FREDHOLM EIGENVALUES AND QUASICONFORMAL MAPPING

BY

GEORGE SPRINGER

University of Kansas (1)

1. Introduction

Let $\tilde{D}$ be a region of connectivity n in the z-plane which contains the point of infinity and whose boundary C consists of n smooth Jordan curves $C_1, C_2, \ldots, C_n$. Each curve C_j is the boundary of a bounded simply connected region D_j and we write $D = \bigcup_{j=1}^{n} D_j$. The Neumann–Poincaré integral equation is

$$f(s) = \lambda \int_C K(s, t) f(t)\, dt, \tag{1}$$

where s and t represent the arc length parameter on C, $z(s)$ is a parametric representation of C in terms of its arc length, $\partial/\partial n_t$ represents differentiation in the direction of the inward normal at $z(t)$, and

$$K(s, t) = \frac{\partial}{\partial n_t} \log \frac{1}{|z(s) - z(t)|}. \tag{2}$$

This integral equation plays an important role in potential theory and conformal mapping. It can be solved by iteration and the Neumann–Liouville series so obtained converges like a geometric series whose ratio is $1/|\lambda|$ where λ is the lowest eigenvalue of (1) whose absolute value is greater than one. The eigenvalues of (1) are known as the Fredholm eigenvalues of C. They are all real, satisfy $|\lambda| \geqslant 1$, and those for which $|\lambda| > 1$ lie symmetrically about the origin. Those of modulus one are referred to as the trivial eigenvalues. In order to have an estimate for the rate of convergence of the Nenmann–Liouville series, it has been an important problem to estimate from below the lowest non-trivial positive Fredholm eigenvalue, which will be denoted by λ in what follows.

(1) This work was supported by the National Science Foundation.

Ahlfors [1] showed how quasiconformal mapping leads to a very practical method for obtaining such estimates when C is the boundary of a simply connected region $\tilde{D}$. In particular, he showed that if $\tilde{D}$ admits a quasiconformal reflection [3] of maximal eccentricity k, i.e., a quasiconformal mapping f of $\tilde{D}$ onto its exterior D which leaves C pointwise fixed and for which $|f_z| < k|f_{\bar{z}}|$, then $\lambda \geqslant 1/k$.

Since a quasiconformal reflection is not possible for multiply connected regions in the plane, Royden [10] embedded the given region $\tilde{D}$ in a compact Riemann surface on which the exterior of $\tilde{D}$ had the same topological structure as $\tilde{D}$. An integral equation analogous to (1) is then studied and its lowest non-trivial eigenvalue can be estimated by using a quasiconformal relfection. However the kernel K is no longer the Dirichlet kernel (2) but involves the Green's function of the Riemann surface which generally is unknown.

Returning to simply connected domains, Warschawski [15, 16] showed how eigenvalue estimates can be obtained for a domain which is "close" to a domain for which such estimates are known, for example, for "nearly-circular" or "nearly-convex" domains. Schiffer [11] used variational methods to obtain such estimates for simply connected regions and similar methods are used in [12, 13] to obtain estimates for certain multiply connected regions.

In this paper, a generalization of the Ahlfors method is presented which is applicable to multiply connected regions, and which gives a practical method for obtaining estimates for the lowest non-trivial positive eigenvalue λ for the Neumann–Poincaré equation (1). Let $\tilde{D}$ be a domain of connectivity n containing ∞ and having Jordan curves $C_1, C_2, \ldots, C_n$ as its boundary. Each C_k is the boundary of a simply connected bounded domain D_k, and $D = \bigcup_{k=1}^{n} D_k$, $C = \bigcup_{k=1}^{n} C_k$. We shall prove the following theorem.

THEOREM 1. *Let $\zeta(z)$ be a quasiconformal homeomorphism of the whole z-plane onto the whole ζ-plane with $\zeta(\infty) = \infty$ which is K-quasiconformal in D and M-quasiconformal in $\tilde{D}$. Let ζ map the curve system C onto a curve system C^*. We shall assume that the Jordan curves in C and C^* have continuous curvature. If λ and λ^* denote the Fredholm eigenvalue of C and C^* respectively, then*

$$\frac{1}{KM}\frac{\lambda+1}{\lambda-1} \leqslant \frac{\lambda^*+1}{\lambda^*-1} \leqslant KM\frac{\lambda+1}{\lambda-1}. \tag{3}$$

Two curve systems C and C^* are called quasiconformally equivalent if there is a quasiconformal homeomorphism of the whole plane which takes C onto C^*. We denote by $\mathfrak{D}^*$ a class of canonical domains for domains of connectivity n, and assume that each domain $\tilde{D}^* \in \mathfrak{D}^*$ is bounded by a curve system C^* for which $\lambda^* > 1$. Let $\mathfrak{J}^*$ denote the class of curve systems C^* which are boundaries of domains $\tilde{D}^* \in \mathfrak{D}^*$. We then have:

THEOREM 2. *The conformal mapping of $\tilde{D}$ onto a domain $\tilde{D}^* \in \mathfrak{D}^*$ can be extended to a quasiconformal homeomorphism of the whole plane if, and only if, $\lambda > 1$.*

COROLLARY. *The curve system C is quasiconformally equivalent to a curve system $C^* \in \mathfrak{J}^*$ if, and only if, $\lambda > 1$. In particular, C is quasiconformally equivalent to a system of circles if, and only if, $\lambda > 1$.*

Finally the cross-ratio condition given by Ahlfors in [3] can be extended to curve systems C consisting of n-Jordan curves. We shall prove

THEOREM 3. *A curve system C is quasiconformally equivalent to a system of circles if, and only if, there is a constant A such that*

$$(\overline{P_1P_2}\cdot\overline{P_4P_3})/(\overline{P_1P_3}\cdot\overline{P_4P_2}) \leqslant A < \infty$$

for any four points P_1, P_2, P_3, P_4 which follow each other in this order on any C_k, $k = 1, 2, \ldots, n$.

2. The Fredholm eigenvalues

Let us assume that the curve system C is such that the spectrum of the integral equation (1) consists of at most a countable sequence $\{\lambda_n\}$. This is the case, for example, when C is given parametrically in terms of arc length by a function $z(s)$ which is of class C^2, that is, when C has continuous curvature. We then denote by λ the smallest Fredholm eigenvalue satisfying $\lambda > 1$.

The eigenvalue λ may be characterized by the following extremal property. If $\tilde{h}$ is any function which is harmonic in $\tilde{D}$, (regular at ∞), and has a single valued harmonic conjugate, and if h is harmonic in D and satisfies $h = \tilde{h}$ on C, then we shall call $\tilde{h}$ and h an admissible pair of harmonic functions for C. Then for any admissible pair $\tilde{h}$ and h, we have

$$\frac{\lambda-1}{\lambda+1} \leqslant \frac{\iint_D (\nabla h)^2 d\tau}{\iint_{\tilde{D}} (\nabla \tilde{h})^2 d\tau} \leqslant \frac{\lambda+1}{\lambda-1}, \tag{4}$$

where $d\tau$ is the area element. Equality holds on the right when, and only when, $\tilde{h}$ is the harmonic conjugate of the double-layer potential with density on C equal to an eigenfunction belonging to λ. Equality holds on the left when, and only when, $\tilde{h}$ is the harmonic conjugate of the double-layer potential with density on C equal to an eigenfunction belonging to $-\lambda$. Those admissible pairs for which equality holds are called extremal for λ and $-\lambda$ respectively.

Now let $\zeta(z)$ be a quasiconformal homeomorphism of the whole z-plane onto the whole ζ-plane which carries infinity into infinity. If $\zeta \in C^1$, then the partial derivatives ζ_z and $\zeta_{\bar{z}}$ are defined by

$$\zeta_z = \frac{\partial \zeta}{\partial z} = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} - i\,\frac{\partial \zeta}{\partial y}\right), \quad \zeta_{\bar{z}} = \frac{\partial \zeta}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} + i\,\frac{\partial \zeta}{\partial y}\right).$$

Every quasiconformal homeomorphism ζ of the whole plane has generalized derivatives ζ_z and $\zeta_{\bar{z}}$ [6, 8, 9] which are locally integrable and satisfy

$$\iint \zeta_z \phi\, d\tau = -\iint \zeta \phi_z\, d\tau, \quad \iint \zeta_{\bar{z}} \phi\, d\tau = -\iint \zeta \phi_{\bar{z}}\, d\tau$$

for all $\phi \in C^1$ with compact support, the integration extended over the whole plane. It will be convenient to write $p = \zeta_z$ and $q = \zeta_{\bar{z}}$. We say that ζ has maximal eccentricity k in a region Ω if there is a number $k < 1$ such that $\|q/p\|_\infty \leqslant k$, that is, if $|q| < k|p|$ holds almost everywhere on Ω. If we set $K = (1+k)/(1-k)$, we call K the maximal dilatation of ζ, and ζ is called K-quasiconformal. Note that if the maximal dilatation is equal to 1 in a region Ω, then ζ is conformal in Ω.

We now assume that ζ is K-quasiconformal in D and M-quasiconformal in $\tilde{D}$. The mapping ζ carries the region $\tilde{D}$ into a region $\tilde{D}^*$ and each $D_j\,(j = 1, \ldots, n)$ goes into a region D_j^*. Likewise, we shall write C^* for the image of C and C_j^* for the image of each C_j. We shall let λ^* denote the lowest positive non-trivial Fredholm eigenvalue for the curve system C^*.

3. The case $M=1$

We now consider the special case in which $M = 1$, that is, $\zeta(z)$ is conformal in $\tilde{D}$ and K-quasiconformal in D. Any admissible harmonic function $\tilde{h}$ in $\tilde{D}$ (i.e., the first member of an admissible pair for C) transforms into a function $\tilde{h}^*$ defined by

$$\tilde{h}^*(\zeta(z)) = \tilde{h}(z). \tag{5}$$

Since $\zeta(z)$ is conformal, $\tilde{h}^*$ is also harmonic, is regular at infinity, and has a single-valued harmonic conjugate in $\tilde{D}^*$. Thus $\tilde{h}^*$ is eligible to be the first member of an admissible pair for C^*, and

$$\iint_{\tilde{D}} (\nabla \tilde{h})^2 \, d\tau_z = \iint_{\tilde{D}^*} (\nabla \tilde{h}^*)^2 \, d\tau_\zeta. \tag{6}$$

For the harmonic function h in D forming an admissible pair with $\tilde{h}$, we shall likewise define its transform h^* by

$$h^*(\zeta(z)) = h(z). \tag{7}$$

Since ζ is not conformal in D, we cannot assert that h^* is harmonic in D^*. On the other hand, the generalized derivatives ζ_z and $\zeta_{\bar{z}}$ satisfy the usual chain rule and integrals transform according to the classical rule in which the Jacobian of ζ is taken to be $|p|^2 - |q|^2$; c.f. [4]. We then have

$$\begin{aligned}
\iint_D (\nabla h)^2 d\tau &= 2\iint_D \left(\left|\frac{\partial h}{\partial z}\right|^2 + \left|\frac{\partial h}{\partial \bar{z}}\right|^2 \right) d\tau_z = 2\iint_D \left| \frac{\partial h^*}{\partial \zeta} p + \frac{\partial h^*}{\partial \bar{\zeta}} \bar{q} \right|^2 + \left| \frac{\partial h^*}{\partial \zeta} q + \frac{\partial h^*}{\partial \bar{\zeta}} \bar{p} \right|^2 d\tau_z \\
&\leqslant 2\iint_D \left(\left|\frac{\partial h^*}{\partial \zeta}\right|^2 + \left|\frac{\partial h^*}{\partial \bar{\zeta}}\right|^2 \right) (|p| + |q|)^2 d\tau_z = 2\iint_{D^*} \left(\left|\frac{\partial h^*}{\partial \zeta}\right|^2 + \left|\frac{\partial h^*}{\partial \bar{\zeta}}\right|^2 \right) \frac{(|p|+|q|)}{(|p|-|q|)} d\tau_\zeta \\
&\leqslant K \iint_{D^*} (\nabla h^*)^2 d\tau_\zeta.
\end{aligned} \tag{8}$$

Since the inverse of $\zeta(z)$ also has maximal dilatation K in D^*, we also have

$$\iint_{D^*} (\nabla h^*)^2 d\tau_\zeta \leqslant K \iint_D (\nabla h)^2 d\tau_z. \tag{9}$$

The pair $\tilde{h}$ and h is an admissible pair for C, so from (4), (6), and (9), we conclude that

$$\frac{\lambda+1}{\lambda-1} \geqslant \frac{\iint_D (\nabla h)^2 d\tau_z}{\iint_{\tilde{D}} (\nabla \tilde{h})^2 d\tau_z} \geqslant \frac{1}{K} \frac{\iint_{D^*} (\nabla h)^2 d\tau_\zeta}{\iint_{\tilde{D}^*} (\nabla \tilde{h}^*)^2 d\tau_\zeta} \tag{10}$$

holds. If we now let g^* be the function which is harmonic in D^* and has the same boundary values as h^* (and hence the same as $\tilde{h}^*$), the Dirichlet principle tells us that

$$\iint_{D^*} (\nabla h^*)^2 d\tau_\zeta \geqslant \iint_{D^*} (\nabla g^*)^2 d\tau_\zeta \tag{11}$$

holds. Therefore, the combination of (10) and (11) gives us

$$\frac{\lambda+1}{\lambda-1} \geqslant \frac{1}{K} \frac{\iint_{D^*} (\nabla g^*)^2 d\tau_\zeta}{\iint_{\tilde{D}^*} (\nabla \tilde{h}^*)^2 d\tau_\zeta}, \tag{12}$$

where $\tilde{h}^*$ and g^* are an admissible pair for C^*.

Thus to each admissible pair $\tilde{h}$, h for C corresponds an admissible pair $\tilde{h}^*$, g^* such that (11) holds. In particular, we shall let h_1^* be the harmonic conjugate of the double-layer potential with density on C^* equal to a Fredholm eigenfunction belonging to λ^*, and we take as $\tilde{h}$ the function $\tilde{h}_1^*(\zeta(z))$. For this $\tilde{h}$, there is a function h in D which has the same boundary values as $\tilde{h}$ and is harmonic in D (the solution of a suitable Dirichlet problem). To this admissible pair $\tilde{h}$, h for C corresponds the admissible pair $\tilde{h}_1^*$, g_1^* for C^* which is extremal for λ^*, and we have

$$\frac{\lambda+1}{\lambda-1} \geqslant \frac{1}{K} \frac{\iint_{D^*} (\nabla g_1^*)^2 d\tau_\zeta}{\iint_{\tilde{D}^*} (\nabla \tilde{h}_1^*)^2 d\tau_\zeta} = \frac{1}{K} \frac{\lambda^*+1}{\lambda^*-1}. \tag{13}$$

Since the inverse of $\zeta(z)$ is also conformal in $\tilde{D}$ and K-quasiconformal in D, this inequality (13) also holds when λ^* and λ are interchanged. Then we can write

$$\frac{1}{K} \frac{\lambda+1}{\lambda-1} \leqslant \frac{\lambda^*+1}{\lambda^*-1} \leqslant K \frac{\lambda+1}{\lambda-1}. \tag{14}$$

This proves (3) in the special case $M=1$. Looking back over the proof, we see that it was only for convenience that we normalized the problem at infinity, so that the restriction $\zeta(\infty)=\infty$ can be relaxed to allow ζ to be any quasiconformal homeomorphism of the whole sphere. Then the inequality (12) can already be used to deduce the fact that λ remains invariant under a linear fractional transformation of the plane, for in this case, $M=K=1$ and $\lambda=\lambda^*$. This fact was observed by Bergman and Schiffer in [5].

4. The case $K=1$

When $K=1$, ζ is conformal in D and we shall assume that ζ is M-quasiconformal in $\tilde{D}$. From the left inequality in (3), we get that

$$\frac{\lambda+1}{\lambda-1} \geqslant \frac{\iint_{\tilde{D}} (\nabla \tilde{h})^2 d\tau_z}{\iint_{D} (\nabla h)^2 d\tau_z} \tag{15}$$

holds for each admissible pair of functions $\tilde{h}$, h for C. Using (7) to define h^* and the conformality of ζ in D, we have

$$\iint_{D^*} (\nabla h^*)^2 \, d\tau_\zeta = \iint_D (\nabla h)^2 \, d\tau_z. \tag{16}$$

If we use (5) to define $\tilde{h}^*$, we have from an argument similar to (8) and (9) that

$$\iint_{\tilde{D}} (\nabla \tilde{h})^2 \, d\tau_z \geqslant \frac{1}{M} \iint_{\tilde{D}^*} (\nabla \tilde{h}^*)^2 \, d\tau_\zeta. \tag{17}$$

If we replace $\tilde{h}^*$ by the harmonic function $\tilde{f}^*$ in $\tilde{D}^*$ which is regular at infinity and which assumes the same boundary values as $\tilde{h}^*$ on C, we may use the Dirichlet principle to conclude that

$$\iint_{\tilde{D}^*} (\nabla \tilde{h}^*)^2 \, d\tau_\zeta \geqslant \iint_{\tilde{D}^*} (\nabla \tilde{f}^*)^2 \, d\tau_\zeta. \tag{18}$$

It is now possible that the function $\tilde{f}^*$ does not have a single-valued harmonic conjugate in $\tilde{D}^*$. Since $\tilde{f}^*$ is regular at infinity, we have

$$\int_{C^*} \frac{\partial \tilde{f}^*}{\partial n} \, ds = 0.$$

Let ω_j denote the harmonic measure of the contour C_j^* relative to $\tilde{D}^*$, and

$$p_{jk} = \int_{C_k^*} \frac{\partial \omega_j}{\partial n} \, ds.$$

Since the matrix (p_{jk}), $k = 1, \ldots, n-1$, $j = 1, \ldots, n-1$, is positive definite, it is possible to solve the system of equations

$$\sum_{j=1}^{n-1} \alpha_j p_{jk} = \int_{C^*} \frac{\partial \tilde{f}^*}{\partial n} \, ds, \quad k = 1, 2, \ldots, n-1,$$

for the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. We then define the function

$$\tilde{g}^* = \tilde{f}^* - \sum_{j=1}^{n-1} \alpha_j \omega_j.$$

Now $\tilde{g}^*$ has a single-valued harmonic conjugate. Furthermore, each ω_j is Dirichlet orthogonal to harmonic functions $\tilde{g}^*$ which have single-valued harmonic conjugates, for we have

$$\iint_{\tilde{D}} (\nabla \omega_j) \cdot (\nabla \tilde{g}^*) d\tau_\zeta = \int_{C^*} \omega_j \frac{\partial \tilde{g}^*}{\partial n} ds = \int_{C_j^*} \frac{\partial \tilde{g}^*}{\partial n} ds = 0.$$

Therefore

$$\iint_{\tilde{D}} (\nabla \tilde{f}^*)^2 d\tau_\zeta = \iint_{\tilde{D}^*} (\nabla \tilde{g}^*)^2 d\tau_\zeta + \iint_{\tilde{D}^*} \left(\Delta \sum_{j=1}^{n-1} \alpha_j \omega_j\right)^2 d\tau_\zeta \geqslant \iint_{\tilde{D}^*} (\nabla \tilde{g}^*)^2 d\tau_\zeta \tag{19}$$

has been established.

The new function g^* differs from $\tilde{f}^*$ on C_j^* by a constant for each $j = 1, 2, \ldots, n$. Hence $\tilde{g}^*$ differs from h^* on each C_j by some constant, say $\tilde{g}^* = h^* + c_j$ on C_j^* $j = 1, 2, \ldots, n$. We then define $g^* = h^* + c_j$ in D_j^*. The functions g^* and h^* have the same Dirichlet integrals in D^*, so we have

$$\frac{\lambda+1}{\lambda-1} \geqslant \frac{1}{M} \frac{\iint_{\tilde{D}^*} (\nabla \tilde{g}^*)^2 d\tau_\zeta}{\iint_{D^*} (\nabla g^*)^2 d\tau_\zeta}. \tag{20}$$

If we now take for $\tilde{g}^*$ the harmonic conjugate $\tilde{g}_1^*$ of the double-layer potential with density on C^* a Fredholm eigenfunction corresponding to $-\lambda^*$, we can define

$$f_1(z) = \tilde{g}^*(\zeta(z)) \quad z \in C.$$

We next solve the Dirichlet problem in $\tilde{D}$ and D for functions $\tilde{h}_1$ and h_1 respectively, both of which assume the boundary values f_1 on C. The function $\tilde{h}_1$ may not have a single-valued conjugate, but by adding a suitable linear combination of harmonic measures to $\tilde{h}_1$, we get an admissible fucntion $\tilde{g}_1$ in $\tilde{D}$ and by adding suitable constants to h_1 in each component of D, we get a function g_1 in D such that $\tilde{g}_1, g_1$ form an admissible pair for C. Since $\tilde{g}_1$ and f_1 differ by constants on each component of C_j, the chain of operations going from (15) to (20) starting with $\tilde{h} = \tilde{g}_1$ and $h = g_1$ lead to the extremal admissible pair $\tilde{g}^* = \tilde{g}_1^*$ and $g^* = g_1^*$. This allows us to write

$$\frac{\lambda+1}{\lambda-1} \geqslant \frac{\iint_{\tilde{D}} (\nabla \tilde{g}_1)^2 d\tau_z}{\iint_{D} (\nabla g_1)^2 d\tau_z} \geqslant \frac{\frac{1}{M} \iint_{\tilde{D}^*} (\nabla \tilde{g}_1^*)^2 d\tau_\zeta}{\iint_{D^*} (\nabla g_1^*)^2 d\tau_\zeta} = \frac{1}{M} \frac{\lambda^*+1}{\lambda^*-1}. \tag{21}$$

Thus we have proved (3) for the case in which $K = 1$.

5. Factorization of quasiconformal mappings

To complete the proof of Theorem 1, we shall use the following lemma:

LEMMA 1. *Every homeomorphism f of the plane which is K-quasiconformal in D and M-quasiconformal in $\tilde{D}$ can be factored to the composition of two mappings $f = h \circ g$ where g is conformal in $\tilde{D}$ and K-quasiconformal in D and h is conformal in $g(D)$ and M-quasiconformal in $g(\tilde{D})$.*

Since f is quasiconformal, it has generalized derivatives f_z and $f_{\bar z}$ which satisfy $f_z = \mu f_{\bar z}$ where $\|\mu\|_\infty \leqslant k = (K-1)/(K+1)$ in D and $\|\mu\|_\infty \leqslant m = (M-1)/(M+1)$ in $\tilde{D}$; cf. [6]. In general, a mapping ϕ is called ν-conformal if its generalized derivatives satisfy $\phi_z = \nu \phi_{\bar z}$. For convenience, we shall again normalize f so that $f(\infty) = \infty$. For any measurable function ν, we can use the fundamental existence theorem [4] which says that there exists a unique ν-conformal homeomorphism of the whole plane with fix points 0, 1, and ∞.

There is a ν-conformal homeomorphism of the whole z-plane onto the whole w-plane satisfying

(a) $g(\infty) = \infty$,

(b) $\nu(z) = \mu(z)$ for $z \in D$,

(c) $\nu(z) = 0$ for $z \in \tilde{D}$.

Thus g is conformal in $\tilde{D}$ and K-quasiconformal in D. The conformality of g in $\tilde{D}$ assures us that $g_z(z) = g'(z) \neq 0$ holds for all $z \in \tilde{D}$, and we may define $\varrho(z) = g'(z)/\overline{g'(z)}$ for $z \in \tilde{D}$. We then have $|\varrho(z)| \equiv 1$ for $z \in \tilde{D}$.

Now let Ψ be a ν^*-conformal homeomorphism of the whole w-plane onto the ζ-plane satisfying

(a*) $\Psi(\infty) = \infty$,

(b*) $\nu^*(w) = 0$ if $w \in g(D)$,

(c*) $\nu^*(w) = \mu(g^{-1}(w))\, \varrho(g^{-1}(w))$ if $w \in g(\tilde{D})$.

The mapping Ψ is conformal in $g(D)$ and since $|\varrho| = 1$, Ψ is M-quasiconformal in $g(\tilde{D})$.

Using the formulas [4], $(\Psi \circ g)_z = \Psi_w \cdot g_z + \Psi_{\bar w} \bar g_z$ and $(\Psi \circ g)_{\bar z} = \Psi_w g_{\bar z} + \Psi_{\bar w} \bar g_{\bar z}$ along with $\bar g_{\bar z} = \overline{(g_z)}$ and $\bar g_z = \overline{(g_{\bar z})}$, we deduce that for $z \in D$,

$$(\Psi \circ g)_{\bar z}/(\Psi \circ g)_z = g_{\bar z}/g_z = \mu$$

holds and for $z \in \tilde{D}$,

$$(\Psi \circ g)_{\bar z}/(\Psi \circ g)_z = |\Psi_{\bar w}/\Psi_w|\, |\bar g_{\bar z}/g_z| = \mu \varrho \frac{\bar g'}{g'} = \mu$$

holds. Thus f and $\Psi \circ g$ are both μ conformal in the whole plane. Let L be a similarity transformation $L(\zeta)=a\zeta+b$ with a and b selected so that f and $L\circ(\Psi\circ g)$ agree at $0, 1$, and ∞. Since $L_\zeta=a$ and $L_{\bar\zeta}=0$, we have

$$\frac{(L\circ(\Psi\circ g))_{\bar z}}{(L\circ(\Psi\circ g))_z}=\frac{(\Psi\circ g)_{\bar z}}{(\Psi\circ g)_z}=\mu.$$

Now f and $L\circ(\Psi\circ g)$ are both μ conformal and they agree on $0, 1$ and ∞, which makes them identical. If we set $h=L\circ\Psi$, we have the decomposition $f=h\circ g$ required in the lemma.

Theorem 1 can now be deduced from Lemma 1 by factoring the mapping which is K-quasiconformal in D and M-quasiconformal in D into the composition of g and h as given in the lemma. The mapping g carries C into a curve system C' in the w-plane having Fredholm eigenvalue λ'. The mapping h then carries the curve system C' into the curve system C^* with eigenvalue λ^*. From (14) and (21) we deduce

$$\frac{\lambda^*+1}{\lambda^*-1}\leqslant M\frac{\lambda'+1}{\lambda'-1}\leqslant KM\frac{\lambda+1}{\lambda-1},$$

which is the right side of (2). The same argument applied to the inverse of ζ proves the left side of (2) and Theorem 1 is proved.

6. Simply connected regions

When $\tilde D$ is simply connected, it may be mapped conformally on to the exterior of the unit circle with infinity going into infinity. Let us now assume that this mapping ζ can be extended to be a homeomorphism of the whole plane onto the whole plane which is K-quasiconformal in D, the exterior of $\tilde D$. Then if λ is the Fredholm eigenvalue of C, the boundary of $\tilde D$, and if λ^* is the Fredholm eigenvalue of $\zeta(C)=C^*$, Theorem 1 tells us that

$$\frac{\lambda+1}{\lambda-1}\leqslant K\frac{\lambda^*+1}{\lambda^*-1}.$$

For a circle, $\lambda^*=\infty$, so we have

$$\frac{\lambda+1}{\lambda-1}\leqslant K=\frac{1+k}{1-k},$$

where k is the maximal eccentricity of ζ in D. This yields the inequality

$$\lambda \geqslant \frac{1}{k} \tag{22}$$

as an estimate for λ.

Let σ^* represent the mapping $\sigma(z) = 1/\bar{z}$ which is a reflection in the unit circle. The composite mapping $\sigma = \zeta^{-1} \circ \sigma^* \circ \zeta$ is a mapping of $\tilde{D}$ onto D which leaves the points of C fixed. Hence ζ is conformal in $\tilde{D}$, K-quasiconformal in D, and since σ^* is anticonformal (maximal dilatation 1 and sense reversing), we see that σ is also K-quasiconformal. Ahlfors has called such mapping K-quasiconformal reflections of $\tilde{D}$ onto D, cf. [13].

The existence of a K-quasiconformal reflection enables us to define the homeomorphism of the plane taking $\tilde{D}$ onto $|\zeta|>1$ conformally and D onto $|\zeta|<1$ K-quasiconformally. We simply let ζ be the conformal mapping of $\tilde{D}$ onto $|\zeta|>1$ with $\zeta(\infty)=\infty$, and we set $\zeta = \sigma^* \circ \zeta \circ \sigma^{-1}$ in D.

We now see that when D is a simply connected region, we can choose D^* to be the unit circle and we obtain as a special case of Theorem 1 the following theorem of Ahlfors [1]: *If $\tilde{D}$ admits a K-quasiconformal reflection, then its Fredholm eigenvalue satisfies*

$$\lambda \geqslant \frac{1}{k},$$

where $k=(K-1)/(K+1)$.

7. Eigenvalue estimates

The mapping $\zeta = z + 1/z$ maps a circle $|z| = a$, $a>1$, onto an ellipse C_a with foci at $\zeta = 2$ and $\zeta = -2$, semi-major axis of length $a + 1/a$ and semiminor axis of length $a - 1/a$. (Any ellipse is similar to such an ellipse.) This mapping ζ takes the region $|z|>a$ conformally onto the exterior $\tilde{D}_a$ of C_a. The mapping ζ can be extended to give a quasiconformal mapping of $|z|<a$ onto the interior D_a of C_a by the following definition: if $|z|<1$,

$$\zeta(z) = z + \frac{\bar{z}}{a^2}. \tag{23}$$

For this mapping $\zeta(z)$ in $|z|<1$, we have $\zeta_z = 1$ and $\zeta_{\bar{z}} = 1/a^2$, so $\zeta_{\bar{z}}/\zeta_z = 1/a^2$ and we see that ζ has maximal eccentricity $1/a^2$ in $|z|<1$. If λ^* represents the Fredholm eigenvalue for the ellipse, we have

$$\frac{\lambda^*+1}{\lambda^*-1} \leqslant \frac{1+\dfrac{1}{a^2}}{1-\dfrac{1}{a^2}},$$

or simply
$$\lambda^* \geqslant a^2. \tag{24}$$

According to Schiffer [11], the exact value of λ^* for the ellipse C_a is actually a^2, so our method has yielded a sharp estimate in this case.

We now consider the doubly connected region $\tilde{D}$ contained between two confocal ellipses. By a similarity mapping, these may be brought into the standard position with foci at -2 and 2. The semi-major axes of the two ellipses can then be written in the form $a+1/a$ and $b+1/b$ where $b>a>1$. Thus the region $\tilde{D}$ is the conformal image of the annulus $a<|z|<b$ under the mapping $\zeta = z+1/z$. Let D_a represent the interior of the ellipse C_a and D_b the exterior of the ellipse C_b. The function $\zeta = z+1/z$ also gives us a conformal mapping of $|z|>b$ onto D_b and the extension to $|z|<a$ defined by (23) gives us a mapping of $|z|<a$ onto D_a with maximal eccentricity $1/a^2$. The Fredholm eigenvalue λ for the annulus $a<|z|<b$ is $\lambda=(b/a)^2$; cf. [13]. Then Theorem 1 gives us the estimate

$$\frac{\lambda^*+1}{\lambda^*-1} \leqslant \frac{1+\dfrac{1}{a^2}\,\dfrac{\dfrac{b^2}{a^2}+1}{}}{} $$

or simply
$$\lambda^* \geqslant 1 + \frac{(a^2-1)\,(b^2-a^2)}{b^2+a^4}. \tag{25}$$

This lower bound for the region between confocal ellipses can be compared with the lower bound
$$\lambda^* \geqslant 1 + \frac{(a^2-1)\,(b-a)}{b+a^3} \tag{26}$$

obtained by variational methods in [13]. It is readily shown that (25) gives a larger lower bound for λ^* than does (26).

Another way that Theorem 1 can be used to get estimates for the lowest positive, non-trivial, Fredholm eigenvalue is demonstrated in the following. The affine mapping $\zeta = az+b\bar{z}$ has maximal eccentricity $k=|b/a|$. This mapping carries an annulus $k<|z|<1$ into the region $\tilde{D}$ between concentric similar ellipses. Since λ for the annulus is $1/R^2$, we can obtain an estimate for the eigenvalue λ of the boundary of $\tilde{D}$. We have
$$\frac{\lambda^*+1}{\lambda^*-1} \leqslant \frac{(|a|+|b|)^2}{(|a|-|b|)^2}\,\frac{1+R^2}{1-R^2}.$$

This same kind of argument can be applied to any region which is the affine (or more generally, quasiconformal) image of a region whose eigenvalue λ is known. In particular, estimates are known for such regions as circular regions (i.e., regions whose boundary components are circles), or regions bounded by the n components of a limniscate (i.e., the level curves of $|P(z)|=m$, where $P(z)$ is an nth degree polynomial with n simple zeros and m is sufficiently small so that each level curve encloses just one zero of P and no critical points of P). These may be found in [13]. For example, in the case of a circular domain bounded by circles $|z-a_i|=R_i$, $i=1,2,\ldots,n$,

$$\lambda \geqslant \min_{i\neq j}\left(\frac{|a_i-a_j|}{R_i+R_j}\right)^2. \tag{27}$$

We shall close this section with an estimate for λ for a curve system each of whose curves C_j, $j=1,2,\ldots,n$, is smooth and star-shaped with respect to a point a_j in this interior region D_j. We shall further suppose that each closed set $D_j \cup C_j$ is contained in a disk $|z-a_j|<R_j$ and that the n disks $|z-a_j|<R_j$ are disjoint.

If a curve C, which is star-shaped with respect to the origin, is given in polar coordinates by the equation $r=g(\theta)$, we can easily define a quasiconformal homeomorphism F of the whole plane which carries a circle $|z|=\varrho<R$ onto C and which is the identity mapping in $|z|>R$. Such a mapping is given by

$$F(re^{i\theta};\, g, R, \varrho)=\begin{cases} \dfrac{g(\theta)}{\varrho}\, re^{i\theta} & \text{for } r\leqslant \varrho \\[2mm] \left[R-\dfrac{R-r}{R-\varrho}\,(R-g(\theta))\right] e^{i\theta} & \text{for } \varrho<r<R \\[2mm] re^{i\theta} & \text{for } r\geqslant R. \end{cases}$$

Using the facts that

$$\frac{\partial r}{\partial z}=\frac{r}{2z},\quad \frac{\partial r}{\partial \bar z}=\frac{z}{2r},\quad \frac{\partial \theta}{\partial z}=\frac{1}{2iz},\quad \frac{\partial \theta}{\partial \bar z}=\frac{-1}{2i\bar z},$$

we can compute expressions for $|F_{\bar z}/F_z|$. In order to express the results in geometrical terms, let us observe that $g'(\theta)/g(\theta)=\tan \nu(\theta)$, where $\nu(\theta)$ is the angle between the radius vector from 0 to $g(\theta)e^{i\theta}$ and the normal vector to C at $g(\theta)e^{i\theta}$. We have

$$|F_{\bar z}/F_z|^2=\begin{cases} \dfrac{\tan^2 \nu(\theta)}{4+\tan^2 \nu(\theta)} & \text{for } r<\varrho \\[3mm] \dfrac{\left(1-\dfrac{\varrho}{g(\theta)}\right)^2+\left(1-\dfrac{r}{R}\right)^2\tan^2\nu(\theta)}{\left[\left(1-\dfrac{\varrho}{g(\theta)}\right)+2\dfrac{r}{R}\left(\dfrac{R}{g(\theta)}-1\right)\right]^2+\left(1-\dfrac{r}{R}\right)^2\tan^2\nu(\theta)} & \text{for } \varrho<r>R \\[3mm] 0 & \text{for } r>R. \end{cases}$$

Let us introduce the following notation:

$$\alpha = \underset{|\theta|\leqslant\pi}{\text{g.l.b.}} \frac{\varrho}{g(\theta)}, \quad \beta = \underset{|\theta|\leqslant\pi}{\text{l.u.b.}} \frac{\varrho}{g(\theta)}, \quad \gamma = \underset{|\theta|\leqslant\pi}{\text{l.u.b.}} |\tan \nu(\theta)|, \quad \mu = \frac{\varrho}{R}.$$

We can then show that if ϱ is chosen such that $\alpha \leqslant 1$ ($|z| = \varrho$ lies inside of C)

$$|F_{\bar z}/F_z|^2 \leqslant \begin{cases} \dfrac{\gamma^2}{4+\gamma^2} & \text{for } r < \varrho \\ \dfrac{(1-\alpha)^2 + (1-\mu)^2\gamma^2}{(1-\alpha)^2 + (1-\mu)^2\gamma^2 + 4(1-\mu)(\alpha-\mu)} & \text{for } \varrho < r < R \\ 0 & \text{for } r > R. \end{cases}$$

On the other hand, if ϱ is chosen so that $\alpha \geqslant 1$ (C lies inside of $|z| = \varrho$), then

$$|F_{\bar z}/F_z|^2 \leqslant \begin{cases} \dfrac{\gamma^2}{4+\gamma^2} & \text{for } r < \varrho \\ \dfrac{(\beta-1)^2 + (1-\mu)^2\gamma^2}{(\beta-1)^2 + (1-\mu)^2\gamma^2 + 4(1-\mu)(\alpha-\mu)} & \text{for } \varrho < r < R \\ 0 & \text{for } \varrho > R. \end{cases}$$

Now if the curve C_j of a curve system C is star-shaped with respect to a point a_j, the curve C_j is given by a polar equation $|z - a_j| = g_j(\theta)$, $\theta = \arg(z - a_j)$. We have assumed that there are radii R_j, $j = 1, \ldots, n$ such that $g_j(\theta) < R_j$ for all θ and the disks $|z - a_j| < R_j$, $j = 1, \ldots, n$ are mutually disjoint. Select ϱ_j such that $0 < \varrho_j < R_j$. We define a quasiconformal mapping f of the whole plane which takes the system of circles $|z - a_j| = \varrho_j$ onto the curve system C as follows:

$$f(z) = \begin{cases} F(z - a_j; g_j, R_j, \varrho_j) & \text{for } |z - a_j| < R_j, \quad j = 1, \ldots, n, \\ z & \text{elsewhere.} \end{cases}$$

The desired estimate for the eigenvalue λ of the system C of star-shaped curves is then

$$\frac{\lambda+1}{\lambda-1} \leqslant \frac{1+k}{1-k}\,\frac{1+m}{1-m}\,\frac{L+1}{L-1}, \tag{28}$$

where $$L = \min_{i,j=1,\ldots,n} \left(\frac{|a_i - a_j|}{\varrho_i + \varrho_j}\right)^2, \quad k = \max_{j=1,\ldots,n} k_j, \quad m = \max_{j=1,\ldots,n} m_j,$$

$$k_j = \gamma_j/\sqrt{4+\gamma_j^2}, \quad \gamma_j = \underset{|\theta|\leqslant\pi}{\text{l.u.b.}} |\tan \nu_j(\theta)| = \underset{|\theta|\leqslant\pi}{\text{l.u.b.}} \left|\frac{g_j'(\theta)}{g_j(\theta)}\right|$$

$$\alpha_j = \underset{|\theta|\leqslant\pi}{\text{g.l.b.}}\, \frac{\varrho_j}{g_j(\theta)}, \quad \beta_j = \underset{|\theta|\leqslant\pi}{\text{l.u.b.}}\, \frac{\varrho_j}{g_j(\theta)}, \quad \mu_j = \frac{\varrho_j}{R_j}$$

and

$$m_j^2 = \begin{cases} \dfrac{(1-\alpha_j)^2 + (1-\mu_j)^2\gamma_j^2}{(1-\alpha_j)^2 + (1-\mu_j)^2\gamma^2 + 4(1-\mu_j)(\alpha_j-\mu_j)} & \text{if } \beta_j \leqslant 1 \\[2ex] \dfrac{(\beta_j-1)^2 + (1-\mu_j)^2\gamma_j^2}{(\beta_j-1)^2 + (1-\mu_j)^2\gamma_j^2 + 4(1-\mu_j)(\alpha_j-\mu_j)} & \text{if } \alpha_j \geqslant 1. \end{cases}$$

The alternative $\beta_j \leqslant 1$ holds when the circle $|z-a_j| = \varrho_j$ has been selected so as to lie within C_j and the alternative $\alpha_j \geqslant 1$ holds when C_j lies within the circle $|z-a_j| = \varrho_j$. When $\gamma_j < \infty$, the numbers m_j are all less than 1, for $g_j(\theta) < R_j$ means that $\alpha_j > \mu_j$, while $\mu_j < 1$. Thus we have a lower bound for λ which is greater than 1. Each of the quantities appearing in the estimate for λ has a simple geometrical significance; for example $\nu_j(\theta)$ is the angle between the vector $z - a_j$ and the normal to C_j at z, where $z - a_j = g_j(\theta)\, e^{i\theta}$.

8. Quasiconformal equivalence of curve systems

Curve systems C and C^* are called quasiconformally equivalent if there is a quasiconformal homeomorphism of the whole plane taking C onto C^*. Let us now remove the restriction in the definition of λ that the system C have continuous curvature. For an arbitrary system of Jordan curves bounding a region $\tilde{D}$ of connectivity n, with $\infty \in \tilde{D}$, we shall define Λ as the greatest lower bound of all numbers L such that

$$L^{-1} \leqslant \frac{\displaystyle\iint_D (\nabla h)^2\, d\tau}{\displaystyle\iint_{\tilde{D}} (\nabla \tilde{h})^2\, d\tau} \leqslant L$$

holds for all admissible pairs of harmonic functions h and $\tilde{h}$ for C. We then set $\lambda = (\Lambda + 1)/(\Lambda - 1)$.

A glance at the proof of Theorem 1 will convince one that if f is a quasiconformal homeomorphism of the whole plane which is K-quasiconformal in D and M-quasiconformal in $\tilde{D}$, then to each admissible pair of functions h, $\tilde{h}$ for C, there corresponds in a one-to-one fashion (by transplanting the boundary values) an admissible pair h^* and $\tilde{h}^*$ for C^* such that

$$\frac{\displaystyle\iint_D (\nabla h)^2\, d\tau}{\displaystyle\iint_{\tilde{D}} (\nabla \tilde{h})^2\, d\tau} \leqslant KM \frac{\displaystyle\iint_{D^*} (\nabla h^*)^2\, d\tau}{\displaystyle\iint_{\tilde{D}^*} (\nabla \tilde{h}^*)^2\, d\tau} \leqslant KM\Lambda^*.$$

Thus $\Lambda \leqslant KM\Lambda^*$, and application of the same reasoning to the inverse function shows that (3) holds for arbitrary finite systems of Jordan curves. Consequently $\lambda^* > 1$ holds for the eigenvalue of C^* (i.e., $\Lambda^* < \infty$), if, and only if, $\lambda > 1$ (i.e., $\Lambda < \infty$) holds for the eigenvalue of C.

Let $\mathcal{D}^*$ represent a class of canonical domains for domains of connectivity n, and $\mathcal{J}^*$ denote the class of curve systems which are boundaries of domains in $\mathcal{D}^*$. We shall assume that each curve system $C^* \in \mathcal{J}^*$ has $\lambda^* > 1$. The circular domains discussed at the end of the preceding section are examples of such domains. We shall now prove the following theorem, suggested to the author by L. V. Ahlfors, who showed by a similar argument that $\lambda > 1$ is sufficient for the existence of a quasiconformal reflection in the simply connected case.

THEOREM 2. *The function f which maps $\tilde{D}$ conformally onto $\tilde{D}^* \in \mathcal{D}^*$ can be extended to a quasiconformal homeomorphism of the whole plane if, and only if, $\lambda > 1$.*

If the extension is possible, the fact that $\lambda^* > 1$ implies that $\lambda > 1$ is shown in the preceding paragraphs. The proof that $\lambda > 1$ implies the possibility of such an extension draws heavily from the work of Ahlfors and Beurling [7]. Let us suppose that $\lambda > 1$. In order to prove that f can be extended to a quasiconformal homeomorphism of the whole plane, it suffices to focus our attention on each component D_k and prove that f can be extended into D_k to give a quasiconformal mapping of D_k onto D_k^*.

Consider four distinct points P_1, P_2, P_3, P_4 in counterclockwise orientation on C_k. Let α denote the arc $\widehat{P_1 P_2}$ between P_1 and P_2 on C_k and $\beta = \widehat{P_3 P_4}$. Let $d(\alpha, \beta)$ denote the extremal distance between the arcs α and β relative to D_k. Then

$$d(\alpha, \beta) = \left(\iint_{D_k} (\nabla h)^2 \, d\tau \right)^{-1}, \tag{29}$$

where h is the real part of the holomorphic function which maps D_k onto the rectangle with P_1, P_2, P_3, P_4 going into vertices, α going into the edge $h = 0$, and β going into the edge $h = 1$. (If C is smooth, h is characterized as the harmonic function in D_k satisfying $h = 0$ on α, $h = 1$ on β, and $\partial h / \partial n = 0$ on $C_k - (\alpha \cup \beta)$.) Let $\tilde{h}$ be the harmonic function in $\tilde{D}$ which has a single-valued conjugate harmonic function, assumes some constant values b_j on each C_j, $j \neq k$, and has the same values as h on C_k. If we extend h to D by $h \equiv b_j$ in D_j then h and $\tilde{h}$ form an admissible pair for C and we conclude that

$$\Lambda^{-1} \leqslant \frac{\iint_{D_k} (\nabla h)^2 \, d\tau}{\iint_{\tilde{D}} (\nabla \tilde{h})^2 \, d\tau} \leqslant \Lambda. \tag{30}$$

If we transplant $\tilde{h}$ to $\tilde{D}^*$ by means of the conformal mapping $\tilde{f}$ (i.e., $\tilde{h}^*(\tilde{f}(z)) = \tilde{h}(z)$) we have

$$\iint_{\tilde{D}^*} (\nabla \tilde{h}^*)^2 \, d\tau = \iint_{\tilde{D}} (\nabla \tilde{h})^2 \, d\tau.$$

This gives us

$$\Lambda^{-1} \leqslant \frac{\iint_{D_k} (\nabla h)^2 \, d\tau}{\iint_{\tilde{D}^*} (\nabla h^*)^2 \, d\tau} \leqslant \Lambda. \tag{31}$$

We next observe that $\tilde{f}$ takes the points P_1, P_2, P_3, P_4 on C_k into four points $P_1^*, P_2^*, P_3^*, P_4^*$ on the curve C_k^*; the arc α goes into α^* and β into β^*.

The harmonic function $\tilde{h}^*$ has value 0 on α^* and 1 on β^*. Let h^* denote the harmonic function in D^* which assumes the same boundary values as $\tilde{h}^*$ on C^*. (Thus h^* is constant in each region D_j^*, $j \neq k$.) Then if λ^* denotes the eigenvalue of C^*, we have for the admissible pair h^*, $\tilde{h}^*$:

$$(\Lambda^*)^{-1} \leqslant \frac{\iint_{\tilde{D}^*} (\nabla \tilde{h}^*)^2 \, d\tau}{\iint_{D_k^*} (\nabla h^*)^2 \, d\tau} \leqslant \Lambda^*. \tag{32}$$

Multiplication of (31) and (32) yields

$$(\Lambda\Lambda^*)^{-1} \leqslant \frac{\iint_{D_k} (\nabla h)^2 \, d\tau}{\iint_{D_k^*} (\nabla h^*)^2 \, d\tau} \leqslant \Lambda\Lambda^*. \tag{33}$$

It is easily demonstrated (by the standard argument using $|\nabla h^*| \, ds$ as a competing metric in the definition of extremal distance) that if g^* is the harmonic function in D_k^* which assumes values 0 on α^*, 1 on β^*, and is the real part of the holomorphic function which maps D_k^* onto a rectangle with $P_1^*, P_2^*, P_3^*, P_4^*$ going into vertices, then

$$\frac{1}{d(\alpha^*, \beta^*)} = \iint_{D_k^*} (\nabla g^*)^2 \, d\tau \leqslant \iint_{D_k^*} (\nabla h^*)^2 \, d\tau, \tag{34}$$

where $d(\alpha^*, \beta^*)$ represents the extremal distance between the arcs α^* and β^* relative to the region D_k^*. From (29), (33) and (34), we conclude that

$$d(\alpha^*, \beta^*)/d(\alpha, \beta) \geqslant (\Lambda\Lambda^*)^{-1}.$$

The same result applies to the complementary arcs $\hat{\alpha} = \widehat{P_2 P_3}$ and $\hat{\beta} = \widehat{P_4 P_1}$, so we have

$$d(\hat{\alpha}^*, \hat{\beta}^*)/d(\hat{\alpha}, \hat{\beta}) \geqslant \Lambda\Lambda^{-1},$$

but $d(\alpha, \beta)\, d(\hat{\alpha}, \hat{\beta}) = 1$, and $d(\alpha^*, \beta^*)\, d(\hat{\alpha}^*, \hat{\beta}^*) = 1$, so we have obtained

$$0 < B^{-1} \leqslant d(\alpha^*, \beta^*)/d(\alpha, \beta) \leqslant B < \infty, \tag{35}$$

where
$$B = \Lambda\Lambda^* = \frac{(\lambda+1)\,(\lambda^*+1)}{(\lambda-1)\,(\lambda^*-1)}. \tag{36}$$

The condition (35) guarantees that the mapping $\tilde{f}$ of $\tilde{D}$ onto $\tilde{D}^*$ can be extended to a quasiconformal homeomorphism of D_k onto D_k^*. The proof of the following lemma will show how (35) can be transformed into the condition given in [7] which assures us that a boundary correspondence on the real axes can be extended into the upper half planes.

LEMMA 2. *If there is a constant $B < \infty$ such that $B^{-1} < d(\alpha^*, \beta^*) < B$ for any arcs α and β on C_k for which $d(\alpha, \beta) = 1$, then f can be extended to give a quasiconformal mapping of D_k onto D_k^*.*

The region D_k can be mapped conformally onto the upper half plane U. Then the points P_1, P_2, P_3, P_4 correspond to some points Q_1, Q_2, Q_3, Q_4 in increasing order on the real axis. Likewise we can map D_k^* onto the upper half plane U^*. The points $P_1^*, P_2^*, P_3^*, P_4^*$ correspond to four points $Q_1^*, Q_2^*, Q_3^*, Q_4^*$ on the real axis. Both mappings shall be chosen to that $Q_4 = \infty$ and $Q_4^* = \infty$. The boundary correspondence between C_k and C_k^* defined by $\tilde{f}$ induces a boundary correspondence ϕ of the real axis onto itself such that $Q_i^* = \phi(Q_i)$, $i = 1, 2, 3, 4$.

The extremal distances $d(\alpha, \beta)$ and $d(\alpha^*, \beta^*)$ are conformal invariants. If $\phi(\alpha) = \gamma$ and $\phi(\beta) = \delta$, we have $d(\gamma, \delta) = 1$ and

$$0 < B^{-1} \leqslant d(\gamma^*, \delta^*) \leqslant B < \infty. \tag{37}$$

Let us denote the cross ratio $\overline{Q_2 Q_3} \cdot \overline{Q_1 Q_4} / \overline{Q_1 Q_2} \cdot \overline{Q_3 Q_4}$ by χ. It is pointed out in [7] that $d(\gamma, \delta) = P(\chi)$, where P is a monotone increasing function of χ satisfying

$P(0)=0$, $P(1)=1$, $P(\infty)=\infty$ and $P(\chi^{-1})=(P(\chi))^{-1}$. If we select the points Q_1, Q_2, Q_3, Q_4 to have the coordinates $x-h$, x, $x+h$, ∞, then $\chi=1$ and $d(\gamma,\delta)=P(\chi)=1$. We now have

$$\chi^* = \frac{\overline{Q_2^* Q_3^*}\cdot\overline{Q_1^* Q_4^*}}{\overline{Q_1^* Q_2^*}\cdot\overline{Q_3^* Q_4^*}} = \frac{\phi(x+h)-\phi(x)}{\phi(x)-\phi(x-h)}$$

and from (37),
$$0 < \frac{1}{P^{-1}(B)} \leqslant \frac{\phi(x+h)-\phi(x)}{\phi(x)-\phi(x-h)} \leqslant P^{-1}(B) < \infty. \tag{38}$$

This is just the necessary and sufficient condition given in [7] for the existence of a quasiconformal mapping of U onto U^* with the boundary correspondence ϕ. Thus the conformal mapping f can be continued quasiconformally into each D_k to give us a quasiconformal homeomorphism of the whole plane, and Theorem 2 is proved.

It is furthermore shown in [7] that there is an extension of f into each component of D which has maximal dilatation K not greater than $[P^{-1}(B)]^2$. Since $P(\varrho) = 1+\theta(\varrho)\log\varrho$, where $\theta(\varrho)$ increases from $\theta(1)=.2284$ to $\theta(\infty)=1/\pi=.3183$, we obtain the estimate

$$K \leqslant e^{(B-1)/(.1142)}. \tag{39}$$

An immediate consequence of Theorem 2 is the following corollary.

COROLLARY. *The curve system C is quasiconformally equivalent to a curve system $C^*\in\mathfrak{J}^*$ if, and only if, $\lambda>1$. In particular, C is quasiconformally equivalent to a system of circles if, and only if, $\lambda>1$.*

In [3] Ahlfors gave a geometrical condition on a simple closed curve C which is necessary and sufficient for C to admit a quasiconformal reflection. This condition can also be extended to a curve system C consisting of n Jordan curves to give us the following theorem.

THEOREM 3. *A curve system C is quasiconformally equivalent to a system of circles if, and only if, there is a constant A such that*

$$(\overline{P_1P_2}\cdot\overline{P_3P_4})/(\overline{P_1P_3}\cdot\overline{P_2P_4}) \leqslant A < \infty \tag{40}$$

for any four points P_1, P_2, P_3, P_4 which follow each other in this order on any C_k, $k=1,2,\dots,n$.

The necessity of the condition (40) can be deduced easily as follows. If C is quasiconformally equivalent to a system of circles C^*, then each C_k is mapped onto a circle C_k^*. The quasiconformal mapping from D_k to D_k^*, followed by the reflection in C_k^*, and this followed by the quasiconformal mapping of D_k^* onto D_k gives us a quasiconformal reflection in C_k. The condition (40) is just the necessary (and sufficient) condition for the existence of a quasiconformal reflection in C_k (see [3]).

In order to show the sufficiency of the condition (40). we shall show, using Ahlfors' argument in [3], that (40) implies the condition given in Lemma 2, where the constant B depends upon A. We denote by $\widehat{P_i P_j}$ the arc of C_k between P_i and P_j. Then we set

$$\alpha=\widehat{P_2 P_3},\quad \beta=\widehat{P_4 P_1},\quad \hat{\alpha}=\widehat{P_1 P_2},\quad \hat{\beta}=\widehat{P_3 P_4}.$$

As before, we can map $\tilde{D}$ conformally onto a circular domain, so that the four points P_1, P_2, P_3, P_4 on C_k go into four points $P_1^*, P_2^*, P_3^*, P_4^*$ on the circle C_k^*. The arcs corresponding to $\alpha, \beta, \hat{\alpha}, \hat{\beta}$ are $\alpha^*, \beta^*, \hat{\alpha}^*, \hat{\beta}^*$ respectively. A linear fractional transformation can be used to take $P_4 \rightarrow \infty$. This leaves the cross ratio invariant, so that (40) says

$$\overline{P_1 P_2} \leqslant A\, \overline{P_1 P_3}, \tag{41}$$

using the same letters for points after the linear fractional transformation.

We select α and β so that $d_{D_k}(\alpha,\beta)=1$, where $d_D(\alpha,\beta)$ represents the extremal distance between α and β relative to D. Then for any point P on β, we have $\overline{P_2 P_1} \leqslant A\,\overline{P_2 P}$ or $\overline{PP_2} \geqslant A^{-1}\,\overline{P_1 P_2}$. For any point Q on α, we have $\overline{QP_2} \leqslant A\overline{P_2 P_3}$ so we see that the points of α are at most at a distance $r_1 = A\,\overline{P_2 P_3}$ from P_2.

We shall next prove that $\overline{P_1 P_2} \leqslant A^2 e^{2\pi}\,\overline{P_2 P_3}$. If $\overline{P_1 P_2}/\overline{P_2 P_3} > A^2 e^{2\pi}$ were to hold, then $\overline{PP_2}/\overline{QP_2} > e^{2\pi}$ would hold for any points $P \in \beta$ and $Q \in \alpha$. Thus α and β would be separated by an annulus whose radii have ratio $e^{2\pi}$. The extremal distance between the two circles of such an annulus is 1, so $d_{D_k}(\alpha,\beta) > 1$, a contradiction. Thus we must have $\overline{P_1 P_2} \leqslant A^2 e^{2\pi}\,\overline{P_2 P_3}$. Likewise, interchanging P_1 and P_3 yields $\overline{P_2 P_3} \leqslant A^2 e^{2\pi}\,\overline{P_1 P_2}$. If $Q_1 \in \alpha$ and $Q_2 \in \beta$, then

$$\overline{Q_1 Q_2} \geqslant A^{-1}\,\overline{Q_1 P_1} \geqslant A^{-2}\,\overline{P_1 P_2} \geqslant A^{-4} e^{2\pi}\,\overline{P_2 P_3}.$$

The minimum distance from α to β is thus at least $r_2 = A_4 e^{-2\pi}\,\overline{P_2 P_3}$.

As a competing metric in the definition of $d_{\tilde{D}}(\alpha,\beta)$, we now use $\varrho|dz|$ where $\varrho=1$ in the circular disk of radius r_1+r_2 about P_2 and $\varrho=0$ elsewhere. Then for any curve γ in $\tilde{D}$ which goes from α to β, $\int_\gamma \varrho|dz| \geqslant r_2$ while $\iint_{\tilde{D}} \varrho^2 d\tau \leqslant \pi(r_1+r_2)^2$, so

$$d_{\tilde{D}}(\alpha,\beta) \geqslant \frac{r_2^2}{\pi(r_1^2+r_2^2)^2} = \pi^{-1}(1+A^5 e^{2\pi})^{-2}.$$

The same estimate applies to $d_{\tilde{D}}(\hat{\alpha},\hat{\beta})$, and since $d_{\tilde{D}}(\hat{\alpha},\hat{\beta})\cdot d_{\tilde{D}}(\alpha,\beta)=1$, we also obtain an upper bound for $d_{\tilde{D}}(\alpha,\beta)$; i.e.

$$\pi^{-1}(1+A^5e^{2\pi})^{-1}\leqslant d_{\tilde{D}}(\alpha,\beta)\leqslant\pi(1+A^5e^{2\pi}). \tag{42}$$

Since $\tilde{D}$ was mapped conformally onto the circular domain $\tilde{D}^*$ and extremal distance is a conformal invariant, we have $d_{\tilde{D}}(\alpha,\beta)=d_{\tilde{D}^*}(\alpha^*,\beta^*)$. We now make use of two facts: (1) the extremal distance is decreased if the domain $\tilde{D}^*$ is expanded to be the whole complement of D_k^*; and (2), the extremal distances between α^* and β^* relative to D_k^* and relative to the complement of D_k^* are the same. Thus

$$d_{D_k^*}(\alpha^*,\beta^*)\leqslant(1+A^5e^{2\pi}).$$

The same argument applied to $\hat{\alpha}^*,\hat{\beta}^*$ yields a lower bound for $d_{D_k^*}(\alpha^*,\beta^*)$, so that

$$\pi^{-1}(1+A^5e^{2\pi})^{-1}\leqslant d_{D_k^*}(\alpha^*,\beta^*)\leqslant\pi(1+A^5e^{2\pi}) \tag{43}$$

when $d_{D_k}(\alpha,\beta)=1$. Use of Lemma 2 completes the proof of Theorem 3.

References

[1]. Ahlfors, Lars V., Remarks on the Neumann–Poincaré integral equation. *Pacific J. Math.*, 2 (1952), 271–280.

[2]. —— Conformality with respect to Riemannian metrics. *Ann. Acad. Sci. Fenn.*, Ser. A, I, 206 (1955), 1–22.

[3]. —— Quasiconformal reflections. *Acta Math.*, 109 (1963), 291–301.

[4]. Ahlfors, Lars V., & Bers, Lipman, Riemann's Mapping theorem for variable metrics. *Ann. of Math.*, 72 (1960), 385–404.

[5]. Bergman, S. & Schiffer, M., Kernel functions and conformal mapping. *Compositio Math.*, 8 (1951), 205–249.

[6]. Bers, L., On a theorem of Mori. *Trans. Amer. Math. Soc.*, 84 (1957), 78–84.

[7]. Beurling, A., & Ahlfors, L., The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96 (1956), 125–142.

[8]. Mori, Akira, On quasiconformality and pseudoanalyticity. *Trans. Amer. Math. Soc.*, 84 (1957), 56–77.

[9]. Pfluger, A., Über die Äquivalenz der geometrischen und der analytischen Definition quasikonformer Abbildungen. *Comment. Math. Helv.*, 33 (1959), 23–33.

[10]. Royden, H. L., A modification of the Neumann–Poincaré method for multiply connected regions. *Pacific J. Math.*, 2 (1952), 385–394.

[11]. Schiffer, M., Fredholm eigenvalues of plane domains. *Pacific J. Math.*, 7 (1957), 1187–1225.

[12]. —— Fredholm eigenvalues of multiply connected domains. *Pacific J. Math*, 9 (1959) 211–269.

[13]. Schiffer, M. & Springer G., Fredholm eigenvalues and conformal mapping of multiply connected domains. To appear in *J. Analyse Math.*

[14]. Tienari, M., Fortsetzung einer quasikonformen Abbildung über einen Jordanbogen. *Ann. Acad. Sci. Fenn.*, Ser. A, I, 321 (1962), 31 pp.

[15]. Warschawski, S. E., On the solution of the Lichtenstein-Gershgorin integral equation in conformal mapping. *Experiments in the Computation of Conformal Maps*, National Bureau of Standards Applied Math. Series, 42 (1955), 7–30.

[16]. —— On the effective determination of conformal maps. *Contributions to the Theory of Riemann Surfaces*, Annals of Math. Studies, 30, Princeton University Press, Princeton, N.J., (1953), 177–188.

Received December 30, 1963.