ON THE MEAN VALUE OF THE ERROR TERM FOR A CLASS OF ARITHMETICAL FUNCTIONS

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§ 1. Introduction

In a recent paper [2] we studied the average order of a large class of arithmetical functions which occur as the coefficients of Dirichlet series which satisfy a functional equation. In this paper we obtain an estimate, in mean, for the error-term associated with such arithmetical functions. Apart from obtaining a number of classical results as special cases, we obtain some new results on certain arithmetical functions in algebraic number theory.

If $\mathfrak{K}$ is an ideal class in an algebraic number field K of degree n, the Dedekind zeta-function of the class $\mathfrak{K}$ is defined by

$$\zeta_K(s, \mathfrak{K}) = \sum_{\mathfrak{a} \in \mathfrak{K}} \frac{1}{(N\mathfrak{a})^s},$$

where the summation is over all non-zero integral ideals in $\mathfrak{K}$, and if we consider the arithmetical function

$$\sum_{k \leqslant x} a_k(\mathfrak{K}),$$

where $a_k(\mathfrak{K})$ denotes the number of ideals in $\mathfrak{K}$ of norm k, then it is known, after Weber and Landau [9] that

$$E(x) \equiv \sum_{k \leqslant x} a_k(\mathfrak{K}) - \lambda x = O(x^{(n-1)/(n+1)}),$$

where λ is the residue of $\zeta_K(s, \mathfrak{K})$ at $s=1$. In this paper we shall show, for example, that if $n=2$, then

$$\frac{1}{x}\int_1^x |E(y)|\, dy = O(x^{\frac{1}{4}}).$$

In view of the fact [2, p. 128] that

$$\overline{\underline{\lim}}\, \frac{E(x)}{x^{\frac{1}{4}}} = \pm\infty,$$

this result seems to be the best possible.

Our main theorem implies this as well as several other results. If $r_2(n)$ denotes the number of representations of the integer n as a sum of two squares, and

$$P(x) = \sum_{n \leqslant x} r_2(n) - \pi x,$$

then $P(x)$ is the error-term in the lattice-point problem for the circle. Although the conjecture that

$$P(x) = O(x^{\frac{1}{4}+\varepsilon}),$$

for every positive ε, is yet to be proved, it was shown by Hardy [5] that

$$\frac{1}{x}\int_1^x |P(y)|\, dy = O(x^{\frac{1}{4}+\varepsilon}),$$

for every $\varepsilon > 0$. This was sharpened by Cramér [4] into

$$\frac{1}{x}\int_1^x |P(y)|\, dy = O(x^{\frac{1}{4}}), \tag{1}$$

which he obtained as a consequence of an asymptotic formula for the error-term, in mean-square, namely

$$\frac{1}{x}\int_1^x |P(y)|^2\, dy = c_1 x^{\frac{1}{2}} + O(x^{\frac{1}{4}+\varepsilon}), \tag{2}$$

for every $\varepsilon > 0$. Here c_1 is a constant given by

$$c_1 = \frac{1}{3\pi^2}\sum_{n=1}^{\infty}\left(\frac{r_2(n)}{n^{\frac{3}{4}}}\right)^2.$$

Cramér also obtained a formula similar to (1) in the case of the error-term in Dirichlet's divisor problem. If $d(n)$ denotes the number of divisors of n, and

$$\Delta(x) = \sum_{n \leqslant x} d(n) - x \log x - (2\gamma - 1)\, x,$$

where γ is Euler's constant, then Cramér's result is that

$$\frac{1}{x}\int_1^x |\Delta(y)|^2\, dy = c_2 x^{\frac{1}{2}} + O(x^{\frac{1}{4}+\varepsilon}), \quad c_2 = \frac{1}{6\pi^2}\sum_{n=1}^{\infty}\left(\frac{d(n)}{n^{\frac{3}{4}}}\right)^2, \tag{3}$$

which implies that
$$\frac{1}{x}\int_1^x |\Delta(y)|\,dy = O(x^{\frac{1}{4}}).$$

Landau [8, Satz 548], and later Walfisz [12] improved the error-term in (2) by showing that
$$\frac{1}{x}\int_1^x |P(y)|^2\,dy = c_1 x^{\frac{1}{2}} + O(\log^3 x).$$

There is a reference in the literature [11] to a similar improvement of (3), although we have not had access to that paper. Both these results, however, will emerge as corollaries to our main theorem. So will the following result, due to Walfisz [13], on Ramanujan's τ-function:
$$\frac{1}{x}\int_1^x |T(y)|^2\,dy = c_3 x^{11\frac{1}{2}} + O(x^{11}\log^2 x),$$

where
$$T(x) = \sum_{n\leqslant x} \tau(n), \quad \text{and} \quad c_3 = \frac{1}{50\pi^2}\sum_{n=1}^{\infty}\frac{\tau^2(n)}{n^{12\frac{1}{2}}}.$$

§ 2. Preliminaries

The functional equation we are concerned with is set up as follows.

Let $\{a_n\}$, $\{b_n\}$ be two sequences of complex numbers, not all zero, and $\{\lambda_n\}$, $\{\mu_n\}$ be two sequences of real numbers such that
$$0<\lambda_1<\lambda_2<\dots<\lambda_n\to\infty,$$
$$0<\mu_1<\mu_2<\dots<\mu_n\to\infty.$$

Let δ be a real number, s a complex number with $s=\sigma+it$. Let
$$\Delta(s) = \prod_{\nu=1}^{N}\Gamma(\alpha_\nu s+\beta_\nu),$$

where $N\geqslant 1$, β_ν is a complex number, and $\alpha_\nu>0$. Let $A=\sum_{\nu=1}^{N}\alpha_\nu$. We say that the functional equation
$$\Delta(s)\,\varphi(s) = \Delta(\delta-s)\,\psi(\delta-s) \tag{4}$$

holds, if φ and ψ can be represented by the Dirichlet series
$$\varphi(s) = \sum_{n=1}^{\infty} a_n\lambda_n^{-s}, \quad \psi(s) = \sum_{n=1}^{\infty} b_n\mu_n^{-s},$$

each of which converges absolutely in some right half-plane, and if there exists in the s-plane a domain D, which is the exterior of a bounded, closed set S, in which there exists a holomorphic function χ with the property $\lim_{|t|\to\infty} \chi(\sigma+it)=0$, uniformly in every interval $-\infty<\sigma_1\leqslant\sigma\leqslant\sigma_2<+\infty$, and

$$\chi(s)=\Delta(s)\,\varphi(s), \quad \text{for} \quad \sigma>c_1,$$

$$\chi(s)=\Delta(\delta-s)\,\psi(\delta-s), \quad \text{for} \quad \sigma<c_2,$$

where c_1, c_2 are some constants.(1)

For $\varrho\geqslant 0$ we define

$$A_\lambda^\varrho(x)=\frac{1}{\Gamma(\varrho+1)}\sum_{\lambda_n\leqslant x}{}' a_n(x-\lambda_n)^\varrho,$$

the dash indicating that the last term has to be multiplied by $\frac{1}{2}$, if $\varrho=0$ and $x=\lambda_n$.

It is known [3, formula (4)] that functional equation (4) implies the identity

$$A_\lambda^\varrho(x)-S_\varrho(x)=\sum_{n=1}^{\infty}\frac{b_n I_\varrho(\mu_n x)}{\mu_n^{\delta+\varrho}}, \tag{5}$$

for $x>0$, and $\varrho\geqslant 2A\beta-A\delta-\frac{1}{2}$, where β is such that $\sum_{n=1}^{\infty}|b_n|\,\mu_n^{-\beta}<\infty$. We assume ϱ to be an integer, in which case

$$S_\varrho(x)=\frac{1}{2\pi i}\int_{C}\frac{\Gamma(s)\,\varphi(s)}{\Gamma(s+\varrho+1)}x^{s+\varrho}\,ds,$$

where C is a curve enclosing all the singularities of the integrand, and

$$I_\varrho(x)=\frac{1}{2\pi i}\int_{C'}\frac{\Gamma(\delta-s)\,\Delta(s)}{\Gamma(\varrho+1+\delta-s)\,\Delta(\delta-s)}x^{\delta+\varrho-s}\,ds.$$

Here C' consists of the lines

$$\sigma=c_\varrho+it \quad \text{with} \quad |t|>R,$$

where

$$c_\varrho=\frac{A\delta+\varrho}{2A}-\varepsilon, \quad 0<\varepsilon<\frac{1}{4A}, \quad c_\varrho>c'\equiv\max\left(-\operatorname{Re}\frac{\beta_\nu}{\alpha_\nu}\right), \quad \nu=1,2,\ldots,N,$$

together with three sides of the rectangle whose vertices are $c_\varrho-iR$, $c_\varrho+r-iR$, $c_\varrho+r+iR$, and $c_\varrho+iR$. We choose r and R such that all the poles of the integrand are to the left of C'.

(1) $c, c_1, c_2, \ldots, c', c'', \ldots$ are constants which do not necessarily have the same value at all occurrences.

If ϱ is an integer, $\lambda > 0$, and $0 < \varrho\lambda < x$, the ϱ^{th} finite difference of the function $F(x)$ is defined as

$$\Delta_\lambda^\varrho F(x) = \sum_{\nu=0}^{\varrho} (-1)^{\varrho-\nu} \binom{\varrho}{\nu} F(x+\nu\lambda).$$

If F has ϱ derivatives, then

$$\Delta_\lambda^\varrho F(x) = \int_x^{x+\lambda} dt_1 \int_{t_1}^{t_1+\lambda} dt_2 \dots \int_{t_{\varrho-1}}^{t_{\varrho-1}+\lambda} F^{(\varrho)}(t_\varrho)\, dt_\varrho,$$

where $F^{(\varrho)}$ is the ϱ^{th} derivative of F. Since

$$\Delta_\lambda^\varrho A_\lambda^\varrho(y) = \sum_{\lambda_n \leqslant y}' a_n \frac{\Delta_\lambda^\varrho (y-\lambda_n)^\varrho}{\Gamma(\varrho+1)} + \sum_{\nu=0}^{\varrho} \frac{1}{\Gamma(\varrho+1)} (-1)^{\varrho-\nu} \binom{\varrho}{\nu} \sum_{y<\lambda_n \leqslant y+\nu\lambda} a_n (y+\nu\lambda-\lambda_n)^\varrho,$$

and since

$$\{\Gamma(\varrho+1)\}^{-1} \Delta_\lambda^\varrho (y-\lambda_n)^\varrho = \lambda^\varrho,$$

we have

$$\Delta_\lambda^\varrho A_\lambda^\varrho(y) = \lambda^\varrho A_\lambda^0(y) + O(\lambda^\varrho \sum_{y<\lambda_n \leqslant y+\varrho\lambda} |a_n|). \tag{6}$$

Again

$$\Delta_\lambda^\varrho S_\varrho(y) = \int_y^{y+\lambda} dt_1 \int_{t_1}^{t_1+\lambda} dt_2 \dots \int_{t_{\varrho-1}}^{t_{\varrho-1}+\lambda} S_0(t_\varrho)\, dt_\varrho,$$

and if the only singularities of φ are assumed to be poles, then

$$S_0(y) = \sum_\xi c_\xi y^\xi (\log y)^{r_\xi - 1},$$

where r_ξ is the order of the pole at $s=\xi$, so that

$$\Delta_\lambda^\varrho S_\varrho(y) = S_0(y)\,\lambda^\varrho + O(\lambda^{\varrho+1} y^{q-1} \log^{r-1} y), \tag{7}$$

where q is the maximum of the real parts of the poles of φ, and r is the maximum order of a pole with real part q.

From (6) and (7) we have

$$A_\lambda^0(y) - S_0(y) = \lambda^{-\varrho} \Delta_\lambda^\varrho [A_\lambda^\varrho(y) - S_\varrho(y)] + O(\lambda y^{q-1} \log^{r-1} y) + O(\sum_{y<\lambda_n \leqslant y+\varrho\lambda} |a_n|). \tag{8}$$

On the left-hand side of (8) is the "error-term" which we wish to estimate in "mean square". If we write

$$\left.\begin{aligned} E(y) &\equiv A^0(y) - S_0(y), \\ W(y) &\equiv \Delta_\lambda^\varrho [A_\lambda^\varrho(y) - S_\varrho(y)], \\ V(y) &\equiv O(\lambda y^{q-1} \log^{r-1} y) + O(\sum_{y<\lambda_n \leqslant y+\varrho\lambda} |a_n|), \end{aligned}\right\} \tag{9}$$

then we have

$$\int_1^x |E(y)|^2\,dy = \int_1^x |V(y)|^2\,dy + \int_1^x \lambda^{-2\varrho}|W(y)|^2\,dy + \int_1^x \lambda^{-\varrho}(W\bar{V} + \bar{W}V)\,dy. \tag{10}$$

Our problem now reduces to estimating the integrals on the right-hand side of (10). The form of V is such that in addition to the assumptions already made on the nature and location of the singularities of φ, we need assume only an order condition on the a_n in order to estimate the first integral. To estimate the second integral, we assume functional equation (4), apply the difference-operator Δ_λ^ϱ to identity (5) which results from it [3, (4)], and take the square of the absolute value on both sides. This would involve estimating the integral

$$\int_1^x \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_m \bar{b}_n \Delta_\lambda^\varrho I_\varrho(\mu_m y)\cdot \Delta_\lambda^\varrho \bar{I}_\varrho(\mu_n y)}{(\mu_m \mu_n)^{\delta+\varrho}} \right) \lambda^{-2\varrho}\,dy, \tag{11}$$

where the bars denote complex conjugates. An essential element of the method is to choose λ not as a constant but as a function of y, namely $\lambda = y^{1-1/2A-\eta}$, $\eta > 0$, where the A comes from the gamma-factors in the functional equation, and to choose η suitably in the resulting estimate. Since ϱ may be chosen as large as we require, the estimation of (11) depends on an estimate of the integral

$$\int_1^x \lambda(y)^{-2\varrho}\cdot \Delta_\lambda^\varrho I_\varrho(\mu_m y)\cdot \Delta_\lambda^\varrho \bar{I}_\varrho(\mu_n y)\,dy \tag{12}$$

for different ranges of μ_m and μ_n.

Now the asymptotic expansion of I_ϱ is known in a convenient form [3]. If m is any positive integer, we have

$$I_\varrho(x) = \sum_{\nu=0}^{m} e_\nu(\varrho)\, x^{\omega-\nu/2A} \cos(hx^{1/2A} + k_\nu \pi) + O(x^{\omega-(m+1)/2A}),$$

as $x \to \infty$, where $e_\nu(\varrho)$ and k_ν are constants,

$$\left.\begin{aligned}
&\omega = \omega_0 + \varrho\left(1 - \frac{1}{2A}\right), \quad \omega_0 = \frac{\delta}{2} - \frac{1}{4A},\\
&h = 2e^{-\theta/2A}, \quad \theta = 2\left\{\sum_{\nu=1}^{N} \alpha_\nu \log \alpha_\nu - A \log A\right\},\\
&k = Aa - \mu, \quad \mu = \tfrac{1}{2} + \sum_{\nu=1}^{N} (\beta_\nu - \tfrac{1}{2}), \quad a = -\left(\frac{\delta}{2} + \frac{\varrho}{2A} + \frac{1}{4A}\right),\\
&k_\nu = k + \frac{\nu}{2},\\
&e_0(\varrho) = e^{B-a\theta}(A\pi^{\frac{1}{2}})^{-1}, \quad B = -\delta \sum_{\nu=1}^{N} \alpha_\nu \log \alpha_\nu + (A\delta + \varrho + 1)\log A.
\end{aligned}\right\} \tag{13}$$

If this asymptotic formula is used in (12), the estimate of (12) is reduced to that of the integral

$$U_{m,n}(x) \equiv \int_1^x \lambda(y)^{-2\varrho} \cdot \Delta_\lambda^\varrho(y^\omega e^{ih\mu_m^{1/2A} y^{1/2A}}) \cdot \Delta_\lambda^\varrho(y^\omega e^{-ih\mu_n^{1/2A} y^{1/2A}})\, dy, \tag{14}$$

and this, in turn, depends on estimates for

$$\Delta_\lambda^\varrho(y^\omega e^{i\mu_m^{1/2A} y^{1/2A}}), \tag{15}$$

and for
$$e^{i\mu_m^{1/2A} y^{1/2A}} \frac{d}{dy}[e^{-i\mu_m^{1/2A} y^{1/2A}} \Delta_\lambda^\varrho(y^\omega e^{i\mu_m^{1/2A} y^{1/2A}})]. \tag{16}$$

These we proceed to obtain in the sequel, so that the second integral on the right-hand side of (10) is also taken care of. It is this integral which gives rise, in some cases, to an asymptotic formula for

$$\int_1^x |E(y)|^2\, dy$$

with a "main" term and an O-term, and in other cases to an O-term only. Our choice of η, and therefore of λ, will be different in these different cases.

An estimate for the third integral on the right-hand side of (10) results from the estimates for the first and second integrals by Schwarz's inequality. But in some cases it would be advantageous directly to use the known estimates for W (see [2], p. 110, (4.20)) and for V.

§ 3. Estimates for the finite differences

We shall now obtain estimates for (15) and (16), and use them to estimate the integral in (14).

Let ϱ be a fixed integer, $\lambda > 0$, and $0 < \varrho\lambda < y$. Let ω and μ be real numbers. Then

$$\begin{aligned} \Delta_\lambda^\varrho(y^\omega e^{i\mu y^{1/2A}}) &= y^\omega \sum_{\nu=0}^{\varrho} (-1)^{\varrho-\nu} \binom{\varrho}{\nu} \left(1 + \frac{\nu\lambda}{y}\right)^\omega e^{i\mu(y+\nu\lambda)^{1/2A}} \\ &= y^\omega \Delta_\lambda^\varrho e^{i\mu y^{1/2A}} + y^\omega \sum_{\nu=0}^{\varrho} \sum_{k=1}^{\varrho-1} (-1)^{\varrho-\nu} \binom{\varrho}{\nu} \binom{\omega}{k} \left(\frac{\nu\lambda}{y}\right)^k e^{i\mu(y+\nu\lambda)^{1/2A}} + O(\lambda^\varrho y^{\omega-\varrho}). \end{aligned} \tag{17}$$

Set
$$(\nu, k) \equiv \nu(\nu-1)\dots(\nu-k+1),$$

for k integral, and $0<k\leqslant\nu$, with $(\nu,0)=1$. Then

$$(\nu,k)\binom{\varrho}{\nu}=(\varrho,k)\binom{\varrho-k}{\nu-k}.$$

We can find constants $\alpha_j^{(k)}$, such that

$$\nu^k\equiv\alpha_1^{(k)}(\nu,k)+\ldots+\alpha_k^{(k)}(\nu,1),$$

for $\nu\geqslant1$, with $\alpha_1^{(k)}=1$. We have

$$\sum_{\nu=0}^{\varrho}(-1)^{\varrho-\nu}\binom{\varrho}{\nu}\nu^k e^{i\mu(y+\nu\lambda)^{1/2A}}=\sum_{l=1}^{k}\alpha_{k-l+1}^{(k)}\cdot(\varrho,l)\,\Delta_\lambda^{\varrho-l}\,e^{i\mu(y+l\lambda)^{1/2A}}, \tag{18}$$

for
$$\sum_{\nu=0}^{\varrho}(-1)^{\varrho-\nu}\binom{\varrho}{\nu}[\alpha_1^{(k)}(\nu,k)+\ldots+\alpha_k^{(k)}(\nu,1)]\,e^{i\mu(y+\nu\lambda)^{1/2A}}$$
$$=\sum_{\nu=0}^{\varrho}\sum_{r=0}^{k-1}\alpha_{r+1}^{(k)}(\nu,k-r)\cdot\binom{\varrho}{\nu}(-1)^{\varrho-\nu}e^{i\mu(y+\nu\lambda)^{1/2A}}$$
$$=\sum_{\nu=0}^{\varrho}\sum_{l=1}^{k}\alpha_{k-l+1}^{(k)}\cdot(\varrho,l)\cdot\binom{\varrho-l}{\nu-l}\cdot(-1)^{\varrho-\nu}e^{i\mu(y+\nu\lambda)^{1/2A}}$$
$$=\sum_{l=1}^{k}\sum_{r=0}^{\varrho-l}\alpha_{k-l+1}^{(k)}\cdot(\varrho,l)\cdot\binom{\varrho-l}{r}\cdot(-1)^{\varrho-l-r}e^{i\mu[y+(l+r)\lambda]^{1/2A}}.$$

From (17) and (18) we get

$$\Delta_\lambda^\varrho(y^\omega e^{i\mu y^{1/2A}})=y^\omega\Delta_\lambda^\varrho e^{i\mu y^{1/2A}}+y^\omega\sum_{k=1}^{\varrho-1}\sum_{l=1}^{k}\alpha_{k-l+1}^{(k)}\cdot(\varrho,l)\cdot\left(\frac{\lambda}{y}\right)^k\Delta_\lambda^{\varrho-l}e^{i\mu(y+l\lambda)^{1/2A}}+O(\lambda^\varrho y^{\omega-\varrho}), \tag{19}$$

uniformly in λ, μ, y.

From the definition of Δ_λ^ϱ we have

$$\Delta_\lambda^\varrho(y^\omega e^{i\mu y^{1/2A}})=O(y^\omega). \tag{20}$$

On the other hand, we can prove that, for $|\mu|>c$,

$$\Delta_\lambda^\varrho(y^\omega e^{i\mu y^{1/2A}})=O(\lambda^\varrho y^{\omega-\varrho+\varrho/2A}|\mu|^\varrho). \tag{21}$$

This follows from the fact that if $f(y)=y^\omega e^{i\mu y^{1/2A}}$, and $f^{(\varrho)}$ denotes the ϱ^{th} derivative of f, then we have

$$f^{(\varrho)}(y)=y^\omega\left(\frac{i\mu}{2A}y^{1/2A-1}\right)^\varrho e^{i\mu y^{1/2A}}+\sum_{k=0}^{\varrho-1}c_k y^{\omega-\varrho+k}(\mu y^{1/2A-1})^k e^{i\mu y^{1/2A}}, \tag{22}$$

and
$$\Delta_\lambda^\varrho f(y) = O(\lambda^\varrho \sup_{y<t<y+\varrho\lambda} |f^{(\varrho)}(t)|). \tag{23}$$

We shall use (20) or (21) according to convenience.

The next step is to estimate (16). We shall now make a *further* assumption on λ, namely
$$\lambda \equiv \lambda(y) = c_1 \cdot y^c, \quad c_1 > 0, \tag{24}$$
so that the derivative $\lambda'(y) = O(\lambda/y)$. Set
$$D(y) \equiv e^{i\mu y^{1/2A}} \frac{d}{dy}[e^{-i\mu y^{1/2A}} \Delta_\lambda^\varrho (y^\omega e^{i\mu y^{1/2A}})]. \tag{25}$$

If we use the rule
$$\frac{d}{dy}\{\Delta_\lambda^\varrho F(y)\} = \Delta_\lambda^\varrho F'(y) + \varrho\lambda'(y)\cdot\Delta_\lambda^{\varrho-1} F'(y+\lambda),$$

which is easily deduced from the definition of Δ_λ^ϱ, we obtain
$$D(y) = -\frac{i\mu}{2A} y^{1/2A-1}\Delta_\lambda^\varrho(y^\omega e^{i\mu y^{1/2A}}) + \omega\Delta_\lambda^\varrho(y^{\omega-1} e^{i\mu y^{1/2A}}) + \frac{i\mu}{2A}\Delta_\lambda^\varrho(y^{\omega-1+1/2A} e^{i\mu y^{1/2A}})$$
$$+ \varrho\lambda'(y)\cdot\Delta_\lambda^{\varrho-1}\left[\omega(y+\lambda)^{\omega-1} + \frac{i\mu}{2A}(y+\lambda)^{\omega-1+1/2A}\right]e^{i\mu(y+\lambda)^{1/2A}}.$$

On using (19) together with the facts that $(\lambda/y)^k = O(\lambda/y)$, and $\lambda'(y) = O(\lambda/y)$, we have
$$D(y) = -\frac{i\mu}{2A} y^{1/2A-1}[y^\omega \Delta_\lambda^\varrho e^{i\mu y^{1/2A}} + O(y^{\omega-1}\lambda)] + \frac{i\mu}{2A} y^{\omega-1+1/2A}\Delta_\lambda^\varrho e^{i\mu y^{1/2A}}$$
$$+ O(y^{\omega-2+1/2A}\lambda\cdot|\mu|) + \omega y^{\omega-1}\Delta_\lambda^\varrho e^{i\mu y^{1/2A}} + \dots,$$

in which the "later" terms are of lower order than the ones retained. Thus, if $\lambda\cdot|\mu|\, y^{1/2A-1} > c_2$, then
$$D(y) = \omega y^{\omega-1}\Delta_\lambda^\varrho e^{i\mu y^{1/2A}} + O(y^{\omega-2+1/2A}\cdot\lambda\cdot|\mu|) = O(y^{\omega-2+1/2A}\lambda\cdot|\mu|). \tag{26}$$

If, on the other hand, $\lambda\cdot|\mu|\cdot y^{1/2A-1} \leqslant c_2$, then on using (19), together with (21) with $\omega = 0$, we obtain
$$D(y) = O\left(\left|(\mu y^{1/2A-1})\left\{y^\omega \sum_{k=1}^{\varrho-1}\sum_{l=1}^{k}\left(\frac{\lambda}{y}\right)^k \alpha_{k-l+1}^{(k)}\cdot(\varrho, l)\cdot\Delta_\lambda^{\varrho-l} e^{i\mu(y+l\lambda)^{1/2A}}\right\}\right|\right) + O(\lambda^\varrho y^{\omega-\varrho+1/2A-1}|\mu|)$$
$$= O\left(|\mu|\cdot y^{\omega+1/2A-1}\sum_{k=1}^{\varrho-1}\sum_{l=1}^{k}\left(\frac{\lambda}{y}\right)^k\cdot\left(\frac{\lambda}{y}\right)^{\varrho-l}(|\mu|\cdot y^{1/2A})^{\varrho-l}\right) + O(\lambda^\varrho y^{\omega-\varrho+1/2A-1}|\mu|)$$
$$= O[(|\mu|\cdot\lambda\cdot y^{1/2A-1})^\varrho\, y^{\omega-1}]. \tag{27}$$

In the range $|\mu| \leqslant c_2 \cdot y^{1-1/2A} \lambda^{-1}$, however, we have

$$(|\mu| \cdot \lambda \cdot y^{1/2A-1})^{\varrho}\, y^{\omega-1} \leqslant c_3 (|\mu| \cdot \lambda \cdot y^{1/2A-1})\, y^{\omega-1},$$

so that (26) is also valid in this range. Similarly in the range $|\mu| > c_2 y^{1-1/2A} \lambda^{-1}$, we have

$$y^{\omega-2+1/2A} \cdot \lambda \cdot |\mu| \leqslant c_4 \cdot (|\mu| \cdot \lambda \cdot y^{1/2A-1})^{\varrho}\, y^{\omega-1},$$

so that both (26) and (27) are, in fact, valid in both ranges, though (26) gives us a better estimate for $|\mu| > c_2 \cdot y^{1-1/2A} \lambda^{-1}$, as (27) does for $|\mu| \leqslant c_2 y^{1-1/2A} \lambda^{-1}$.

Estimates (26) and (27) are enough for dealing with (16). We shall now use them to estimate the integral in (14).

LEMMA 1. *Let $\{\mu_n\}$ be the sequence given in functional equation* (4). *Let δ, ω, ω_0, and h be the real numbers given in* (13), $h > 0$, $\delta > 0$. *For $y > 0$, let $\lambda = \lambda(y) = y^{1-1/2A-\eta}$, $A \geqslant 1$, $\eta > 0$. Let $z > 1$, and*

$$U_{m,n}(x) \equiv \int_1^x \{\lambda(y)\}^{-2\varrho} \cdot \Delta_\lambda^{\varrho}\big(y^{\omega} e^{ih\mu_m^{1/2A} y^{1/2A}}\big) \cdot \Delta_\lambda^{\varrho}\big(y^{\omega} e^{-ih\mu_n^{1/2A} y^{1/2A}}\big)\, dy.$$

Then we have for $m > n$,

$$
\begin{aligned}
|U_{m,n}(x)| &\leqslant \frac{c_1 \cdot x^{2\omega_0+1-1/2A} (\mu_m \mu_n)^{\varrho/2A}}{\mu_m^{1/2A} - \mu_n^{1/2A}}, \quad \textit{for} \quad \mu_n < \mu_m \leqslant z,\\
&\leqslant \frac{c_2 \cdot \{\lambda(x)\}^{-\varrho} \cdot \mu_n^{\varrho/2A} x^{\omega_0+\omega+1-1/2A} (1 + \lambda(x) \cdot \mu_m^{1/2A} x^{1/2A-1})}{\mu_m^{1/2A} - \mu_n^{1/2A}}, \quad \textit{for} \quad \mu_n \leqslant z < \mu_m,\\
&\leqslant \frac{c_3 \cdot \{\lambda(x)\}^{-2\varrho} \cdot x^{2\omega} (x^{1-1/2A} + \lambda(x)\, \mu_m^{1/2A})}{\mu_m^{1/2A} - \mu_n^{1/2A}}, \quad \textit{for} \quad z < \mu_n < \mu_m. \qquad (28)
\end{aligned}
$$

Proof. If we write

$$G_m(y) \equiv e^{-ih\mu_m^{1/2A} y^{1/2A}}\, \Delta_\lambda^{\varrho}\big(y^{\omega} e^{ih\mu_m^{1/2A} y^{1/2A}}\big), \quad \text{and} \quad F(y) \equiv \{\lambda(y)\}^{-2\varrho} y^{1-1/2A} G_m(y) \cdot \overline{G_n(y)},$$

it is easy to see that

$$
\begin{aligned}
U_{m,n} &= \int_1^x \{\lambda(y)\}^{-2\varrho} G_m(y) \cdot \overline{G_n(y)} \cdot e^{ihy^{1/2A}(\mu_m^{1/2A} - \mu_n^{1/2A})}\, dy\\
&= \frac{2A}{ih(\mu_m^{1/2A} - \mu_n^{1/2A})} \int_1^x \{\lambda(y)\}^{-2\varrho} y^{1-1/2A} G_m \cdot \bar{G}_n \cdot \frac{d}{dy}\big(e^{ihy^{1/2A}(\mu_m^{1/2A} - \mu_n^{1/2A})}\big)\, dy\\
&= O\left(\frac{1}{\mu_m^{1/2A} - \mu_n^{1/2A}}\right) \left[|F(x)| + |F(1)| + \int_1^x |F'(y)|\, dy\right]. \qquad (29)
\end{aligned}
$$

If $\mu_n < \mu_m \leqslant z$, then because of (21), we have

$$|F(y)| = O[y^{1-1/2A}\lambda^{-2\varrho}\{\lambda^{2\varrho}y^{2\omega-2\varrho}y^{\varrho/A}(\mu_m\mu_n)^{\varrho/2A}\}] = O[y^{2\omega_0+1-1/2A}(\mu_m\mu_n)^{\varrho/2A}], \tag{30}$$

and because of (27) and (21) we have

$$|F'(y)| = O[y^{-1}|F(y)| + \lambda^{-2\varrho}y^{1-1/2A}\{y^{\omega-\varrho-1}\lambda^{\varrho}(\mu_m y)^{\varrho/2A}\lambda^{\varrho}y^{\omega-\varrho}(\mu_n y)^{\varrho/2A}\}]. \tag{31}$$

Now (29), (30), and (31) lead to the proof of the first part of (28), if we note that $\delta > 0$, $A \geqslant 1$. As to the second part, we have $\mu_n \leqslant z < \mu_m$, so that from (20), (21), (26) and (27) we obtain

$$F(y) = O[\lambda^{-2\varrho}y^{1-1/2A}\{y^{\omega}\cdot y^{\omega-\varrho}\lambda^{\varrho}(\mu_n y)^{\varrho/2A}\}] = O[\lambda^{-\varrho}\cdot\mu_n^{\varrho/2A}\cdot y^{\omega_0+\omega+1-1/2A}],$$

and

$$\begin{aligned}|F'(y)| &= O[y^{-1}|F(y)| + \lambda^{-2\varrho}y^{1-1/2A}\{y^{\omega}\cdot y^{\omega-\varrho-1}\lambda^{\varrho}(\mu_n y)^{\varrho/2A} \\ &\qquad + (\mu_m^{1/2A}\cdot\lambda\cdot y^{\omega-2+1/2A})(\mu_n y)^{\varrho/2A}y^{\omega-\varrho}\lambda^{\varrho}\}] \\ &= O[\lambda^{-\varrho}\mu_n^{\varrho/2A}y^{\omega_0+\omega-1/2A}\{1+\lambda\mu_m^{1/2A}y^{1/2A-1}\}],\end{aligned}$$

and these inequalities lead to the proof of the second part of (28). If $z < \mu_n < \mu_m$, we have again from (20),

$$|F(y)| = O(\lambda^{-2\varrho}y^{1-1/2A+2\omega}),$$

and

$$\begin{aligned}|F'(y)| &= O(y^{-1}|F(y)| + \lambda^{-2\varrho}y^{1-1/2A}y^{\omega-2+1/2A}\cdot y^{\omega}\cdot\lambda[\mu_m^{1/2A}+\mu_n^{1/2A}]) \\ &= O(y^{-1}|F(y)| + y^{2\omega-1}\lambda^{-2\varrho+1}[\mu_m^{1/2A}+\mu_n^{1/2A}]),\end{aligned}$$

because of (26), and these inequalities lead to a proof of the third part of (28). Thus Lemma 1 is proved.

§ 4. Estimate of the error-term

We have already seen in (10) that the error-term in mean square is given by

$$\int_1^x |E(y)|^2\,dy = \int_1^x |V(y)|^2\,dy + \int_1^x |W(y)|^2\lambda^{-2\varrho}\,dy + \int_1^x \lambda^{-\varrho}(W\overline{V} + \overline{W}V)\,dy, \tag{32}$$

where

$$E(y) = A_\lambda^0(y) - S_0(y), \quad W(y) = \Delta_\lambda^\varrho[A_\lambda^\varrho(y) - S_\varrho(y)],$$

$$V(y) = O(\lambda y^{q-1}\log^{r-1}y) + O\Big(\sum_{y<\lambda_n\leqslant y+\varrho\lambda}|a_n|\Big).$$

If we assume that functional equation (4) holds, then we have [2, (4.6)] identity (5), from which it is immediate that

$$|W(y)|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_m \bar{b}_n \Delta_\lambda^\varrho I_\varrho(\mu_m y) \cdot \Delta_\lambda^\varrho \bar{I}_\varrho(\mu_n y)}{(\mu_m \mu_n)^{\delta+\varrho}}. \tag{33}$$

Let α_1 and β_1 be such that

$$\sum_{n=1}^{\infty} |a_n| \cdot \lambda_n^{-\alpha_1} < \infty, \quad \sum_{n=1}^{\infty} |b_n| \cdot \mu_n^{-\beta_1} < \infty. \tag{34}$$

Let us assume, in addition, that

$$\sum_{\mu_n \leqslant x} |b_n|^2 = O(x^{2\beta-1} \log^{\beta'} x), \quad \beta' \geqslant 0. \tag{35}$$

Since not all the b_n are zero, it follows that $\beta \geqslant \frac{1}{2}$. We shall sometimes use a similar assumption on the a_n, namely

$$\sum_{\lambda_n \leqslant x} |a_n|^2 = O(x^{2\alpha-1} \log^{\alpha'} x), \quad \alpha' \geqslant 0. \tag{36}$$

For simplicity we further assume that

$$\mu_n = c_1 \cdot n, \quad \lambda_n = c_2 \cdot n, \tag{37}$$

even though our final result would require only an assumption on the density of $\{\mu_n\}$, $\{\lambda_n\}$ like $\mu_{n+1} - \mu_n \geqslant c_1 > 0$, $\lambda_{n+1} - \lambda_n \geqslant c_2 > 0$. As before, we choose

$$\lambda = \lambda(y) = y^c, \quad c = 1 - \frac{1}{2A} - \eta, \quad \eta > 0, \tag{38}$$

and

$$z = x^{2A\eta}. \tag{39}$$

To estimate $\int_1^x \lambda^{-2\varrho} |W(y)|^2 \, dy$, we write

$$|W(y)|^2 = \sum_{n=1}^{\infty} \frac{|b_n|^2 \cdot |\Delta_\lambda^\varrho I_\varrho(\mu_n y)|^2}{\mu_n^{2(\delta+\varrho)}} + \sum_{\substack{m,n \\ m \neq n}} \frac{b_m \bar{b}_n \Delta_\lambda^\varrho I_\varrho(\mu_m y) \, \Delta_\lambda^\varrho \bar{I}_\varrho(\mu_n y)}{(\mu_m \mu_n)^{\delta+\varrho}}$$

$$= W_1(y) + W_2(y), \text{ say,}$$

and estimate $\int_1^x \lambda^{-2\varrho} W_1 \, dy$ and $\int_1^x \lambda^{-2\varrho} W_2 \, dy$ separately. In the former integral, we split the series for W_1 into two parts, according as $\mu_n \leqslant z$ or $\mu_n > z$. In the first part, we use the estimate

$$|\Delta_\lambda^\varrho I_\varrho(\mu_n y)|^2 = (\mu_n \lambda)^{2\varrho} |I_0(\mu_n y)|^2 + O(\mu_n^{2\varrho+2\omega_0+1/2A} \lambda^{2\varrho+1} y^{2\omega_0-(1-1/2A)})$$
$$+ O(\mu_n^{2\varrho+2} \lambda^{2\varrho+2} (\mu_n y)^{2\omega_0-2(1-1/2A)}), \tag{40}$$

which follows from the observation that

$$\Delta_\lambda^\varrho I_\varrho(\mu_n y) = \int_y^{y+\lambda} dt_1 \int_{t_1}^{t_1+\lambda} dt_2 \ldots \int_{t_{\varrho-1}}^{t_{\varrho-1}+\lambda} \mu_n^\varrho \cdot I_0(\mu_n t)\, dt,$$

and therefore

$$\Delta_\lambda^\varrho I_\varrho(\mu_n y) = (\mu_n \lambda)^\varrho I_0(\mu_n y) + O\{\mu_n^{\varrho+1}\lambda^{\varrho+1}(\mu_n y)^{\omega_0-(1-1/2A)}\}, \tag{41}$$

since $I_{-1}(y) = O(y^{\omega_0-(1-1/2A)})$, and $I_0(y) = O(y^{\omega_0})$, [cf. (13)]. We also use the simpler estimate

$$\Delta_\lambda^\varrho I_\varrho(\mu_n y) = O[(\mu_n \lambda)^\varrho (\mu_n y)^{\omega_0}].$$

Thus
$$\int_1^x \lambda^{-2\varrho} \sum_{\mu_n \leqslant z} \frac{|b_n|^2 \cdot |\Delta_\lambda^\varrho I_\varrho(\mu_n y)|^2}{\mu_n^{2(\delta+\varrho)}}\, dy = \int_1^x \lambda^{-2\varrho} \left(\sum_{\mu_n \leqslant y^{2A\eta}} + \sum_{y^{2A\eta} < \mu_n \leqslant z} \right) dy, \tag{42}$$

say, where

$$\int_1^x \lambda^{-2\varrho} \cdot \sum_{y^{2A\eta} < \mu_n \leqslant z} \cdot\, dy = \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{2(\delta+\varrho)}} \int_1^{\mu_n^{1/2A\eta}} \lambda^{-2\varrho} |\Delta_\lambda^\varrho I_\varrho(\mu_n y)|^2\, dy$$
$$= O\left[\sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{\delta+1/2A}} \int_1^{\mu_n^{1/2A\eta}} y^{2\omega_0}\, dy \right].$$

Now

$$\sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{\delta+1/2A-(2\omega_0+1)/2A\eta}} = \begin{cases} O(x^{2\omega_0+1} \log^{\beta'} x \cdot x^{2A\eta(2\beta-\delta-1-1/2A)}), & \text{if } 2\beta - \delta - 1 - \dfrac{1}{2A} > 0, \\ O(x^{2\omega_0+1} \log^{\beta'+1} x), & \text{if } 2\beta - \delta - 1 - \dfrac{1}{2A} = 0. \\ O(x^{2\omega_0+1}), & \text{if } 2\beta - \delta - 1 - \dfrac{1}{2A} < 0, \text{ and } \eta = \dfrac{1}{2A}, \\ O(1), & \text{if } 2\beta - \delta - 1 - \dfrac{1}{2A} < 0, \text{ and } \eta \text{ is sufficiently large.} \end{cases}$$

We shall choose $\eta = 1/2A$ if $2\beta - \delta - 1/A > 0$, and η sufficiently large if $2\beta - \delta - 1/A \leqslant 0$. According to this choice of η, we have

$$\int_1^x \lambda^{-2\varrho} \sum_{y^{2A\eta} < \mu_n \leqslant z} dy = \begin{cases} O(1), & \text{if } 2\beta - \delta - \dfrac{1}{A} \leqslant 0, \\ O(x^{2\omega_0+1}), & \text{if } 2\beta - \delta - \dfrac{1}{A} > 0, \text{ and } 2\beta - \delta - 1 - \dfrac{1}{2A} < 0, \\ O(x^{2\beta-1/A} \log^{\beta'} x), & \text{if } 2\beta - \delta - \dfrac{1}{A} > 0, \text{ and } 2\beta - \delta - 1 - \dfrac{1}{2A} > 0, \\ O(x^{2\omega_0+1} \log^{\beta'+1} x), & \text{if } 2\beta - \delta - \dfrac{1}{A} > 0, \text{ and } 2\beta - \delta - 1 - \dfrac{1}{2A} = 0. \end{cases} \tag{43}$$

On the other hand,

$$\int_1^x \lambda^{-2\varrho} \cdot \sum_{\mu_n \leqslant y^{2A\eta}} dy = \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{2(\delta+\varrho)}} \int_{\mu_n^{1/2A\eta}}^x \lambda^{-2\varrho} (\mu_n \lambda)^{2\varrho} \cdot |I_0(\mu_n y)|^2 \, dy$$

$$+ O\left(\sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{2(\delta+\varrho)}} \cdot \mu_n^{2\varrho+2\omega_0+1/2A} \int_{\mu_n^{1/2A\eta}}^x \lambda \cdot y^{2\omega_0-(1-1/2A)} \, dy \right)$$

$$+ O\left(\sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{2(\delta+\varrho)}} \cdot \mu_n^{2\varrho+2\omega_0+1/A} \int_{\mu_n^{1/2A\eta}}^x \lambda^2 \cdot y^{2\omega_0-2(1-1/2A)} \, dy \right)$$

$$= M_1 + M_2 + M_3, \text{ say.}$$

To estimate M_2, we observe that

$$\sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^\delta} \int_{\mu_n^{1/2A\eta}}^x y^{2\omega_0-\eta} \, dy = c \cdot \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^\delta} \left(x^{2\omega_0+1-\eta} - \mu_n^{(2\omega_0+1-\eta)/2A\eta} \right)$$

since $2\omega_0 + 1 - \eta \neq 0$, by assumption, and

$$\sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^\delta} \cdot x^{2\omega_0+1-\eta} = \begin{cases} O(x^{2A\eta(2\beta-1-\delta)} \log^{\beta'} x \cdot x^{2\omega_0+1-\eta}), & \text{if } \; 2\beta - 1 - \delta > 0, \\ O(x^{2\omega_0+1-\eta} \log^{\beta'+1} x), & \text{if } \; 2\beta - 1 - \delta = 0, \\ O(x^{2\omega_0+1-\eta}), & \text{if } \; 2\beta - 1 - \delta < 0. \end{cases}$$

This, together with the estimate immediately preceding (43), implies that

$$M_2 = \begin{cases} O(1), & \text{if } \; 2\beta - \delta - \dfrac{1}{A} \leqslant 0, \\ O(x^{2\omega_0+1}), & \text{if } \; 2\beta - \delta - \dfrac{1}{A} > 0, \text{ and } \; 2\beta - \delta - 1 - \dfrac{1}{2A} < 0, \\ O(x^{2\beta-1/A} \log^{\beta'} x), & \text{if } \; 2\beta - \delta - \dfrac{1}{A} > 0, \text{ and } \; 2\beta - \delta - 1 - \dfrac{1}{2A} > 0, \\ O(x^{2\omega_0+1} \log^{\beta'+1} x), & \text{if } \; 2\beta - \delta - \dfrac{1}{A} > 0, \text{ and } \; 2\beta - \delta - 1 - \dfrac{1}{2A} = 0, \end{cases} \tag{44}$$

according to our choice of η.

The estimate of M_3 is the same as that of M_2, provided that $2\omega_0 + 1 - 2\eta \neq 0$, as can be seen from the fact that

$$\sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^\delta} \cdot \mu_n^{1/2A} \int_{\mu_n^{1/2A\eta}}^x y^{2\omega_0-2\eta} \, dy = O\left(z^{1/2A} \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^\delta} \cdot x^{2\omega_0+1-2\eta} \right) + O\left(\sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{\delta+1/2A-(2\omega_0+1)/2A\eta}} \right).$$

If, however, $2\omega_0 + 1 - 2\eta = 0$, then we have

$$M_3 = O\left(\log x \cdot x^{\eta} \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{\delta}}\right)$$

$$= O\left(\log x \cdot x^{2\omega_0+1-\eta} \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{\delta}}\right)$$

$$= \begin{cases} O(x^{2A\eta(2\beta-1-\delta)} \cdot \log^{\beta'+1} x \cdot x^{2\omega_0+1-\eta}), & \text{if } 2\beta-1-\delta>0, \\ O(\log^{\beta'+2} x \cdot x^{2\omega_0+1-\eta}), & \text{if } 2\beta-1-\delta=0, \\ O(x^{2\omega_0+1-\eta} \log x), & \text{if } 2\beta-1-\delta<0. \end{cases}$$

Thus, in any case, we have

$$M_3 = \begin{cases} O(1), & \text{if } 2\beta-\delta-\frac{1}{A} \leqslant 0, \\ O(x^{2\omega_0+1}), & \text{if } 2\beta-\delta-\frac{1}{A}>0, \text{ and } 2\beta-\delta-1-\frac{1}{2A}<0, \\ O(x^{2\beta-1/A} \log^{\beta'+1} x), & \text{if } 2\beta-\delta-\frac{1}{A}>0, \text{ and } 2\beta-\delta-1-\frac{1}{2A}>0, \\ O(x^{2\omega_0+1} \log^{\beta'+1} x), & \text{if } 2\beta-\delta-\frac{1}{A}>0, \text{ and } 2\beta-\delta-1-\frac{1}{2A}=0. \end{cases} \tag{45}$$

We now estimate M_1. We have from (13),

$$I_0(y) = \sum_{\nu=0}^{m} e_\nu y^{\omega_0-\nu/2A} \tfrac{1}{2}\left[e^{i(hy^{1/2A}+k_\nu\pi)} + e^{i(hy^{1/2A}+k_\nu\pi)}\right] + O(y^{\omega_0-(m+1)/2A}), \quad e_\nu \equiv e_\nu(0),$$

so that the first term in the asymptotic expansion for $|I_0(\mu_n y)|^2$ leads us to consider

$$\tfrac{1}{2} e_0^2 \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{2\delta}} \int_{\mu_n^{1/2A\eta}}^{x} (\mu_n y)^{2\omega_0} dy = c_1 \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{\delta+1/2A}} (x^{2\omega_0+1} - \mu_n^{(2\omega_0+1)/2A\eta}), \quad c_1 = \tfrac{1}{2} e_0^2 \cdot \frac{1}{2\omega_0+1}.$$

An estimate of the second term here was already given immediately ahead of (43). The first term gives

$$c_1 \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{\delta+1/2A}} \cdot x^{2\omega_0+1}$$

$$= \begin{cases} c_2 x^{2\omega_0+1} + O(x^{2\omega_0+1} \cdot \log^{\beta'} x \cdot x^{2A\eta(2\beta-1-\delta-1/2A)}), & \text{if } 2\beta-\delta-1-\frac{1}{2A}<0, \\ O(x^{2\omega_0+1} \cdot \log^{\beta'} x \cdot x^{2A\eta(2\beta-\delta-1-1/2A)}), & \text{if } 2\beta-\delta-1-\frac{1}{2A}>0, \\ O(x^{2\omega_0+1} \log^{\beta'+1} x), & \text{if } 2\beta-\delta-1-\frac{1}{2A}=0, \end{cases} \tag{46}$$

where, in the first case,

$$c_2 = \tfrac{1}{2} e_0^2 \cdot \frac{1}{2\omega_0+1} \cdot \sum_{n=1}^{\infty} |b_n|^2 \mu_n^{-\delta-1/2A}.$$

Hence

$$\tfrac{1}{2} e_0^2 \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{2\delta}} \int_{\mu_n^{1/2A}\eta}^{x} (\mu_n y)^{2\omega_0}\, dy$$

$$= \begin{cases} c_2 x^{2\omega_0+1} + O(1), & \text{if } \ 2\beta-\delta-\dfrac{1}{A} \leqslant 0. \\ O(x^{2\omega_0+1}), & \text{if } \ 2\beta-\delta-\dfrac{1}{A} > 0, \ \text{ and } \ 2\beta-\delta-1-\dfrac{1}{2A} < 0, \\ O(x^{2\beta-1/A} \log^{\beta'} x), & \text{if } \ 2\beta-\delta-\dfrac{1}{A} > 0, \ \text{ and } \ 2\beta-\delta-1-\dfrac{1}{2A} > 0, \\ O(x^{2\omega_0+1} \log^{\beta'+1} x), & \text{if } \ 2\beta-\delta-\dfrac{1}{A} > 0, \ \text{ and } \ 2\beta-\delta-1-\dfrac{1}{2A} = 0. \end{cases} \tag{47}$$

The other terms in the asymptotic expansion for $|I_0(\mu_n y)|^2$ lead to

$$\sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{2\delta}} \int_{\mu_n^{1/2A}\eta}^{x} (\mu_n y)^{2\omega_0-1/2A}\, dy = \sum_{\mu_n \leqslant z} \frac{|b_n|^2}{\mu_n^{\delta+1/A}} \int_{\mu_n^{1/2A}\eta}^{x} y^{2\omega_0-1/2A}\, dy = O(x^{2\omega_0+1-1/2A} \log^{\beta'+1} y),$$

which is of smaller order than (47). Thus relations (42) to (47) give us

$$\int_1^x \lambda^{-2\varrho} \sum_{\mu_n \leqslant z} \frac{|b_n|^2 \cdot |\Delta_\lambda^\varrho I_\varrho(\mu_n y)|^2}{\mu_n^{2(\delta+\varrho)}}\, dy$$

$$= \begin{cases} c_2 x^{2\omega_0+1} + O(x^{2\omega_0+1-1/2A} \log^{\beta'+1} x), & \text{if } \ 2\beta-\delta-\dfrac{1}{A} \leqslant 0, \\ O(x^{2\omega_0+1}), & \text{if } \ 2\beta-\delta-\dfrac{1}{A} > 0, \ \text{ and } \ 2\beta-\delta-1-\dfrac{1}{2A} < 0, \\ O(x^{2\beta-1/A} \log^{\beta'+1} x), & \text{if } \ 2\beta-\delta-\dfrac{1}{A} > 0, \ \text{ and } \ 2\beta-\delta-1-\dfrac{1}{2A} > 0, \\ O(x^{2\omega_0+1} \log^{\beta'+1} x), & \text{if } \ 2\beta-\delta-\dfrac{1}{A} > 0, \ \text{ and } \ 2\beta-\delta-1-\dfrac{1}{2A} = 0. \end{cases} \tag{48}$$

On the other hand, the second part of the series for W_1 leads us to consider

$$\int_1^x \sum_{\mu_n > z} \frac{|b_n|^2 \cdot |\Delta_\lambda^\varrho I_\varrho(\mu_n y)|^2}{\mu_n^{2(\delta+\varrho)}} \lambda^{-2\varrho}\, dy$$

$$= \sum_{\mu_n > z} \frac{|b_n|^2}{\mu_n^{2(\delta+\varrho)}} \int_1^x \lambda^{-2\varrho} |\Delta_\lambda^\varrho I_\varrho(\mu_n y)|^2\, dy, \quad \text{for } \varrho \text{ large enough,}$$

$$= O\left(\sum_{\mu_n > z} \frac{|b_n|^2}{\mu_n^{2(\delta+\varrho)-2\omega}} \int_1^x \lambda^{-2\varrho} y^{2\omega}\, dy \right)$$

$$= O\left(\sum_{\mu_n > z} \frac{|b_n|^2}{\mu_n^{2(\delta+\varrho-\omega)}} \cdot x^{2\omega_0+1+2\varrho\eta} \right)$$

$$= O(z^{2\beta-1-2(\delta+\varrho-\omega)} \log^{\beta'} x \cdot x^{2\omega_0+1+2\varrho\eta})$$

$$= O(x^{2\omega_0+1} \cdot \log^{\beta'} x \cdot x^{2A\eta(2\beta-1-\delta-1/2A)}). \tag{49}$$

From (48) and (49) we obtain

$$\int_1^x \lambda^{-2\varrho} W_1(y)\, dy$$

$$= \begin{cases} c_2 x^{2\omega_0+1} + O(x^{2\omega_0+1-1/2A} \log^{\beta'+1} x), & \text{if } 2\beta - \delta - \dfrac{1}{A} \leqslant 0, \\ O(x^{2\omega_0+1}), & \text{if } 2\beta - \delta - \dfrac{1}{A} > 0, \text{ and } 2\beta - \delta - 1 - \dfrac{1}{2A} < 0, \\ O(x^{2\omega_0+1} \log^{\beta'+1} x \cdot x^{2\beta-\delta-1-1/2A}), & \text{if } 2\beta - \delta - \dfrac{1}{A} > 0, \text{ and } 2\beta - \delta - \dfrac{1}{2A} - 1 \geqslant 0. \end{cases} \tag{50}$$

Let us now estimate $\int_1^x \lambda^{-2\varrho} W_2(y)\, dy$, We have

$$W_2(y) = \sum_{\substack{m,\, n \\ m \neq n}} \frac{b_m \bar{b}_n \Delta_\lambda^\varrho I_\varrho(\mu_m y) \cdot \Delta_\lambda^\varrho \bar{I}_\varrho(\mu_n y)}{(\mu_m \mu_n)^{\delta+\varrho}}.$$

It will be sufficient to estimate the part W_2' of this sum for which $\mu_m > \mu_n$, since $W_2(y) = 2 \operatorname{Re} W_2'(y)$. We shall write

$$W_2'(y) = W_{2,1}(y) + W_{2,2}(y) + W_{2,3}(y),$$

where
$$W_{2,1} \equiv \sum_{\substack{\mu_n \leqslant z \\ \mu_n < \mu_m \leqslant 2z}}, \quad W_{2,2} \equiv \sum_{\substack{\mu_n \leqslant z \\ \mu_m \geqslant 2z}}, \quad W_{2,3} \equiv \sum_{\mu_m > \mu_n > z}$$

Now
$$\int_1^x \lambda^{-2\varrho} W_{2,1}(y)\, dy = \sum_{\substack{\mu_n \leqslant z \\ \mu_n < \mu_m < 2z}} \frac{b_m \bar{b}_n}{(\mu_m \mu_n)^{\delta+\varrho}} \int_1^x \lambda^{-2\varrho} \Delta_\lambda^\varrho I_\varrho(\mu_m y) \cdot \Delta_\lambda^\varrho \bar{I}_\varrho(\mu_n y)\, dy$$

$$= O\left(\sum_{\mu_n < \mu_m < 2z} \frac{|b_m b_n|}{(\mu_m \mu_n)^{\delta+\varrho-\omega}} \cdot \frac{x^{2\omega_0+1-1/2A} (\mu_m \mu_n)^{\varrho/2A}}{\mu_m^{1/2A} - \mu_n^{1/2A}} \right)$$

because of (13) and (28). Note that the first term in the asymptotic expansion for I_ϱ leads to this estimate, the later terms being of smaller order. By assumption (37), $c\mu_n = n$, and for $0 \leqslant \xi < 1$, we have $1-\xi \leqslant c'(1-\xi^{1/2A})$. Thus

$$\begin{aligned}\int_1^x \lambda^{-2\varrho} W_{2,1}(y)\,dy &= O\left(x^{2\omega_0+1-1/2A} \sum_{n<m<2cz} \frac{|b_m b_n|}{(mn)^{\delta+\varrho-\omega-\varrho/2A}} \cdot \frac{m^{1-1/2A}}{m-n}\right) \\ &= O\left(x^{2\omega_0+1-1/2A} \sum_{n<m<2cz} \frac{|b_m b_n|}{m^{1/2A}(mn)^{\delta+\varrho-\omega-\varrho/2A}}\left(1+\frac{n}{m-n}\right)\right) \\ &= O[x^{2\omega_0+1-1/2A} \log^{\beta'+1} x\,(x^{2A\eta(2\beta-\delta-1/A)} + \log x)], \qquad (51)\end{aligned}$$

for

$$\begin{aligned}&\sum_{n<m<2cz} \frac{|b_m b_n|}{(mn)^{\delta+\varrho-\omega-\varrho/2A}\, n^{-1}\, m^{1/2A}} \cdot \frac{1}{(m-n)} \\ &= \sum_{k<2cz} \frac{1}{k} \sum_{n<2cz-k} \frac{|b_n b_{n+k}|}{n^{\beta-u-1}(n+k)^{\beta-u+1/2A}}, \qquad \left(u = \beta - \frac{\delta}{2} - \frac{1}{4A}\right) \\ &\leqslant \sum_{k<2cz} \frac{1}{k} \sum_{n<2cz-k} \frac{|b_n b_{n+k}|}{n^{\beta-u-\frac{1}{2}+1/4A}(n+k)^{\beta-u-\frac{1}{2}+1/4A}}, \\ &\leqslant \sum_{k<2cz} \frac{1}{k} \sum_{n<2cz} \frac{|b_n|^2}{n^{2\beta-2u-1+1/2A}} \\ &= \begin{cases} O(\log^{\beta'+2} x), & \text{if} \quad 2\beta-\delta-\dfrac{1}{A}=0, \\ O(x^{2A\eta(2\beta-\delta-1/A)} \log^{\beta'+1} x), & \text{if} \quad 2\beta-\delta-\dfrac{1}{A}>0, \\ O(\log x), & \text{if} \quad 2\beta-\delta-\dfrac{1}{A}<0, \end{cases}\end{aligned}$$

and similarly

$$\begin{aligned}&\sum_{n<m<2cz} \frac{|b_m b_n|}{n^{\delta+\varrho-\omega-\varrho/2A}\, m^{\delta+\varrho-\omega-\varrho/2A+1/2A}} \\ &= O\left[\left(\sum_{n<2cz} \frac{|b_n|}{n^{\delta+\varrho-\omega-\varrho/2A+1/4A}}\right)^2\right] \\ &= O\left(\sum_{n<2cz} \frac{|b_n|^2}{n^{2(\beta-u)-1+1/2A}} \cdot \sum_{n<2cz} \frac{1}{n}\right) \\ &= \begin{cases} O(\log^{\beta'+2} x), & \text{if} \quad 2\beta-\delta-\dfrac{1}{A}=0, \\ O(x^{2A\eta(2\beta-\delta-1/A)} \log^{\beta'+1} x), & \text{if} \quad 2\beta-\delta-\dfrac{1}{A}>0, \\ O(\log x), & \text{if} \quad 2\beta-\delta-\dfrac{1}{A}<0. \end{cases}\end{aligned}$$

Next let us, in view of (28), consider

$$\int_1^x \lambda^{-2\varrho} W_{2,2}(y)\,dy$$
$$= x^{-\varrho(1-1/2A-\eta)} \cdot x^{\omega_0+\omega+1-1/2A} \cdot O\Bigg(\sum_{\substack{\mu_n \leqslant z \\ \mu_m \geqslant 2z}} \frac{|b_m b_n|}{(\mu_m \mu_n)^{\delta+\varrho-\omega}} \frac{\mu_n^{\varrho/2A}(\mu_m^{1-1/2A} + z^{-1/2A}\mu_m)}{\mu_m - \mu_n} \Bigg).$$

The O-term gives

$$O\left[\sum_{\mu_n \leqslant z} \frac{|b_n|}{\mu_n^{\beta-u}} \sum_{\mu_m \geqslant 2z} \frac{|b_m|}{\mu_m^{\beta-u+\varrho/2A}} \left(\frac{\mu_m^{1-1/2A} + z^{-1/2A}\mu_m}{\mu_m - \mu_n} \right) \right], \quad u = \beta - \frac{\delta}{2} - \frac{1}{4A}.$$

Since $\mu_m z^{-1/2A} \geqslant \mu_m^{1-1/2A}$, this is

$$O\left[\sum_{\mu_n \leqslant z} \frac{|b_n|}{\mu_n^{\beta-u}} \sum_{\mu_m \geqslant 2z} \frac{|b_m|}{\mu_m^{\beta-u+\varrho/2A}} \cdot \frac{z^{-1/2A}\mu_m}{\mu_m - \mu_n} \right] = O\left[\sum_{\mu_n \leqslant z} \frac{|b_n|}{\mu_n^{\beta-u}} \sum_{\mu_m \geqslant 2z} \frac{|b_m|}{\mu_m^{\beta-u+\varrho/2A}} \cdot z^{-1/2A} \right],$$

since $\mu_m = \mu_m - \mu_n + \mu_n \leqslant \mu_m - \mu_n + z \leqslant 2(\mu_m - \mu_n)$. Now hypothesis (35) implies that

$$\sum_{\mu_n \leqslant x} |b_n| = O(x^{\beta} \log^{\frac{1}{2}\beta'} x),$$

hence the O-term gives

$$O(z^u \log^{\beta'/2} z \cdot z^{u-\varrho/2A} \log^{\beta'/2} z \cdot z^{-1/2A}),$$

since $u > 0$, as a consequence of the functional equation [2, (5.1)], and this is

$$O(z^{2u-\varrho/2A-1/2A} \log^{\beta'} z) = O(x^{2A\eta(2u-\varrho/2A-1/2A)} \log^{\beta'} x).$$

Hence
$$\int_1^x \lambda^{-2\varrho} W_{2,2}(y)\,dy = O(x^{2\omega_0+1-1/2A} \log^{\beta'} x \cdot x^{2A\eta(2\beta-\delta-1/A)}). \tag{52}$$

Finally we have, from (28),

$$\int_1^x \lambda^{-2\varrho} W_{2,3}(y)\,dy = x^{-2\varrho(1-1/2A-\eta)+2\omega} \cdot O\left[\sum_{z<\mu_n<\mu_m} \frac{|b_m b_n|}{(\mu_m \mu_n)^{\delta+\varrho-\omega}} \left(\frac{x^{1-1/2A} + x^{1-1/2A-\eta}\mu_m^{1/2A}}{\mu_m^{1/2A} - \mu_n^{1/2A}} \right) \right].$$

Since $\mu_m^{1/2A} \geqslant z^{1/2A} = x^{\eta}$, the O-factor is

$$O\left[\sum_{k=1}^{\infty} \frac{1}{k} \sum_{n>cz} \frac{|b_n b_{n+k}|}{n^{\delta+\varrho-\omega}(n+k)^{\delta+\varrho-\omega}} x^{1-1/2A-\eta}(n+k) \right].$$

By hypothesis (35) we have

$$B(k,t) \equiv \sum_{n \leqslant t} |b_n b_{n+k}| = O[t^{\beta-\frac{1}{2}}(t+k)^{\beta-\frac{1}{2}} \log^{\beta'}(t+k)].$$

Hence
$$\sum_{n>cz} \frac{|b_n b_{n+k}|}{n^{d+\beta}(n+k)^{d+\beta-1}}, \quad d=\delta+\varrho-\omega-\beta$$
$$= O\left(\int_{cz}^{\infty} \frac{\log^{\beta'}(t+k)\,dt}{t^{d+\frac{1}{2}+1}(t+k)^{d-\frac{1}{2}}}\right) = O\left(\frac{\log^{\beta'}(z+k)}{(z+k)^{d-\frac{1}{2}} z^{d+\frac{1}{2}}}\right),$$
so that
$$x^{1-1/2A-\eta} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n>cz} \frac{|b_n b_{n+k}|}{n^{\delta+\varrho-\omega}(n+k)^{\delta+\varrho-\omega-1}} = O\left(x^{1-1/2A-\eta} \sum_{k=1}^{\infty} \frac{\log^{\beta'}(z+k)}{k(z+k)^{d-\frac{1}{2}} z^{d+\frac{1}{2}}}\right)$$
$$= O(x^{1-1/2A-\eta} z^{-2d} \log^{\beta'+1} z),$$
since ϱ is as large as we require. Thus
$$\begin{aligned}\int_1^x \lambda^{-2\varrho} W_{2,3}(y)\,dy &= O(x^{-2\varrho(1-1/2A-\eta)+2\omega} x^{1-1/2A-\eta} x^{2A\eta(2\beta+2\omega-2\varrho-2\delta)} \log^{\beta'+1} x)\\ &= O(x^{2\omega_0+2\varrho\eta} x^{1-1/2A-\eta} x^{2A\eta(2\beta-\delta-1/2A-\varrho/A)} \log^{\beta'+1} x)\\ &= O(x^{2\omega_0+1-1/2A} \log^{\beta'+1} x \cdot x^{2A\eta(2\beta-\delta-1/A)}). \qquad (53)\end{aligned}$$
From (51), (52), and (53), and the definition of $W_2'(y)$, we obtain
$$\int_1^x \lambda^{-2\varrho} W_2(y)\,dy = O[x^{2\omega_0+1-1/2A} \log^{\beta'+1} x\,(x^{2A\eta(2\beta-\delta-1/A)} + \log x)]. \qquad (54)$$
We now consider (50) and (54), and, as before, choose η large and positive if $2\beta - \delta - 1/A \leqslant 0$ (which implies that $2\beta-\delta-1-1/2A<0$, since $A \geqslant 1$), and $\eta = 1/2A$ if $2\beta-\delta-1/A>0$. We then obtain
$$\int_1^x \lambda^{-2\varrho} |W(y)|^2\,dy$$
$$= \begin{cases} c_2 x^{2\omega_0+1} + O(x^{2\omega_0+1-1/2A} \log^{\beta'+2} x), & \text{if } 2\beta-\delta-\dfrac{1}{A} \leqslant 0,\\ O(x^{2\omega_0+1}) + O(x^{2\beta+1-2/A} \log^{\beta'+1} x), & \text{if } 2\beta-\delta-\dfrac{1}{A}>0, \text{ and } 2\beta-\delta-1-\dfrac{1}{2A}<0,\\ O(x^{2\beta+1-2/A} \log^{\beta'+1} x), & \text{if } 2\beta-\delta-\dfrac{1}{A}>0, \text{ and } 2\beta-\delta-1-\dfrac{1}{2A} \geqslant 0. \end{cases} \qquad (55)$$
We next estimate $\int_1^x |V(y)|^2\,dy$ in (32). We note that $V(y) = V_1(y) + V_2(y)$, where $V_1(y) \equiv O(\lambda y^{q-1} \log^{r-1} y)$, $V_2(y) \equiv O(\sum_{y<\lambda_n \leqslant y+\varrho\lambda} |a_n|)$, according to (9). Here q is the maximum of the real parts of the poles of φ, and r is the maximum order of a pole with maximum real part. It is obvious that

$$\int_1^x |V_1(y)|^2\,dy = O(x^{2q-1+2c}\log^{2r-2} x),$$

where c is defined as in (38), while

$$\Bigl(\sum_{y<\lambda_n\leqslant y+\varrho\lambda} |a_n|\Bigr)^2 = O\Bigl(y^{c_1}\sum_{y<\lambda_n\leqslant y+\varrho\lambda}|a_n|^2\Bigr), \quad c_1 = 1-\frac{1}{A},$$

because of assumption (37) so that

$$\int_1^x |V_2(y)|^2\,dy = O\Bigl(\int_1^x y^{c_1}\sum_{y<\lambda_n\leqslant y+\varrho\lambda}|a_n|^2\,dy\Bigr) = O\Bigl(\sum_{1<\lambda_n\leqslant x+\varrho x^c}|a_n|^2\int_{\lambda_n-\varrho\lambda_n^c}^{\lambda_n} y^{c_1}\,dy\Bigr)$$

$$= O\Bigl(\sum_{1<\lambda_n\leqslant x+\varrho x^c}|a_n|^2\cdot\lambda_n^{c+c_1}\Bigr) = O\Bigl(\sum_{1<\lambda_n\leqslant x+\varrho x^c}|a_n|^2\cdot\lambda_n^{2-3/2A-\eta}\Bigr).$$

If $2\beta-\delta-1/A\leqslant 0$, we have chosen η large and positive, so that

$$\int_1^x |V_2(y)|^2\,dy = O(1).$$

If $2\beta-\delta-1/A>0$, we have chosen $\eta = 1/2A$, in which case

$$\int_1^x |V_2(y)|^2\,dy = \begin{cases} O(x^{2\alpha+2c-1}\log^{\alpha'} x), & \text{if}\quad 2\alpha+2c-1\neq 0,\\ O(\log^{\alpha'+1} x), & \text{if}\quad 2\alpha+2c-1=0,\end{cases}$$

provided that we assume not only (35) but also (36). Thus we have, in any case,

$$\int_1^x |V(y)|^2\,dy = O(x^{2q-1+2c}\log^{2r-2} x) + O[(x^{2\alpha-1+2c}+\log x)\log^{\alpha'} x]. \tag{56}$$

Finally we have to consider

$$\int_1^x \lambda^{-\varrho}(W\overline{V}+\overline{W}V)\,dy = O\Bigl(\int_1^x \lambda^{-\varrho}|W|\cdot|V|\,dy\Bigr).$$

We first assume that $2\beta-\delta-1/A\leqslant 0$. Then $2u\equiv 2\beta-\delta-1/2A\leqslant 1/2A$, or $2Au\leqslant\frac{1}{2}$. From a previous paper [2, (4.20)] we have the estimate

$$W(y) = O(\lambda^{\varrho-2Au}\,y^{\delta/2-1/4A+(2A-1)u}),$$

so that
$$\lambda^{-\varrho}W(y) = O(y^{\delta/2-1/4A+2A\eta u}) = O(y^{\omega_0+2A\eta u}),$$

since $\lambda\equiv\lambda(y) = y^{1-1/2A-\eta} = y^c$. Hence

$$\int_1^x \lambda^{-\varrho}|W|\cdot \sum_{y<\lambda_n\leqslant y+\varrho\lambda}|a_n|\cdot dy = \sum_{1<\lambda_n\leqslant x+\varrho x^c}|a_n|\cdot\int_{\lambda_n-\varrho\lambda_n^c}^{\lambda_n} y^{\omega_0+2A\eta u}\,dy$$
$$= \sum_{1<\lambda_n\leqslant x+\varrho x^c}|a_n|\,\lambda_n^{\omega_0+2A\eta u}\,\lambda_n^{1-1/2A-\eta} = O(1),$$

since $2Au-1<0$, and η is large and positive (irrespective of the precise order of $\sum_{\lambda_n\leqslant x}|a_n|$). Similarly

$$\int_1^x \lambda^{-\varrho}|W|\cdot V_1(y)\,dy = O(1),$$

in case η is sufficiently large. Thus the order of magnitude of $\int_1^x \lambda^{-\varrho}|W|\cdot|V|\,dy$ is smaller than that of $\int_1^x \lambda^{-2\varrho}|W(y)|^2\,dy$.

We next consider the case $2\beta-\delta-1/A>0$. In this case we have chosen $\eta=1/2A$. We have

$$\int_1^x \lambda^{-\varrho}|W|\cdot|V|\,dy \leqslant \left\{\int_1^x \lambda^{-2\varrho}|W(y)|^2\,dy\right\}^{\frac{1}{2}}\left\{\int_1^x |V(y)|^2\,dy\right\}^{\frac{1}{2}}$$
$$= O(x^{2\omega_0+1}+x^{2\beta+1-2/A}\log^{\beta'+1}x)^{\frac{1}{2}}\cdot O(x^{2\alpha+1-2/A}\log^{\alpha'+1}x)^{\frac{1}{2}},$$

on using (55) and (56), if $\alpha'\geqslant 2(r-1)$, since $\alpha\geqslant q$. If $\alpha=\beta$, $\alpha'=\beta'$, then this term is of the same or smaller order than $\int_1^x \lambda^{-2\varrho}|W(y)|^2\,dy$.

Hence we have the following

Theorem 1. *If functional equation (4) is satisfied with $\delta>0$, $A\geqslant 1$, and $\mu_n=c'n$, $\lambda_n=c''n$, and the only singularities of φ are poles, and*

$$\sum_{\mu_n\leqslant x}|b_n|^2 = O(x^{2\beta-1}\log^{\beta'}x),$$

then for $2\beta-\delta-1/A\leqslant 0$, we have

$$\int_1^x |E(y)|^2\,dy = c_2 x^{2\omega_0+1} + O(x^{2\omega_0+1/2A}\log^{\beta'+2}x), \tag{57}$$

where $\omega_0=\delta/2-1/4A$, and the error-term $E(y)$ is defined by (9).

If $2\beta-\delta-1/A>0$, then on the basis of the further assumptions that

$$\sum_{\lambda_n\leqslant x}|a_n|^2 = O(x^{2\beta-1}\log^{\beta'}x),$$

and that $\beta'\geqslant 2(r-1)$, where r is the maximum order of a pole with maximum real part, we have

$$\int_1^x |E(y)|^2\,dy = O(x^{2\omega_0+1}) + O(x^{2\beta+1-2/A}\log^{\beta'+1}x). \tag{58}$$

Thus, if $0 < 2\beta - \delta - 1/A < 1/2A$, *we have*

$$\int_1^x |E(y)|^2 \, dy = O(x^{2\omega_0+1}),$$

and if $2\beta - \delta - 1/A \geqslant 1/2A$, *then*

$$\int_1^x |E(y)|^2 \, dy = O(x^{2\beta+1-2/A} \log^{\beta'+1} x).$$

The above theorem yields a number of results on the mean value of the error-term associated with several arithmetical functions. When applied to the Dedekind zeta-function, it gives new results on the number of ideals with a given norm.

Let K be an algebraic number field of degree n, and $\mathfrak{K}$ an ideal class in K. The Dedekind zeta-function of the class $\mathfrak{K}$ is defined by

$$\zeta_K(s, \mathfrak{K}) = \sum_{\mathfrak{a} \in \mathfrak{K}} (N\mathfrak{a})^{-s},$$

where the summation is over all non-zero integral ideals in $\mathfrak{K}$. We may write

$$\zeta_K(s, \mathfrak{K}) = \sum_{k=1}^{\infty} \frac{a_k(\mathfrak{K})}{k^s}, \tag{59}$$

where $a_k(\mathfrak{K})$ is the number of ideals in $\mathfrak{K}$ of norm k. It is known that $\zeta_K(s, \mathfrak{K})$ is a meromorphic function with a simple pole at $s=1$, with residue, say, λ, which is independent of $\mathfrak{K}$, and satisfies the functional equation

$$\xi(s, \mathfrak{K}) = \xi(1-s, \tilde{\mathfrak{K}}), \tag{60}$$

where $$\xi(s, \mathfrak{K}) = \Gamma^{r_1}(\tfrac{1}{2}s)\, \Gamma^{r_2}(s)\, B^{-s} \zeta_K(s, \mathfrak{K}),$$

with $$B = 2^{r_2} \pi^{n/2} (|\Delta|)^{-\frac{1}{2}}.$$

Here r_1 is the number of real conjugates of K, $2r_2$ the number of imaginary conjugates, Δ is the discriminant, and $\tilde{\mathfrak{K}}$ is the class conjugate to $\mathfrak{K}$.

The Dedekind zeta-function of K is defined by $\zeta_K(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s}$, where the summation is over all non-zero integral ideals in K. Clearly $\zeta_K(s) = \sum_{\mathfrak{K}} \zeta_K(s, \mathfrak{K})$, and satisfies the functional equation

$$\xi(s) = \xi(1-s),$$

where $\xi(s) = \Gamma^{r_1}(\frac{1}{2}s)\, \Gamma^{r_2}(s)\, B^{-s} \zeta_K(s)$. It has again a simple pole at $s=1$, with residue λh, where h is the class number of K. We may write

$$\zeta_K(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s},$$

where a_k is the number of integral ideals of norm k.

From a previous paper [3] we have, for $n \geqslant 2$,

$$\sum_{k \leqslant x} a_k^2 = O(x \log^{n-1} x),$$

and, if the field K is Galois, then

$$\sum_{k \leqslant x} a_k^2 \sim c \cdot x \log^{n-1} x.$$

If we apply Theorem 1 to the function $\zeta_K(s, \mathfrak{L})$, which satisfies equation (60), and note that $\delta = 1$, $A = \frac{1}{2}n$, $\beta = 1$, $\beta' = n-1$, $r = 1$, we obtain the following

THEOREM 2. *Let K be an algebraic number field of degree n, $\mathfrak{L}$ an ideal class in K, and $\zeta_K(s, \mathfrak{L})$ the Dedekind zeta-function of the class $\mathfrak{L}$, given by*

$$\zeta_K(s, \mathfrak{L}) = \sum_{k=1}^{\infty} \frac{a_k(\mathfrak{L})}{k^s}.$$

Let $E(x) \equiv \sum_{k \leqslant x} a_k(\mathfrak{L}) - \lambda x$, where λ is the residue of $\zeta_K(s, \mathfrak{L})$ at $s = 1$. Then, if $n = 2$, we have

$$\int_1^x |E(y)|^2 \, dy = c_2 x^{3/2} + O(x \log^3 x), \tag{61}$$

and, if $n > 2$, then

$$\int_1^x |E(y)|^2 \, dy = O(x^{3-4/n} \log^n x). \tag{62}$$

Remarks. I. Let us consider the case $n = 2$. Relation (61) implies that

$$\frac{1}{x} \int_1^x |E(y)| \, dy = O(x^{\frac{1}{4}}). \tag{63}$$

From a previous paper [2, p. 128] we know that

$$\overline{\underline{\lim}} \frac{E(x)}{x^{\frac{1}{4}}} = \pm \infty, \tag{64}$$

so that (63) seems to be "best possible".

If $n > 2$, then we have

$$\frac{1}{x} \int_1^x |E(y)| \, dy = O(x^{1-2/n} \log^{n/2} x), \tag{65}$$

as against the Ω-result [2, (8.18)]

$$\overline{\underline{\lim}}\, \frac{E(x)}{x^{(n-1)/2n}} = \pm\infty. \tag{66}$$

We notice that for $n=3$, the two results (65) and (66) seem to fit in, whereas for $n \geqslant 4$ it is difficult to maintain that either of them is "best possible".

II. It is obvious that Theorem 1 applies also to L-series, and to Hecke's zeta-function with Grössencharacters, cf. [2, § 8]. We get mean-value theorems for the character-sums. We do not write down the actual results, since they are very similar to (61) and (62).

§ 5. Applications to classical arithmetical functions

If $r_2(n)$ denotes the number of representations of n as a sum of two squares, the properties of the generating function $\zeta_2(s) = \sum_{n=1}^{\infty} r_2(n)/n^s$ are well known [1]. We have, further, the property [12, p. 84]

$$\sum_{n \leqslant x} r_2^2(n) = O(x \log x).$$

Hence, by Theorem 1,

$$\int_1^x |P_2(y)|^2\, dy = c_1 x^{3/2} + O(x \log^3 x),$$

where $P_2(x) = \sum_{n \leqslant x} r_2(n) - \pi x$, a result which is due to Cramér, Landau, and Walfisz, as stated in § 1.

We can similarly consider $r_3(n)$, the number of representations of n as a sum of three squares. It is known [1, p. 502] that $r_3(4n) = r_3(n)$, and that if $n = g^2 q$, where q is square-free, and $4 \nmid n$, then

$$r_3(n) \leqslant c_\varepsilon \cdot g^{1+\varepsilon} r_3(q), \quad 0 < \varepsilon < 1.$$

It can also be proved from the explicit formula for $r_3(n)$ that $r_3(q) = O(\sqrt{q} \log q)$. Further we have [2, Th. 4.1], $\sum_{n \leqslant x} r_3(n) = O(x^{3/2})$. Hence

$$\begin{aligned}\sum_{n \leqslant x} r_3^2(n) = \sum_{g^2 q \leqslant x} r_3^2(g^2 q) &= O\left(\sum_{g^2 q \leqslant x} g^{2+2\varepsilon} r_3^2(q)\right)\\ &= O\left[\sum_{g^2 \leqslant x} \frac{g^{2+2\varepsilon}}{g} \sum_{q \leqslant x/g^2} r_3(q)\right] \cdot O(x^{\frac{1}{2}} \log x)\\ &= O(x^{\frac{1}{2}} \log x) \cdot O\left(x^{\frac{3}{2}} \sum_{g^2 \leqslant x} \frac{g^{2\varepsilon}}{g^2}\right) = O(x^2 \log x).\end{aligned}$$

Theorem 1 will then yield the result

$$\int_1^x |P_3(y)|^2\, dy = O(x^2 \log^2 x),$$

where $P_3(x) = \sum_{n \leqslant x} r_3(n) - \pi^{\frac{3}{2}} x^{\frac{3}{2}}/\Gamma(5/2)$. Jarnik [7] has, however, shown that

$$\int_1^x |P_3(y)|^2\, dy = c_1 x^2 \log x + O(x^2 \log^{\frac{1}{2}} x).$$

Similar results can be obtained for $r_k(n)$ for $k \geqslant 4$.

If $d(n)$ denotes the number of divisors of n, then $\sum_{n=1}^{\infty} d(n)\, n^{-s} = \zeta^2(s)$, the square of Riemann's zeta-function; and we have [10, p. 133]

$$\sum_{n \leqslant x} d^2(n) = O(x \log^3 x),$$

so that Theorem 1 gives

$$\int_1^x |\Delta(y)|^2\, dy = c_2 \cdot x^{\frac{3}{2}} + O(x \log^5 x),$$

where $\Delta(x) = \sum_{n \leqslant x} d(n) - x \log x - (2\gamma - 1)\, x$, which is an improvement on Cramér's result (3). We have found a reference to a similar improvement in [11], though we have not seen that paper.

If $\tau(n)$ denotes Ramanujan's function, then it is well known that $\sum_{n=1}^{\infty} \tau(n)\, n^{-s}$ satisfies the functional equation (4) with $\lambda_n = \mu_n = 2\pi n$, $\delta = 12$, $a_n = b_n = \tau(n)$, and that [6, p. 172]

$$\sum_{n \leqslant x} \tau^2(n) = O(x^{12}).$$

Hence, by Theorem 1,

$$\int_1^x |T(y)|^2\, dy = c_3 x^{12\frac{1}{2}} + O(x^{12} \log^2 x),$$

where $T(x) = \sum_{n \leqslant x} \tau(n)$, a result which is due to Walfisz [13].

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Received November 4, 1963