ has the following interpretation (see § 5): p is a vector field along the fibres on the fundamental fibering and $\bar\omega$ is a component of the canonical complex connexion in the fibering $T^{2a}\to X\to\hat{X}$. Since the connexion is *right*-invariant, $L_p\bar\omega=0$ and because $[\mathfrak{p},\mathfrak{p}]=0$, it follows that A^1 is Abelian. Now $B^1\cong\mathfrak{g}\otimes\mathbf{C}^a$ and an indecomposable element $g\otimes\bar\omega\in B^1$ has the following interpretation: g is induced by the action of $\exp(\mathrm{tg})\in G$ on $X=G/U$ and again $\bar\omega$ is a connexion form. We recall that $\mathfrak{g}\cong\pi^*H^0(X,\Theta)$ transforms under M by Ad and that the forms $\bar\omega$ are M-invariant.

LEMMA 9.1. *The* $(0,1)$ *form* $L_g\bar\omega$ *on* X *is* $\bar\partial$*-closed and transforms under* M *by* Ad.

Proof. We recall that $L_g\bar\omega=i(g)\,d\bar\omega+di(g)\,\bar\omega=i(g)\,d\bar\omega$ (since $\bar\omega$ is of type $(0,1)$. Thus $L_g\bar\omega=i(g)\,\bar\Xi$ where Ξ is a component of the curvature form of the canonical complex connexion; $\bar\Xi$ is thus M-invariant. The proof follows from this and the following fact: if Y is a manifold, T an automorphism of Y, φ, v respectively a form and a vector field on Y, then $T^*(i(v)\,\varphi)=i(T_*^{-1}v)\,T^*\varphi$.

LEMMA 9.2. $L_g\bar\omega$ *is* $\bar\partial$ *cohomologous to zero.*

Proof.(1) $L_g\bar\omega\in H^1(X,\Omega)$ (a priori); however, $H^1(X,\Omega)$ transforms invariantly by M whereas $L_g\bar\omega$ transforms strictly non-invariantly. Now equation (1.8) in § 1 coupled with the non-invariance of $L_g\bar\omega$ tells us that $L_g\bar\omega\sim 0$.

COROLLARY. *If* $g,g'\in\mathfrak{g}$; $\bar\omega,\bar\omega'\in H^1(X,\Omega)$ *then* $g\otimes\bar\omega'\wedge L_{g'}\bar\omega\sim 0$.

The proof of (9.9) now follows immediately from (9.8).

(1) This proof may be done alternatively as follows. Since $\partial\bar\omega=\bar\Xi$ is the conjugated curvature tensor of the canonical complex connexion on $\hat{X}$ in the homogeneous bundle $T^{2a}\to X\to\hat{X}$, $\bar\Xi$ is a $(1,1)$ form on $\hat{X}$. Thus $L_g\bar\omega=i(g)\,\bar\Xi$ is a $\bar\partial$-closed $(0,1)$ form on $\hat{X}$, and, since $H^1(X,\Omega)=0$, $L_g\bar\omega=\bar\partial f$ for some function f on $\hat{X}$. We may now lift everything back up to X.

10. Deformation Theory; Part II

In § 9, we obtained an infinitesimal deformation space D_Y where $Y=\hat{X}^a$ or X; we shall now construct explicitly a local family which is infinitesimally parametrized by D_Y. In the Kähler case, the family will even be global.

(i) The Kähler Case

The most general homogeneous compact Kähler manifold is of the form $\hat{X}^a = X\times T^{2a}$ where X is a Kähler C-space. We shall see that the question of deforming these manifolds falls in the general pattern described by the relationship between the automorphisms of a compact Kähler manifold Y and the deformations of a complex fibre bundle over Y. We begin therefore with a discussion of this latter topic.

We first recall some results from [19]. Let Y be a compact Kähler manifold, A a connected complex Lie group, and $A\to P\to X$ a holomorphic principal bundle. Then one may vary the bundle structure of P while holding the complex structure on Y "fixed", and, in the same manner as $H^1(Y,\Theta)$ parametrizes the variation of structure on $Y, H^1(Y,\mathfrak{L}(P))$ parametrizes the bundle deformations of $A\to P\to Y$. (Here $\mathfrak{L}=P\times_A\mathfrak{a}$ where A acts on $\mathfrak{a}$ by Ad.) To see this, we let $\{U_i\}$ be a suitable covering of Y such that P has transition functions $f_{ij}: U_i\cap U_j\to A$. If we have a 1-parameter variation $P(t)$, $P(0)=P$, of P, then $P(t)$ is described by transition functions $f_{ij}(t): U_i\cap U_j\to A$, $f_{ij}(0)=f_{ij}$. The infinitesimal cocycle tangent to this family is given by $\xi=\{\xi_{ij}\}\in H^1(Y,\mathfrak{L}(P))$ where

$$\xi_{ij}=\operatorname{Ad} f_{ij}\left(f_{ij}(t)^{-1}\left[\frac{\partial f_{ij}(t)}{\partial t}\right]_{t=0}\right). \tag{10.1}$$

There is a bracket (without differentiation) $\{\ ,\ \}: H^1(Y,\mathfrak{L}(P)\otimes H^1(Y,\mathfrak{L}(P))$, and in order that $\xi\in H^1(Y,\mathfrak{L}(P))$ be tangent to a deformation, it is necessary that $\{\xi,\xi\}\sim 0$.

DEFINITION. We say that Y satisfies *condition* N if the following holds: There exists a basis $\bar{\omega}_1\ldots,\bar{\omega}_s$ of $H^1(Y,\Omega)$ ($\cong H^{0,1}(Y,\mathbf{C})$) such that, if $\bar{\omega}$ and $\bar{\omega}'$ in $H^1(Y,\Omega)$ are written in terms of this basis and $\bar{\omega}\wedge\bar{\omega}\sim 0$, then $\bar{\omega}\wedge\bar{\omega}'\equiv 0$.

Let $\mathfrak{l}=H^0(Y,\mathfrak{L}(P))$ and assume henceforth that Y satisfies condition N. Then the mapping $\mathfrak{l}\otimes H^1(Y,\Omega)\to H^1(Y,\mathfrak{L}(P))$ is an injection, and, letting $\mathfrak{k}\subseteq\mathfrak{l}$ be a maximal abelian sub-algebra, we may state

THEOREM 11. *The subspace* $\mathfrak{k}\otimes H^1(Y,\Omega)$ *globally parametrizes a family of deformations of* $A\to P\to Y$. *Furthermore, it is maximal in* $\mathfrak{l}\otimes H^1(Y,\Omega)$ *if* $\dim H^1(Y,\Omega)>1$.

Proof. Let H_1^* be the torsion-free part of $H^1(Y, \mathbf{Z})$, construct the covering fibration $H_1^* \to Y^* \to Y$, and lift P to a bundle $A \to P^* \to Y^*$. If $\lambda \in \mathfrak{l}$, $\bar{\omega} \in H^1(Y, \Omega)$, we shall construct geometric transformations $\varrho_{(\lambda, \bar{\omega})}(t)(\gamma) = \varrho(t)(\gamma)$ $(\gamma \in H_1^*)$ on P^* such that $\varrho(0)(\gamma)$ is the canonical action of H_1^* on P^* and such that the bundles $P(t) = P^*/\varrho(t) H_1^*$ give a deformation of $P = P(0)$ with infinitesimal tangent $\lambda \otimes \bar{\omega}$.

(a) *Construction of the bundles* $P(t)$. Connecting the fiberings $A \to P^* \to Y^*$ and $H_1^* \to Y \to Y$, we have a diagram

$$\begin{array}{ccccc} H_1^* & \to & P^* & \to & P \\ \| & & \downarrow \sigma^* & & \downarrow \sigma \\ H_1^* & \to & Y^* & \xrightarrow{\tau} & Y \end{array}$$

which we now make more explicit.

By definition, $P^* \subseteq Y^* \times P = \{(y^*, p) \in Y^* \times P \mid \tau(y^*) = \sigma(p)\}$, and then $\sigma^*(y^*, p) = y^*$. For $\gamma \in H_1^*$, $y^* \in Y^*$, the action $\gamma \cdot y^*$ is just the covering-transformation by γ, and if $p^* = (y^*, p) \in P^*$, then $\gamma \cdot p^* = (\gamma y^*, p)$.

The Lie algebra $\mathfrak{l} = H^0(Y, \mathbf{L}(P))$ is just the algebra of infinitesimal bundle automorphisms of P which project to the trivial action on Y; for $\lambda \in \mathfrak{l}$, the 1-parameter group $\exp(t\lambda)$ of bundle automorphisms of P is defined. We define the transformation $\varrho(t)(\gamma)$ $(\gamma \in H_1^*)$ on P^* by the equation:

$$\varrho(t)(\gamma)\, p^* = \left(\gamma y^*,\ \exp\left(\left(t \int_\gamma \bar{\omega}\right) \lambda\right)(p)\right) \quad (p^* = (y^*, p)). \tag{10.2}$$

Furthermore, we define the complex manifold $P(t)$ by

$$P(t) = P^*/\varrho(t)(H_1^*). \tag{10.3}$$

It is clear that $P(t)$ is an analytic principal bundle with group A over Y and $P(0) = P$. In fact, the family $P(t)$ obviously gives a deformation of P.

(b) *The Transition Functions of* $P(t)$. If $\{U_i\}$ is a suitable covering of Y, the vector field λ is given in U_i by a holomorphic function $\lambda_i : U_i \to \mathfrak{a}$ such that $\lambda_i = \operatorname{Ad} f_{ij}(\lambda_j)$ in $U_i \cap U_j$. For fixed t, we define mappings $a_i(t) : U_i \to A$ by $a_i(t) = \exp(t\lambda_i)$; then $a_i(t) f_{ij} = f_{ij} a_j(t)$ in $U_i U_j$, and the action of $a(t) = \exp(t\lambda)$ on P is given locally in U_i by $a(t)(z, \alpha) = (z, a_i(t)(z)\alpha)$ $(\alpha \in A)$.

Now, since Y is Kähler, we may assume that we have chosen the covering $\{U_i\}$ and, for each i, a point $Z(i) \in U_i$, such that; $U_i \cap U_j \cap U_k \neq \emptyset$ implies

$$\int_{Z(i)}^{Z(j)} \bar{\omega} + \int_{Z(j)}^{Z(k)} \bar{\omega} = \int_{Z(i)}^{Z(k)} \bar{\omega}. \tag{10.4}$$

With this in mind, we assert that the transition functions $\{f_{ij}(t)\}$ of $P(t)$ are given by

$$f_{ij}(t)=\left(\exp\left(t\int_{Z(i)}^{Z(j)}\bar{\omega}\right)\lambda_i\right)\cdot f_{ij}. \tag{10.5}$$

For example, we check that $f_{ij}(t)\,f_{jk}(t)=f_{ik}(t)$;

$$\begin{aligned} f_{ij}(t)\cdot f_{jk}(t) &= \left(\exp\left(t\int_{Z(i)}^{Z(j)}\bar{\omega}\right)\lambda_i\right)\cdot f_{ij}\cdot\left(\exp\left(t\int_{Z(j)}^{Z(k)}\bar{\omega}\right)\lambda_j\right)\cdot f_{jk} \\ &= \exp\left(t\left(\int_{Z(i)}^{Z(j)}\bar{\omega}\right)\lambda_i\right)\cdot\left(\exp\left(t\int_{Z(j)}^{Z(k)}\bar{\omega}\right)\lambda_i\right)\cdot f_{ij}\cdot f_{jk} \\ &= \exp\left(t\left(\int_{Z(i)}^{Z(k)}\bar{\omega}\right)\cdot\lambda_i\right)\cdot f_{ik}. \end{aligned}$$

Using this, the rest of the calculation may be modeled after the discussion given in [17], § 2.

(c) *The Infinitesimal Deformation.* If $f_{ij}(t)$ is given by (10.5), then

$$\begin{aligned} f_{ij}(t)^{-1}\left[\frac{\partial f_{ij}(t)}{\partial t}\right]_{t=0} &= f_{ij}^{-1}\left(\exp\left(-t\int_{Z(i)}^{Z(j)}\bar{\omega}\right)\lambda_i\right)\cdot\frac{\partial}{\partial t}\left[\exp(t)\left(\left(\int_{Z(i)}^{Z(j)}\bar{\omega}\right)\lambda_i\right]_{t=0}\right)\cdot f_{ij} \\ &= f_{ij}^{-1}\left(\left(\int_{Z(i)}^{Z(j)}\bar{\omega}\right)\cdot\lambda_i\right)f_{ij}=\operatorname{Ad} f_{ij}^{-1}\left(\left(\int_{Z(i)}^{Z(j)}\bar{\omega}\right)\lambda_i\right). \end{aligned}$$

Thus, by (10.1), the infinitesimal tangent $\xi=\{\xi_{ij}\}$ to the deformation $P(t)$ is given by $\xi_{ij}=(\int_{Z(i)}^{Z(j)}\bar{\omega})\,\lambda_i$. To complete the proof of the first part of the theorem, it will suffice to show that, under the Dolbeault isomorphism, ξ corresponds to $\lambda\otimes\bar{\omega}\in H^1(Y,\mathbf{L}(P))$. To do this, it will suffice to show that $\bar{\omega}$ corresponds to the 1-cocycle $\tau_{ij}=\int_{Z(i)}^{Z(j)}\bar{\omega}$ in $H^1(Y,\Omega)$. Since Y is Kähler, $\bar{\omega}=df_i=\bar{\partial}f_i$ in U_i, and the sheaf cocycle representing $\bar{\omega}$ may be taken to be f_i-f_j in $U_i\cap U_j$. But we may take $f_i(Z)=\int_{Z(i)}^{Z}\bar{\omega}$. $(Z\in U_i)$.

(d) *Completion of the Proof.* The general element of $\mathfrak{k}\otimes H^1(Y,\Omega)$ is of the form $\xi=\sum_{j=1}^{s}\lambda_j\otimes\bar{\omega}_j$. The condition $[\lambda_i,\lambda_j]=0$ clearly allows us to make the same construction for ξ as we did above for $\lambda\otimes\bar{\omega}$. Finally, the fact that Y satisfies condition N will assure that $\mathfrak{k}\otimes H^1(Y,\Omega)$ is maximal if $\dim H^1(Y,\Omega>1$. Q.E.D.

Remarks. (i) If P is the trivial bundle $Y\times A$, then $H^0(Y,\mathbf{L}(P))\cong\mathfrak{a}$; if $\xi=\sum_{j=1}^{s}\lambda_j\otimes\bar{\omega}_j\in\mathfrak{k}\otimes H^1(Y,\Omega)$, then the above construction amounts to defining a representation $\varrho_\xi(t)\colon H_1^*\to A$ by $\varrho_\xi(t)(\gamma)=\prod_{j=1}^{s}\exp(t)(\int_\gamma\bar{\omega})\,\lambda_j)$, and then setting $P(t)=Y^*\times_{H_1^*}A$. (The fact that $\varrho_\xi(t)$ is a representation is guaranteed by the assumption $\xi \in \mathfrak{k} \otimes H^1(Y, \Omega)$.) If $\xi = \lambda \otimes \bar{\omega}$, then the transition functions of $P(t)$ are given by $f_{ij}(t) = \exp(t(\int_{z(i)}^{z(j)} \bar{\omega})\lambda)$, and, since the $f_{ij}(t)$ are constant, the bundles $P(t)$ all have holomorphic connexions.

(ii) If $\dim A = 1$, $A = \mathbf{C}^*$, then the above construction reduces to the construction of the *Picard Variety* $\mathfrak{P}$ of the compact Kähler manifold Y.

(iii) Suppose now that $Y = X^a = G/\hat{U} \times T^{2a}$ is a Kähler homogeneous space where $X = G/\hat{U}$ is a C-space. Then: $H^1(\hat{X}^a, \Theta_{\hat{X}^a}) \cong \{\mathfrak{g} \otimes H^1(T^{2a}, \Omega)\} \oplus \{\mathbf{C}^a \otimes H^1(T^{2a}, \Omega)\} = A \oplus B$. The elements in B correspond simply to the variations of the complex structure on T^{2a}; the resulting manifolds are all homogeneous. The manifold $\hat{X}^a$ satisfies condition N, and, if $a > 1$, a maximal abelian subspace of A is of the form $\mathfrak{h} \otimes H^1(T^{2a}, \Omega)$ where $\mathfrak{h}$ is a Cartan sub-algebra of $\mathfrak{g}$.

THEOREM. *The subspace* $\{\mathfrak{h} \otimes H^1(T^{2a}, \Omega)\} \oplus B$ *gives a global deformation space of* $\hat{X}^a$ *which is locally universal. If* $h \in \mathfrak{h}$, $\bar{\omega} \in H^1(T^{2a}, \Omega)$, *the manifold* $\hat{X}^a(h, \bar{\omega})$ *corresponding to* $h \otimes \bar{\omega}$ *is non-homogeneous and is constructed as follows: From the trivial bundle* $G \to T^{2a} \times G \to T^{2a}$, *one constructs by Theorem 11 a family of bundles* $P_{(h, \bar{\omega})}(t) = P(t)$ *deforming the trivial bundle, and then* $\hat{X}^a(h, \bar{\omega}) = P(1)/U$.

Proof. All statements in the theorem are immediate except perhaps the non-homogeneity of $\hat{X}^a$. This is implied by

PROPOSITION 10.1. *The connected automorphism group of* X^a *is* $G_h \otimes T^{2a}$ *where* $G_h = \{g \in G \mid \operatorname{Ad} g(h) = h\}$. *By considering* $\hat{X}^a(h, \bar{\omega})$ *as a bundle over* T^{2a} *with fibre* $G/\hat{U}$, *the automorphisms of* T^{2a} *lift to* $\hat{X}^a$ *and this is how* T^{2a} *acts. The group* G_h *is the identity component of the complex Lie group of bundle automorphisms of* $G/U \to \hat{X}^a \to T^{2a}$ *which induce the identity automorphism on the base space.*

Proof. We have a fibering $H_1^* \to G/\hat{U} \times \mathbf{C}^a \to \hat{X}^a$, and the automorphisms of $\hat{X}^a$ consist of those automorphisms which are invariant under H_1^*. From this, it follows easily that $A^0(X^a) \cong G^1 \times T^{2a}$ where G^1 is a complex subgroup of G. Then we have that $G^1 = \{g \in G \mid g \cdot \exp h = \exp h \cdot g\}$ and thus $G^1 = G_h$.

Remarks. (i) The deformations of $\hat{X}^a$ may be thought of as parametrized by a family of "cones" over $B(\cong \mathbf{C}^{a^2})$ where this "sides" of the cones correspond to the Cartan sub-algebras of $\mathfrak{g}$. It follows from Proposition 10.1 that, if h is a semi-simple element in $\mathfrak{g}$ (so that G_h is abelian), then the manifold $\hat{X}^a(h, \bar{\omega})$ is unobstructed. Thus the obstructions occur on a lower dimensional sub-variety of the deformation space constructed above.

(ii) If $g \in \mathfrak{g}$, $\bar{\omega} \in H^1(T^{2a}, \Omega)$, then we may always construct a family of manifolds $\hat{X}^a(g, \bar{\omega}; t)$ with infinitesimal tangent $g \otimes \bar{\omega} \in H^1(\hat{X}^a, \Theta_{\hat{X}^a})$. If $\gamma \in G$, then γ will in general be only a real-analytic automorphism of $\hat{X}^a(h, \bar{\omega})$, but we have

PROPOSITION 10.2. *Consider the variations of $\hat{X}^a$ described above as all having the same underlying C^∞ structure. Then the C^∞ automorphism determined by the action of $\gamma \in G$ on $\hat{X}^a \cong_{C^\infty} \hat{X}^a(g, \bar{\omega})$ establishes a complex analytic equivalence between the families $X^a(g, \bar{\omega}; t)$ and $X^a(\mathrm{Ad}\,\gamma(g), \bar{\omega}, t)$. Briefly: The infinitesimal action of γ on $H^1(X^a, \Theta_{\hat{X}^a})$ can be covered by a mapping between the deformation families.*

Proof. The proof will follow from the proof of Theorem 13 below.

(ii) The non-Kähler Case

Quite clearly the same construction as above will not yield the variations of a non-Kähler C-space $X(=M/V=G/U)$, this is due primarily to the fact that if $\bar{\omega} \in H^1(X, \Omega)$, then $\partial\bar{\omega} \neq 0$ in general. Before beginning the construction of the deformations of X, we record a few preliminary remarks. The space $C^{0,1}(X, \mathbf{C}) = \Gamma_\infty(X, T(X)')$ of $C^\infty(0,1)$ forms on X is on M-representation space, and from § 1 we have

$$C^{0,1}(X, \mathbf{C}) \sim \sum_{\lambda \in D(\mathfrak{g})} V_\lambda \otimes ((\mathfrak{n})' \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^0}. \tag{10.6}$$

The forms in each component $V^\lambda \otimes ((\mathfrak{n})' \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^0}$ are *real-analytic* on X, and M acts on this subspace by $\lambda \otimes 1$. Letting $\mathfrak{n} = \hat{\mathfrak{n}} \oplus \mathfrak{p}$ $(\hat{\mathfrak{n}} = \mathfrak{c}(e_{-\alpha} : \alpha \in \Sigma^+ - \Psi^+))$, the element $\bar{\omega} \in H^1(X, \Omega)$ is represented in $V^0 \otimes ((\mathfrak{n})' \otimes V)^{\mathfrak{v}^0} \cong ((\mathfrak{n})')^{\tilde{\mathfrak{v}}^0}$ by the dual p' of some $p \in \mathfrak{p}$. Furthermore, in the notation of § 9, $L_g\bar{\omega}$ $(g \in \mathfrak{g})$ transforms by Ad. Thus $L_g\bar{\omega} \in \mathfrak{g} \otimes ((\mathfrak{n})' \otimes \mathfrak{g}')^{\tilde{\mathfrak{v}}^0}$ ($'$ = contragredient representation). On the other hand (see § 1),

$$\begin{aligned} d\bar{\omega} = \partial\bar{\omega} = dp' &= \sum_{\alpha \in \Sigma^+ - \Psi^+} \langle \alpha, p \rangle\, \omega^\alpha \wedge \bar{\omega}^\alpha \\ &= -\sum_{\alpha \in \Sigma^+ - \Psi^+} \langle \alpha, p \rangle\, \bar{\omega}^\alpha \otimes (e_\alpha)' \end{aligned} \tag{10.7}$$

(since $\mathfrak{p} \subseteq$ centralizer of $\tilde{\mathfrak{v}}^0$ in $\mathfrak{g}$).

PROPOSITION 10.3.

$$L_g\bar{\omega} \in \mathfrak{g} \otimes ((\mathfrak{n})' \otimes \mathfrak{g}')^{\tilde{\mathfrak{v}}^0}$$

is real-analytic and is equal to

$$-g \otimes \sum_{\alpha \in \Sigma^+ - \Psi^+} \langle \alpha, p \rangle\, (\bar{\omega}^\alpha \otimes (e_\alpha)').$$

Furthermore, the element $-g \otimes p' \in \mathfrak{g} \otimes ((\mathfrak{g})')^{\mathfrak{v}^0}$ *represents a real-analytic function* $f = f(g, \bar{\omega})$ *an* X *and* $\bar{\partial} f = L_g \bar{\omega}$.

Remark. The Frobenius reciprocity law reads:

$$C^\infty(M/V) \sim \sum_{\lambda \in D(\mathfrak{g})} V^\lambda \otimes (V^{-\lambda})^{\bar{\mathfrak{v}}^0},$$

and the statement $g \otimes (p)' \in C^\infty(X)$ is to be interpreted in this sence.

We shall explicitly construct a 1-parameter family $X(g, \bar{\omega}; t) = X(t)$ of non-homogeneous manifolds (except for $t=0$) and show that $g \otimes \bar{\omega} \in H^1(X, \Theta)$ infinitesimally represents the deformation $X(t)$ of $X = X(0)$.

It is convenient to take a slightly broader point of view of deformations than that adopted at the beginning of § 9. A family of complex structures Y_t ($Y_0 = Y$) defined on a single C^∞ manifold Y_d is given as follows:

Let $\{U_j\}$ be a suitable covering of Y_d by coordinate neighborhoods. The family Y_t is given by

(i) n C^∞ complex-valued functions $\zeta_j^1(y,t), \ldots, \zeta_j^n(y,t)\} = \{\zeta_j^\alpha(y,t)\}$ defined on U_j such that the $\zeta_j^\alpha(y, t_0)$ define holomorphic coordinates in $U_j \subset Y_{t_0}$. (10.8)

(ii) transition functions $h_{ij}^\alpha(\zeta_j(y,t), t)$ which are holomorphic in the $\zeta_j^\beta(y,t)$ and the complex variable t and which define the coordinate changes in Y_t. (See [20], § 1.) (10.9)

We denote by the symbol "$\cdot$" the derivative of anything with respect to t; the symbol "$\cdot$"$_0$ means "$\cdot$" taken at $t=0$. As in § 9, the 1-cocycle $\theta = \{\theta_{ij}\} \in H^1(N(U), \Theta)$ representing the family Y_t is given by

$$\theta_{ij} = (\theta_{ij}^1, \ldots, \theta_{ij}^n) = \{\theta_{ij}^\alpha\},$$

where $\theta_{ij}^\alpha = (\dot{h}_{ij}^\alpha)_0$. We wish to find a Dolbeault representative for θ.

PROPOSITION 10.4. *A Dolbeault representative for* θ *is given by the vectorial-*(0, 1)*-form* $\Phi \in H^{0,1}(Y, \Theta)$ *where* $\Phi \mid U_i = (\Phi_i^1, \ldots, \Phi_i^n)$ *and*

$$\Phi_i^\alpha(y) = \bar{\partial}(\dot{\zeta}_i^\alpha(y,t))_0. \quad (\textit{Here } \bar{\partial} \textit{ is taken on } Y_0 = Y.) \tag{10.10}$$

Proof. Since

$$\dot{\zeta}_i^\alpha(y,t) = \sum_\beta \frac{\partial h_{ij}^\alpha(\zeta_j(y,t), t)}{\partial \zeta_j^\beta(y,t)} \dot{\zeta}_j^\beta(y,t) + \dot{h}_{ij}^\alpha(\zeta_j(y,t), t),$$

we have $$(\dot\zeta_i^\alpha(y,t))_0 - \sum_\beta \frac{\partial h_{ij}^\alpha(\zeta_j(y,t),t)}{\partial \zeta_j^\beta(y,t)}\Big|_{t=0} (\dot\zeta_j^\beta(y,t))_0 = (\dot h_{ij}(\zeta_j(y,t),t))_0.$$

By definition of the Dolbeault isomorphism, $\bar\partial(\dot\zeta_i^\alpha(y,t))_0 = \Phi_i^\alpha(y)$ represents θ.

Let $H^1(Y,\Theta)$, $H^0(Y,\Theta)$, $H^1(Y,\Omega)$ have the usual meaning and suppose that $\mathfrak{A} \subseteq H^0(Y,\Theta)$ is a subalgebra.

PROPOSITION 10.5. *Let* $\theta, \theta' \in \mathfrak{A}$; $\bar\omega, \bar\omega' \in H^1(Y,\Omega)$. *Then, under the Dolbeault isomorphism,* $\{\theta\otimes\bar\omega, \theta'\otimes\bar\omega'\}$ *is represented by*

$$\theta\otimes\bar\omega'\wedge L_{\theta'}\bar\omega + \theta'\otimes\bar\omega\wedge L_\theta\bar\omega' + [\theta,\theta']\otimes\bar\omega\wedge\bar\omega'. \tag{10.11}$$

Remark. This proposition was promised in the proof of Theorem 10.

Proof. If $\{U_i\}$ is a suitable covering of Y, then there exist C^∞ functions f_i defined on U_i such that $\bar\omega \mid U_i = \bar\omega_i = \bar\partial f_i$; $f_i \cdot f_j \in Z^1(N(U),\Omega)$ is the Čech representative of $\bar\omega$, In the same way, we find f_i' for $\bar\omega'$. Denote $f_i - f_j$ by $\bar\omega_{ij}$ and $f_i - f_j$ by $\bar\omega_{ij}$. Then, if $\theta \mid U_i = \theta_i$, by definition

$$\begin{aligned}\{\theta\otimes\bar\omega, \theta'\otimes\bar\omega'\}_{ijk} &= \tfrac12([\bar\omega_{ij}\theta_i, \bar\omega_{jk}'\theta_i'] + [\bar\omega_{ij}'\theta_i', \bar\omega_{jk}\theta_i])\\ &= \tfrac12(\bar\omega_{ij}\bar\omega_{jk}' - \bar\omega_{ij}'\bar\omega_{jk})[\theta_i,\theta_i'] + \tfrac12\bar\omega_{ij}(\theta_i\cdot\bar\omega_{jk}')\theta_i'\\ &\qquad - \tfrac12\bar\omega_{jk}(\theta_i'\cdot\bar\omega_{ij})\theta_i.\end{aligned}$$

Because of the alternating principle (i.e., we may always skew-symmetrize cochains), and the (easily verified) fact that $(\bar\omega\wedge\bar\omega')_{ijk} = \tfrac12(\bar\omega_{ij}\bar\omega_{jk}' - \bar\omega_{ij}'\bar\omega_{jk})$, we find that

$$\begin{aligned}\{\theta\otimes\bar\omega, \theta'\otimes\bar\omega'\}_{ijk} &= (\bar\omega\wedge\bar\omega')_{ijk}[\theta_i,\theta_i']\\ &\quad + \tfrac12(\bar\omega_{ij}(\theta_i\cdot\bar\omega_{jk}') - \bar\omega_{jk}(\theta_i\cdot\bar\omega_{ij}'))\theta_i\\ &\quad + \tfrac12(\bar\omega_{ij}'(\theta_i'\cdot\bar\omega_{ik}) - \bar\omega_{jk}'(\theta_i'\cdot\bar\omega_{ij}))\theta_i.\end{aligned} \tag{10.12}$$

From (10.12) it will suffice to show that, under the Dolbeault isomorphism, $(L_\theta\bar\omega')_{ij} = (\theta_i\cdot\bar\omega_{ij}')$. But in U_i, $\bar\omega_i' = \bar\partial f_i'$ and

$$\begin{aligned}L_\theta\bar\omega_i' &= \frac{d}{dt}\left(\frac{(\exp t\theta)^*\bar\omega_i' - \bar\omega_i'}{t}\right)\Big]_{t=0} = \frac{d}{dt}\left(\frac{(\exp t\theta)^*\bar\partial f_i' - \bar\partial f_i'}{t}\right)\Big]_{t=0}\\ &= \bar\partial\frac{d}{dt}\left(\frac{(\exp t\theta)^* f_i' - f_i'}{t}\right)\Big]_{t=0} = \bar\partial(\theta\cdot f_i').\end{aligned}$$

Thus $(L_\theta\bar\omega')_{ij}$ is represented by

$$\theta_i\cdot f_i' - \theta_j\cdot f_j' = \theta_i\cdot(f_i' - f_j') = \theta_i\cdot\bar\omega_{ij}'. \qquad \text{Q.E.D.}$$

DEFINITION. The compact complex manifold Y is said to satisfy Condition D with respect to $\mathfrak{A} \subseteq H^0(Y, \Theta)$ if the following hold:

(i) j ; $\mathfrak{A} \otimes H^1(Y, \Omega) \to H^1(Y, \Theta)$ is injective

(ii) if $\theta \in \mathfrak{A}$, $\bar{\omega} \in H^1(X, \Omega)$, then there exists a C^∞ function $f = f(\theta, \bar{\omega})$ such that $L_\theta \bar{\omega} = \bar{\partial} f$.

We now see what Condition (ii) in the above definition means geometrically. Let Y be a compact complex manifold, and let $0 \to \mathbf{Z} \to \Omega \xrightarrow{\exp} \Omega^* \to 0$ be the canonical exact sheaf sequence (here Ω^* = sheaf of non-zero holomorphic functions). There is the cohomology mapping $\eta : H^1(Y, \Omega) \to H^1(Y, \Omega^*)$, and each $\bar{\omega} \in H^1(Y, \Omega)$ determines a line bundle $\eta(\bar{\omega}) \in H^1(Y, \Omega^*)$. The Atiyah sequence for the principal bundle $P(\bar{\omega})$ of $\eta(\bar{\omega})$ is $0 \to \mathbf{1} \to \mathbf{Q}(\bar{\omega}) \to T(Y) \to 0$, and we have the connecting homomorphism

$$H^0(Y, \Theta) \xrightarrow{\delta^0} H^1(Y, \Omega). \tag{10.13}$$

PROPOSITION 10.6. *For* $\theta \in H^0(Y, \Theta)$,

$$L_\theta \bar{\omega} = -\delta^0(\theta) \ \textit{in} \ (10.13).$$

Proof. The following was proven in [17] (and may be easily checked directly): Let Y be any complex manifold, and $A \to P \to Y$ an analytic principal bundle with Atiyah sequence $0 \to \mathbf{L}(P) \to \mathbf{Q}(P) \to T(Y) \to 0$. Then, if Ξ is the $(1,1)$ curvature form arising from a connexion of type $(1,0)$ in P (i.e. a connexion respecting the complex structure), and if $\theta \in H^0(Y, \Theta)$, then $\delta^0(\theta) = i(\theta)\,\Xi$ where $\delta^0 : H^0(Y, \Theta) \to H^1(Y, \mathbf{L}(P))$ is the connecting map. Recall that a $(1,0)$ connexion ω in P is given by a certain collection of $(1,0)$ forms $\{\omega_i\}$ (ω_i in U_i) with values in $\mathbf{L} \mid U_i$, and then $\Xi = \{\bar{\partial}\omega_i\}$ is the (global) curvature form. For a line bundle with transition functions $\{f_{ij}\}$, we may find in each U_i a C^∞ $(1,0)$ form ω_i such that $\partial \log f_{ij} = \omega_i - \omega_j$, and then $\{\bar{\partial}\omega_i\} = \Xi$ is a suitable curvature form.

Returning to the proposition, if $\{U_i\}$ is a suitable covering of Y, we let $\bar{\omega} \in H^1(Y, \Omega)$ be given by the Čech cocycle $\{\omega_{ij}\}$; we may find C^∞ functions ξ_i in U_i such that $\omega_{ij} = \xi_i - \xi_j$ in $U_i \cap U_j$, and then $\{\bar{\partial}\xi_i\}$ is a Dolbeault representative for $\bar{\omega}$. The line bundle $\eta(\bar{\omega})$ transition functions $f_{ij} = \exp \omega_{ij}$, and $\partial \log f_{ij} = \partial\omega_{ij} = \partial\xi_i - \partial\xi_j$. Hence a curvature form for $P(\bar{\omega})$ is given by $\bar{\partial}\partial\xi_i = -\partial\bar{\partial}\xi_i = -\partial\bar{\omega}$. On the other hand, $L_\theta \bar{\omega} = i(\theta)\, d\bar{\omega} = i(\theta)\, \partial\bar{\omega}$, and the Proposition follows from this.

COROLLARY. $L_\theta \bar{\omega} \sim 0$ *if and only if the action of* $\exp(t\theta)$ *on* Y *lifts to bundle action in* $P(\bar{\omega})$.

THEOREM 12. *Let Y satisfy Condition D with respect to a subalgebra $\mathfrak{A} \subseteq H^0(Y, \Theta)$, and assume that the forms $\bar{\omega} \in H^1(Y \cdot \Omega)$ and the functions $f = f(\theta, \bar{\omega})$ are chosen to be real analytic.*(1)

(i) *Every indecomposable element $\theta \otimes \bar{\omega} \in H^1(Y, \Omega)$ is tangent to a 1-parameter amily of deformations $Y_t = Y(\theta, \bar{\omega}; t)$.*

(ii) *If $\mathfrak{h} \subseteq \mathfrak{A}$ is maximal abelian, then $\mathfrak{h} \otimes H^1(Y, \Omega)$ parametrizes a local deformation space, which is maximal in $\mathfrak{A} \otimes H^1(Y, \Omega)$ if $\dim H^1(Y, \Omega) > 1$.*

Proof. Let $\{U_i\}$ be a covering of Y with coordinates $Z_i = (Z_i^1, -, Z_i^n)$ in U_i. We shall construct real-analytic functions $\zeta_i^\alpha(Z, t)$ in U_i (with $\zeta_i^\alpha(Z, 0) = Z_i^\alpha$) and transition functions $h_{ij}^\alpha(\zeta_j(Z, t), t)$ satisfying (10.8)–(10.9) with the further condition that

$$\theta \otimes \bar{\omega} \mid U_i = \sum_\alpha \bar{\partial}\,(\dot{\zeta}_i^\alpha(Z, t))_0 \frac{\partial}{\partial Z_j^\alpha}.$$

In view of Proposition 10.4, this will prove (i) in Theorem 12. In order to construct the $\zeta_i(Z, t) = (\zeta_i^1(Z, t, \ldots, \zeta_i^n(Z, t))$, we shall use the *Frobenius Theorem* in the real analytic case, which we now state in a convenient form. (Of course, the Newlander–Nirenberg Theorem ([23]) would do in the C^∞ case, but the full strength of this is not necessary here.)

Suppose that we have given a global section $\Phi(t)$ of $T(Y) \otimes \overline{T(Y)}'$ (i.e., a vector-valued $(0, 1)$ form) with $\Phi(0) = 0$ which is real-analytic in Y and in the variable t. We consider the system of partial differential equations

$$\left.\begin{aligned} &\bar{\partial}\varphi - \sum_\beta \Phi_i^\beta(Z, t_0)\frac{\partial \varphi}{\partial Z_i^\beta} = 0, \\ &\frac{\partial \varphi}{\partial \bar{t}} = 0 \end{aligned}\right\} \tag{10.14}$$

and we seek n functionally independent real-analytic solutions $\varphi = \zeta_i^\alpha(Z, t)$. The Frobenius Theorem states that the *integrability condition* for (10.14) is

$$\bar{\partial}\Phi(t) - \{\Phi(t), \Phi(t)\} \equiv 0 \ \text{(in } t), \tag{10.15}$$

and when (10.15) is satisfied, a solution exists. Furthermore, we see from (10.14) that, for a solution $\zeta_i^\alpha(Z, t)$, $\bar{\partial}(\dot{\zeta}_i^\alpha(Z, t))_0 - (\dot{\Phi}_i^\alpha(Z, t))_0 = 0$; i.e., the global vector-valued (10.1) from $\dot{\Phi}(t)]_0$ gives the infinitesimal deformation to the family of manifolds $Y_{\Phi(t)} = Y_t$ defined by (10.14). Thus, to prove (i), we must produce a real-analytic

(1) By using a real-analytic metric, this is always possible. Also, the forms with which we are working on non-Kähler C-spaces satisfy this condition by Proposition 10.3.

$$\Phi(\theta, \bar{\omega}; t) = \Phi(t) \text{ with } \Phi(0) = 0 \text{ and } \left[\frac{d}{dt}\Phi(t)\right]_{t=0} = \theta \otimes \bar{\omega}$$

which satisfies (10.15). Clearly we must write $\Phi(t)$ as a series in t with leading term $(\theta \otimes \bar{\omega})\, t$ and determine the higher coefficients to be real-analytic and such that $\Phi(t)$ satisfies (10.15).

Suppose we write formally

$$\Phi(t) = \sum_{j=0}^{\infty} q_j (\theta \otimes f^j \bar{\omega})\, t^{j+1} \quad (L_\theta \bar{\omega} = \bar{\partial} f)$$

and try to determine the coefficients q_j to meet our requirements. Setting $\sigma_n(t) = \sum_{j=0}^{n} q_j (\theta \otimes f^j \bar{\omega})\, t^{j+1}$, then (10.15) is equivalent to

$$\bar{\partial}\sigma_n(t) - \{\sigma_n(t), \sigma_n(t)\} \equiv 0 \bmod t^{n+2}. \tag{10.16}$$

The following formulae may be checked inductively:

$$\{(\theta \otimes f^j \bar{\omega})\, t^{j+1}, (\theta \otimes f^k \bar{\omega})\, t^{k+1}\} = \frac{2}{j+k+1} \bar{\partial}(\theta \otimes f^{j+k+1} \bar{\omega})\, t^{j+k+2}, \tag{10.17}$$

$$\{\sigma_n(t), \sigma_n(t)\} = \sum_{j,k=0}^{n} \frac{2 q_j q_k}{j+k+1} \bar{\partial}(\theta \otimes f^{j+k+1} \bar{\omega})\, t^{j+k+2}. \tag{10.18}$$

Setting $q_0 = 1$, we may thus determine the q_j $(j \geqslant 1)$ inductively from (10.17) and (10.18). Indeed, we have

$$q_n = \sum_{j+k=n-1} \frac{2 q_j q_k}{j+k+1}. \tag{10.19}$$

From this it follows that $q_n = 2^n$. This is true for $n = 0$, and, if true for $n-1$, then

$$q_n = \sum_{i+k=n-1} \frac{2 q_j q_k}{n} = \frac{2(n)\, 2^{n-1}}{n} = 2^n.$$

Thus, if we set $\|f\| = \operatorname{Sup}_{y \in Y} |f(y)|$, then the series $\Phi(t) = \sum_{j=0}^{\infty} 2^j (\theta \otimes f^j \bar{\omega})\, t^{j+1}$ converges to a vector-valued form $\Phi(t)$ satisfying all our requirements for $|t| < (2\|f\|)^{-1}$. This completes the proof of (i).

Now if we have a general element $\xi = \sum h_j \otimes \bar{\omega}_j \in \mathfrak{h} \otimes H^1(Y, \Omega)$, then the proof is, in principal, the very same as above for $h \otimes \bar{\omega}$. The conditions $[h_i, h_j] = 0$ will guarantee that we may recursively determine the coefficients q_j as above (i.e., all the obstructions vanish). The rest of the theorem follows in the same way as Theorem 11. Q.E.D.

Remarks. (i) The above Theorem clearly yields the local variations of structure of a non-Kähler C-space (by Proposition 10.4). This construction has a slightly different flavor from the examples of deformations known to the author. There are two differences which we mention:

(a) These non-Kähler C-spaces are the only examples known to the author where an infinite series is definitely needed to define a deformation through an element $\xi \in H^1(Y, \Theta)$. Indeed, for algebraic curves, complex tori, hypersurfaces in P_N, Hopf surfaces, etc., the vector forms $\Phi(t)$ defining the deformations are *polynomials* in t. (That the series in our case is infinite follows from $f(\theta, \bar{\omega}) \not\equiv 0$, which has the geometric interpretation that, on a Kähler C-space, there is no linear connexion invariant under the automorphism group.)

(b) To the author's knowledge, the known examples in deformation have been constructed in what might be called an extrinsic fashion; i.e., the construction has used on auxiliary space such as a projective space in which the variety is embedded (hypersurfaces in P_N) or the universal covering space (complex tori and Hopf manifolds) or, for algebraic curves, the Siegel space. These auxiliary spaces help one get information about the deformed manifolds (e.g. the jumps of structure on the Hirzebruch examples), whereas we do not know much to say about the manifolds $X(h, \bar{\omega}, t)$. We do know that these manifolds are non-homogeneous, and what we shall do is to determine, to some extent, their automorphism groups.

If X is a non-Kähler C-space, then there is a fibering $T^{2a} \to X \to \hat{X}$ and the complex Lie group T^{2a} of automorphisms of X will clearly "live on" to the manifolds $X(h, \bar{\omega}, t)$ (essentially because these automorphisms leave $h \otimes \bar{\omega}$ fixed); what we shall see now is that the subgroup $G_h = \{g \in G \mid \operatorname{Ad} g(h) = h\}$ is the maximal subgroup of G which still acts on $X(h, \bar{\omega}, t)$, and this shows that the manifolds $X(h, \bar{\omega}, t)$ are no longer homogeneous.

(iii) The Question of Equivalences

Let $X = G/U = M/V$ be a non-Kähler C-space. From § 10, (ii) we have associated to each $g \in \mathfrak{g} \subset H^0(X, \Theta)$, $\bar{\omega} \in H^1(X, \Omega)$, and $t(|t|$ small) a non-homogeneous complex manifold $X(g, \bar{\omega}; t)$. Letting "$\leftrightarrow$" denote biregular equivalence, it may well happen that $X(g, \bar{\omega}; t) \leftrightarrow X(g, \bar{\omega}; t)$ for distinct triples $(g, \bar{\omega}; t)$ and $(g', \bar{\omega}'; t')$. We shall now show that this equivalence occurs whenever $t = t'$, $\bar{\omega} = \bar{\omega}'$, and $g' = \operatorname{Ad} m(g)$ for any $m \in M$; it is our feeling that these are essentially the only equivalences.

THEOREM 13. $X(g, \bar{\omega}; t) \leftrightarrow X(\operatorname{Ad} m(g), \bar{\omega}; t)$.

Proof. The proof will be done in three steps.

(i) Let D, D' be two domains in $\mathbf{C}^n$, $f: D \to D'$ a bi-holomorphic mapping. Suppose given in D a vector field θ, a function h, and a 1-form φ; also assume that we have similarly θ', h', and φ' in D' and that $f_*(\theta)=\theta'$, $f^*(h')=h$, $f^*(\varphi')=\varphi$.

LEMMA 10.1. *If w' is a function in D' such that $h'\theta'\otimes\varphi'(w')=h'(\theta'(w'))\varphi'=\bar\partial' w'$, then*

$$h\theta\otimes\varphi(w'\circ f)=h(\theta(w'\circ f))\varphi=\bar\partial(w'\circ h).$$

Proof. Let $p\in D$, $p'=f(p)\in D'$. Then, by assumption, $h'(p')\theta'(w')(p')\varphi'(p')=\bar\partial w'(p')$. We then have:

$$\begin{aligned}h(p)\,\theta(w'\circ f)(p)\,\varphi(p)&=(f^*h')(p)\,f_*\theta(w')(p')\,\varphi(p)\\&=h'(p')\,\theta'(w')(p')\,\varphi(p).\end{aligned}$$

If t is any tangent to D at p,

$$\begin{aligned}h(p)\,\theta(w'\circ f)(p)\,\langle\varphi,t\rangle_p&=h'(p')\,\theta'(w')(p')\,\langle f^*\varphi',t\rangle_p\\&=h'(p')\,\theta'(w')(p')\,\langle\varphi',f_*t\rangle_{p'}\\&=\langle\bar\partial' w',f_*t\rangle_{p'}=\langle f^*(\bar\partial' w'),t\rangle_p\\&=\langle\bar\partial(w'\circ f),t\rangle_p,\end{aligned}$$

the last step being because f is holomorphic. However, the equation

$$h(p)\,\theta(w'\circ f)(p)\,\langle\varphi,t\rangle_p=\langle\bar\partial(w'f),t\rangle_p$$

it what was to be proven.

(ii) Let $g\in\mathfrak{g}\subset H^0(X,\Theta)$, $\bar\omega\in H^1(X,\Omega)$. In § 10, (ii) we associated to g and $\bar\omega$ a C^∞ function f_g on X defined (up to a constant) by $\bar\partial f_g=L_g(\bar\omega)=i(g)\,\partial\bar\omega$.

LEMMA 10.2. *If $g'=\operatorname{Ad} m(g)$ for some $m\in M$, then $m^*f'_g=f_g$ where m acts as an automorphism on X.*

Proof. It will suffice to show that $m^*\bar\partial f_{g'}=\bar\partial f_g$. If t is any tangent to X, then

$$\begin{aligned}\langle m^*\bar\partial f_{g'},t\rangle&=\langle\bar\partial f_{g'},m_*t\rangle=\langle i(g')\,\partial\bar\omega,m_*t\rangle\\&=\langle\partial\bar\omega,m_*(m_*^{-1}g'\wedge t)\rangle=\langle m^*\partial\bar\omega,\operatorname{Ad} m^{-1}(g')\wedge t\rangle\\&=\langle\partial\bar\omega,g\wedge t\rangle=\langle i(g)\,\partial\bar\omega,t\rangle\\&=\langle\bar\partial f_g,t\rangle.\quad\text{Q.E.D.}\end{aligned}$$

(iii) We consider the complex manifolds $Y=X(g,\bar\omega;t)$ and $Y'=X(g',\bar\omega;t)$ where $g'=\operatorname{Ad} m(g)$. Both manifolds have the same underlying differentiable structure and we may define a C^∞ mapping $m: Y \to Y'$ to be simply the action of $m \in M$ on M/V; it will suffice to show that m is complex analytic.

In § 10, (ii) we constructed global vector-valued forms $\Phi = \Phi(g, \bar\omega; t)$ and $\Phi' = \Phi(g', \bar\omega; t)$ such that the local differential equations

$$\left.\begin{aligned} \bar\partial w - \sum_{\beta=1}^{n} \Phi^\beta \frac{\partial w}{\partial z^\beta} = 0, \\ \bar\partial' w' - \sum_{\beta=1}^{n} \Phi'^\beta \frac{\partial w'}{\partial z'^\beta} = 0 \end{aligned}\right\} \tag{10.20}$$

defined local complex analytic coordinates w and w' on Y and Y' respectively. From (10.11) it follows that Φ and Φ' are of the form $f_g g \otimes \bar\omega$ and $f_{g'} g' \otimes \bar\omega$ where $m_* g = g'$ and, by Lemma 10.3, $m^* f_{g'} = f_g$.

Let $D \subset Y$ be a coordinate neighborhood, $D' = m(D)$ and we may assume that D' is a coordinate neighborhood on Y'. Since the equations (10.18) define the respective complex structures in D and D', to prove Theorem 13 we may show: if w' on D' is a solution of

$$\bar\partial' w' - \sum_{\beta=1}^{n} \Phi'^\beta \frac{\partial w'}{\partial z'^\beta} = 0,$$

then the function $w = w' \circ m$ on D is a solution of

$$\bar\partial w - \sum_{\beta=1}^{n} \Phi^\beta \frac{\partial w}{\partial z^\beta} = 0.$$

However, this follows from Lemma 10.2 and the above remarks concerning Φ and Φ'. Q.E.D.

11. Some General Results on Homogeneous Vector Bundles

(i) On the Equivalence Question for Homogeneous Vector Bundles

X is taken to be a C-space G/U and all bundles are analytic bundles over X. Up to now, we have defined homogeneous bundles extrinsically as being associated to the fibering $U \to G \to X$ by a holomorphic representation $\varrho: U \to U'$; we now give an intrinsic definition. If $A \to P \xrightarrow{\bar\omega} X$ is any principal bundle, we have defined the complex Lie group $F(P)$ of bundle automorphisms of P:

$$F(P) = \{\text{biregular mappings } f: P \to P \,|\, f(pa) = f(p)a,\ p \in P,\ a \in A\}.$$

There is a natural homomorphism $\tilde{\omega}: F(P) \rightarrow A(X)$ and P may be said to be homogeneous if $\tilde{\omega}(F(P))$ is transitive on X; clearly extrinsic homogeneity $\Rightarrow$ intrinsic homogeneity. If conversely, $\tilde{\omega}(F(P))$ is transitive on X, we pick a fixed point $p_0 \in P$ and let $A' \subset F(P)$ be the stability group of the fibre $\tilde{\omega}^{-1}(\tilde{\omega}(p_0))$. There is then a homomorphism $\sigma: A' \rightarrow A$ defined by $\sigma(a') = a$ if $a' p_0 = p_0 a$; the bundle $A \rightarrow P \rightarrow X$ is then associated to $A' \rightarrow F(P) \rightarrow F(P)/A \cong X$. We have

PROPOSITION (Matsushima). *The two definitions of homogeneity are equivalent.*

DEFINITION 11.1. If $\mathbf{E}$, $\mathbf{F}$ are vector bundles of the same fibre dimension then $\mathbf{E}$ is *equivalent* to $\mathbf{F}$, written $\mathbf{E} \sim \mathbf{F}$, if there is a $\varphi \in \Gamma(\mathrm{Hom}(\mathbf{E}, \mathbf{F}))$ which is an isomorphism on fibres.

DEFINITION 11.2. If $\mathbf{E}^\varrho$, $\mathbf{E}^\tau$ are homogeneous vector bundles, then they are *homogeneously equivalent,* written $\mathbf{E}^\varrho \approx \mathbf{E}^\tau$, if $\mathbf{E}^\varrho \sim \mathbf{E}^\tau$ and φ may be chosen to be M-invariant.

LEMMA 11.1. $\mathbf{E}^\varrho \approx \mathbf{E}^\tau \Leftrightarrow E^\varrho$ *is equivalent to* E^τ *as a* U-*module.*

Proof. (1.9)

The converse is not true, for if ϱ is any non-trivial representation of G, then $\varrho | U = \varrho'$ gives rise to $\mathbf{E}^{\varrho'}$ and $\mathbf{E}^{\varrho'} \sim \mathbf{1}^m$ ($m = \dim E^{\varrho'}$) but $\mathbf{E}^{\varrho'} \not\approx \mathbf{1}^m$. In view of this, we should only speak of $\mathbf{E}^\varrho$ as being a *homogeneous representation* of the class of bundles $\mathbf{E}$ such that $\mathbf{E} \sim \mathbf{E}^\varrho$. This representation is unique in the following case.

PROPOSITION 11.1. *If* $\mathbf{E}^\varrho$, $\mathbf{E}^\tau$ *are line bundles, then* $\mathbf{E}^\varrho \sim \mathbf{E}^\tau \Leftrightarrow \mathbf{E}^\varrho \approx \mathbf{E}^\tau$.

Proof. Lemma 11.1 and § 6.

We discuss briefly the two following questions:

(i) Does an exact sequence of homogeneous vector bundles necessarily arise from an exact sequence of U-modules?

(ii) If $j: \mathbf{E} \rightarrow \mathbf{F}$ is an injection of homogeneous bundles, does it arise from an injection $E^\varrho \rightarrow F^\tau$ of U-modules?

Clearly (ii) $\Rightarrow$ (i); we shall, however, give a counter-example to (ii) and two examples to support (i) although we do not know if it is true in general.

Let $X = P_2(\mathbf{C}) = G/U$ where $U =$ unimodular matrices of the form

$$\left\{ u = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \right\};$$

then the line bundle $\mathbf{H}=\mathbf{H}^{\varrho}$of a hyperplane is given by $\varrho(u)=a_{11}$. By § 15, (i) and (ii) we have $H^0(X, \mathbf{H}^s) \neq 0$ if $s>0$. Choose integers α, r_1, r_2, r_3 such that $-\alpha+r_i>0$ $(i=1,2,3)$; we then get non-zero mappings $\gamma_j: \mathbf{H}^{\alpha} \to \mathbf{H}^{r_j} (j=1,2,3)$. Since the divisors of γ_j are curves in general position, we may assume that the γ_j are never simultaneously zero; this defines an injection $j: \mathbf{H}^{\alpha} \to \mathbf{H}^{r_1} \oplus \mathbf{H}^{r_2} \oplus \mathbf{H}^{r_3}$. Because of Lemma 11.3 and the Krull–Schmidt theorem, j does not arise from an injection of U-modules.

We now give an example of an exact sequence of homogeneous vector bundles which, under one representation, does not arise from an exact sequence of U-modules, but which, under another representation, does. Let X be Kähler and let $\mathbf{E}^{\varrho} \to X$ be a homogeneous vector bundle with sufficiently many sections (§ 8). In the notation of that §, we have:

$$0 \to \mathbf{F}^{\varrho} \to H^0(X, \mathcal{E}^{\varrho}) \times X \to \mathbf{E}^{\varrho} \to 0.$$

The middle bundle may be obtained either by the trivial action of U, or by restricting the action of G on $H^0(X, \mathcal{E}^{\varrho})$ to U. In the first case, we do not get an exact sequence of U-modules, whereas in the second case we do.

Finally, we have the following proposition which will be proven in § 11 (iii).

PROPOSITION. *If we have* $0 \to \mathbf{E}^{\varrho} \to \mathbf{E} \to \mathbf{E}^{\tau} \to 0$, $\dim E^{\varrho} = \dim E^{\tau} = 1$, *then* $\mathbf{E}$ *is homogeneous* $\Leftrightarrow \mathbf{E} \sim \mathbf{E}^{\sigma}$ *for some* E^{σ} *such that* $0 \to E^{\varrho} \to E^{\sigma} \to E^{\tau} \to 0$ *is* U*-exact.*

We formalize (i) by the following definition:

DEFINITION 11.3. Let $(S): 0 \to \mathbf{E}^{\varrho} \to \mathbf{E} \to \mathbf{E}^{\tau} \to 0$ be an exact sequence of homogeneous vector bundles. We say that (S) is *strongly homogeneous* if $\mathbf{E} \sim \mathbf{E}^{\sigma}$ for some σ such that $0 \to E^{\varrho} \to E^{\sigma} \to E^{\tau} \to 0$ is U-exact.

(ii) Extension Theory of Homogeneous Vector Bundles

Let Y be an arbitrary complex manifold; suppose that $E \to \mathbf{E} \to Y$, $H \to \mathbf{H} \to Y$ are analytic vector bundles over Y.

DEFINITION 11.4. Ext $(\mathbf{H}, \mathbf{E})$, the classes of extensions of $\mathbf{H}$ by $\mathbf{E}$, consist of those analytic vector bundles $F \to \mathbf{F} \to Y$ such that we have the exact sequence

$$(S): 0 \to \mathbf{E} \to \mathbf{F} \to \mathbf{H} \to 0. \tag{11.3}$$

If we consider the vector space Hom (H, E), then $GL(E) \times GL(H)$ is represented on this vector space as follows: for $\xi \in \text{Hom}\ (H, E)$ $e \in GL(E)$, $h \in GL(H)$,

$$(e \times h)(\xi) = e\xi h^{-1}. \tag{11.4}$$

It is known that Ext $(\mathbf{H}, \mathbf{E})$ may be given a vector space such that

$$\text{Ext}(\mathbf{H}, \mathbf{E}) \cong H^1(Y, \widetilde{\text{Hom}}(\mathbf{H}, \mathbf{E})). \tag{11.5}$$

This isomorphism may be made explicit as follows. Let $U = \{U_j\}$ be a suitable covering of Y, set $N = N(U) =$ nerve of U, and suppose that $\mathbf{E}, \mathbf{H}$ have transition functions $e_{ij}: U_i \cap U_j \to GL(E)$, $h_{ij}: U_i \cap U_j \to GL(H)$ respectively. If $\mathbf{F} \in \text{Ext}(\mathbf{H}, \mathbf{E})$ has transition functions $f_{ij}: U_i \cap U_j \to GL(F)$, then we may write

$$f_{ij} = \begin{pmatrix} e_{ij} & g_{ij} \\ 0 & h_{ij} \end{pmatrix}.$$

Define a mapping

$$\zeta : \text{Ext}(\mathbf{H}, \mathbf{E}) \to H^1(N(U), \widetilde{\text{Hom}}(\mathbf{H}, \mathbf{E}))$$

as follows: $\zeta(\mathbf{F}) =$ cocycle $\{\gamma_{ij}\}$ given by

$$\gamma_{ij} = g_{ij} h_{ij}^{-1} : U_i \cap U_j \to \text{Hom}(\mathbf{H}, \mathbf{E}) \mid U_i \cap U_j.$$

It is not hard to check that ζ sets up the isomorphism (11.5).

Suppose now that $X = G/U$ is a C-space and let $\varrho: U \to GL(E^\varrho)$, $\sigma: U \to GL(E^\sigma)$ give homogeneous vector bundles $E^\varrho \to \mathbf{E}^\varrho \to X$, $E^\sigma \to \mathbf{E}^\sigma \to X$, respectively. Then we have

$$H^1(X, \widetilde{\text{Hom}}(\mathbf{E}^\sigma, \mathbf{E}^\varrho)) = \sum_{\lambda \in D(\mathfrak{g})} V^\lambda \otimes H^1(\mathfrak{n}, \text{Hom}(E^\sigma, E^\varrho) \otimes V^{-\lambda})^{\tilde{\mathfrak{b}}^\circ} \tag{11.6}$$

(see § 1) where $\mathfrak{g}$ acts on $V^\lambda \otimes H^1(\mathfrak{n}, \text{Hom}(E^\sigma, E^\varrho) \otimes V^{-\lambda})^{\tilde{\mathfrak{b}}^\circ}$ by $\lambda \otimes 1$.

THEOREM 14. *Let X be a C-space and let $\mathbf{E}^\varrho, \mathbf{E}^\sigma$ be homogeneous vector bundles. Then $\mathbf{E} \in \text{Ext}(\mathbf{E}^\sigma, \mathbf{E}^\varrho)$ is strongly homogeneous $\Leftrightarrow \zeta(\mathbf{E}) \in H^1(\mathfrak{n}, \text{Hom}(E^\sigma, E^\varrho))^{\tilde{\mathfrak{b}}^\circ}$; i.e. $\zeta(\mathbf{E})$ is M-invariant.*

Proof. If $\mathbf{E} \in \text{Ext}(\mathbf{E}^\sigma, \mathbf{E}^\varrho)$ is homogeneous, then by definition, there exists a holomorphic representation $\tau: U \to GL(E^\tau)$ such that $\mathbf{E}^\tau \sim \mathbf{E}$ and $0 \to E^\varrho \to E^\tau \to E^\sigma \to 0$ is an exact sequence of U-modules.

Conversely, such an exact sequence of U-modules gives a homogeneous element $\mathbf{E} \in \text{Ext}(\mathbf{E}^\sigma, \mathbf{E}^\varrho)$. The proof is completed by the following two lemmas together with the above discussion of the extension cocycle.

LEMMA 11.2. *Let $\mathfrak{k}$ be a Lie algebra and let $\mathfrak{a}$, $\mathfrak{b}$ be $\mathfrak{k}$-modules. Then the classes of exact sequences of $\mathfrak{k}$-modules $0 \to \mathfrak{a} \to \mathfrak{c} \to \mathfrak{b} \to 0$ are in a one-to-one correspondence with* $H^1(\mathfrak{k}, \mathrm{Hom}(\mathfrak{b}, \mathfrak{a}))$

LEMMA 11.3. *Given the $\mathfrak{u}$-modules E^ϱ, E^σ and $f \in H^1(\mathfrak{n}, \mathrm{Hom}(E^\sigma, E^\varrho))$, we form by Lemma 11.2 the exact sequence of $\mathfrak{n}$-modules $0 \to E^\varrho \to E^\tau \to E^\sigma \to 0$. Considering this as the trivial exact sequence of $\tilde{\mathfrak{v}}^0$-modules, then E^τ is a $\mathfrak{u} = \mathfrak{n} \oplus \tilde{\mathfrak{v}}^0$-module $\Leftrightarrow f \in H^1(\mathfrak{n}, \mathrm{Hom}(E^\sigma, E^\varrho))^{\mathfrak{v}^0}$ in which case $0 \to E^\varrho \to E^\tau \to E^\sigma \to 0$ is $\mathfrak{u}$-exact.*

Remark. Lemma 11.2 follows the usual pattern of extension theorems; we shall use the constructions in the proof several times in the sequel.

Proof of Lemma 11.2 (Outline). Let $\varrho: \mathfrak{k} \to gl(\mathfrak{a})$, $\sigma: \mathfrak{k} \to gl(\mathfrak{b})$ be the representations defining the $\mathfrak{k}$-modules $\mathfrak{a}$, $\mathfrak{b}$ respectively, if $f \in H^1(\mathfrak{k}, \mathrm{Hom}(\mathfrak{b}, \mathfrak{a}))$, then $f \in C^1(\mathfrak{k}, \mathrm{Hom}(\mathfrak{b}, \mathfrak{a}))$ and $df = 0$. If $k \in \mathfrak{k}$, $\alpha \in \mathrm{Hom}(\mathfrak{b}, \mathfrak{a})$

$$k \circ \alpha = \varrho(k)\,\alpha - \alpha\,\sigma(k), \tag{11.7}$$

and for $k, k' \in \mathfrak{k}$,

$$k' \circ f(k) - k \circ f(k') = -f[k, k']. \tag{11.8}$$

Define $\mathfrak{c} = \mathfrak{a} \oplus \mathfrak{b}$ and let $\gamma_f: \mathfrak{k} \to gl(\mathfrak{c})$ be defined by

$$\gamma_f(k) = \begin{pmatrix} \varrho(k) & f(k) \\ 0 & \sigma(k) \end{pmatrix}. \tag{11.9}$$

The fact that $\gamma_f[k, k'] = \gamma_f(k)\,\gamma_f(k') - \gamma_f(k')\,\gamma_f(k)$ follows by a simple computation from (11.8).

To complete the lemma, we must show: $f = dg$ for some $g \in C^0(\mathfrak{k}, \mathrm{Hom}(\mathfrak{b}, \mathfrak{a})) = \mathrm{Hom}(\mathfrak{b}, \mathfrak{a}) \Leftrightarrow \mathfrak{c}$ is equivalent to $\mathfrak{a} \oplus \mathfrak{b}$ as a $\mathfrak{k}$-module. If $f = dg$, it follows from (11.7) that

$$\begin{pmatrix} I & g \\ 0 & I \end{pmatrix} \begin{pmatrix} \varrho(k) & f(k) \\ 0 & \sigma(k) \end{pmatrix} = \begin{pmatrix} \varrho(k) & 0 \\ 0 & \sigma(k) \end{pmatrix} \begin{pmatrix} I & g \\ 0 & I \end{pmatrix}$$

for all $k \in \mathfrak{k}$. On the other hand, if there exist $x \in \mathrm{Hom}(\mathfrak{a}, \mathfrak{a})$, $y \in \mathrm{Hom}(\mathfrak{b}, \mathfrak{b})$, $z \in \mathrm{Hom}(\mathfrak{b}, \mathfrak{a})$ such that for all $k \in \mathfrak{k}$,

$$\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \begin{pmatrix} \varrho(k) & f(k) \\ 0 & \sigma(k) \end{pmatrix} = \begin{pmatrix} \varrho(k) & 0 \\ 0 & \sigma(k) \end{pmatrix} \begin{pmatrix} x & z \\ 0 & y \end{pmatrix},$$

then if follows easily that $f = d(x^{-1}z)$. Q.E.D.

Proof of Lemma 11.3. We know that $\mathfrak{u} = \mathfrak{n} \oplus \tilde{\mathfrak{v}}^0$ and, in the above notation, we define γ_f on $\mathfrak{u}$ by

$$\left.\begin{aligned} \gamma_f(n) &= \begin{pmatrix} \varrho(n) & f(n) \\ 0 & \sigma(n) \end{pmatrix} \quad n \in \mathfrak{n}, \\ \gamma_f(v) &= \begin{pmatrix} \varrho(v) & 0 \\ 0 & \sigma(v) \end{pmatrix} \quad v \in \tilde{\mathfrak{v}}^0. \end{aligned}\right\} \tag{11.10}$$

Since $[\tilde{\mathfrak{v}}^0, \mathfrak{n}] \subseteq \mathfrak{n}$, it will suffice to show:

$$\gamma_f[v, n] = [\gamma_f(v), \gamma_f(n)] \Leftrightarrow f \in C^1(\mathfrak{n}, \operatorname{Hom}(E^\sigma, E^\varrho))^{\tilde{\mathfrak{v}}^0}.$$

This is done by a straightforward computation which we omit.

This concludes the proof of Theorem 14.

Example. We give an example of the construction made in the proof of Theorem 14. Let $X = G/U$ be a *non*-Kähler C-space with fundamental fibering $T^{2a} \to G/U \xrightarrow{\pi} G/\hat{U} = \hat{X}$. Writing $\mathfrak{p} = \hat{\mathfrak{u}}/\mathfrak{u}$, $\hat{\mathfrak{n}}^* = \mathfrak{g}/\hat{\mathfrak{u}}$, $\mathfrak{n} = \mathfrak{g}/\mathfrak{u}$, we have an exact sequence of U-modules

$$0 \to \mathfrak{p} \to \mathfrak{n}^* \to \hat{\mathfrak{n}}^* \to 0. \tag{11.11}$$

Following the notations of § 9, (11.11) gives the exact sequence of vector bundles

$$0 \to \mathfrak{p} \to (\mathfrak{n}^*_{\mathrm{Ad}}) \to (\hat{\mathfrak{n}}^*_{\mathrm{Ad}}) \to 0 \tag{11.12}$$

where $\mathfrak{p}$ is now the trivial bundle $\mathfrak{p} \times X$. Note that $(\hat{\mathfrak{n}}^*_{\mathrm{Ad}}) = \pi^{-1}(\hat{\mathfrak{n}}^*_{\mathrm{Ad}}) = \pi^{-1}(T(\hat{X}))$ and $\mathfrak{p}$ is the bundle along the fibres of the fundamental fibering.

Now $\zeta((\mathfrak{n}^*)_{\mathrm{Ad}}) \in H^1(X, \widetilde{\operatorname{Hom}}((\hat{\mathfrak{n}}^*_{\mathrm{Ad}}), p)) \cong H^1(X, \operatorname{Hom}((\hat{\mathfrak{n}}^*_{\mathrm{Ad}}), \mathbf{C})) \otimes \mathfrak{p}$; furthermore $\zeta((\mathfrak{n}^*)_{\mathrm{Ad}}) \neq 0$ since (11.12) doesn't split analytically. Let $\hat{\Omega}^1 =$ sheaf of germs of holomorphic $(1,0)$ forms on $\hat{X}$, we have from Proposition 5.3

$$H^1(X, \operatorname{Hom}((\hat{\mathfrak{n}}_{\mathrm{Ad}}), \mathbf{C})) \cong H^1(X, \pi^{-1}(\hat{\Omega}^1)) \cong H^1(\hat{X}, \hat{\Omega}^1).$$

The structure of $H^1(\hat{X}, \hat{\Omega}^1)$ was given in § 4 and may be described as follows: let $[\tilde{\mathfrak{v}}^0] = [\tilde{\mathfrak{v}}^0, \tilde{\mathfrak{v}}^0]$ and let $\lambda \in \mathfrak{h}'$ be orthogonal to $[\tilde{\mathfrak{v}}^0] \cap \mathfrak{h}$; setting $f_\lambda = \sum_{\alpha \in \Sigma^+ - \psi^+} \langle \lambda, h_\alpha \rangle \omega^\alpha \wedge \bar{\omega}^\alpha$, the f_λ generate $H^1(\hat{X}, \hat{\Omega}^1) \cong H^2(\hat{X}, \mathbf{C})$. Written in $C^1(\mathfrak{n}, \operatorname{Hom}(\hat{\mathfrak{n}}^*, \mathfrak{p}))^{\tilde{\mathfrak{v}}^0}$, the general element of $H^1(X, \operatorname{Hom}((\hat{\mathfrak{n}}^*_{\mathrm{Ad}}), \mathfrak{p}))$ is of the form

$$\sum_{\lambda; p \in \mathfrak{p}} f_\lambda \otimes p. \tag{11.13}$$

Let $p_1, \dots, p_a$ be a basis of $\mathfrak{p}$, $\lambda_1, \dots, \lambda_a$ a dual basis; then the λ_j are orthogonal to $[\tilde{\mathfrak{v}}^0] \cap \mathfrak{h}$.

PROPOSITION 11.2. *Writing* $T(X) = (\mathfrak{n}^*_{\mathrm{Ad}})$, *the element* $\zeta((\hat{\mathfrak{n}}^*_{\mathrm{Ad}})) \in H^1(X\ \mathrm{Hom}\ ((\hat{\mathfrak{n}}^*_{\mathrm{Ad}}), \mathfrak{p}))$ *is given by the M-invariant form*

$$\zeta^{\#} = -\sum_{j=1}^{a} \sum_{\alpha \in \Sigma^+ - \psi^+} \langle \lambda_j, h_\alpha \rangle\, \omega^\alpha \wedge \bar{\omega}^\alpha \otimes p_j. \tag{11.14}$$

Proof. The proof consists of applying the proof of Lemma 11.2 and tracing through a few isomorphisms—we only give the outline. Notation: if V is a vector space, $V^{\#} \subset V$ a subspace with a splitting $V = V^{\#} \oplus V^b$ the projection of $v \in V$ on $V^{\#}$ along V^b is denoted by $v]_{V^{\#}}$. What we must show is the following: for $p \in \mathfrak{p}$, $n^* \in \hat{\mathfrak{n}}^*$, $n \in \mathfrak{n}$, then

$$n \circ (p + n^*) = \mathrm{ad}\ n(p)]_{\mathfrak{p}} + \zeta^{\#}(n)\,(n^*) + \mathrm{ad}\ n(n^*)]_{\hat{\mathfrak{n}}^*}. \tag{11.15}$$

The notation $\zeta^{\#}(n)\,(n^*)$ needs a little explanation: $\zeta^{\#} \in \mathrm{Hom}\ (\mathfrak{n} \otimes \mathfrak{n}^*, \mathfrak{p})$ and by definition, $\zeta^{\#}(n)\,(n^*) = \zeta^{\#}(n \otimes n^*)$. If $n \in \mathfrak{p}^*$, $\zeta(p) = 0$ and (11.15) is trivial; we may assume $n = e_{-\alpha}$ for some $\alpha \in \sum^+ - \psi^+$. Then

$$n \circ (p + n^*) = [n, n^*]_{\mathfrak{n}^*} = [n, n^*]_{\mathfrak{p}} + [n, n^*]_{\hat{\mathfrak{n}}^*}$$

and we are done unless $n^* = e_\alpha$. Then $n \circ n = e_{-\alpha} \circ (e_\alpha) = h_\alpha]_{\mathfrak{p}}$: in the right side of (11.15) only the middle term is $\neq 0$ and

$$\zeta^{\#}(e_{-\alpha})\ (e_\alpha) = \zeta^{\#}(e_{-\alpha} \otimes e_\alpha) = +\sum_{j=1}^{a} \langle \lambda_j, h_\alpha \rangle\, p_j = (h_\alpha)_{\mathfrak{p}}. \quad \text{Q.E.D.}$$

(iii) On the Deformation Theory of Homogeneous Vector Bundles

Let Y be an arbitrary complex manifold, $E \to \mathbf{E} \to Y$ a vector bundle associated to the principal bundle $A \to P \to Y$ where A is a complex Lie group. In § 10, (i) we briefly discussed the deformation theory of the bundle P. There we were varying the bundle structure of P keeping A as the group; if A is a subgroup of A', from $A \to P \to Y$ we get $A' \to P' \to Y$ and the deformation theory for P' is in general quite different from that of P. For example, if $A = GL(r, \mathbf{C}) \times GL(s, \mathbf{C})$ and $A' = GL(r+s, \mathbf{C})$, then deforming $\mathbf{E}$ in A maintains a direct sum decomposition $\mathbf{E} = \mathbf{E}' \oplus \mathbf{E}''$ while such is not in general the case for A'. Thus as a preliminary to studying the full variation of P, we shall restrict the size of the group within which the deformation is taking place.

(α) *The Kähler Case*

Let $X=G/U$ be a Kähler C-space; for simplicity, U-modules will in general be rational U-modules (considering U as an algebraic group). We shall now give a cohomological description of homogeneous vector bundles over X. Let $E \to \mathbf{E} \to X$ be a vector bundle with principal bundle $A \to P \to X$; as usual, we have in this situation the Atiyah sequence $0 \to \mathbf{L}(P) \to \mathbf{Q}(P) \to T(X) \to 0$.

THEOREM 15. *A necessary and sufficient condition that P be homogeneous is that the structure group of P be reducible to a subgroup $A' \subset A$ such that $\mathbf{L}(P')$ constructed from $A' \to P' \to X$ should satisfy*

$$H^q(X, \widetilde{\mathbf{L}(P')}) = 0 \quad (q>0). \tag{11.16}$$

Thus the homogeneous bundles are those bundles which, with a suitable structure group, are locally rigid.

Proof. If (11.16) is satisfied, then we have $H^0(X, \widetilde{\mathbf{Q}(P')}) \to H^0(X, \Theta) \to 0$ and since $H^0(X, \widetilde{\mathbf{Q}(P')})$ is the Lie algebra of infinitesimal bundle automorphisms, P' is homogeneous (§ 11 (i)).

We prove that homogeneous bundles satisfy (11.16). Let $\varrho: U \to GL(E^\varrho)$ be defined so that $\mathbf{E}^\varrho \sim \mathbf{E}$ and set $U' = \varrho(U) \subseteq GL(E^\varrho)$. Then we have exact sequences of U-modules

$$\left.\begin{array}{l} 0 \to \hat{\mathfrak{u}} \to \mathfrak{u} \xrightarrow{\varrho_*} \mathfrak{u}' \to 0, \\ 0 \to \mathfrak{u}' \to gl(E^\varrho) \to gl(E^\varrho)/\mathfrak{u}' \to 0, \end{array}\right\} \tag{11.17}$$

where $\hat{\mathfrak{u}} = \ker \varrho_*: \mathfrak{u} \to gl(E^\varrho)$. Since, for $u \in U$, $\mathfrak{u}^* \in \mathfrak{u}$, Ad $\varrho(u)\,\varrho_*(u^*) = \varrho_*$ Ad $u(u^*)$, we have from (11.17) the exact bundle sequences

$$\left.\begin{array}{l} 0 \to \hat{\mathbf{L}} \to \mathbf{L} \to \mathbf{L}' \to 0, \\ 0 \to \mathbf{L}' \to \mathbf{L}^\varrho \to \mathbf{L}^\varrho/\mathbf{L}' \to 0, \end{array}\right\} \tag{11.18}$$

where $\hat{\mathbf{L}} = G \times_U \hat{\mathfrak{u}}$, $\mathbf{L} = G \times_U \mathfrak{u}$, $\mathbf{L}' = G \times_U \mathfrak{u}'$, $\mathbf{L}^\varrho = G \times_U gl(E^\varrho)$ and all actions are adjoint action or its composition with ϱ. From Theorem 2 we have that $H^{q-1}(X, \tilde{\mathbf{L}}') \cong H^q(X, \tilde{\hat{\mathbf{L}}})$ $(q>0)$ and $H^0(X, \tilde{\hat{\mathbf{L}}}) = 0$. Theorem 15 will be proven if we prove

LEMMA 11.4. $H^q(X, \tilde{\hat{\mathbf{L}}}) = 0$ $(q>1)$ *and* $\dim H^1(X, \tilde{\hat{\mathbf{L}}}) = n(s,d)$ *where* $n(s,d) = \{$*number of negative simple roots in* $\hat{\mathfrak{u}}\} - \{\dim \hat{\mathfrak{u}} \cap \mathfrak{h}\}$.

Proof. We refer to the proof of Theorem 2, § 4; as was done there, we assume that $\mathfrak{u}$ is solvable (see the remark below). The weights of U acting on $\hat{\mathfrak{u}}$ are the 0-weight with multiplicity $=\{\dim \hat{\mathfrak{u}} \cap \mathfrak{h}\}$ and some of the negative roots. We assert that if $h \in \hat{\mathfrak{u}} \cap \mathfrak{h}$, then all root vectors $e_{-\alpha} (\alpha \in \sum^{+})$ with $\langle \alpha, h \rangle \neq 0$ also lie in $\hat{\mathfrak{u}}$. Indeed, $u^* \in \mathfrak{u}$ lies in $\hat{\mathfrak{u}} \Leftrightarrow \varrho_*(u^*) = 0$; since

$$\varrho_*(e_{-\alpha}) = -\frac{1}{\langle \alpha, h \rangle} \varrho_*([e_{-\alpha}, h]) = -\frac{1}{\langle \alpha, h \rangle} [\varrho_*(e_{-\alpha}), \varrho_*(h)] = 0,$$

our assertion is proven. Thus, $\dim(\mathfrak{h} \cap \hat{\mathfrak{u}}) \leqslant \{$number of simple roots α such that $\varrho_*(e_{-\alpha}) = 0\}$ and consequently $n(s, d) \geqslant 0$. Now by using Proposition 4.2, we conclude again as in the proof of Theorem 2 that $\dim H^1(X, \hat{\tilde{\mathbf{L}}}) = n(s, d)$.

COROLLARY. $H^1(X, \tilde{\mathbf{L}}^{\varrho}) \cong H^1(X, \tilde{\mathbf{L}}^{\varrho}/\tilde{\mathbf{L}}')$.

Remark. The proof of the lemma when $X = M/\hat{V}$ and $\hat{V}$ is not abelian is done in the same manner as when $\hat{V} = T$ using the following observation: if $\tilde{\mathfrak{v}}^0 = \mathfrak{z} \oplus \tilde{\mathfrak{v}}_1^0 \oplus \dots \oplus \tilde{\mathfrak{v}}_r^0$ where $\mathfrak{z}$ is abelian and the $\tilde{\mathfrak{v}}_j^0$ are simple, then either $\varrho_*(\tilde{\mathfrak{v}}_j^0) = 0$ or ϱ_* is injective on $\tilde{\mathfrak{v}}_j$.

As an application, we prove

PROPOSITION 11.3. *Let X be Kähler and let* $(S): 0 \to \mathbf{E}^{\varrho} \to \mathbf{E} \to \mathbf{E}^{\tau} \to 0$ *be an exact sequence of line bundles. Then* $\mathbf{E}$ *is homogeneous* $\Leftrightarrow \zeta(\mathbf{E}) \in H^1(X, \widetilde{\mathbf{E}^{-\tau} \mathbf{E}^{\varrho}})$ *is M-invariant.*

Proof. We consider first the following general situation:

Let Y be a complex manifold, $U' \subset GL(r, \mathbf{C})$, $U'' \subset GL(s, \mathbf{C})$ complex linear groups, and $\mathbf{E}' \to Y$, $\mathbf{E}'' \to Y$ analytic vector bundles with groups U', U'' respectively. Then if we have $(S): 0 \to \mathbf{E}' \to \mathbf{E} \to \mathbf{E}'' \to 0$. $\mathbf{E}$ has group $G(U', U'') \subset GL(r+s, \mathbf{C})$ where

$$G(U', U'') = \left\{ g \in GL(r+s, \mathbf{C}) \mid g = \begin{pmatrix} u' & \xi \\ 0 & u'' \end{pmatrix} \right\},$$

where $u' \in U'$, $u'' \in U''$, and $\xi \in \operatorname{Hom}(\mathbf{C}^r, \mathbf{C}^s)$. Thus, as a vector space, $\mathfrak{g}(\mathfrak{u}', \mathfrak{u}'') = \operatorname{Hom}(\mathbf{C}^r, \mathbf{C}^s) \oplus \mathfrak{u}' \oplus \mathfrak{u}''$ and for

$$g = \begin{pmatrix} u' & \xi \\ 0 & u'' \end{pmatrix} \in G(U', U''), \ \gamma = (\eta \oplus \gamma' \oplus \gamma'') \in \mathfrak{g}(\mathfrak{u}', \mathfrak{u}''),$$

$$\begin{aligned} \operatorname{Ad} g \circ \eta = u' \eta (u'')^{-1} - (\operatorname{Ad} u' \circ \gamma')(\xi (u'')^{-1}) \\ + \xi (u'')^{-1} \operatorname{Ad} u'' \circ \gamma'' \oplus \operatorname{Ad} u' \circ \gamma' \oplus \operatorname{Ad} u'' \circ \gamma''. \end{aligned} \tag{11.19}$$

From this we have $0 \to \mathrm{Hom}\,(\mathbf{E}'', \mathbf{E}') \to \mathbf{L} \to \mathbf{L}' \oplus \mathbf{L}'' \to 0$ where $\mathbf{L}, \mathbf{L}', \mathbf{L}''$ refer to $\mathbf{E}, \mathbf{E}'\ \mathbf{E}''$ respectively.

If X is Kähler, $\mathbf{E}', \mathbf{E}''$ are homogeneous and U', U'' are chosen to satisfy (11.16), then we have

$$H^0(X, \mathcal{L}') \oplus H^0(X, \mathcal{L}'') \xrightarrow{\delta_0} H^1(X, \widetilde{\mathrm{Hom}}\,(\mathbf{E}'', \mathbf{E}')) \to H^1(X, \mathcal{L}) \to 0. \tag{11.20}$$

In particular, in the situation where $\mathbf{E}', \mathbf{E}''$ are line bundles, $H^0(X, \mathcal{L}') \cong \mathbf{C} \cong H^0(X, \mathcal{L}'')$; by the result in [12] coupled with (11.19), $\delta_0(0 \oplus 1) = \zeta(\mathbf{E})$, $\delta_0(1 \oplus 0) = -\zeta(\mathbf{E})$. If $\zeta(\mathbf{E})$ is M-invariant, then dim $H^1(X, \widetilde{\mathbf{E}^{-\tau}\mathbf{E}^{\varrho}}) = 1$ and $H^1(X, \mathcal{L}) = 0$ for the group $G(U', U'')$ and $\mathbf{E}$ is homogeneous.

If $\zeta(\mathbf{E})$ is not M-invariant, then dim $H^1(X, \mathbf{E}^{-\tau}\mathbf{E}^{\varrho}) > 1$ and dim $H^1(X, \mathcal{L}) > 0$. In order to prove the proposition, we need only observe that, since dim $E^{\tau} = 1 = \dim E^{\varrho}$, there exists no group A to which the group of $\mathbf{E}$ can be reduced and such that $\mathfrak{u}' \oplus \mathfrak{u}'' \subset \mathfrak{a} \subset \mathfrak{g}(\mathfrak{u}', \mathfrak{u}'')$ (proper inclusions). Q.E.D.

(β) *The non-Kähler Case*

The analogue of Theorem 15 is not true in the non-Kähler case (e.g. line bundles) and there are in general obstructions (§ 9); we shall give only a brief outline of the picture. Let $X = G/U$ be non-Kähler with fundamental fibering $T^{2a} \to X \to \hat{X} = G/\hat{U}$. Let $\hat{\varrho}: \hat{U} \to \hat{U}' \subset GL(E^{\varrho})$ be a rational representation and set $\hat{\varrho} \mid U = \varrho: U \to U' \subset GL(E^{\varrho})$. If $\hat{\mathfrak{u}}^* = \ker \hat{\varrho}_*$, we set $n(s, d) = \{$number of simple roots in $\hat{\mathfrak{u}}^*\} - \{\dim \mathfrak{u}^* \cap \mathfrak{h}\}$. Writing $\hat{\mathfrak{u}} = \mathfrak{u} \oplus \mathfrak{p}$, we let $\mathfrak{p}^* = \mathfrak{p} \cap \mathfrak{u}^*$, $\mathfrak{p}' = \hat{\varrho}_*(\mathfrak{p}) \subseteq \hat{\mathfrak{u}}'$, $\mathfrak{u}^* = \mathfrak{u} \cap \hat{\mathfrak{u}}^*$, and $a^* = \dim\ \mathfrak{p}^*$ so that $a - a^* = a' = \dim \mathfrak{p}'$.

PROPOSITION 11.4.

(i) *$H^q(X, \mathcal{L}')$ is a trivial M-module.*

(ii) *If $\mathfrak{p}' = 0$, then* dim $H^1(X, \mathcal{L}) = a(n(s, d))$. *(Such is the case if ϱ is the complexification of a representation of V.)*

(iii) *If $n(s, d) = 0$, then* dim $H^1(X, \mathcal{L}') = a'$. *(Such is the case if ϱ is the identity and then $a' = a$.)*

Postponing the proof a moment, we describe a corollary. Because of (i), $H^q(X, \mathcal{L}') = H^q(\mathfrak{n}, \mathfrak{u}')^{\tilde{\mathfrak{v}}^0}$; there is thus a natural pairing $\langle\, , \rangle$:

$$H^p(X, \mathcal{L}') \otimes H^q(X, \mathcal{L}') \to H^{p+q}(X, \mathcal{L}')$$

obtained from that in $H^*(\mathfrak{n}, \mathfrak{u}')^{\tilde{\mathfrak{v}}^0}$ by bracketing elements in $\mathfrak{u}'$.

Corollary. *If* $\{,\}: H^1(X, \mathfrak{L}') \otimes H^1(X, \mathfrak{L}') \to H^2 X, \mathfrak{L}')$ *is the obstruction bracket* (§ 9), *then* $\{,\} = \langle,\rangle$. *(This shows that there are, in general, obstructions.)*

Proof of Proposition 11.4. The following is easily checked to be an exact diagram of $\hat{U}$ modules, the actions in the last column being trivial:

$$\left.\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & \mathfrak{u}' & \to & \hat{\mathfrak{u}}' & \to & \mathfrak{p}' & \to 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & \mathfrak{u} & \to & \hat{\mathfrak{u}} & \to & \mathfrak{p} & \to 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & \mathfrak{u}^* & \to & \hat{\mathfrak{u}}^* & \to & \mathfrak{p}^* & \to 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
\end{array}\right\} \tag{11.21}$$

Letting $\mathfrak{u}^{\#}$ be any of the symbols in (11.21) involving a $\mathfrak{u}$, we let $\mathbf{L}^{\#}$ be the corresponding homogeneous vector bundle on $\hat{X}$; the trivial bundles arising from the last column of (11.21) are denoted by the same symbols as the modules. Writing $H^q(\hat{X}, \cdot) = H^q(\cdot)$, we have from Theorem 2, Proposition 9.2, Theorem 3, and Lemma 11.4 the following: $H^q(\hat{\mathfrak{L}}) = 0$ for all q, $H^q(\mathfrak{L}) = 0$ $(q \neq 1)$ and $H^1(\mathfrak{L}) \cong \mathfrak{p}$, $H^0(\tilde{\mathfrak{p}}^{\#}) = \mathfrak{p}^{\#}$ and $H^q(\tilde{\mathfrak{p}}^{\#}) = 0$ for $q > 0$ where $\mathfrak{p}^{\#} = \mathfrak{p}, \mathfrak{p}^*$, or $\mathfrak{p}$, $H^q(\hat{\mathfrak{L}}^*) = 0$ for $q \neq 1$ and $\dim H^1(\hat{\mathfrak{L}}^*) = \dim H^0(\hat{\mathfrak{L}}') = n(s, d)$, and finally $H^q(\hat{\mathfrak{L}}') = 0$ for $q > 0$. Using these, we get from (11.21):

$$\left.\begin{array}{cccccccccc}
 & & 0 & & 0 & & & & & \\
 & & \uparrow & & \uparrow & & & & & \\
0 & \to & \mathfrak{p}' & \to & H^1(\mathfrak{L}') & \to & 0 & & & \\
 & & \uparrow & & \uparrow & & & & & \\
0 & \to & \mathfrak{p} & \to & \mathfrak{p} & \to & 0 & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & \\
0 & \to & \mathfrak{p}^* & \to & H^1(\mathfrak{L}^*) & \to & H^1(\hat{\mathfrak{L}}^*) & \to & 0 & \\
 & & \uparrow & & \uparrow & & \uparrow & & & \\
 & & 0 & \to & H^0(\mathfrak{L}') & \to & H^0(\hat{\mathfrak{L}}') & \to & \mathfrak{p}' \to H^1(\mathfrak{L}') & \to 0 \\
 & & & & \uparrow & & \uparrow & & & \\
 & & & & 0 & & 0 & & &
\end{array}\right\} \tag{11.22}$$

From § 5, there exists a spectral sequence $\{E_r\}$ such that E_∞ belongs to $H^*(X, \mathfrak{L}')$ and $E_2^{p,q} = H^p(\mathfrak{p}) \otimes H^q(\mathfrak{L}')$; in any event, $H^*(X, \mathfrak{L}')$ is a trivial M-module. In general both $H^0(\mathfrak{L}')$ and $H^1(\mathfrak{L}')$ are $\neq 0$ so that their spectral sequence is non-trivial. If (ii) is satisfied, then $H^1(\mathfrak{L}')=0$ and $H^1(X,\mathfrak{L}')\cong H^1(\mathfrak{p})\otimes H^0(\mathfrak{L}')$, which proves (ii). If (iii) is satisfied, then $H^0(\mathfrak{L}')=0$ and dim $H^1(\mathfrak{L}')=a'$ from (11.22); this proves (iii).

Remark. The corollary follows from the fact that, *contrary* to the case for $H^1(X,\Theta)$, the obstruction bracket in $H^1(X,\mathfrak{L}')$ is over the sheaf of holomorphic functions Ω and involves no differentiation.

12. Some applications of § 11

We give some general geometric applications of Theorem 14. Let $\hat{X}=G/\hat{U}=M/\hat{V}$ be a Kähler C-space; suppose that $(\alpha_1,\dots,\alpha_r)$ is a system of simple roots of $(\mathfrak{h},\mathfrak{g})$ such that $(\alpha_1,\dots,\alpha_s)$ $(s<r)$ are the simple roots of $(\mathfrak{h},\tilde{\mathfrak{v}}^0)$.

THEOREM 16.

(i) *Let $E\to\mathbf{E}\to\hat{X}$ be an indecomposable vector bundle with complex nilpotent group N as structure group. Then* $\mathbf{E}$ *is a homogeneous line bundle.*

(ii) *If* $\dim_{\mathbf{C}} E=2$, *there exists an indecomposable vector bundle $E\to\mathbf{E}\to\hat{X}$ with solvable structure group $\Leftrightarrow$ there exists an α_j $(j>s)$ such that $(\alpha_j,\alpha_i)=0$ for $1\leqslant i\leqslant s$.*

COROLLARY 1. ([*16*]). *Any vector bundle over $\hat{X}$ with nilpotent structure group is homogeneous.*

COROLLARY 2. *If $b_2(\hat{X})=1$, then every bundle $E\to\mathbf{E}\to\hat{X}$ with solvable structure group is decomposable into a sum of line bundles.*

COROLLARY 3. *There exists a Kähler C-space $\hat{X}$ (any flag with dim >1) and a non-homogeneous plane bundle over $\hat{X}$ with solvable structure group.*

Proofs. (i) Let $\{U_j\}$ be a suitable covering of $\hat{X}$ so that $\mathbf{E}$, assumed indecomposable, has transition functions

$$n_{ij}=\begin{pmatrix} a_{ij} & & \cdots \\ 0 & & \cdots \\ \vdots & & \\ 0 & \cdots\ 0 & a_{ij}\end{pmatrix}.$$

Suppose for a moment that $\dim_{\mathbf{C}} E=2$; then $n_{ij}=\begin{pmatrix} a_{ij} & \cdot \\ 0 & a_{ij}\end{pmatrix}$ where the a_{ij} are the transition functions of a (homogeneous) line bundle $\mathbf{A}$ and we have $0\to\mathbf{A}\to\mathbf{E}\to\mathbf{A}\to 0$. But then $\zeta(\mathbf{E})\in H^1(\hat{X},\mathbf{A}^{-1}\mathbf{A})=H^1(X,\mathbf{C})=0$ and $\mathbf{E}\sim\mathbf{A}\oplus\mathbf{A}$. An obvious induction completes the proof.

(ii) Let $\dim_{\mathbf{C}} \mathbf{E}=2$ and $E \to \mathbf{E} \to \hat{X}$ be indecomposable. Then $\mathbf{E}$ has transition functions

$$s_{ij} = \begin{pmatrix} a_{ij} & \cdot \\ 0 & b_{ij} \end{pmatrix};$$

we have line bundles $\mathbf{A}$, $\mathbf{B}$ with transition functions (a_{ij}), (b_{ij}) and from the exact sequence $0 \to \mathbf{A} \to \mathbf{E} \to \mathbf{B} \to 0$ and $\zeta(\mathbf{E}) \in H^1(\hat{X}, \widetilde{\mathbf{B}^{-1}\mathbf{A}})$. The proof is completed by the following lemma:

LEMMA 12.1. *There exists a line bundle* $E^\sigma \to \mathbf{E}^\sigma \to \hat{X}$ *such that* $H^1(\hat{X}, \mathcal{E}^\sigma) \neq 0 \Leftrightarrow$ *there exists an* $\alpha_j (j > s)$ *such that* $(\alpha_j, \alpha_i) = 0$ *for* $1 \leqslant i \leqslant s$.

Proof. Any line bundle $\mathbf{E}^\sigma$ on $\hat{X}$ is given by a weight σ on $\mathfrak{h}$ such that $(\sigma, \alpha_i) = 0$ for $1 \leqslant i \leqslant s$. There exists a σ with $H^1(\hat{X}, \mathcal{E}^\sigma) \neq 0 \Leftrightarrow$ there exists an $\alpha_j (s < j \leqslant r)$ such that $\tau_{\alpha_j}(\sigma + g) - g \in D(\mathfrak{g})$. (See § 1 and Theorem B.) Let $\tilde{\omega}_1, \dots \tilde{\omega}_r$ be the fundamental weights of $(\mathfrak{h}, \mathfrak{g})$; they are characterized by $q(\tilde{\omega}_j, \alpha_i) = 2(\tilde{\omega}_j, \alpha_i)/(\alpha_i, \alpha_i) = \delta_j^i$ (for all i, j). If a σ exists with $H^1(\hat{X}, \mathcal{E}^\sigma) \neq 0$, then $\tau_{\alpha_j}(\sigma + g) - g = n_1 \tilde{\omega}_1 + \dots + n_r \tilde{\omega}_r$ where all the n_i are non-negative integers. But then

$$\begin{aligned} \sigma + g - g + \alpha_j &= \tau_{\alpha_j}(\tau_{\alpha_j}(\sigma + g) - g) \\ &= \tau_{\alpha_j}(n_1 \tilde{\omega}_1 + \dots + n_r \tilde{\omega}_r) \\ &= n_1 \tilde{\omega}_1 + \dots + n_r \tilde{\omega}_r - n_j \alpha_j; \end{aligned}$$

i.e.

$$\sigma = n_1 \tilde{\omega}_1 + \dots + n_r \tilde{\omega}_r - (n_j + 1) \alpha_j.$$

But for $\alpha_i \in \psi^+ (1 \leqslant i \leqslant s)$,

$$0 = (\sigma, \alpha_i) = 2 n_i / (\alpha_i, \alpha_i) - (n_j + 1)(\alpha_i, \alpha_j)$$

and since $n_j + 1 > 0$, $(\alpha_i, \alpha_j) \leqslant 0$, we conclude that $(\alpha_i, \alpha_j) = 0$. The argument is reversible and we are done.

Now let X be a *non*-Kähler C-space and $E \to \mathbf{E} \to X$ an indecomposable analytic vector bundle.

THEOREM 16'. *If* $\mathbf{E}$ *has a complex nilpotent group* N *as structure group, then* $\mathbf{E}$ *is homogeneous but is not in general a sum of line bundles.*

COROLLARY. *Any analytic vector bundle* $E \to \mathbf{E} \to X$ *with nilpotent structure group is homogeneous.*

Proof. Let $\{U_j\}$ be a suitable covering of X such that $\mathbf{E}$ has transition functions

$$n_{ij} = \overbrace{\begin{pmatrix} a_{ij} & & \cdots \\ 0 & & \cdots \\ \vdots & & \\ 0 \ldots 0 & a_{ij} & \end{pmatrix}}^{n} \Bigg\} n.$$

By § 6, the theorem is true if $n=1$; we assume the result for $n-1$. Then we have the exact sequence $0 \to \mathbf{A} \to \mathbf{E} \to \mathbf{E}' \to 0$ where $\mathbf{A}$ is a line bundle with transition functions a_{ij} and $\mathbf{E}'$ is an $(n-1)$-dimensional bundle with transition functions

$$n'_{ij} = \overbrace{\begin{pmatrix} a_{ij} & & \cdots \\ 0 & & \cdots \\ \vdots & & \\ 0 \ldots 0 & a_{ij} & \end{pmatrix}}^{n-1} \Bigg\} n-1.$$

By the induction assumption, $\mathbf{E}'$ is homogeneous and $\zeta(\mathbf{E}) \in H^1(X, \widetilde{\mathrm{Hom}}\,(\mathbf{E}', \mathbf{A}))$. The bundle Hom $(\mathbf{E}', \mathbf{A})$ is an $(n-1)$-dimensional vector bundle with transition functions

$$\tilde{n}_{ij} = \begin{pmatrix} 1 & \cdots\cdots & \\ 0 & & \vdots \\ \vdots & & \vdots \\ 0 \cdots 0 & 1 & \end{pmatrix};$$

the results of § 3 tell us that $H^1(X, \widetilde{\mathrm{Hom}}\,(\mathbf{E}', \mathbf{A}))$ is a trivial M-module and Theorem 14 gives the result.

Remarks. The proof of Theorem 16 yields Corollary 2 to Theorem 16 only if dim $\hat{X} > 1$; the result for dim $\hat{X} = 1$ (i.e., $\hat{X} = P_1(\mathbf{C})$) is due to Grothendieck. Corollary 3 to Theorem 16 also holds in the non-Kähler case.

The above results were general geometric statements; we now give some specific examples; the general purpose is to construct "parameter varieties" for classes of bundles over C-spaces.

Let X be a non-Kähler C-space with fundamental fibering $T^{2a} \to X \to \hat{X}$.

PROPOSITION 12.1. *The class of indecomposable bundles over X with structure group* $N = \left\{ \begin{pmatrix} 1 & \cdot \\ 0 & 1 \end{pmatrix} \right\}$ *consists entirely of homogeneous entries and is parametrized by* $P_{a-1}(\mathbf{C})$.

Proof. Letting $\mathbf{1}$ = trivial line bundle and $\Omega = \tilde{\mathbf{1}}$ = sheaf of germs of holomorphic functions on X, we have Ext $(\mathbf{1}, \mathbf{1}) \cong H^1(X, \Omega) = \mathbf{C}^a$. The rest is easy. Q.E.D.

Let $E^\sigma \to \mathbf{E}^\sigma \to X$ be a homogeneous plane bundle over X with structure N where N is the same as in proposition 12.1. Then we have exact sequences

$$\begin{cases} 0 \to 1 \to E^\sigma \to 1 \to 0 & \text{(of } U\text{-modules)} \\ 0 \to \mathbf{1} \to \mathbf{E}^\sigma \to \mathbf{1} \to 0 & \text{(of vector bundles).} \end{cases}$$

From the exact cohomology sequence of the exact sheaf sequence $0 \to \Omega \to \mathcal{E}^\sigma \to \Omega \to 0$, it follows that $H^*(X, \mathcal{E}^\sigma)$ is a trivial M-module and $H^*(X, \mathcal{E}^\sigma) \cong H^*(\mathfrak{n}, E^\sigma)^{\tilde{\mathfrak{v}}^0}$. As an illustration of § 3, we give

PROPOSITION 12.2. *$H^*(X, \mathcal{E}^\sigma)$ is a trivial M-module and*

$$\left.\begin{aligned} \dim H^q(X, \mathcal{E}^\sigma) &= \binom{a}{q} \quad (q \leqslant a), \\ \dim H^q(X, \mathcal{E}^\sigma) &= 0 \quad (q > a). \end{aligned}\right\} \tag{12.1}$$

Proof. We follow previous notations. Then $\zeta(\mathbf{E}^\sigma) \in H^1(X, \Omega) \cong H^{0,1}(X, \mathbf{C})$ and thus $\zeta(\mathbf{E}^\sigma)$ is represented by a global $(0, 1)$ form $\bar{\omega}_1$ (see § 5). Let $\bar{\omega}_1, \ldots, \bar{\omega}_a$ be a basis of $H^1(X, \Omega)$ which at the origin gives a basis of $(\mathfrak{p}^*)'$ $(\mathfrak{p} = \hat{\mathfrak{u}}/\mathfrak{u},\ \mathfrak{n} = \hat{\mathfrak{n}} \oplus \mathfrak{p}^*)$. From the proof of Lemma 11.2, we may choose an isomorphism between E^σ and $\mathbf{C}^2$ such that $\sigma : \mathfrak{u} \to gl(E^\sigma)$ is of the following form:

$$\left.\begin{aligned} \sigma(v) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (v \in \tilde{\mathfrak{v}}^0), \\ \sigma(n) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (n \in \tilde{\mathfrak{n}}), \\ \sigma(p_j^*) &= \begin{pmatrix} 0 & \delta_j^i \\ 0 & 0 \end{pmatrix}. \end{aligned}\right\} \tag{12.2}$$

where $p_1^*, \ldots, p_a^*$ are a basis of $\mathfrak{p}^*$ dual to $\bar{\omega}_1, \ldots, \bar{\omega}_a$. It will suffice to compute $H^q(\mathfrak{n}, E^\sigma)^{\tilde{\mathfrak{v}}^0}$ from the exact cohomology sequence of $\mathfrak{n}$-modules $0 \to 1 \to E^\sigma \to 1 \to 0$. This cohomology sequence is

$$\cdots \to H^{q-1}(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \xrightarrow{\delta^{q-1}} H^q(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \to H^q(\mathfrak{n}, E^\sigma)^{\tilde{\mathfrak{v}}^0} \to H^q(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \xrightarrow{\delta^q} H^{q+1}(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \to \cdots \tag{12.3}$$

LEMMA 12.2. *Let τ be a $(0, r)$ form in $H^r(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \cong \Lambda^r(\mathfrak{p}^*)'$. Then*

$$\delta^r(\tau) = \bar{\omega}_1 \wedge \tau \in H^{r+1}(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \cong \Lambda^{r+1}(\mathfrak{p}^*)'.$$

Proof.

(i) $r=0$. In this case τ is a complex number. In the exact sequence

$$0 \to C^0(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \to C^0(\mathfrak{n}, E^\sigma)^{\tilde{\mathfrak{v}}^0} \to C^0(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \to 0,$$

we lift τ back to $\hat{\tau} = \begin{pmatrix} 0 \\ \tau \end{pmatrix} \in C^0(\mathfrak{n}, E^\sigma)^{\tilde{\mathfrak{v}}^0}$. Then, for $n \in \mathfrak{n}$,

$$d\hat{\tau}(n) = n \circ \hat{\tau} = n \circ \begin{pmatrix} 0 \\ \tau \end{pmatrix} = \begin{cases} 0 & n \in \hat{\mathfrak{n}} \\ \tau(\delta_j^1) & n = p_j^* \end{cases} \quad \text{by (12.2).}$$

(ii) r arbitrary. It suffices to take $\tau = \bar{\omega}_{i_1 \dots i_r} = \bar{\omega}_{i_1} \wedge \dots \wedge \bar{\omega}_{i_r}$; in

$$0 \to C^r(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \to C^r(\mathfrak{n}, E^\sigma)^{\tilde{\mathfrak{v}}^0} \to C^r(\mathfrak{n})^{\tilde{\mathfrak{v}}^0} \to 0,$$

we lift back to $\quad \hat{\tau} = \bar{\omega}_{i_1 \dots i_r} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in (\Lambda^r(\mathfrak{n}) \otimes E^\sigma)^{\tilde{\mathfrak{v}}^0} = C^r(\mathfrak{n}, E^\sigma)^{\tilde{\mathfrak{v}}^0}.$

Then, $\quad d\hat{\tau}(p_{j_1}^* \wedge \dots \wedge p_{j_{r+1}}^*) = \sum (-1)^{k+1} \delta_{i_1 \dots i_r}^{j_1 \dots \hat{j}_k \dots j_{r+1}} p_k^* \circ \begin{pmatrix} 0 \\ 1 \end{pmatrix},$

i.e., $d\hat{\tau} = \bar{\omega}_1 \wedge \bar{\omega}_{i_1 \dots i_r}$. Q.E.D.

Completion of the proof of (12.1). From (12.3) we see that

$$\begin{aligned} \dim H^q(\mathfrak{n}, E^\sigma)^{\tilde{\mathfrak{v}}^0} &= \binom{a}{q} - \dim(\delta^{q-1}) + \dim(\ker \delta^q) \\ &= \binom{a}{q} - \left(\binom{a}{q-1} - \dim(\ker \delta^{q+1}) \right) + \dim(\ker \delta^q) \\ &= \binom{a}{q} - \binom{a}{q-1} + \binom{a-1}{q-2} + \binom{a-1}{q-1} = \binom{a}{q}. \quad \text{Q.E.D.} \end{aligned}$$

Remark. $\dim H^1(X, \mathcal{E}^\sigma) = a$ and a basis for this vector space is given (in the above notation) by

$$\bar{\omega}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \bar{\omega}_j \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (j > 1). \tag{12.4}$$

We use Proposition 12.2 together with Theorem 14 to construct one more parameter variety. Let $\hat{H}$ be the complex unipotent group of complex matrices of the form

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

and denote by $G(2, a)$ the Grassmann variety of 2-planes in $\mathbf{C}^a$.

THEOREM 17. *The space of indecomposable (homogeneous) vector bundles over X with structure group H is parametrized by the disjoint union of the following three varieties:*

(i) B,

(ii) $G(2, a)$,

(iii) $G(2, a)$

where B is a vector bundle over $P_{a-1}(\mathbf{C})$ with fibre $\mathbf{C}^{a-1}$.

Proof. Let $\sigma : \mathfrak{u} \to \hat{\mathfrak{h}}$ where

$$\sigma(u) = \begin{pmatrix} 0 & a(u) & b(u) \\ 0 & 0 & c(u) \\ 0 & 0 & 0 \end{pmatrix}$$

give rise to $E^{\sigma} \to \mathbf{E}^{\sigma} \to X$. If H^* is the unipotent group of complex matrices of the form

$$\left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right\},$$

then, from σ, we construct $\tau : \mathfrak{u} \to \mathfrak{h}^*$ by setting

$$\tau(u) = \begin{pmatrix} 0 & a(u) \\ 0 & 0 \end{pmatrix}.$$

We have $E^{\tau} \to \mathbf{E}^{\tau} \to X$ and $0 \to \mathbf{E}^{\tau} \to \mathbf{E}^{\sigma} \to \mathbf{1} \to 0$ and thus

$$\zeta(\mathbf{E}^{\sigma}) \in H^1(X, \mathcal{E}^{\tau}) \cong H^1(\mathfrak{n}, E^{\tau})^{\tilde{\mathfrak{v}}^0}.$$

We must compute this group and pick out those bundles which are indecomposable when we allow τ to vary over representations of U in H^*. We treat cases:

(i) $\mathbf{E}^{\tau}$ indecomposable; then $\tau \neq 0$ (τ is not trivial) and

$$\tau(u) = \begin{pmatrix} 0 & a(u) \\ 0 & 0 \end{pmatrix} \quad (u \in \mathfrak{u}).$$

Writing $\mathfrak{u} = \hat{\mathfrak{n}} \oplus \mathfrak{p}^* \oplus \tilde{\mathfrak{v}}^0$, then

$$\begin{cases} a(v) = 0 & (v \in \tilde{\mathfrak{v}}^0) \\ a(n) = 0 & (n \in \hat{\mathfrak{n}}) \\ a(p_j^*) = \gamma_j & (p_1^*, \dots, p_a^* = \text{ basis of } p^*). \end{cases}$$

Thus a is in $(p^*)'$ and we may write $a = \sum \gamma_j \bar{\omega}_j$. Since (by assumption) $a \neq 0$, we may pick a new basis $\bar{\xi}_1\ldots,\bar{\xi}_a$ of $(\mathfrak{p}^*)'$ such that $a=\bar{\xi}_1$; by the remark following Proposition 12.2, a basis for $H^1(\mathfrak{n}, E^\tau)^{\tilde{\mathfrak{v}}^0}$ is $\bar{\xi}_1\begin{pmatrix}0\\1\end{pmatrix}, \bar{\xi}_j\begin{pmatrix}0\\1\end{pmatrix}$ $(j>1)$. Using this basis for $H^1(\mathfrak{n}, E^\tau)^{\tilde{\mathfrak{v}}^0}$ and letting $\varphi_1^*,\ldots,\varphi_a^*$ be a dual basis of $\mathfrak{p}^*$, if $(\varrho_1,\ldots,\varrho_a)\in H^1(\mathfrak{n}, E^\tau)^{\tilde{\mathfrak{v}}^0}\cong \operatorname{Ext}(\mathbf{E}^\tau, \mathbf{1})$, it follows that the bundle $\zeta^{-1}(\varrho_1,\ldots,\varrho_a)$ with structure group $\hat{H}$ is given by $\sigma:\mathfrak{u}\to\hat{\mathfrak{h}}$ where

$$\left.\begin{aligned}
&\sigma(v)=0 \qquad (v\in\tilde{\mathfrak{v}}^0),\\
&\sigma(n)=0 \qquad (n\in\hat{\mathfrak{n}}),\\
&\sigma\left(\sum\lambda_j\varphi_j^*\right)=\begin{pmatrix}0 & \lambda_1 & \sum_{j>1}\lambda_j\varrho_j\\ 0 & 0 & \lambda_1\varrho_1\\ 0 & 0 & 0\end{pmatrix}.
\end{aligned}\right\} \tag{12.5}$$

If $\varrho_1\neq 0$, we have the following:

The bundles $E^\sigma\to\mathbf{E}^\sigma\to X$ which have no decomposable sub- or quotient-bundle are parametrized by a space B which is a vector bundle with fibre $\mathbf{C}^{a-1}$ (parameters $\varrho_2,\ldots,\varrho_a$) over $P_{a-1}(\mathbf{C})\cong H^1(\mathfrak{n}, E^\tau)^{\tilde{\mathfrak{v}}^0}-0/\mathbf{C}^*$ (corresponding to indecomposable E^τ).

(ii) Since a bundle with group

$$\left\{\begin{pmatrix}1 & 0 & a\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

is decomposable, the only possibilities not covered in (i) are bundles with group

$$\hat{H}'=\left\{\begin{pmatrix}1 & 0 & e\\ 0 & 1 & f\\ 0 & 0 & 1\end{pmatrix}\right\}\ (f\neq 0)\quad\text{or}\quad \hat{H}^*=\left\{\begin{pmatrix}1 & f & e\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right\}\ (f\neq 0).$$

The two situations are dual and it suffices to treat H^*. Letting $\tau:\mathfrak{u}\to\hat{\mathfrak{h}}^*$ be given by $\tau(u)=\begin{pmatrix}0 & f(u)\\ 0 & 0\end{pmatrix}$, we have as usual $0\to\mathbf{E}^\tau\to\mathbf{E}^\sigma\to\mathbf{1}\to 0$ and $\zeta(\mathbf{E}^\sigma)\in H^1(\mathfrak{n}, E^\tau)^{\tilde{\mathfrak{v}}^0}$. Taking $\varrho_1=0$ in (12.5), to have indecomposability we must have $(\varrho_2,\ldots,\varrho_a)\neq(0,\ldots,0)$ and thus the indecomposable bundles with group H^* are parametrized by $G(2, a)$ (vectors $(1,0,\ldots,0)$ and $(0,\varrho_2,\ldots,\varrho_a)$). We omit further details and conclude the proof of Theorem 17.

We close this section with a statement about flags and a discussion relating homogeneous extensions to the outer automorphisms of certain complex unipotent Lie algebras.

Let $X=G/U=M/T$ be a flag manifold.

PROPOSITION 12.3. *If* $\mathbf{E}^\sigma$, $\mathbf{E}^\tau$ *are homogeneous line bundles over* X, *there are three mutually distinct possibilities for* Ext $(\mathbf{E}^\tau, \mathbf{E}^\sigma)$:

(i) Ext $(\mathbf{E}^\tau, \mathbf{E}^\sigma) = 0$ (*i.e., only trivial extension*).

(ii) Ext $(\mathbf{E}^\tau, \mathbf{E}^\sigma)$ *is composed entirely of non-homogeneous extensions*

$$\Leftrightarrow \dim H^1(X, \widetilde{\mathbf{E}^{-\tau}\,\mathbf{E}^\sigma}) > 1.$$

(iii) Ext $(\mathbf{E}^\tau, \mathbf{E}^\sigma)$ *is composed entirely of homogeneous extensions*

$$\Leftrightarrow \dim H^1(X, \widetilde{\mathbf{E}^{-\tau}\,\mathbf{E}^\sigma}) = 1.$$

If $\mathfrak{k}$ is any Lie algebra, then it is known and easily checked that $H^1(\mathfrak{k}, \mathfrak{k}) \cong$ space of outer automorphisms of $\mathfrak{k}$ (modulo inner automorphisms). Consider now the class of all unipotent complex Lie algebras $\mathfrak{n}$ which have the property that there exists a Kähler C-space $\hat{X} = G/\hat{U} = M/\hat{V}$ such that $\hat{\mathfrak{u}} = \mathfrak{n} \oplus \tilde{\hat{\mathfrak{v}}}^0$. Then $H^1(\mathfrak{n}, \mathfrak{n})$ is a $\hat{V}$-module (since $[\tilde{\hat{\mathfrak{v}}}^0, \mathfrak{n}] \subseteq \mathfrak{n}$) and thus

$$H^1(\mathfrak{n}, \mathfrak{n}) = \sum_{\gamma \in D(\mathfrak{v}^0)} m_\gamma E^\gamma, \tag{12.6}$$

where $m_\gamma =$ multiplicity of the irreducible $\hat{V}$-representation space E^γ with highest weight γ in $H^1(\mathfrak{n}, \mathfrak{n})$. Thus

$$\dim H^1(\mathfrak{n}, \operatorname{Hom}(\mathfrak{n}^*, E^\gamma))^{\tilde{\hat{\mathfrak{v}}}^0} = \dim (H^1(\mathfrak{n}, \mathfrak{n}) \otimes E^\gamma)^{\tilde{\hat{\mathfrak{v}}}^0} = m_{-\gamma} \text{ (Schur's lemma).}$$

On the other hand, $\dim H^1(\mathfrak{n}, \operatorname{Hom}(\mathfrak{n}^*, E^\gamma))^{\tilde{\mathfrak{v}}^0} =$ multiplicity of the trivial representation of M on $H^1(\hat{X}, \widetilde{\operatorname{Hom}}(T(\hat{X}), \mathbf{E}^\gamma)) =$ (Theorem 14) the dimension of the space of homogeneous extensions of the form

$$0 \to \mathbf{E}^\gamma \to \mathbf{E}^\sigma \to T(\hat{X}) \to 0. \tag{12.7}$$

PROPOSITION 12.4. *Let* $\mathfrak{n}$ *be a unipotent complex Lie algebra as described above. Then the outer isomorphisms of* $\mathfrak{n}$ *are in a one-to-one correspondence with the exact sequences (12.7) of homogeneous vector bundles as* E^γ *varies over the irreducible* $\hat{V}$*-modules.*

COROLLARY. *The outer isomorphisms of* $\mathfrak{n}$ *which commute with the action of* $\hat{V}$ *on* $\mathfrak{n}$ *are paired to the exact sequences*

$$0 \to \mathbf{1} \to \mathbf{E}^\sigma \to T(\hat{X}) \to 0 \tag{12.8}$$

($\mathbf{1} =$ *trivial bundle*) *and these form a vector space equal in dimension to the second Betti number* $b_2(\hat{X})$.

Remark. From Proposition 12.4, it is easily checked that if $\mathfrak{n} = \mathfrak{c}(e_{-\alpha} : \alpha \in \sum^+ - \psi^+)$, then the automorphisms corresponding to (12.8) are of the form $f_\tau(e_{-\alpha}) = -(\tau, \alpha) e_{-\alpha}$ where τ runs through the weights on $\mathfrak{h}$ which are orthogonal to ψ^+. A full description of $H^1(\mathfrak{n}, \mathfrak{n})$ is to appear in a paper by B. Kostant.

13. Bundles over Arbitrary Homogeneous Kähler Manifolds

Let X be a compact but not necessarily simply-connected homogeneous Kähler manifold; then $X = \hat{X}^a = \hat{X} \times T^{2a}$ where $\hat{X}$ is a Kähler C-space and $T^{2a} = \mathbf{C}^a/\Gamma$ where $\Gamma \subset C^a$ is a suitable lattice. We shall examine the geometry of homogeneous bundles over such an X. A few preparatory remarks which will be used later are helpful here.

LEMMA 13.1. *Let A, B, C be complex connected Lie groups with A a closed complex normal subgroup of B, $C = A/B$. Then the fibering $A \to B \to C$ has a holomorphic connexion.*

Proof. ([22]) Let $\mathfrak{a}$ = complex Lie algebra of A; the vertical space V_b at $b \in B$ is given by $L_b(\mathfrak{a}) = R_b R_b^{-1} L_b(\mathfrak{a}) R_b(\mathfrak{a})$ since $\mathfrak{a}$ is invariant in $\mathfrak{b}$. In the exact sequence

$$0 \to \mathfrak{a} \xrightarrow{j} \mathfrak{b} \xrightarrow{\mu} \mathfrak{c} \to 0. \tag{13.1}$$

choose a linear splitting map $\gamma : \mathfrak{c} \to \mathfrak{b}$ ($\mu \circ \gamma$ = identity). Then $\mathfrak{b} = j(\mathfrak{a}) \oplus \gamma(\mathfrak{c})$ and we may take the horizontal space H_b at $b \in B$ to be $R_b(\gamma(\mathfrak{c}))$. Q.E.D.

We denote this holomorphic connexion by γ and observe that the curvature Ξ_λ is a $(2, 0)$ form given at the identity by

$$\Xi_\gamma(c, c') = j^{-1}([\gamma(c), \gamma(c')]) \tag{13.2}$$

(c, $c' \in \mathfrak{c}$). It follows that

$$j(\Xi_\gamma(c, c')) = ([\gamma(c), \gamma(c')] - \gamma[c, c']); \tag{13.3}$$

henceforth we shall omit reference to j.

If now $\mathfrak{a}$ is abelian, the exact sequences (13.1) are in a one-to-one correspondence with $H^2(\mathfrak{c}, \mathfrak{a})$ (see the exercises in [6]) where $\mathfrak{c}$ acts on $\mathfrak{a}$ by "ad" ($\mathfrak{a}$ is an ideal in $\mathfrak{b}$ and ad $\mathfrak{a} \circ \mathfrak{a} = 0$). To get the obstruction to splitting (13.1), one chooses $\gamma : \mathfrak{c} \to \mathfrak{b}$ as above and then the obstruction cocycle $f_\gamma \in C^2(\mathfrak{c}, \mathfrak{a})$ is given by

$$f_\gamma(c, c') = ([\gamma(c), \gamma(c')] - \gamma[c, c']) = \Xi_\lambda(c, c'). \tag{13.4}$$

Thus

LEMMA 13.2. *The sequence (13.1) can be split* $\Leftrightarrow$ *the connexion given in Lemma (13.1) may be chosen to be integrable; and, in this case, the fibering* $A \to B \to C$ *is associated to a representation of* $\pi_1(C)$ *in* A.

We shall actually be interested in the case when C is abelian (and is in fact a torus); as will be seen below, it will be sufficient for our purposes to assume that A is also abelian. The following example shows that, even when $\dim A = 1$, the above connexion may not be integrable.

Example. Let B = group of matrices of the form

$$\left\{\begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}\right\};$$

A = subgroup of matrices of the form

$$\left\{\begin{pmatrix} 1 & 0 & z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}.$$

Then A is normal in B and it is easily seen that $A/B \cong \mathbf{C}^2$ parameters z_1, z_2). The group B is abelian and

$$\begin{pmatrix} 1 & 0 & z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to z_2$$

given an isomorphism of B with $\mathbf{C}$; thus the fibering $0 \to A \to B \to C \to 0$ is a principal $\mathbf{C}$-bundle over $\mathbf{C}^2$. It is easily checked that, in our framework, $\mathfrak{c}$ = algebra of matrices of the form

$$\left\{\begin{pmatrix} 0 & c_1 & 0 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

$c_1, c_2 \in \mathbf{C}$), and, by choosing the obvious splitting γ,

$$\Xi_\gamma \left\{\begin{pmatrix} 0 & c_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} 0 & 0 & c_1 c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{13.5}$$

Thus (Lemma 13.2), the connexion is not integrable.

To put this example "over a torus", we let $G \subset B$ be the subgroup of matrices of the form

$$\left\{\begin{pmatrix} 1 & g_1 & g_3 \\ 0 & 1 & g_2 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

where the g_i are Gaussian integers. Since

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & g_1 & g_3 \\ 0 & 1 & g_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_1+g_1 & g_3+z_1\, g_2+z_3 \\ 0 & 1 & z_2+g_2 \\ 0 & 0 & 1 \end{pmatrix},$$

it follows that (Iwasawa) $G/B=X$ is a bundle of complex 1-tori (parameter z_3) over a complex 2-torus (parameters z_1, z_2). The above remarks on curvature still hold for X. However, these remarks are better phrased as follows: the holomorphic 1-forms $\omega_1=dz_1$, $\omega_2=dz_3$, $\omega_3=dz_2-z_3\,dz_1$ form a basis for $H^{1,0}(X, \mathbf{C})$. The forms ω_1, ω_2 are simply the inverse images on X of the forms on the base space; the form ω_3 is seen to be the connexion form of the holomorphic connexion described above. The curvature of this connexion is given by

$$\Xi_\gamma = d(\omega_3) = dz_1 \wedge dz_3 \tag{13.6}$$

which exactly corresponds to (13.5). We remark that to turn X into a $\mathbf{C}$-bundle over the base space, we simply replace G by G' = set of matrices of the form

$$\begin{pmatrix} 1 & g_1 & 0 \\ 0 & 1 & g_2 \\ 0 & 0 & 1 \end{pmatrix}$$

(g_1, g_2 Gaussian integers).

Remark. This example is in some sense the worst that can happen. For, if $0\to\mathfrak{a}\to\mathfrak{b}\to\mathfrak{c}\to 0$ is exact and if $\mathfrak{a}$, $\mathfrak{c}$ are abelian, then $[\mathfrak{b}, \mathfrak{b}]\subseteq\mathfrak{a}$ (by (13.4)) and thus $[[\mathfrak{b}, \mathfrak{b}], [\mathfrak{b}, \mathfrak{b}]]=0$ which puts an "upper bound" on $\mathfrak{b}$ which is actually realized by the above example. The following lemma seems to be about the best we can hope for:

LEMMA 13.3. *Let A, B, C be as above and let A, C be abelian. Assume that $\mathfrak{b}$ is given a linear Lie algebra ($\subset gl(V)$ for some V) and that, in this representation, $\mathfrak{c}$ is semi-simple. Then the sequence (13.1) splits.*

Remark. The above conditions are met, for example, if $A=\mathbf{C}^*$.

Proof. We may first assume that $\mathfrak{b}$ is solvable; then the following is well- known: if $\mathfrak{n}\subseteq\mathfrak{b}$ is any ideal and if $\varphi\in\mathfrak{n}'$ then the subspace $V_\varphi=\{v\in V: n\circ v=\varphi(n)v$ for all $n\in\mathfrak{n}\}$ reduces $\mathfrak{b}$. Since $\mathfrak{a}$ is semi-simple, we may use this fact to put $\mathfrak{b}$ in triangular form with $\mathfrak{a}$ on the diagonal. The obstruction to splitting is given by a commutor $[\gamma(c), \gamma(c')] \subseteq \mathfrak{a}$ and because $\mathfrak{a} \cap [\mathfrak{b}, \mathfrak{b}] = 0$, we are done.

Returning to the sequence (13.1) where $\mathfrak{a}$ is abelian but $\mathfrak{c}$ is arbitrary, we may write $\mathfrak{b} = \mathfrak{r} \cdot \hat{\mathfrak{s}}$ where $\mathfrak{r}$ = radical of $\mathfrak{b}$ and $\hat{\mathfrak{s}}$ is semi-simple. Then $\mu(\hat{\mathfrak{s}}) = 0$ and we have the following lemma, due to Matsushima:

LEMMA 13.4. *The homogeneous bundles over a complex torus are of the form $A \to B \to T^{2a}$ where B is a complex solvable Lie group; all such fiberings have holomorphic connexions.*

Returning again to our original problem, we let $X = \hat{X}^a = \hat{X} \times T^{2a} = G/\hat{U} \times B/A$; if $\varrho : \hat{U} \times A \to GL(E^\varrho)$ defines a homogeneous vector bundle $E^\varrho \to \mathbf{E}^\varrho \to \hat{X}^a$, we wish to determine the $G \times B$-module $H^*(\hat{X}^a\, \mathcal{E}^\varrho)$. Since $\varrho(A) \subseteq \varrho(\hat{U} \times A) \subseteq GL(E^\varrho)$ is normal in $\varrho(\hat{U} \times A)$, the subspaces of E^ϱ reducing $\varrho(A)$ reduce $\varrho(\hat{U} \times A)$. As in § 5, we assume here that, if $\mathfrak{k} \subseteq \mathfrak{a}$ is maximal abelian, $\varrho \mid \mathfrak{k}$ is semi-simple. Under this assumption, we may, as in § 3, get a series of exact sequences

$$0 \to \mathbf{E}^{\varrho_1} \to \mathbf{E}^\varrho \to \mathbf{E}^{\varrho^1} \to 0$$
$$0 \to \mathbf{E}^{\varrho_2} \to \mathbf{E}^{\varrho^1} \to \mathbf{E}^{\varrho^2} \to 0$$
$$0 \to \mathbf{E}^{\varrho_n} \to \mathbf{E}^{\varrho^{n-1}} \to \mathbf{E}^{\varrho^{n+1}} \to 0$$

where $\varrho_j \mid \mathfrak{a}$ is irreducible $(j = 1, \ldots, n+1)$. Then, theoretically at least, using Proposition 2 in [12], we may calculate $H^*(\hat{X}^a, \mathcal{E}^\varrho)$ knowing the $H^*(\hat{X}^a, \mathcal{E}^{\varrho_j})$ $(j = 1, \ldots, n+1)$. If $\varrho \mid \mathfrak{a}$ is irreducible, it is clear that $\mathbf{E}^\varrho \sim \mathbf{E}^{\hat{\varrho}} \otimes \mathbf{E}^{\tau(\varrho)}$ where $E^{\hat{\varrho}} \to \mathbf{E}^{\hat{\varrho}} \to \hat{X}$ is a homogeneous vector bundle and $E^{\tau(\varrho)} \to \mathbf{E}^{\tau(\varrho)} \to T^{2a}$ is a homogeneous bundle. Since

$$H^p(\hat{X}^0, \mathcal{E}^\varrho) \cong \sum_{r+s=p} H^r(\hat{X}, \mathcal{E}^{\hat{\varrho}}) \otimes H^s(T^{2a}, \mathcal{E}^{\tau(\varrho)}), \tag{13.7}$$

it will suffice to determine the A-modules $H^*(T^{2a}, \mathcal{E}^\tau)$ when $E^\tau \to \mathbf{E}^\tau \to T^{2a}$ runs through the homogeneous line bundles on T^{2a}.

The structure of the bundles $\mathbf{E}^\tau$ is well known and was given in § 10. Writing $\Gamma \to \mathbf{C}^a \to T^{2a}$, the bundles are (uneffectively) parametrized by $H^{0,1}(T^{2a}, \mathbf{C})$ and are given by *unitary* representations $\varrho\,\bar{\omega} : \Gamma \to \mathbf{C}^*$ $(\bar{\omega} \in H^{0,1}(T^{2a}, \mathbf{C}))$ where, in fact, $\varrho\bar{\omega}(\gamma) = \exp\left(\int_\gamma \omega + \bar{\omega}\right)(\gamma \in \Gamma)$. A global section of $\mathbf{E}^\tau = \mathbf{E}^{\tau(\bar{\omega})}$ is given by an entire function f on $\mathbf{C}^a$ such that $f(z+\gamma) = \varrho\bar{\omega}(\gamma) f(z)$. Thus f is bounded, hence constant, and the constants are inadmissible unless $\omega + \bar{\omega} \in H^1(T^{2a}, \mathbf{Z})$; i.e. $\mathbf{E}^{\tau(\bar{\omega})} \sim \mathbf{1}$. (This is due again to Matsushima in [22].)

Remark. The above statement is more general. Namely, let D be any Stein variety and let Γ be any group acting discontinuously and without fixed points on D, such that D/Γ is compact. Then if $\varrho : \Gamma \to GL(n, \mathbf{C})$ is any unitary representation giving a vector bundle $E^{\varrho} \to \mathbf{E}^{\varrho} \to D/\Gamma$, $H^0(D/\Gamma, \mathcal{E}^{\varrho}) = 0$ unless ϱ is trivial. For example, if D is the upper half plane and D/Γ is an algebraic curve (Riemann surface), then, for any line bundle $E^{\varrho} \to \mathbf{E}^{\varrho} \to D/\Gamma$ with deg $\mathbf{E}^{\varrho} = c_1(\mathbf{E}^{\varrho}) = 0$, $H^0(D/\Gamma, \mathcal{E}^{\varrho}) = 0$ unless $\mathbf{E}^{\varrho} \sim 1$. In this case, if g = genus of D/Γ,

$$\dim H^1(D/\Gamma, \mathcal{E}^{\varrho}) = \begin{cases} g & \text{if } \mathbf{E}^{\varrho} \sim 1 \\ g-1 & \text{if } \mathbf{E}^{\varrho} \nsim 1 \end{cases}$$

(by the Riemann-Roch; the elements in $H^1(D/\Gamma, \mathcal{E}^{\varrho}) \cong H^0(D/\Gamma, \Omega^1 \otimes \mathcal{E}^{\varrho})$ are just the Prym differentials.)

THEOREM 18. *If $E^{\varrho} \to \mathbf{E}^{\varrho} \to T^{2a}$ is any line bundle with $c_1(\mathbf{E}^{\varrho}) = 0$ over T^{2a} where we write $A \to B \to T^{2a}$, then $H^*(T^{2a}, \mathcal{E}^{\varrho})$ is a trivial B-module. Furthermore,*

$$\dim H^q(T^{2a}, \mathcal{E}^{\varrho}) = \begin{cases} \binom{a}{q} & \textit{if } \mathcal{E}^{\varrho} \sim 1 \\ 0 & \textit{otherwise.} \end{cases} \tag{13.8}$$

Proof. We need the following simple lemma in potential theory:

LEMMA 13.5. *Let X be any compact manifold and $V \to \mathbf{V} \xrightarrow{\pi} X$ a vector bundle over X with a metric structure $(\ ,\ )$. Suppose that we have a fixed covering*

$$\{U_i\} \textit{ of } X,\ \pi^{-1}(U_i) \cong U_i \times V,$$

and suppose that we have an elliptic operator Δ on the $\mathbf{V}$-valued forms on X such that Δ in U_i is equal to the Euclidean laplacian for all i. Then if $S : X \to \mathbf{V}$ is any section such that $\Delta S = 0$, then S is locally constant.

Proof. Let S be any non-constant section with $\Delta S = 0$ and let $x_0 \in X$ be a maximum point for $|S|^2 = (S, S)$. Let $H(S)$ = Hessian matrix of S (i.e., $H(S)_{ij} = \partial^2 S / \partial x^i \partial x^j$). Then $\operatorname{tr} H(S) = \Delta S = 0$ on X; but at x_0, $\operatorname{tr} H(S)_{x_0} < 0$ if S is non-constant. Q.E.D.

We apply this lemma to $\mathbf{E}^{\varrho} \otimes \Lambda^q \overline{T}'$ (where $\overline{T}'$ bundle of $(0, 1)$ forms on T^{2a}) and to the elliptic operator $\square$ constructed from an invariant Kähler metric on T^{2a} and the metric given by the Hermitian structure in $\mathbf{E}^{\varrho}$. On the one hand, we know a priori that

$$H^q(T^{2a}, \mathcal{E}^{\varrho}) \cong H^{0,q}(T^{2a}, \mathbf{E}^{\varrho}) \cong H^q(\mathbf{E}^{\varrho}),$$

where $H^q(\mathbf{E}^\varrho)$ = kernel of $\square$ on $\mathbf{E}^\varrho \otimes \Lambda^q \bar{T}'$. On the other hand, since the transition functions of $\mathbf{E}^\varrho$ are constant and since T^{2a} is a torus, Lemma 13.5 applies and ker $\square = 0$. Q.E.D.

We give an application of Theorem 18.

PROPOSITION 13.1. (Matsushima). *Let $E \to \mathbf{E} \to T^{2a}$ be an indecomposable homogeneous vector bundle. Then $\mathbf{E} \sim \mathbf{L} \otimes \mathbf{N}$ where $\mathbf{L}$ is a homogeneous line bundle and $\mathbf{N}$ is a homogeneous bundle with structure group*

$$N = \begin{pmatrix} 1 \dots & \\ 0 \dots & \\ \cdot \dots & \\ 0 \dots & 1 \end{pmatrix}.$$

Proof. It will suffice to assume $\dim_{\mathbf{C}} E = 2$; induction will give the general result. Then, relative to a suitable convering $\{U_i\}$, $\mathbf{E}$ has transition functions

$$e_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & c_{ij} \end{pmatrix}$$

where a_{ij}, c_{ij} are the transition functions of homogeneous line bundles $\mathbf{A}$, $\mathbf{C}$, respectively, and we have $0 \to \mathbf{A} \to \mathbf{E} \to \mathbf{C} \to 0$. In the notation of § 11, $\zeta(\mathbf{E}) \in H^1(T^{2a}, \mathbf{C}^{-1}\mathbf{A})$; but $H^1(T^{2a}, \mathbf{C}^{-1}\mathbf{A}) = 0$ unless $\mathbf{A} = \mathbf{C}$. Q.E.D.

COROLLARY. (i) (Matsushima). *The space of indecomposable homogeneous bundles $E \to \mathbf{E} \to T^{2a}$ with $\dim_{\mathbf{C}} E = 2$ is parametrized by $\mathfrak{P}(T^{2a}) \times P_{a-1}(\mathbf{C})$, where $\mathfrak{P}(T^{2a})$ = Picard variety of T^{2a}.*

(ii) (Morimoto). *The space of indecomposable homogeneous bundles $E \to \mathbf{E} \to T^{2a}$ with $\dim_{\mathbf{C}} E = 3$ is parametrized in the same manner as was given in Theorem 17.*

Let $X = G/U$ be a non-Kähler C-space with fibering $T^{2a} \to G/U \to G/\hat{U}$ and let $\tau: U \to GL(E^\tau)$ give a homogeneous line bundle $E^\tau \to \mathbf{E}^\tau \to G/U$. If we restrict $\mathbf{E}^\tau$ to a fibre in the above fibering, then $\mathbf{E}^\tau$ is a homogeneous line bundle over T^{2a}; in the terminology of § 5, $E^\tau \to \mathbf{E}^\tau \to G/U$ is a *rational* homogeneous line bundle $\Leftrightarrow$ $\mathbf{E}^\tau$ restricted to a fibre is analytically trivial $\Leftrightarrow$ $\mathbf{E}^\tau$ restricted to a fibre corresponds to the zero point in the Picard variety of the fibre. Thus Theorem 4 is a Künneth relation between Theorems B and 18 ((13.8)). This "explains" Theorem 4 but, of course, does not prove it.

14. Examples and Counterexamples

We shall now give some examples illustrating the general theory and further examples showing why certain theorems given above are not true under more general circumstances.

(i) An Illustration of the General Theory

We shall describe how the theorems given above apply to a simple example; since the easiest representations to describe explicitly are those of the full linear groups, we shall choose our example from those of type 1 in Wang's list ([24]). We now recall the Lie algebra structure of $\mathfrak{g} = sl(n, \mathbf{C}) = \{g \in gl(n, \mathbf{C}) : \operatorname{tr} g = 0\}$. Letting e_{ij} be the matrix with 1 in the $i-j$ position, zeroes elsewhere, a Cartan sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ is given by $\mathfrak{h} = \{\lambda \in \mathfrak{g} : \lambda = \sum_{j=1}^n \lambda_j e_{jj}, \sum_j \lambda_j = 0\}$. The roots of $(\mathfrak{h}, \mathfrak{g})$ are the linear forms φ_{ij} $(i \neq j)$ defined by $\langle \varphi_{ij}, \lambda \rangle = \lambda_i - \lambda_j$; a rational basis for $\mathfrak{h}$ consists of $\mu_1, \ldots, \mu_{n-1}$ where $\mu_j = e_{jj} - e_{j+1, j+1}$. Relative to this rational basis, the positive roots $\sum^+ = \{\varphi_{ij}\ i < j\}$ and $-(\varphi_{ij}) = \varphi_{ji}$. If $\lambda = \sum \lambda_j e_{jj}$, $\mu = \sum \mu_k e_{kk}$, one easily checks that

$$\operatorname{Tr}(\operatorname{ad}\lambda \operatorname{ad}\mu) = 2n \operatorname{Tr}(\lambda\mu) - (\operatorname{Tr}\lambda)(\operatorname{Tr}\mu)$$

and it follows that if $\lambda, \mu \in \mathfrak{h}$ and $(\,,)$ is the Killing form

$$(\lambda, \mu) = 2n \sum_{k=1}^{n} \lambda_k \mu_k. \tag{14.1}$$

For $\varphi_{ij} \in \sum^+$, we define $h_{\varphi_{ij}} \in \mathfrak{h}$ $(h_{\varphi_{ij}}, \lambda) = \langle \varphi_{ij}, \lambda \rangle$ for all $\lambda \in \mathfrak{h}$); from (14.1) we have that $h_{\varphi_{ij}} = (1/2n)(e_{ii} - e_{jj})$. Since the Weyl normalization requires that $[e_{\varphi_{ij}}, e_{-\varphi_{ij}}] = h_{\varphi_{ij}}$, we find that $e_{\varphi_{ij}} = (1/2n)\, e_{ij}$; the condition $(e_{\varphi_{ij}}, e_{\varphi_{kl}}) = \delta^i_l \delta^j_k$ is then satisfied.

Let $\mathbf{Z}^n = \underbrace{\mathbf{Z} \times \ldots \times \mathbf{Z}}_{n}$; we define a homomorphism $\mu : \mathbf{Z}^n \to Z(\mathfrak{g})$ as follows: for $r = (r_1, \ldots, r_n) \in \mathbf{Z}^n$, $\lambda \in \mathfrak{h}$, $\langle \mu(r), \lambda \rangle = \sum_{j=1}^n r_j \lambda_j$. Then $\ker \mu = \mathbf{z}((1, \ldots, 1))$ and we must work modulo this subspace. The Weyl group $W(\mathfrak{g})$ is isomorphic to S_n = permutations on n-symbols and for $\sigma \in S_n$, $\lambda \in \mathfrak{h}$, $\sigma(\lambda) = \sigma(\sum \lambda_j e_{jj}) = \sum \lambda_{\sigma^{-1}(j)} e_{jj}$; from this it follows that $\sigma(r) = \sigma(r_1, \ldots, r_n) = (r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(n)})$. If we set $D^n = \{(r_1, \ldots, r_n) : r_1 \geqslant r_2 \geqslant \ldots \geqslant r_n\}$, then $\mu(D^n) = D(\mathfrak{g})$; furthermore, $\mu^{-1}(\varphi_{ij}) = 0, \ldots, \overset{i}{1}, \ldots, \overset{j}{-1}, \ldots, 0)$ and since $2g = \sum_{i<j} \varphi_{ij}$, $\mu^{-1}(g) = (n, n-1, \ldots, 1)$ and $\mu^{-1}(g)$ is "minimal" in $(D^n)^0$. It is easily checked (from (14.1)) that the fundamental weights $\tilde{\omega}_1, \ldots, \tilde{\omega}_{n-1}$ are given by $\mu^{-1}(\tilde{\omega}_j) = (1, \ldots, \overset{j}{1}, 0, \ldots, 0)$ and then $g = \sum_{j=1}^{n-1} \tilde{\omega}_j$. The involution $\delta \in W(\mathfrak{g})$ is given by $\delta(r_1, \ldots, r_n) = (r_n, r_{n-1}, \ldots, r_1)$.

Finally we recall briefly the elements of Young theory ([26]). Let $\mathfrak{g}(n) = gl(n, \mathbf{C})$ and, for $r = (r_1, \ldots, r_n) \in D^n$, we let $\varrho^r : \mathfrak{g}(n) \to gl(V^{\mu(r)})$ be the unique irreducible representation of $\mathfrak{g}(n)$ on $V^{\mu(r)}$ $(\mu(r) \in D(\mathfrak{g}))$ such that $\varrho^r \mid \mathfrak{g} = \mu(r)$ and $\varrho^r(1^n) = n(r)$ where $n(r) = \sum r_i$. The vector space $V^{\mu(r)}$ is constructed as follows: to $(r_1, \ldots, r_n) \in D^n$ we associate the Young diagram

1, 1	1, 2	...	...	...	...	...	1, r_1
2, 1	2, 2	...	...	...	...	2, r_2	
$\vdots$	$\vdots$						
n, 1	...	n, r_n					

We let $V^j = \underbrace{V \otimes \ldots \otimes V}_{j}$ $(V = \mathbf{C}^n)$ and consider $V^{n(r)}$. Letting $\sigma(r_1, \ldots, r_n)$ be the symmetry operator corresponding to the above diagram, $V^{\mu(r)} = \sigma(r_1, \ldots, r_n)\, V^{n(r)}$.

Let $M = SU(n)$ and let $\hat{V}^{\#}$ = compact group of matrices of the form

$$\left\{ \begin{pmatrix} e^{i\theta_1} & & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & e^{i\theta_k} & \vdots \\ 0 & & & U(n-k) \end{pmatrix} \right\};$$

if $\hat{V} = \hat{V}^{\#} \cap M$, then $M/\hat{V} = \hat{X}$ is a Kähler C-space; if we choose r such that $r \equiv 0 \bmod 2$, then, if $V^{\#}$ = matrices of the form

$$\left\{ \begin{pmatrix} 1 & 0 & \cdots\cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & e^{i\theta_{r+1}} & & \vdots \\ \vdots & & & e^{i\theta_k} & 0 \\ 0 & \cdots\cdots & & 0 & U(n-k) \end{pmatrix} \right\}$$

and $V^{\#} \cap M$, $X = M/V$ is a non-Kähler C-space and we have $T^{r/2} \to X \to \hat{X}$. We shall discuss these manifolds.

In the notations of § 1,

$$\tilde{\mathfrak{v}}^0 = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots \; \lambda_k & \\ 0 & & \mathfrak{g}(n-k) \end{pmatrix} \right\} \cap \mathfrak{g}$$

and

$$\hat{\mathfrak{v}}^0 = \left\{ \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \lambda_{r+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_k \\ & & & & & & \mathfrak{g}(n-k) \end{pmatrix} \right\} \cap \mathfrak{g}.$$

Those $\lambda \in Z(\tilde{\mathfrak{v}}^0)$ = center of $\tilde{\mathfrak{v}}^0$ may be written as $\lambda = (\lambda_1, \ldots, \lambda_k, \tilde{\lambda}, \ldots, \tilde{\lambda})$ where $\sum_{j=1}^{k} \lambda_j = -(n-k)\tilde{\lambda}$. The representations θ_j of $Z(\tilde{\mathfrak{v}}^0)$ defined by $\theta_j(\lambda) = \lambda_j$ $(j = 1, \ldots, k)$ generate (over $\mathbf{Z}$) with respect to $\otimes$ the group of line bundles $L(\hat{X})$; we write $\mathbf{E}^{\theta_j}$ for the line bundle with character θ_j given above. The general line bundle on $\hat{X}$ is of the form $\mathbf{E}^{n_1\theta_1} \ldots \mathbf{E}^{n_k\theta_k}$ and $L(X) \cong \mathbf{Z}^k \cong H^2(\hat{X}, \mathbf{Z})$. The general line bundle on X is of the form $\mathbf{E}^{c_1\theta_1} \ldots \mathbf{E}^{c_r\theta_r} \mathbf{E}^{n_{r+1}\theta_{r+1}} \ldots \mathbf{E}^{n_k\theta_k}$ and $L(X) \cong \mathbf{C}^{r/2} \oplus \mathbf{Z}^{k-r} = A \oplus B$.(1) The line bundles in A are precisely those with Chern class = 0 (although with Atiyah Chern class $\neq 0$). The vector bundles over X defined by an irreducible representation of V are of the form $\mathbf{L} \otimes \mathbf{E}^{\varrho}$ where $\mathbf{L} \in B$ and ϱ is an irreducible representation of $[\tilde{\mathfrak{v}}^0, \tilde{\mathfrak{v}}^0]$; a similar statement holds for $\hat{X}$. These bundles are all indecomposable (§ 8).

For $1 \leqslant j \leqslant r$, $H^*(X, \mathcal{E}^{c_j\theta_j}) = 0$ if c_j is not an integral vector,

$$H^q(X, \mathcal{E}^{n_j\theta_j}) \cong \sum_{p+s=q} H^p(\hat{X}, \mathcal{E}^{n_j\theta_j}) \otimes H^s(X, \Omega_X)$$

(for all j), and $H^s(X, \Omega_X)$ is a trivial M-module of dimension $\binom{r}{s}$. We now determine $I(n_j\theta_j)$ $(n_j \in \mathbf{Z})$. In

$$\mathbf{Z}^n,\ n_j\theta_j = (0, \ldots, \overset{j}{n_j}, \ldots, 0) \quad \text{and} \quad n_j\theta_j + g = (n, n-1, \ldots, \overset{j}{n-j+1+n_j}, \ldots, 1);$$

thus $n_j\theta_j + g$ is regular $\Leftrightarrow$ (i) $n_j > j-1$ or (ii) $n_j < j-n$. In case (i), $|n_j\theta_j + g| = j-1$ and $I(n_j\theta_j) = (n_j + 1 - j, 1, \ldots \overset{j}{1}, 0, \ldots, 0)$. Thus $H^q(\hat{X}, \mathcal{E}^{n_j\theta_j}) = 0$ unless $q = j-1$ and $H^{j-1}(\hat{X}, \mathcal{E}^{n_j\theta_j})$ is the irreducible $\mathfrak{g}$-module given by the Young diagram

$$\begin{array}{|c|c|c|} \hline 1,1 & \cdots & 1, n_j+1-j \\ \hline 2,1 & & \\ \cline{1-1} \vdots & & \\ \cline{1-1} j,1 & & \\ \cline{1-1} \end{array}$$

For example, if $j = 1$ and $n_j = 1$, the induced representation is just the ordinary re-

(1) There are $\frac{1}{2}r$ relations among the c_j.

presentation of $SU(n)$ on $\mathbf{C}^n$. In case (ii), $|n_j\theta_j+g|=n-j$ and $I(n_j\theta_j)=(n,n-1,\ldots,n-j+1,n-j-1,\ldots,1,n_j-j+1)$. A similar statement to the above concerning cohomology groups may be made.

Now if $k=1$ (thus $\hat{X}=P_{n-1}(\mathbf{C})$), the line bundle $\mathbf{E}^{\theta_1}$ gives a projective imbedding (§ 8); we wish to generalize this. To do so, notice that $\langle\tilde{\omega}_j,[\tilde{\mathfrak{v}}^0,\tilde{\mathfrak{v}}^0]\rangle=0$ for $1\leqslant j\leqslant k$ and that $\sum_{1\leqslant j\leqslant k}\tilde{\omega}_j$ gives a character $\chi=\theta_1+\ldots+\theta_k$ and a line bundle $\mathbf{E}^\chi=\mathbf{E}^{k\theta_1}\ldots\mathbf{E}^{\theta_k}$. Since $g=g_1+g_2$ and $g_1=\tilde{\omega}_{k+1}+\ldots+\tilde{\omega}_n$, it follows that $\chi=g_2$; the imbedding mapping $(\chi)^*$ discussed in § 8 is biregular on $\hat{X}$ and maps $\hat{X}$ into the projective space associated to the vector space corresponding to the Young diagram

1, 1	…	…	1, k
2, 1	…	2, $k-1$	
⋮			
k, 1			

We may also speak of the imbedding mapping $(\chi)^*$ on X; this mapping does not separate fibre points in the fibering $X\to\hat{X}$, which is just as it should be.

The question of finding the sheaf cohomology of $\mathbf{E}^\varrho$ where ϱ is an irreducible representation of $[\tilde{\mathfrak{v}}^0,\tilde{\mathfrak{v}}^0]$ is the question of transferring a Young diagram into a Young diagram. For example, if we let $n=6$, $k=4$, then $[\tilde{\mathfrak{v}}^0,\tilde{\mathfrak{v}}^0]\cong sl\,(2,\mathbf{C})$ and $D(\tilde{\mathfrak{v}}^0)=\{r=(0,\ldots,0,r_5,r_6):r_5\geqslant r_6\}$. If we assume that $r_5>r_6\geqslant 0$, then $g+r$ is regular $\Leftrightarrow$ (i) $r_6>5$ or (ii) $r_6=0$, $r_5>4$. We may describe the induced action as follows:

(i) $|r+g|=8$ and

I:

1, 1	…	…	1, r_5
2, 1	…	2, r_6	

→

1, 1	…	…	1, r_5-6
2, 1	…	2, r_6-5	
3, 1	3, 2		
4, 1	4, 2		
5, 1	5, 2		
6, 1	6, 2		

(ii) $|r+g|=4$ and

$$\mathrm{I}:\ \begin{array}{|c|c|c|}\hline 1,1 & \dots & 1,r_5 \\ \hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 1,1 & \dots & \dots & 1,r_5-5 \\ \hline 2,1 \\ 3,1 \\ 4,1 \\ 5,1 \\ 6,1 \\ \hline\end{array}$$

We may ask about extensions of bundles. For example, (§ 11), the space of vector bundles $\mathbf{E}$ such that we have $0 \to \mathbf{E}^{\theta_2} \to \mathbf{E} \to \mathbf{E}^{\theta_1} \to 0$ (over $\hat{X}$) consists entirely of homogeneous entries and is in fact a vector space of dimension 1- This is because $\theta_2-\theta_1=(-1,1,0,\dots,0)$ and $g+\theta_2-\theta_1=(n-1,\,n,\,n-2,\dots,1)$; $|g+\theta_2-\theta_1|=1$ and

$$I(\theta_2-\theta_1)=\tau_{\varphi_{12}}(g+\theta_2-\theta_1)-g=(n,\,n-1,\,n-2,\dots,1)-g=0.$$

The space of bundles $\mathbf{E}$ such that we have $0 \to \mathbf{E}^{n_1\theta_2} \to \mathbf{E} \to \mathbf{E}^{n_2\theta_2} \to 0$ is non-void $\Leftrightarrow$ $n_1-n_2>1$ in which case, all the bundles $\mathbf{E}$ are non-homogeneous and form a vector space of dimension

$$\binom{n+n_1-n_2}{n_1-n_2-1}.$$

This is because $(n_1-n_2)\theta_2=(0,n_1-n_2,0,\dots,0)$ and $(n_1-n_2)\theta_2+g$ is non-singular $\Leftrightarrow$ (i) $n_1-n_2>1$ or (ii) $n_1-n_2<n-1$; furthermore, $|(n_1-n_2)\theta_2+g|=1$ $\Leftrightarrow$ $n_1-n_2>1$ and in this case $I((n_1-n_2)\theta_2)=1,1\ \dots\ 1,n_1-n_2-1$.

Over X, the second situation just described is a bit different. The space of extensions $0 \to \mathbf{E}^{c_1\theta_2} \to \mathbf{E} \to \mathbf{E}^{c_2\theta_2} \to 0$ is non-void $\Leftrightarrow$ c_1-c_2 is integral and (i) $c_1-c_2>1$ or (ii) $c_1-c_2=0$. In case (i) the extensions are non-homogeneous and form a vector space of dimension

$$\binom{n+c_1-c_2}{c_1-c_2-1};$$

in the second case ((ii)), the extensions are all homogeneous and form a vector space of dimension $\frac{1}{2}r$.

The complex dimension of $\hat{X}$ is $nk-k^2$; that of X is $nk-k^2+\frac{1}{2}r$. If we assume, e.g., that $n-k$ is odd, then $H^*(X,\mathbf{Z})$ has no torsion and is in fact given by

$$\frac{K[X_1,\dots,X_r]\otimes S(X_{r+1},\dots,X_k)}{S(X_1,\dots,X_k)}\otimes\wedge\,(X_{2n-1},\dots,X_{2n-(n-k)})$$

(in the usual notation of topology). Letting $m=nk-k^2$, $F[X]=F[\hat{X}]\cong P(z_1, \ldots, z_m)=$ rational homogeneous functions in m-variables. There are $\frac{1}{4}r^2$ parameters varying the homogeneous structure on X. If $r=2$, there are $(n-1)^2$ parameters of non-homogeneous deformation; if $r>2$, there are still $(n-1)^2$ such parameters, there being $(\frac{1}{2}r-1)(n-1)^2$ obstructed parameters in this case.

(ii) An Example Concerning the Semi-Simplicity of Certain Representations

Let $X=G/U$ be a non-Kähler C-space; $\hat{X}=G/\hat{U}$ the associated Kähler C-space with fundamental fibering $T^{2a}\to X\to\hat{X}$. Theorem 4 in § 5 stated the following: Let $\varrho: U\to GL(E^{\varrho})$ be an abelian representation of U which does not extend to $\hat{U}$; then if $\varrho\,|\,\mathfrak{u}\cap\mathfrak{h}$ is semi-simple, $H^*(X, \mathcal{E}^{\varrho})=0$. We now show by an example why the restriction of semi-simplicity was in fact necessary; in fact, this example is in some sense indicative of the only alternative to semi-simplicity. For simplicity we assume that $\hat{U}$ (and hence U) is solvable. We write $\mathfrak{u}=\mathfrak{n}\oplus\mathfrak{h}_{\mathfrak{r}}\oplus\mathfrak{p}$; $\hat{\mathfrak{u}}=\mathfrak{n}\oplus\mathfrak{h}_{\mathfrak{r}}\oplus\mathfrak{p}\oplus\bar{\mathfrak{p}}$ as in § 1, and we choose any $\lambda\in\mathfrak{p}'$ such that $\langle\lambda, \mathfrak{h}_{\mathfrak{r}}\rangle=0$. Then, for $\xi\in\mathfrak{u}\cap\mathfrak{h}$, we define a representation

$$\varrho_\lambda: \mathfrak{u}\cap\mathfrak{h}\to gl(2, \mathbf{C}) \text{ by } \varrho_\lambda(\xi)=\begin{pmatrix}0 & \langle\lambda, \xi\rangle\\ 0 & 0\end{pmatrix}.$$

Clearly ϱ_λ extends to all of $\mathfrak{u}$ and we assume that ϱ_λ is covered by a representation

$$\varrho_\lambda: U\to GL(2, \mathbf{C})\ (\varrho_\lambda(\exp\,\xi)=\begin{pmatrix}1 & \langle\lambda, \xi\rangle\\ 0 & 1\end{pmatrix}.$$

Then ϱ_λ does not in general extend to $\hat{U}$. To compute $H^*(X, \mathcal{E}^{\varrho_\lambda})$, we let $\bar{\omega}_\lambda$ be the $(0, 1)$ form in $H^{0,1}(X, \mathbf{C})$ corresponding to λ and observe that we have the exact sequence of homogeneous vector bundles:

$$0\to\mathbf{1}\to\mathbf{E}^{\varrho_\lambda}\to\mathbf{1}\to 0. \tag{14.2}$$

In the exact cohomology sequence

$$\ldots\to H^{q-1}(X,\Omega)\xrightarrow{\delta_q}H^q(X,\Omega)\to H^q(X,\mathcal{E}^{\varrho_\lambda})\to H^q(X,\Omega)\xrightarrow{\delta_{q+1}}H^{q+1}(X,\Omega)\to\ldots$$

the coboundary maps δ_q are given as follows: for

$$\bar{\eta}\in H^{q-1}(X,\Omega),\ \delta_q(\bar{\eta})=\bar{\omega}_\lambda\wedge\bar{\eta}\in H^q(X,\Omega)$$

(see Lemma 12.2 or also [12]). By the same calculation as in Proposition 12.2, it follows that $H^q(X, \mathcal{E}^{e\lambda})$ is a trivial M-module of dimension $\binom{a}{q}$.

(iii) Line Bundles over P_r (C)

Let V be a vector space and $W \subset V$ a subspace of codimension 1. Letting $G = GL(V)$, $U =$ subgroup of G preserving W, and writing $E = W/V$, we have the exact sequence of U-modules

$$0 \to W \to V \to E \to 0. \tag{14.3}$$

The C-space G/U is a projective space P_r $(r+1 = \dim_{\mathbf{C}} V)$ and $E \to \mathbf{E} \to P_r$ is the line bundle of a hyperplane section. If $(t_1, \ldots, t_{r+1})$ are homogeneous coordinates in P_r, the sets $U_i = \{(t_1, \ldots, t_{r+1}) : t_i \neq 0\}$ give the usual affine covering of P_r with non-homogeneous coordinates $w_i^\alpha = t_\alpha / t_i$ $(\alpha \neq i)$ in U_i. The bundle $\mathbf{E}$ has transition functions $s_{ij} = (t_j/t_i)$ in $U_i \cap U_j$. If U is written as a set of matrices

$$\left\{ u = \begin{pmatrix} u_{11} & \cdots\cdots & u_{1,r+1} \\ 0 & u_{22} & \\ \vdots & \vdots & \\ 0 & u_{r+1,2} & u_{r+1,r+1} \end{pmatrix} \right\},$$

then $\mathbf{E}$ is given by the representation $\lambda(u) = u_{11}$. More generally, the bundle $\mathbf{E}_n$ with transition functions $(t_j/t_i)^n$ is given by $\lambda^n(u) = u_{11}^n$. From theorem B and the above discussion on Young symmetrizers, it follows that

$$H^q(P_r, \mathcal{E}^n) = 0 \quad q > 0 \quad (n \geqslant 0), \ \dim\ H^0(P_r, \mathcal{E}^n) = \binom{r+n-1}{n}$$

and in fact $H^0(P_r, \mathcal{E}^n)$ is the irreducible G-module of symmetric tensors of rank n. One may easily compute all the G-modules $H^q(P_r, \mathcal{E}^n)$ using these results and the duality theorem; observe that for the canonical bundle

$$\mathbf{K} = -(r+1)\,\mathbf{E}.$$

(iv) A New Type of Obstruction

If L is a locally free coherent sheaf of Lie algebras over a compact complex manifold Y, then $H^*(Y, L)$ has the structure of a graded Lie algebra. For certain L, the non-triviality of the mapping $\{,\} : H^1(Y, L) \otimes H^1(Y, L) \to H^2(Y, L)$ gives rise to obstructions to varying the structure of something analytic on Y (see §§ 9, 10, 14 (v)).

In all cases which we have encountered, and indeed in all examples known to the author, the obstructed elements which have arisen lie in the image of $H^0(Y, L) \otimes H^1(Y, \Omega)$ in $H^1(Y, L)$ (under the pairing $L \otimes \Omega \to L$. A general reason for such obstructions was discussed in § 9; roughly speaking, they were of an "ad-hoc" nature. We now give an example of an obstruction which arises in an entirely different manner—this obstruction might be termed "a priori".

We consider a complex l-torus $T = T^{2l}$ and bundles over T with group N-matrices of the form $\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right\}$; if $E \to \mathbf{E} \to T$ is a vector bundle with group N, then we have an exact sequence

$$(S) \quad 0 \to \mathbf{A} \to \mathbf{E} \to \mathbf{C} \to 0, \tag{14.6}$$

where $\mathbf{A}$, $\mathbf{C}$ are line bundles. The bundle $\mathbf{E}$ is homogeneous $\Leftrightarrow$ $\mathbf{A}$, $\mathbf{C}$ are also (§ 13). Furthermore, if $\mathbf{E}$ is homogeneous, then in order that (14.6) not be splittable, it is necessary that $\mathbf{A} = \mathbf{C}$ (§ 13). The obstruction we seek will arise when we try to deform $\mathbf{1} \oplus \mathbf{1}$ into a bundle $\mathbf{E}$ of the above form which is not decomposable and is such that $\mathbf{A} \neq \mathbf{C}$.

To be more precise, we consider $\mathbf{V} = \mathbf{1} \oplus \mathbf{1}$ as having structure group N and construct its principal bundle $N \to P \to T$. As usual, we have associated to P the Atiyah sequence

$$0 \to \mathbf{L} \to \mathbf{Q} \to T(T) \to 0 \quad (\mathbf{L} = \mathbf{L}(P),\ \mathbf{Q} = \mathbf{Q}(P));$$

we must examine $H^1(T, \mathfrak{L})$.

Lemma 14.3. $\mathfrak{L} \cong \mathbf{1} \oplus \mathbf{1} \oplus (\operatorname{Hom}(\mathbf{1}, \mathbf{1})) \cong \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$.

Proof. The proof is easy; see § 11 (iv).

An element in $H^q(T, \mathfrak{L})$ is of the form $\begin{pmatrix} \eta & \tau \\ 0 & \eta' \end{pmatrix}$ where $\eta, \eta', \tau \in H^q(T, \Omega)$.

Proposition 14.2.

$$\left\{\begin{pmatrix} \eta & \tau \\ 0 & \eta' \end{pmatrix}, \begin{pmatrix} \eta & \tau \\ 0 & \eta' \end{pmatrix}\right\} = \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix} \in H^2(T, \mathfrak{L})$$

where $$\sigma = (\eta - \eta') \wedge \tau \in H^2(T, \Omega).$$

Proof. The proposition is quickly proven by a straightforward calculation in local coordinates using the definition of $\{\ \}$ given in § 9 (ii) ((9.7)).

COROLLARY. *If* $\tau \neq 0$, $\begin{pmatrix} \eta & \tau \\ 0 & \eta' \end{pmatrix} \in H^1(T, \mathfrak{L})$ *is unobstructed* $\Leftrightarrow$ $\eta = \eta'$.

To have a deformation of $\mathbf{1} \oplus \mathbf{1}$ represented infinitesimally by $\begin{pmatrix} \eta & \tau \\ 0 & \eta' \end{pmatrix}$ means the following: each of η and η' gives rise to a line bundle $\mathbf{E}_\eta$ or $E_{\eta'}$ on the Picard variety $\mathfrak{P}(T)$; to have $\eta \neq \eta'$ implies that $\mathbf{E}_\eta \neq E_{\eta'}$. To say that $\tau \neq 0$ means that we are to have a non-split extension of $\mathbf{E}_{\eta'}$ by $\mathbf{E}_\eta$ which is possible $\Leftrightarrow$ $\eta = \eta'$, whence the obstruction to deformation.

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