JORDAN ALGEBRAS OF TYPE I

BY

ERLING STØRMER

Oslo Universitet, Oslo, Norway

1. Introduction

Jordan, von Neumann, and Wigner [5] have classified all finite dimensional Jordan algebras over the reals. The present paper is an attempt to do the same in the infinite dimensional case. The following restriction will be imposed: we assume the Jordan algebras are weakly closed Jordan algebras of self-adjoint operators with minimal projections acting on a Hilbert space, i.e. are irreducible JW-algebras of type I.(1) The result is then quite analogous to that in [5], except we do not get hold of the Jordan algebra \( \mathfrak{M}_2 \) of that paper, as should be expected from the work of Albert [1]. We first classify all irreducible JW-algebras of type \( I_n \), \( n \geq 3 \) (Theorem 3.9). These algebras are roughly all self-adjoint operators on a Hilbert space over either the reals, the complexes, or the quaternions. Then all JW-factors of type \( I_n \), \( n \geq 3 \), will be classified (Theorem 5.2). In addition to those in the irreducible case we find an additional JW-factor, namely one which is the \( C^* \)-homomorphic image of all self-adjoint operators on a Hilbert space. JW-factors of type \( I_2 \) are studied separately (Theorem 7.1). They are the spin factors, and except when the dimensions are small, are exactly those JW-factors which are not reversible. Global results of this type are obtained in section 6. Finally we show that the von Neumann algebra generated by a reversible JW-algebra of type I is itself of type I (Theorem 8.2).

A J-algebra is a real linear space \( \mathfrak{J} \) of self-adjoint operators on a (complex) Hilbert space \( \mathcal{H} \) closed under the product \( \mathcal{J} \circ \mathcal{B} = \frac{1}{2}(\mathcal{A} \mathcal{B} + \mathcal{B} \mathcal{A}) \). Then \( \mathfrak{J} \) is closed under products of the form \( \mathcal{A} \mathcal{B} \mathcal{A} \) and \( \mathcal{A} \mathcal{B} + \mathcal{B} \mathcal{A} \), \( A, B, C \in \mathfrak{J} \) (see [4]). A JC-algebra (resp. JW-algebra) is a uniformly (resp. weakly) closed J-algebra. A JW-factor is a JW-algebra with center the scalars (with respect to operator multiplication). A projection \( E \) in a J-algebra \( \mathfrak{J} \) is abelian if \( E \mathfrak{J} E \) is an abelian family of operators. By a symmetry we shall mean a self-adjoint unitary operator. Two projections \( E \) and \( F \) in a JW-algebra \( \mathfrak{J} \) are said to be equivalent if there

---

(1) In a forthcoming paper all irreducible JW-algebras will be shown to be of type I.
exists a symmetry \( S \) in \( \mathcal{A} \) such that \( E = SFS \). The central carrier of a projection \( E \) in \( \mathcal{A} \) is the least central projection in \( \mathcal{A} \) greater than or equal to \( E \). \( \mathcal{A} \) is of type I if there exists an abelian projection in \( \mathcal{A} \) with central carrier the identity. If \( n \) is a cardinal \( \mathcal{A} \) is of type \( I_n \) if there exists an abelian projection in \( \mathcal{A} \) \( n \) equivalent copies of which add up to the identity operator. We shall write \( I_{\omega} \) whenever \( n \) is infinite. A Jordan ideal in a \( J \)-algebra \( \mathcal{A} \) is a \( J \)-algebra \( \mathcal{J} \subset \mathcal{A} \) such that \( A \circ B \in \mathcal{J} \) whenever \( A \in \mathcal{J} \), \( B \in \mathcal{A} \). Let \( \mathcal{R}(\mathcal{A}) \) denote the uniformly closed real algebra generated by \( \mathcal{A} \). If \( n \) is a positive integer then \( \mathcal{A}^n \) is the uniformly closed real linear space generated by products of the form \( \prod_{i=1}^{n} A_i \), \( A_i \in \mathcal{A} \), and \( (\mathcal{A}) \) denotes the \( C^* \)-algebra generated by \( \mathcal{A} \). \( \mathcal{A} \) is reversible if \( \prod_{i=1}^{n} A_i + \prod_{i=1}^{n} A_i \in \mathcal{A} \) whenever \( A_1, \ldots, A_n \in \mathcal{A} \), \( n = 1, 2, \ldots \). If \( \mathcal{A} \) is a \( JC \)-algebra then \( \mathcal{A} \) is reversible if and only if \( \mathcal{A} = \mathcal{R}(\mathcal{A}) \mathcal{Z} \mathcal{A} \) [10].

We denote by \( \mathcal{M}_{SA} \) the set of self-adjoint operators in a family \( \mathcal{M} \) of operators. \( \mathcal{M} \) is said to be self-adjoint if \( \mathcal{M} \) contains the adjoint of each operator in \( \mathcal{M} \). \( \mathcal{M}^* \) is the weak closure and \( \mathcal{M}^* \) the commutant of \( \mathcal{M} \). If \( \mathcal{M} \subset \mathcal{H} \) then \( [\mathcal{M}, \mathcal{H}] \) is the subspace of \( \mathcal{H} \) generated by vectors of the form \( Ax, A \in \mathcal{M}, x \in \mathcal{H} \). We identify subspaces of \( \mathcal{H} \) and their projections. We denote by \( \mathcal{B}(\mathcal{H}) \) the algebra of all bounded operators on \( \mathcal{H} \). Throughout this paper \( \mathbb{R} \) denotes the real numbers, \( \mathbb{C} \) the complex numbers, and \( \mathbb{Q} \) the quaternions. We shall consider \( \mathbb{Q} \) as a subalgebra of \( M_2 \)—the complex \( 2 \times 2 \) matrices.

We are indebted to L. Ingelstam for pointing out an error in an early version of Theorem 2.1.

### 2. Real algebras

As will be seen there is a close relationship between real self-adjoint algebras of operators and \( JW \)-algebras. Kaplansky [7] has classified (up to isomorphisms) the simplest real algebras. We shall need a more detailed description of them.

**Theorem 2.1.** Let \( \mathcal{R} \) be a real self-adjoint algebra of operators on a Hilbert space such that every self-adjoint operator in \( \mathcal{R} \) is a scalar multiple of the identity \( I \). Then \( \mathcal{R} \) is characterized as follows:

1. \( \mathcal{R} = \mathbb{R}I \).
2. \( \mathcal{R} = \mathbb{C}I \).
3. There exists a minimal projection \( P' \in \mathcal{R}^* \) with central carrier \( I \) such that \( P'\mathcal{R} = QP' \).
4. There exist two non zero projections \( P \) and \( Q \) with \( P + Q = I \) such that
   \[
   \mathcal{R} = \{ \lambda P + \lambda Q : \lambda \in \mathcal{C} \}.
   \]

**Proof.** Let \( A \) be a non zero operator in \( \mathcal{R} \) and suppose there exists a sequence \( (B_n) \) of operators in \( \mathcal{R} \) such that \( B_n A \to 0 \) or \( A B_n \to 0 \) uniformly, say \( B_n A \to 0 \). Then \( A^* B_n^* B_n A = \ldots \)
(B_\alpha A)^* B_\alpha A \to 0. By hypothesis \( B_\alpha^* B_\alpha = \alpha_\alpha I \) with \( \alpha_\alpha \in \mathbb{R} \). Thus \( \alpha_\alpha A^* A \to 0, \alpha_\alpha \to 0 \), and \( B_\alpha \to 0 \). Similarly \( A B_\alpha \to 0 \) implies \( B_\alpha \to 0 \). Thus \( \mathfrak{A} \) has no non zero topological divisors of 0. By [7, Theorem 3.1] \( \mathfrak{A} \) is isomorphic to one of \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{Q} \). This could also be shown by an application of [3]. If \( \mathfrak{A} \cong \mathbb{R} \) we have case (1). Let \( \mathfrak{B} \) denote the \( C^* \)-algebra generated by \( \mathfrak{A} \). Since \( \mathfrak{A} \) is finite dimensional being isomorphic to \( \mathbb{C} \) or \( \mathbb{Q} \), \( \mathfrak{B} = \mathfrak{A} + \mathfrak{A} \). Assume \( \mathfrak{A} \cong \mathbb{C} \). Then the dimension of \( \mathfrak{B} \) as a vector space over \( \mathbb{C} \) is 1 or 2. In case \( \dim \mathfrak{B} = 1 \) we have case (2). Assume the dimension is 2. Then there exist two orthogonal non zero projections \( P \) and \( Q \) in \( \mathfrak{B} \) with \( P + Q = I \) such that every operator in \( \mathfrak{B} \) is of the form \( \lambda P + \mu Q \) with \( \lambda, \mu \in \mathbb{C} \). We may thus identify \( \mathfrak{B} \) with \( \mathbb{C} \times \mathbb{C} \). Let \( (\lambda, \mu) \in \mathfrak{A} \). Then \( (\lambda + \mu, \mu + \mu) \in \mathfrak{A} \), hence is a scalar. Thus \( \text{Re} \lambda = \text{Re} \mu \). Moreover, if \( (\zeta, \xi) \in \mathfrak{A} \) then \( (\lambda \xi, \lambda \xi) \in \mathfrak{A} \), hence \( \text{Re} \lambda \xi = \text{Re} \mu \xi \), and \( 1 \lambda \xi - 1 \mu \xi = 1 \mu \xi - 1 \lambda \xi \). In particular, with \( \lambda = \zeta, \mu = \xi \), \( (1 \lambda \xi)^2 = (1 \mu \xi)^2 \). If \( 1 \lambda \xi = 1 \mu \xi = 0 \) then by the above identity \( 1 \lambda = 1 \mu \xi = 0 \) for all \( (\zeta, \xi) \in \mathfrak{A} \), so \( \mathfrak{A} = \{ (\lambda, \lambda) : \lambda \in \mathbb{C} \} \), and we have case (2). Otherwise \( 1 \lambda \xi = -1 \mu \xi \) for all \( (\lambda, \mu) \in \mathfrak{A} \), \( \mathfrak{A} = \{ (\lambda, \lambda) : \lambda \in \mathbb{C} \} \), and we have case (4).

It remains to consider the case when \( \mathfrak{A} \cong \mathbb{Q} \). Then the dimension of \( \mathfrak{A} \) as a real vector space is 4. Let \( \mathfrak{H} \) be the underlying Hilbert space. Since \( \mathfrak{A} \) is a vector subspace of \( \mathfrak{B}(\mathfrak{H}) \) the real dimension of \( \mathfrak{B} = \mathfrak{A} + \mathfrak{A} \) is less than or equal to 8, i.e. \( \dim \mathfrak{B} \leq 4 \) as a complex vector space. Since \( \mathfrak{A} \) is non abelian so is \( \mathfrak{B} \), hence \( \mathfrak{B} \cong \mathbb{M}_2 \). In particular, \( \mathfrak{B} \) being finite dimensional, is a factor of type I, hence \( \mathfrak{B} \) is a factor of type I. Let \( \mathfrak{P} \) be a minimal projection in \( \mathfrak{B} \). Then its central carrier is \( I \). Since \( \mathfrak{P} \) is minimal, \( (\mathfrak{B} \mathfrak{P})' = \mathfrak{P} \mathfrak{B} \mathfrak{P} = \{ \lambda \mathfrak{P} \} \), considered as a von Neumann algebra on the Hilbert space \( \mathfrak{P} \), hence \( \mathfrak{B} \mathfrak{P} = \mathbb{M}_2 \mathfrak{P} \). Thus \( \mathfrak{B} \mathfrak{P} = \mathfrak{Q} \mathfrak{P} \). The proof is complete.

It should be remarked that in the quaternionian case there exist real algebras like those in case (4). However, if there exist projections \( P \) and \( Q \) with sum \( I \) such that \( \mathfrak{A} = \{ \lambda P + \lambda Q : \lambda \in \mathbb{Q} \} \), then \( P \) and \( Q \) belong to \( \mathfrak{B} \) but not to \( \mathfrak{B} \)—the \( C^* \)-algebra generated by \( \mathfrak{A} \). Moreover, the map \( A \to \mathfrak{A} P \) is an isomorphism of \( \mathfrak{A} \) onto \( \mathfrak{B} P \). This should be kept in mind in order to get a full understanding of the classification theorems to follow.

**Corollary 2.2.** Let \( \mathfrak{A} \) be a reversible \( JW \)-factor of type I. Let \( E \) be an abelian projection in \( \mathfrak{A} \). Let \( \mathfrak{A} = \mathfrak{E}(\mathfrak{A}) E \). Then \( \mathfrak{A} \) is a real self-adjoint algebra satisfying the conditions of Theorem 2.1, and \( E(\mathfrak{A}) E \) is characterized as follows:

1. \( E(\mathfrak{A}) E = CE \).
2. There exists a minimal projection \( P' \) in \( \mathfrak{E}(\mathfrak{A}) E' \) such that \( \mathfrak{E}(\mathfrak{A}) E P' = M_2 P' \).
3. There exist two non zero projections \( P \) and \( Q \) with \( P + Q = E \) such that

\[
E(\mathfrak{A}) E = CP \oplus CQ.
\]
Proof. Since $E$ is abelian $E\Xi E = E \Xi$ [11, Corollary 24]. Since $\mathcal{A}$ is reversible it follows that $\mathcal{A}$ satisfies the conditions of Theorem 2.1. The rest is clear from the theorem and its proof.

The next result will be used in section 8.

**Corollary 2.3.** Let $\mathcal{R}$ be an irreducible, uniformly closed, self-adjoint, real algebra with identity $I$ acting on a Hilbert space $\mathcal{H}$. Assume $\mathcal{R}_{sa}$ is abelian. Then $\mathcal{R}$ is one of the following algebras:

1. (resp. 2) $\mathcal{R} = RI$ (resp. $CI$), and $\dim \mathcal{H} = 1$.
2. $\mathcal{R} = QI$, and $\dim \mathcal{H} = 2$.

**Proof.** Let $\mathcal{M}$ be a maximal ideal in $\mathcal{R}_{sa}$. Let $\varphi$ be a pure state of $\mathcal{R}_{sa}$ with kernel $\mathcal{M}$. Let $\varphi$ be a state extension of $\varphi$ to the C*-algebra $\mathcal{A}$ generated by $\mathcal{R}$. Then $f = \omega \varphi$ with $\omega$ a representation of $\mathcal{R}$. Let $\mathcal{M} = \mathcal{R} \cap \ker \varphi$. Then $\mathcal{M}$ is a uniformly closed self-adjoint ideal in $\mathcal{R}$. If $A \in \mathcal{M}$ then $AA^* \in \mathcal{M}$, hence $0 = \omega \varphi(AA^*) - f(AA^*)$, and $AA^* \in \mathcal{M}$. Let $\mathcal{J}$ be a uniformly closed ideal in $\mathcal{R}_{sa}$. Then $f \mathcal{J} \cap \mathcal{M} = \mathcal{M}$, where the $\mathcal{M}$ are constructed as above. Let $A \in f \mathcal{J}$. Then $AA^* \in \mathcal{M}$, hence $AA^* \in \mathcal{M}$ for all $A$. By the above $AA^* \in \mathcal{M}$ for all $A$, hence $AA^* \in \mathcal{M}$.

We show $\mathcal{R}_{sa} = RI$. If not then there exist two non zero uniformly closed ideals $\mathcal{J}$ and $\mathcal{J}$ in $\mathcal{R}_{sa}$ such that $\mathcal{J} \cap \mathcal{J} = 0$. Let $\mathcal{J}$ and $\mathcal{J}$ be ideals in $\mathcal{R}$ constructed as above. Let $A \in \mathcal{J}$ and $B \in \mathcal{J}$. Then $AB \in \mathcal{J} \cap \mathcal{J}$. Thus $AB(AB)^* \in \mathcal{J} \cap \mathcal{J} = 0$, and $AB = 0$. Thus $\mathcal{J} = 0$. Since $\mathcal{R}$ is irreducible $\mathcal{R}$ has no non zero ideal divisors of zero [8, Lemma 2.5]. Thus $\mathcal{J}$ or $\mathcal{J} = 0$, contrary to assumption; $\mathcal{R}_{sa} = RI$. An application of Theorem 2.1 completes the proof.

3. Irreducible JW-algebras

We classify all irreducible JW-algebras of type $I_n$, $n \geq 3$. The key to this and later results lies in the following general lemma on the structure of JW-algebras.

**Lemma 3.1.** Let $\mathcal{A}$ be a JW-algebra such that there exists a family $\{E_\sigma\}_{\sigma \in \mathcal{J}}$ of orthogonal, non zero, equivalent projections in $\mathcal{A}$ with $\sum_{\sigma \in \mathcal{J}} E_\sigma = I$, and card $\mathcal{J} \geq 3$. For $\sigma, \tau \in \mathcal{J}$ let $\mathcal{S}_{\sigma\tau} = E_\sigma \mathcal{A} E_\tau$. Then the following relations hold:
JORDAN ALGEBRAS OF TYPE 1

(1) $\mathcal{E}_{\sigma} \mathcal{E}_{\eta} = \begin{cases} 0 & \text{if } \tau + \eta \\ \mathcal{E}_{\sigma} & \text{if } \tau - \eta, \sigma + \eta \\ \mathcal{E}_{\eta} & \text{for all } \eta + \sigma \text{ if } \sigma = \eta + \tau - \eta. \end{cases}$

(2) $\mathcal{E}_{\sigma} = \mathcal{E}_{\eta} \mathcal{E}_{\sigma} \mathcal{E}_{\eta}$ if $\sigma = \eta$.

(3) If $\sigma + \eta$, let $\mathbb{H}_{\sigma}$ be the uniformly closed real linear space generated by $\mathcal{E}_{\sigma} \mathcal{E}_{\eta}$. Then $\mathbb{H}_{\sigma}$ is a real self-adjoint algebra with identity $\mathcal{E}_{\sigma}$, $\mathbb{H}_{\sigma}$ is independent of $\eta$, and $\mathbb{H}_{\sigma} \mathbb{H}_{\sigma} = \mathcal{E}_{\sigma}$.

(4) $E_{\sigma} \mathbb{H}(\mathbb{H}) E_{\eta} = \mathcal{E}_{\eta}$ if $\sigma = \eta$, $E_{\sigma} \mathbb{H}(\mathbb{H}) E_{\eta} \subset \mathbb{H}_{\sigma}$.

(5) $\mathbb{H}$ is reversible, and $\mathbb{H}(\mathbb{H})^{-1} = \mathbb{H}^{-1}$.

Proof. Clearly $\mathcal{E}_{\sigma} \mathcal{E}_{\eta} = 0$ if $\tau + \eta$. Now $\mathcal{E}_{\sigma} \mathcal{E}_{\eta} \subset \mathcal{E}_{\sigma}$ if $\sigma = \eta$. In fact, take $B, C \in \mathbb{H}$ and put $A = E_{\sigma} BE_{\eta} + E_{\eta} BE_{\sigma} + E_{\sigma} CE_{\eta} + E_{\eta} CE_{\sigma}$. Clearly $A \in \mathbb{H}$, hence $A^2 \in \mathbb{H}$, so that $E_{\sigma} A^2 E_{\eta} \in \mathcal{E}_{\sigma}$. There are three cases, namely, $\tau = \sigma, \tau = \eta$, and $\tau + \sigma$ and $\tau + \eta$. In the two first cases straightforward computations yield $E_{\sigma} BE_{\eta} CE_{\sigma} = E_{\eta} A^2 E_{\eta} \in \mathcal{E}_{\sigma}$. In the third case a similar computation yields $E_{\sigma} BE_{\eta} CE_{\sigma} = E_{\eta} A^2 E_{\eta} \in \mathcal{E}_{\sigma}$, and $\mathcal{E}_{\sigma} \subset \mathcal{E}_{\sigma}$ for all $\tau$ whenever $\sigma = \eta$, as asserted. The opposite inclusion is clear if $\tau = \sigma$ or $\tau = \eta$. Assume therefore $\sigma, \eta, \tau$ all distinct. Let $S$ be a symmetry in $\mathbb{H}$ such that $E_{\sigma} = SE_{\eta} S$. Let $V = E_{\sigma} SE_{\eta}$. Then $V$ is a partial isometry in $\mathbb{H}_{\sigma}$ such that $V^* V = E_{\sigma}$ and $VV^* = E_{\sigma}$. Let $A \in \mathcal{E}_{\sigma}$. Then

$A = A E_{\sigma} = (A V) V^* E_{\sigma} \in (\mathcal{E}_{\sigma} \mathcal{E}_{\eta}) \mathcal{E}_{\sigma} \subset \mathcal{E}_{\sigma} \mathcal{E}_{\eta}$.

by the above. Thus $\mathcal{E}_{\sigma} \subset \mathcal{E}_{\sigma} \mathcal{E}_{\eta}$, and they are equal.

Assume $\sigma = \eta$. Let $\phi + \tau$ in $J$, both different from $\sigma$. Then, by the above, $\mathcal{E}_{\sigma} \subset \mathcal{E}_{\eta} = \mathcal{E}_{\eta} \mathcal{E}_{\eta} \subset \mathcal{E}_{\eta} \mathcal{E}_{\eta}$, and (1) is proved.

If $\phi + \sigma$ let $\tau \notin J$ be distinct from both. Then by repeated applications of (1),

$\mathcal{E}_{\sigma} \subset \mathcal{E}_{\sigma} \mathcal{E}_{\tau} = \mathcal{E}_{\sigma} \mathcal{E}_{\tau} \mathcal{E}_{\tau} \subset \mathcal{E}_{\sigma} \mathcal{E}_{\tau} \mathcal{E}_{\tau} \subset \mathcal{E}_{\sigma} \mathcal{E}_{\tau} \mathcal{E}_{\tau} \mathcal{E}_{\tau} \subset \mathcal{E}_{\sigma} \mathcal{E}_{\tau} \mathcal{E}_{\tau} \mathcal{E}_{\tau} \mathcal{E}_{\tau}$,

and (2) is proved.

It is clear from (1) that $\mathbb{H}_{\sigma}$ is independent of $\eta$. By (2),

$(\mathcal{E}_{\sigma} \mathcal{E}_{\eta})^2 = \mathcal{E}_{\sigma} \mathcal{E}_{\eta} \mathcal{E}_{\eta} \mathcal{E}_{\sigma} = \mathcal{E}_{\sigma} \mathcal{E}_{\eta} \mathcal{E}_{\eta} \mathcal{E}_{\sigma}$,

so $\mathcal{E}_{\sigma} \mathcal{E}_{\eta}$ is multiplicative. Since it is clearly self-adjoint $\mathbb{H}_{\sigma}$ is a real self-adjoint algebra with identity $E_{\sigma}$. Let $A$ be a self-adjoint operator in $\mathbb{H}_{\sigma}$. Then $A$ is a uniform limit of self-adjoint operators of the form $\sum_{i=1}^{n} B_{i}$ with

$B_{i} = E_{\sigma} C_{i} E_{\sigma} D_{i} E_{\sigma} \in (\mathcal{E}_{\sigma} \mathcal{E}_{\eta}) \mathbb{H}_{\sigma} \mathbb{H}_{\sigma} \mathbb{H}_{\sigma}$,

$C_{i}, D_{i} \in \mathbb{H}$. 
 Since \[ B_i = \frac{1}{2}(B_i + B_i^*) = \frac{1}{2} E_i (C_i E_i D_i + D_i E_i C_i) E_i \in \mathcal{A} \mathcal{E}_\mathcal{A} = \mathcal{E}_{\sigma \tau}, \]

\( A \in \mathcal{E}_{\sigma \tau} \). Thus \( \mathcal{A}_{\sigma \tau} \subseteq \mathcal{E}_{\sigma \tau} \). The opposite inclusion follows as soon as we have shown \( \mathcal{E}_{\sigma \tau} \subseteq \mathcal{A}_{\sigma \tau} \). For this let \( P < E_\sigma \) be a non-zero projection in \( \mathcal{A} \). Let \( S \) be a symmetry in \( \mathcal{A} \) such that \( SE_\sigma S = E_\sigma, \sigma = \tau \). Let \( Q = S P S \). Then \( Q < E_\tau \). Let \( R = P + Q \). Let \( T = R S R \). Then \( T \in \mathcal{A} \), and

\[ P = T Q T = E_\sigma T Q T E_\sigma = E_\sigma T E_\sigma T E_\sigma \in \mathcal{E}_{\sigma \tau} \mathcal{E}_{\sigma \tau} \subseteq \mathcal{A}_{\sigma \tau}. \]

Since real linear combinations of projections are uniformly dense in the JW-algebra \( \mathcal{E}_{\sigma \tau} \), and since \( \mathcal{A}_{\sigma \tau} \) is uniformly closed \( \mathcal{E}_{\sigma \tau} \), as asserted, (3) is proved. Notice that the same argument shows \( (\mathcal{A}_{\sigma \tau})_{JM} = \mathcal{E}_{\sigma \tau} \), a fact which will be used below.

We next show \( \mathcal{E}_{\sigma \tau} \) is weakly closed whenever \( \sigma \neq \tau \). Let \( A \in \mathcal{E}_{\sigma \tau} \). Then \( A = E_\sigma A E_\tau \). Let \( \{A_\nu\} \) be a net in \( \mathcal{A} \) such that \( E_\sigma A_\nu E_\tau \to A \) weakly. Since the \(*\)-operation is weakly continuous, \( E_\sigma A_\nu E_\tau \to A \) weakly, hence \( A + A^* \) is the weak limit of the net \( \{E_\sigma A_\nu E_\tau + E_\sigma A_\nu E_\tau\} \), the net consisting of operators in \( \mathcal{A} \). Therefore \( A + A^* \in \mathcal{A} \), as \( \mathcal{A} \) is weakly closed. Thus \( A = E_\sigma (A + A^*) E_\tau \in \mathcal{E}_{\sigma \tau}, \mathcal{E}_{\sigma \tau} \) is weakly closed.

In order to show (4) let \( A \in \mathcal{R}(\mathcal{A}) \). Then \( A \) is a uniform limit of operators of the form \( \sum_{i=1}^{n} \prod_{i=1}^{n} A_{ij} \) with \( A_{ij} \in \mathcal{A} \). Therefore, in order to show \( E_\sigma A E_\tau \in \mathcal{E}_{\sigma \tau} \) if \( \sigma \neq \tau \) (resp. in \( \mathcal{A} \) if \( \sigma = \tau \)), it suffices by linearity and the fact that \( \mathcal{E}_{\sigma \tau} \) and \( \mathcal{A} \) are weakly closed so uniformly closed, to show that any operator of the form

\[ E_\sigma \prod_{j=1}^{n} A_j E_\tau \in \mathcal{E}_{\sigma \tau} \] (resp. \( \mathcal{A} \)), where \( A_j \in \mathcal{A} \).

For \( n = 1 \) this is trivial. Use induction and assume it holds for \( n - 1 \). Then

\[ E_\sigma \prod_{j=1}^{n} A_j E_\tau \approx E_\sigma \prod_{j=1}^{n-1} A_j E_\sigma A_n E_\tau, \]

which is the strong limit of operators

\[ \sum_{j \in J'} E_\sigma \prod_{j=1}^{n-1} A_j E_\sigma A_n E_\tau, \]

with \( J' \) a finite subset of \( J \). By induction hypothesis

\[ E_\sigma \prod_{j=1}^{n-1} A_j E_\tau \in \mathcal{E}_{\sigma \tau} \] if \( \sigma \neq \tau \)

and in \( \mathcal{A} \) if \( \sigma = \tau \). Hence, by (1)

\[ E_\sigma \prod_{j=1}^{n-1} A_j E_\tau \in \mathcal{E}_{\sigma \tau} \] (resp. \( \mathcal{A} \)) if \( \tau + \sigma \)

and if \( \tau = \sigma \) then by (2) and the fact that \( \mathcal{E}_{\sigma \tau} \) is weakly closed,
JORDAN ALGEBRAS OF TYPE I

Thus
$$E\left(\prod_{j=1}^{n-1} A E A_n E\right)\in \mathcal{S}_q$$
for all finite subsets $J'$ of $J$. As $\mathcal{S}_q$ and $\mathfrak{U}_q$ are weakly so strongly closed, it follows that
$$E\left(\prod_{j=1}^{n-1} A E A_n E\right)\in \mathcal{S}_q$$
Thus
$$E_\sigma \mathcal{R}(\mathfrak{U}) E_\sigma \subset \mathcal{S}_q$$
Since clearly $\mathcal{S}_q \subset E_\sigma \mathcal{R}(\mathfrak{U}) E_\sigma$ follows. Notice that $E_\sigma \mathcal{R}(\mathfrak{U})^{-1} E_\sigma \supset \mathcal{S}_q$, so they are equal.

Let $A \in E_\sigma \mathcal{R}(\mathfrak{U})$. Then $E_\sigma A E_\sigma \in \mathcal{S}_q$. If $A \neq 0$ then there exist $B, C \in \mathfrak{U}$ such that

$$E\left(\prod_{j=1}^{n-1} A E A_n E\right)\in \mathcal{S}_q$$
Thus $A \in \mathfrak{U}$. It is reversible.

It is clear that each $\mathcal{S}_q \subset \mathfrak{U}$. Thus $\mathcal{S}_q \subset \mathfrak{U}$. Hence by (4), $\mathcal{R}(\mathfrak{U})^{-1} \supset \mathcal{S}_q$, and they are equal. The proof is complete.

From now on the $E_\sigma$ in Lemma 3.1 will be abelian projections in a JW-factor $\mathfrak{U}$ of type I. If $\mathfrak{U}$ is of type I, we define the $\mathcal{S}_q$ as above. Whenever we write $\mathcal{S}_q$, we shall assume $\sigma = q$.

**Lemma 3.2.** Let $\mathfrak{U}$ be a JW-factor with orthogonal non zero abelian projections $\{E_\sigma\}_{\sigma \in \mathfrak{U}}$, such that $\sum_{\sigma} E_\sigma = I$. Then every operator in $\mathcal{S}_q$ is a scalar multiple of a partial isometry of $E_\sigma$ onto $E_\sigma$. Moreover, if $S, T \in \mathcal{S}_q$, then there exists a real number $\alpha$ such that

$$S^*T + TS^* = \alpha E_\sigma$$

**Proof.** Let $S \in \mathcal{S}_q$. Then $S = E_\sigma A E_\sigma$, $A \in \mathfrak{U}$. Thus

$$S^*S - E_\sigma A E_\sigma A E_\sigma E E_\sigma = R E_\sigma$$

[11, Corollary 24], hence $S^*S = \|S\|^2 E_\sigma$. Similarly $SS^* = \|S\|^2 E_\sigma$, and $S = \beta V$ with $V$ a partial isometry of $E_\sigma$ onto $E_\sigma$, $\beta \in \mathbb{C}$. Let $T$ be another operator in $\mathcal{S}_q$. Since $\mathfrak{U}$ is linear so is $\mathcal{S}_q$. Thus $S + T = \gamma W$ with $W$ a partial isometry of $E_\sigma$ onto $E_\sigma$, $\gamma \in \mathbb{C}$. Therefore,

$$\gamma^2 E_\sigma = (S + T)^* (S + T) = S^* S + T^* T + (S^* T + T^* S) = |\beta|^2 E_\sigma + \delta E_\sigma + (S^* T + T^* S),$$

where $T^* T = \delta E_\sigma$. Thus $S^* T + T^* S = \alpha E_\sigma$. We may assume $T + 0$. Then
\[ TS^* + ST^* = T S^* E_a + E_a S T^* = \delta^{-1} T (S^* T + T^* S) T^* = \alpha E_a T^* = \alpha E_a. \]

The proof is complete.

From now on \( A \) is a \( JW \)-factor of type \( I_n, n \geq 3 \), and the \( E_a \) are as in Lemma 3.2. Then they are all equivalent [11, Corollary 26], and Lemma 3.1 is applicable. We keep the notation in Lemma 3.1.

**Lemma 3.3.** For each pair \( \sigma \neq \varphi \) in \( J \) we can choose one partial isometry \( W_{\sigma \varphi} \in \mathcal{E}_{\sigma \varphi} \) such that whenever \( \varphi, \sigma, \tau \) are three distinct elements in \( J \) then

1. \( W_{\sigma \varphi} = W_{\varphi \sigma}^* \),
2. \( W_{\sigma \varphi} = W_{\sigma \tau} W_{\tau \varphi} \).

**Proof.** Let first \( \varphi, \sigma, \tau \) be three distinct elements in \( J \). Choose partial isometries \( W_{\varphi \sigma} \) and \( W_{\sigma \tau} \) in \( \mathcal{E}_{\varphi \sigma} \) and \( \mathcal{E}_{\sigma \tau} \) respectively (Lemma 3.2). Define \( W_{\sigma \varphi} \) and \( W_{\tau \varphi} \) by (1). Let

\[ W_{\varphi \sigma} W_{\sigma \tau} = W_{\varphi \sigma} W_{\sigma \tau}, \quad W_{\sigma \varphi} = W_{\sigma \tau} W_{\tau \varphi}. \]

By Lemma 3.1 \( W_{\varphi \sigma} \in \mathcal{E}_{\varphi \sigma} \) and \( W_{\sigma \tau} \in \mathcal{E}_{\sigma \tau} \), and \( W_{\varphi \tau} = W_{\varphi \tau}^* \). It is straightforward to check (2) for the different rearrangements of \( \varphi, \sigma, \) and \( \tau \), e.g. \( W_{\varphi \tau} = W_{\tau \varphi} W_{\sigma \varphi} W_{\varphi \sigma} = W_{\varphi \varphi} W_{\varphi \tau} = W_{\varphi \tau} W_{\varphi \varphi} \). Thus the lemma holds for the three elements \( \varphi, \sigma, \) and \( \tau \) in \( J \).

Let \( K \) be a maximal subset of \( J \) containing \( \varphi, \sigma, \) and \( \tau \), and for which the \( W_{\varphi \sigma} \) are chosen so that (1) and (2) hold for all elements \( \eta, \varphi \) in \( K \). Then \( K = J \). If not let \( \eta \in J - K \). Choose \( W_{\varphi \eta} \) and \( W_{\eta \varphi} \) in \( \mathcal{E}_{\varphi \eta} \) and \( \mathcal{E}_{\eta \varphi} \) respectively such that \( W_{\varphi \eta} = W_{\eta \varphi}^* \). Let

\[ W_{\varphi \eta} = W_{\varphi \eta} W_{\varphi \eta}, \quad W_{\eta \varphi} = W_{\eta \varphi} W_{\eta \varphi}, \]

for all \( \varphi \in K, \varphi \neq \eta \). Then \( W_{\eta \varphi}^* = W_{\eta \varphi} \), so (1) holds. Let \( \mu \neq \varphi \) be in \( K \) and distinct form \( \varphi \). This is possible by the preceding paragraph. As above we can show (2) holds for all rearrangements of \( \eta, \varphi \), and \( \varphi \). We show (2) holds for all rearrangements of \( \eta, \mu \), and \( \varphi \). Indeed,

\[ W_{\varphi \eta} = W_{\eta \varphi} W_{\varphi \eta} = W_{\mu \varphi} W_{\varphi \eta} W_{\eta \varphi} = W_{\mu \varphi} E_{\mu} W_{\phi \eta} = W_{\mu \eta} W_{\eta \varphi}, \]

\[ W_{\varphi \eta} = W_{\mu \eta} W_{\eta \varphi}, \]

\[ W_{\mu \varphi} = W_{\varphi \mu} W_{\mu \varphi} = W_{\varphi \mu} E_{\varphi} W_{\varphi \eta} = W_{\varphi \eta} W_{\varphi \eta} W_{\eta \varphi} = W_{\varphi \eta} W_{\eta \varphi}, \]

and (2) holds. Thus \( K \cup \{ \eta \} \) satisfies (1) and (2) for all its elements, contradicting the maximality of \( K \). Thus \( K = J \), the proof is complete.

**Lemma 3.4.** The \( A_{\sigma}, \sigma \in J \), are all spatially isomorphic, and each \( A_{\sigma} \) is one of the following algebras:
JORDAN ALGEBRAS OF TYPE I

(1) (resp. 2) \( \mathfrak{A}_\sigma = R E_\sigma \) (resp. \( CE_\sigma \)).

(3) There exists a projection \( P' \in \mathfrak{A}' \) with central carrier \( I \) such that if \( \mathfrak{A} \) is replaced by \( P' \mathfrak{A} \), then \( \mathfrak{A}_\sigma = Q E_\sigma \).

(4) There exist two orthogonal non-zero projections \( P_\sigma \) and \( Q_\sigma \) with sum \( E_\sigma \) such that \( \mathfrak{A}_\sigma = \{ \lambda P_\sigma + \lambda Q_\sigma : \lambda \in C \} \).

Proof. Choose the \( W_\sigma \in \mathfrak{S}_\sigma \) as in Lemma 3.3. By Lemma 3.1 \( \mathfrak{S}_\sigma = \mathfrak{S}_\sigma \mathfrak{S}_\varphi \mathfrak{S}_\sigma \mathfrak{S}_\sigma \). If \( V \in \mathfrak{S}_\sigma \) then \( V = V E_\sigma = V W_\sigma W_{\sigma \varphi} = E_\sigma V = W_{\sigma \varphi} W_\sigma V \). Thus \( \mathfrak{S}_\sigma = \mathfrak{S}_\sigma \mathfrak{S}_\varphi \mathfrak{S}_\varphi W_{\sigma \varphi} = W_{\sigma \varphi} \mathfrak{S}_\varphi \mathfrak{S}_\varphi \mathfrak{S}_\sigma \). Since \( \mathfrak{S}_\varphi \) is linear \( \mathfrak{S}_\varphi = \mathfrak{A}_\varphi W_{\sigma \varphi} = W_{\sigma \varphi} \mathfrak{A}_\varphi \). In particular \( \mathfrak{A}_\sigma = W_{\sigma \varphi} \mathfrak{A}_\sigma W_{\sigma \varphi} \), and they are spatially isomorphic.

By Lemma 3.1 \( \mathfrak{S}_{\sigma \sigma} = E_\sigma \mathfrak{S}_\sigma = R E_\sigma \). By Theorem 2.1 there are four cases. The cases (1), (2), and (4) of that theorem yield cases (1), (2), and (4) above. Assume \( \mathfrak{A}_\sigma \) is given by (3) in Theorem 2.1 for all \( \sigma \in J \). Then there exists a minimal projection \( P_\sigma \in \mathfrak{S}_\sigma \), where \( \mathfrak{S}_\sigma \) is considered as an algebra on the Hilbert space \( E_\sigma \), such that \( P_\sigma \mathfrak{A}_\sigma = Q P_\sigma ' \). Let \( \mathfrak{B}_\sigma \) denote the C*-algebra generated by \( \mathfrak{A}_\sigma \). Then \( \mathfrak{B}_\sigma \) is a factor of type I. The map \( A \rightarrow AP_\sigma ' \) is an isomorphism of \( \mathfrak{B}_\sigma \). Fix \( \sigma \in J \). For each \( \rho \neq \sigma \) let \( P_\rho ' = W_{\rho \sigma} P_\sigma W_{\rho \sigma} \). Since the map \( \mathfrak{A}_\rho \rightarrow \mathfrak{A}_\sigma \) by \( A \rightarrow W_{\rho \sigma} A W_{\rho \sigma} \) is an isomorphism, \( P_\rho ' \) is a minimal projection in \( \mathfrak{S}_\rho \). Let \( \sigma = \sum \sigma P_\rho ' \). Then \( P' \in \mathfrak{S}_\rho \). In fact, let \( A \in \mathfrak{A}_\rho \). Then \( P' E_\rho A E_\rho - P_\rho ' E_\rho A E_\rho = E_\rho A E_\rho P_\rho ' - E_\rho A E_\rho P_\rho ' \), since \( P_\rho ' \in \mathfrak{S}_\rho \). Since \( \mathfrak{S}_\rho = \mathfrak{A}_\rho W_{\rho \varphi} \) there exists \( A_\rho \in \mathfrak{A}_\rho \) such that \( E_\rho A E_\rho = -A_\rho W_{\rho \varphi} \). Thus, with \( \tau \) and \( \varphi \) distinct from \( \sigma \),

\[
P' E_\rho A E_\rho - P_\rho ' A_\rho W_{\rho \varphi} = A_\rho W_{\rho \varphi} P_\rho ' W_{\rho \varphi} = -A_\rho W_{\rho \varphi} P_\rho ' W_{\rho \varphi},
\]

and \( P' \in \mathfrak{S}_\rho \) as asserted. Let \( Q \) be a central projection in \( \mathfrak{S}_\rho \) such that \( Q > P' \). Then \( Q_\rho = E_\rho Q E_\rho > P_\rho ' \). But \( Q_\rho \) is central in \( E_\rho \mathfrak{S}_\rho E_\rho = \mathfrak{S}_\rho \), so \( Q_\rho = E_\rho \). Thus \( Q = \sum E_\rho = I \), and the central carrier of \( P' \) is \( I \). The proof is complete.

As an immediate application of this lemma and its proof we have

**LEMMA 3.5.** Let the \( W_\sigma \), \( \sigma \neq \rho \in J \), be partial isometries in \( \mathfrak{S}_\sigma \) chosen as in Lemma 3.3. Then \( \mathfrak{S}_\sigma \) is characterized as follows:

(1) \( \mathfrak{S}_\sigma = R W_{\sigma \varphi} \).

(2) \( \mathfrak{S}_\sigma = C W_{\sigma \varphi} \).

(3) There exists a projection \( P' \in \mathfrak{S}_\rho \) with central carrier \( I \) such that if \( \mathfrak{A} \) is replaced by \( P' \mathfrak{A} \), then \( \mathfrak{S}_\sigma = Q W_{\sigma \varphi} \).

(4) There exist orthogonal projections \( P_\rho, Q_\rho, P_\sigma, Q_\sigma \) with \( P_\rho + Q_\rho = E_\rho, P_\sigma + Q_\sigma = E_\sigma \) such that
\[ \mathcal{E}_{\sigma} = \{(\lambda P_\sigma + \lambda Q_\sigma)\cdot W_{\sigma}; \lambda \in \mathbb{C}\} = \{W_{\sigma}(\mu P_\sigma + \mu Q_\sigma); \mu \in \mathbb{C}\}. \]

The remaining part of the proof of Theorem 3.9 consists of eliminating case 4 in Lemma 3.5. We shall need a simple and probably well-known lemma of more general nature.

**Lemma 3.6.** Let \( E, F \) and \( G, H \) be two pairs of orthogonal non-zero projections, let \( V \) be a partial isometry of \( G + H \) onto \( E + F \). Assume that for each \( \lambda \in \mathbb{C} \) there exists \( \mu \in \mathbb{C} \) such that

\[ (1) \ (\lambda E + \lambda F) V = V(\mu G + \mu H). \]

Then either \( V^*EV = G \) and \( V^*FV = H \), or \( V^*EV = H \) and \( V^*FV = G \).

**Proof.** Multiply (1) on the right by \( G \). Then

\[ (2) \ \lambda EVG + \lambda FVG = \mu VG. \]

Multiply (2) on the left by \( E \). Then

\[ \lambda EVG = \mu EVG. \]

Thus either \( \lambda = \mu \) or \( EVG = 0 \). Similarly, multiplication of (2) on the left by \( F \) yields \( \lambda = \mu \) or \( FVG = 0 \). Since (1) holds for all \( \lambda \in \mathbb{C} \) either \( EVG = 0 \) or \( FVG = 0 \). Multiplication of (1) on the right by \( H \) yields \( EVH = 0 \) or \( FVH = 0 \). If \( EVH = 0 \) then \( 0 = EV(G + H) = EVV^* = E \), contrary to assumption. Thus \( FVH = 0 \). Now \( (V^*EV)G = 0 \) hence \( V^*EV \leq H \). Similarly \( V^*FV \leq G \). Since \( V^*EV + V^*FV = H + G \), \( V^*EV = H \), \( V^*FV = G \). The case \( FVG = 0 \) is treated similarly.

**Lemma 3.7.** Assume the \( E_{\sigma} \) are given by (4) in Lemma 3.5. Then the \( P_\sigma \) and \( Q_\sigma \) can be chosen so that \( P_\sigma W_{\sigma} = W_{\sigma} P_\sigma \) and \( Q_\sigma W_{\sigma} = W_{\sigma} Q_\sigma \), where the \( W_{\sigma} \) are chosen as in Lemma 3.3.

**Proof.** Fix \( \tau \in J \) and \( P_\tau \). By relabelling \( P_\tau \) if necessary whenever \( \varrho \neq \tau \) Lemma 3.6 gives

\[ P_\sigma W_{\tau} = W_{\sigma} P_\tau, \]

hence \( W_{\sigma} = (P_\tau W_{\sigma})^* = (W_{\sigma} P_\tau)^* = P_\tau W_{\sigma} \) whenever \( \varrho \neq \tau \). Let \( \sigma \in J \). We may assume \( \sigma \neq \tau \). If \( \varrho \neq \sigma, \tau \), then \( P_\sigma W_{\varrho} = P_\sigma W_{\varrho} W_{\tau} = W_{\sigma} P_\tau W_{\varrho} = W_{\sigma} W_{\tau} P_\sigma = W_{\sigma} P_\sigma \). The proof is complete.

Assume the \( W_{\sigma} \) are chosen as in Lemma 3.3 and the \( P_\sigma \) as in Lemma 3.7.

**Lemma 3.8.** Assume the \( E_{\sigma} \) are given by (4) in Lemma 3.5. Let \( P \) be the projection

\[ P = \sum_{\sigma \in J} P_\sigma. \]

Then \( P \) belongs to the center of \( \mathfrak{H}' \).

**Proof.** Let \( A \in \mathfrak{H} \). Then \( A \) is a strong limit of finite sums of the form

\[ \sum_{\sigma \in J} (\lambda_{\sigma} P_\sigma + \lambda_{\sigma} Q_\sigma)W_{\sigma} + \sum_\sigma \varrho_{\sigma} E_\sigma. \]

Hence \( P \in \mathfrak{H}' \) if we can show

\[ P_\sigma(\lambda P_\sigma + \lambda Q_\sigma)W_{\sigma} = (\lambda P_\sigma + \lambda Q_\sigma)W_{\sigma} P_\sigma. \]
for all \( \sigma, \varphi \in J \). But this is immediate from Lemma 3.7. Since the \( P_\sigma \) are all in \( \mathcal{A} \) (see Corollary 2.2), \( P \in \mathcal{A}' \). Thus \( P \) is central in \( \mathcal{A}' \).

**Theorem 3.9.** Let \( \mathcal{A} \) be an irreducible JW-algebra of type I, \( n \geq 3 \). Let \( \{E_\sigma\}_{\sigma \in J} \) be an orthogonal family of non-zero abelian projections in \( \mathcal{A} \) with \( \sum_{\sigma \in J} E_\sigma = I \). Let \( \mathcal{E}_{\sigma \varphi} = E_\sigma \mathcal{A} E_\varphi \) for \( \sigma \neq \varphi \). Then every operator in \( \mathcal{E}_{\sigma \varphi} \) is a scalar multiple of a partial isometry of \( E_\sigma \) onto \( E_\varphi \).

If \( W_{\sigma \varphi} \) is a partial isometry in \( \mathcal{E}_{\sigma \varphi} \) then one of three cases occurs.

1. \( \mathcal{E}_{\sigma \varphi} = RW_{\sigma \varphi} \) for all \( \sigma \neq \varphi \), and \( \dim E_\sigma = 1 \)
2. \( \mathcal{E}_{\sigma \varphi} = CW_{\sigma \varphi} \) for all \( \sigma \neq \varphi \), and \( \dim E_\sigma = 1 \)
3. \( \mathcal{E}_{\sigma \varphi} = QW_{\sigma \varphi} \) for all \( \sigma \neq \varphi \), and \( \dim E_\sigma = 2 \).

Moreover, \( \mathcal{A} \) is reversible.

*Proof.* Case (4) in Lemma 3.5 cannot occur. Indeed, if it did, let \( P = \sum_{\sigma \in J} P_\sigma \), \( Q = \sum_{\sigma \in J} Q_\sigma \). By Lemma 3.8 both \( P \) and \( Q \) are non-zero orthogonal projections in \( \mathcal{A}' \) contradicting the fact that \( \mathcal{A} \) is irreducible. By Lemma 3.2 every operator in \( \mathcal{E}_{\sigma \varphi} \) is a scalar multiple of a partial isometry of \( E_\sigma \) onto \( E_\varphi \). By Lemma 3.5 \( \mathcal{E}_{\sigma \varphi} \) is one of the three sets described above. By Corollary 2.2 \( \dim E_\sigma \) is also as described. Finally, by Lemma 3.1 \( \mathcal{A} \) is reversible. The proof is complete.

### 4. Abelian projections

One of the difficulties in the study of JW-algebras is due to the poor behaviour of cyclic projections. In this section and in section 8 we shall obtain useful results on such operators. Presently we shall find a formula connecting abelian and cyclic projections.

**Lemma 4.1.** Let \( \mathcal{A} \) be a JC-algebra with identity acting on a Hilbert space \( \mathcal{H} \). Let \( x \) be a vector in \( \mathcal{H} \) and assume \( \|\mathcal{A}x\| = 1 \). Let \( F \) be a projection in \( \mathcal{A} \) such that \( F \leq I - \mathcal{A}x \). Then \( F = 0 \).

*Proof.* By assumption the projections \( \mathcal{A}^n x \) converge strongly to \( I \), \( n = 1, 2, \ldots \). Also \( F[\mathcal{A}x] = 0 \). Use induction and assume \( F[\mathcal{A}^{n-1} x] = 0 \), \( n \geq 2 \). Let \( n \geq 2 \) and \( A_1, \ldots, A_n \in \mathcal{A} \). If \( n = 2 \) then

\[
FA_1 A_2 x = F(FA_1 A_2 + A_2 A_1 F)x \in F[\mathcal{A}x] = 0.
\]

If \( n \geq 3 \), then

\[
F \prod_{i=1}^{n} A_i x = F(FA_1 A_2 + A_2 A_1 F) \prod_{i=3}^{n} A_i x \in F[\mathcal{A}^{n-1} x] = 0.
\]

Thus \( F[\mathcal{A}^n x] = 0 \), \( n = 1, 2, \ldots \). Since \( [\mathcal{A}^n x] \to I \) strongly, \( 0 = F[\mathcal{A}^n x] \to F \) strongly. \( F = 0 \).

For reversible JC-algebras similar techniques give an inequality in the opposite direction.
LEMMA 4.2. Let $\mathcal{A}$ be a reversible JC-algebra. Let $E$ be a projection in $\mathcal{A}$ and $x$ a vector in $E$. Assume $[[\mathcal{A}]]x = I$. Then $I - E \leq [\mathcal{A}x]$.

Proof. Let $F = I - E$. Then $F[[\mathcal{A}x]] = [\mathcal{A}x]F$ for $n = 1, 2, \ldots$. Indeed, $Fx = 0$, so with $A_1, \ldots, A_n \in \mathcal{A}$,

$$F \left( \prod_{i=1}^{n} A_i x \right) = \left( F \prod_{i=1}^{n} A_i \right) \left( 1 + \prod_{i=1}^{n} A_i F \right) x \in [\mathcal{A}x] \leq [\mathcal{A}^n x],$$

and the assertion follows. Since $[\mathcal{A}^n x] \rightarrow I$ strongly, $[\mathcal{A}x]F = F[\mathcal{A}x] \rightarrow F$, strongly. Thus $[\mathcal{A}x]F = F$, the proof is complete.

LEMMA 4.3. Let $\mathcal{A}$ be a JC-algebra acting on a Hilbert space $\mathcal{H}$. Let $\omega_z$ be a pure vector state on $\mathcal{A}$. Let $F$ denote the set of vectors $z \in \mathcal{H}$ such that $\omega_z[[\mathcal{A}]] = \|z\|^2 \omega_z[\mathcal{A}]$. Then $F$ is a subspace of $\mathcal{H}$, and $F \leq [x] + I - [\mathcal{A}x]$.

Proof. Clearly $z \in F$, $\lambda \in \mathbb{C}$ implies $\lambda z \in F$. Let $w$ and $z$ be unit vectors in $F$. Let $A > 0$ be in $\mathcal{A}$, Then

$$\omega_{w+z}(A) = \omega_w(A) + \omega_z(A) + 2 \text{Re}(Aw, z)$$

$$= 2 (\omega_z(A) + \text{Re}(A^w, w^z))$$

$$\leq 2 (\omega_z(A) + \|A^w\| \|w^z\|)$$

$$= 4 \omega_z(A).$$

Since $\omega_z$ is pure and $\omega_{w+z} \leq 4 \omega_z$, $\omega_{w+z} = \|w+z\|^2 \omega_z$ on $\mathcal{A}$, $w+z \in F$. Thus $F$ is a linear manifold. It is clear that $F$ is closed, hence is a subspace. Now $x \in F$. Hence in order to show $F \leq [x] + I - [\mathcal{A}x]$ it suffices to consider a unit vector $y \in F - [x]$. Let $\lambda$ be a complex number of modulus 1. Then $x + \lambda y \in F$, so

$$\omega_{x+\lambda y} = \|x+\lambda y\|^2 \omega_z = (\|x\|^2 + \|y\|^2)\omega_z - 2 \omega_z \quad \text{on} \quad \mathcal{A}.$$}

Thus for all $A \in \mathcal{A}$,

$$2\omega_z(A) = \omega_{x+\lambda y}(A) = 2 \omega_y(A) + 2 \text{Re}(Ay, x),$$

so that $\text{Re}(Ay, x) = 0$ for all complex numbers $\lambda$ of modulus 1. Thus $0 = (Ay, x) = (y, Ax)$ for all $A \in \mathcal{A}$, i.e. $y \in I - [\mathcal{A}x]$. The proof is complete.

THEOREM 4.4. Let $\mathcal{A}$ be a JW-factor acting on a Hilbert space $\mathcal{H}$. Let $E$ be a projection in $\mathcal{A}$ and $x$ a unit vector in $E$. Assume $[[\mathcal{A}]]x = I$. Then $E$ is abelian if and only if

1. $E \leq [x] + I - [\mathcal{A}x]$. 

Moreover, if $\mathcal{A}$ is reversible then the inequality (1) is equality.
JORDAN ALGEBRAS OF TYPE I

177

Proof. Suppose $E$ is abelian. Then $E = EQ = RE$, hence $\omega_\xi$ is pure on $\mathfrak{A}$. Moreover, if $y$ is a unit vector in $E$ then $\omega_y[\mathfrak{A}] = \omega_\xi[\mathfrak{A}]$, so by Lemma 4.3, (1) follows.

Conversely, assume (1) holds. Let $E$ be a projection in $\mathfrak{A}$ with $E \leq F$. If $x \in G$ then $E - G = I - [\mathfrak{A}x]$, hence $E - G = 0$ by Lemma 4.1. If $x \in E - G$ then similar arguments give $G = 0$. Assume $Gx = 0$. Since $G \leq E \leq [x] + I - [\mathfrak{A}x]$, $Gx = \lambda x + y$ with $y \in I - [\mathfrak{A}x]$, $\lambda \neq 0$. Then $y = (G - \lambda I)x \in [\mathfrak{A}x]$, hence $y = 0$. Thus $G = \lambda x$. Since $\lambda \neq 0$, $x \in G$, $G = E$ by the above. Thus $E$ is a minimal projection in $\mathfrak{A}$, hence is abelian. If $\mathfrak{A}$ is reversible then Lemma 4.2 shows that (1) must be equality. The proof is complete.

5. JW-factors of type $I_n$, $n \geq 3$

In Lemma 3.5 we have practically classified all JW-factors of type $I_n$, $n \geq 3$. However, the description of case (4) is incomplete. The present section fills out this gap. For this we shall need an analogue of the result for von Neumann algebras, which states that a factor of type $I$ has a faithful normal representation as all bounded operators on a Hilbert space.

Theorem 5.1. Let $\mathfrak{A}$ be a JW-factor of type $I_n$, $n \geq 3$. Then there exists a representation of $(\mathfrak{A})$ which, when restricted to $\mathfrak{A}$, is a faithful normal representation as an irreducible JW-algebra of type $I_n$.

Proof. Let $E$ be an abelian projection in $\mathfrak{A}$. Let $x$ be a unit vector in $E$. Then $\omega_x$ is a pure state of $\mathfrak{A}$. Let $f$ be a pure state extension of $\omega_x$ to $(\mathfrak{A})$. Then $f = \omega_x \varphi$ with $\varphi$ an irreducible representation of $(\mathfrak{A})$. Thus $\varphi(\mathfrak{A})$ is a irreducible JC-algebra. Moreover, $y \in F = \varphi(E)$ with $F$ an abelian projection in $\varphi(\mathfrak{A})$, hence in $\varphi(\mathfrak{A})^{-\prime}$, which is thus of type $I$. By assumption $\mathfrak{A}$ is of type $I_n$, $n \geq 3$. Since all abelian projections in $\mathfrak{A}$ are equivalent [11, Corollary 26], $\varphi$ is faithful on $\mathfrak{A}$, hence $\varphi(\mathfrak{A})^{-\prime}$ is of type $I_m$, $m \geq n$. If $m$ is finite clearly $m = n$. Otherwise $m = \infty$ in which case $m = \infty$, hence $\varphi(\mathfrak{A})^{-\prime}$ is of type $I_n$. In particular $\varphi(\mathfrak{A})^{-\prime}$ is reversible (Theorem 3.9). We show $\varphi$ is ultra-weakly continuous on $\mathfrak{A}$. There are two cases.

Case 1. $\varphi(\mathfrak{A})^{-\prime}$ is determined by (1) or (2) in Theorem 3.9. Then $\dim F = 1$, so by Theorem 4.4 $[\varphi(\mathfrak{A})] = I$. Let $w = Ay + iBy$ with $A, B \in \varphi(\mathfrak{A})$. Let $S \geq 0$ in $\varphi(\mathfrak{A})$. Then

$$0 \leq (Sw, w)$$

$$= (SAy, Ay) + (SBy, By) - 2 \text{Im}(SBby, Ay)$$

$$\leq (ASAy, y) + (BSBy, y) + 2 \|SBBy\| \|SAy\|$$

$$= ((ASAy, y) + (BSBy, y))_{\frac{1}{2}}.$$

Thus, if $A = \varphi(A_1), B = \varphi(B_1), S = \varphi(S_1)$ then

$$0 \leq (Sw, w)$$

$$= (SAy, Ay) + (SBy, By) - 2 \text{Im}(SBby, Ay)$$

$$\leq (ASAy, y) + (BSBy, y) + 2 \|SBBy\| \|SAy\|$$

$$= ((ASAy, y) + (BSBy, y))_{\frac{1}{2}}.$$
Therefore, if $S_j \geq 0$ in the unit ball $\mathbb{B}_1$ of $\mathcal{A}$ and $S_j \to 0$ weakly, then $(\varphi(S_j), w) \to 0$. Let $z$ be any unit vector in $\mathcal{A}$. Suppose $S_j$ are operators as above. Let $\epsilon > 0$ be given. Choose a unit vector $w = Ay + iBy$ in $\mathcal{A}$, $A, B \in \varphi(\mathcal{A})$, such that $\|w - z\| < \epsilon/4$. By the preceding we can choose $j$ so large that $(\varphi(S_j)w, w) < \epsilon/2$. Then

$$0 < (\varphi(S_j)w, w) < (\varphi(S_j)z, z) + (\varphi(S_j)(z-w), w) + (\varphi(S_j)z, w) + (\varphi(S_j)(z-w), w) + \epsilon/2$$

$$< 2(\varphi(S_j)w, w) + ||z - w|| + \epsilon/2 < 2\epsilon/4 + \epsilon/2 = \epsilon.$$ 

Thus $S_j \to 0$ on the positive part of $\mathcal{A}$ implies $(\varphi(S_j), z) \to 0$ for all unit vectors $z$ in $\mathcal{A}$. As in [6, Remark 2.2.3] it follows that $\varphi$ is ultra-weakly continuous on $\mathcal{A}$.

Case 2. $\varphi(\mathcal{A})^-$ is determined by (3) in Theorem 3.9. Then $\dim F = 2$ so there exists a unit vector $z$ orthogonal to $y$ in $\mathcal{F}$. By Theorem 4.4 $[y] + [z] = F = [y] + [z] - (\varphi(\mathcal{A})y)$, hence $[\varphi(\mathcal{A})y] + [z] = I$. Therefore, every vector $w$ in $\mathcal{A}$ is of the form $w = u + \lambda z$ with $u \in [\varphi(\mathcal{A})y]$, $\lambda \in \mathbb{C}$. Exactly as in case 1 $\omega_\varphi$ is weakly continuous at 0 in $\mathcal{A}_1$. Let $S \geq 0$ be in $\varphi(\mathcal{A})$. Then

$$0 < (Sw, w) = (Su, u) + |\lambda|^2(Sz, z) + 2Re(Slz, u)$$

$$= (Su, u) + |\lambda|^2(Sy, y) + 2Re(Slz, u)$$

$$< (Su, u) + |\lambda|^2(Sy, y) + 2\|Sz\| + \|Slz\|$$

$$< (Su, u) + |\lambda|^2(\|Sy\| + \|Sz\|).$$

As in case 1 we conclude that $\varphi$ is ultra-weakly continuous on $\mathcal{A}$, i.e. $\varphi$ is weakly continuous on $\mathcal{A}$, which is weakly compact. Thus the unit ball in $\varphi(\mathcal{A})$ is weakly compact. As Topping has pointed out the Kaplansky density theorem holds for JC-algebras (see the proof in [2]). Since $\varphi(\mathcal{A})$ is strongly dense in $\varphi(\mathcal{A})^-$ and contains the unit ball in $\varphi(\mathcal{A})^-$ it must be equal to $\varphi(\mathcal{A})^-$, i.e. $\varphi(\mathcal{A})$ is a JW-algebra. The proof is complete.

**Theorem 5.2.** Let $\mathcal{A}$ be a JW-factor of type $I_n$, $n \geq 3$, acting on a Hilbert space $\mathcal{B}$. Let $\{E_\sigma\}_{\sigma \in I}$ be an orthogonal family of non zero abelian projections in $\mathcal{A}$ with $\sum_{\sigma \in I} E_\sigma = I$. For $\sigma \neq \varrho$ let $\varepsilon_{\sigma\varrho} = E_\sigma \mathcal{A} E_{\varrho}$. Let $W_{\sigma\sigma}$ be a partial isometry in $\varepsilon_{\sigma\sigma}$. Then one of the following four cases occurs:

1. $\varepsilon_{\sigma\sigma} = Rw_{\sigma\varrho}$ for all $\sigma \neq \varrho$.
2. $\varepsilon_{\sigma\sigma} = CW_{\sigma\varrho}$ for all $\sigma \neq \varrho$. 


(3) There exists a projection \( P' \in \mathcal{A} \) with central carrier \( I \) such that if \( \mathcal{A} \) is replaced by \( P' \mathcal{A} \) then \( \mathcal{E}_{\mathcal{A}P} = Q_{\mathcal{W}_{\mathcal{A}P}} \).

(4) There exist two non-zero projections \( P_\sigma \) and \( Q_\sigma \) with \( P_\sigma + Q_\sigma = E_\sigma \) such that \( \mathcal{E}_{\mathcal{A}P} = (\lambda P_\lambda + Q_\lambda) \mathcal{W}_{\mathcal{A}P} : \lambda \in \mathbb{C} \). In this case there exist a Hilbert space \( \mathcal{H} \), a normal \(*\)-isomorphism \( \psi_1 \), and a normal \(*\)-anti-isomorphism \( \psi_2 \) of \( \mathcal{B}(\mathcal{H}) \) into \( \mathcal{B}(\mathcal{H}) \) such that

\[
\psi_1(I)\psi_2(I) = 0,
\]

and such that \( \mathcal{A} \) is the image of the \( \mathcal{C}^* \)-isomorphism \( \psi_1 + \psi_2 \) of \( \mathcal{B}(\mathcal{H}) \) into \( \mathcal{B}(\mathcal{H}) \).

**Proof.** If the \( \mathcal{E}_{\mathcal{A}P} \) are determined by (1), (2), (3) in Lemma 3.5 then we have cases (1), (2), (3) above. Assume the \( \mathcal{E}_{\mathcal{A}P} \) are determined by (4) in Lemma 3.5. Let \( P = \sum_\sigma P_\sigma \), \( Q = \sum_\sigma Q_\sigma \), where the \( P_\sigma \) and \( Q_\sigma \) are as in Lemma 3.5. By Lemma 3.8 \( P \) and \( Q \) are central projections in \( \mathcal{A} \) with \( P + Q = I \). Let \( \varphi \) be the representation constructed in Theorem 5.1 of \( \mathcal{A} \). Then \( \varphi \) has an extension to an irreducible representation \( \tilde{\varphi} \) of \( \mathcal{A} \), hence \( \tilde{\varphi}(P) = 0 \) or \( \tilde{\varphi}(Q) = 0 \), say \( \tilde{\varphi}(Q) = 0 \). Then \( \varphi(Q_\sigma) = 0 \) for all \( \sigma \in J \). Consequently \( \varphi(\mathcal{E}_{\mathcal{A}P}) = \varphi(E_P) = \varphi(E_P) \varphi(\mathcal{A}) \varphi(E_P) \). Thus \( \varphi(\mathcal{A}) = \mathcal{B}(\mathcal{A}) \). Let \( \psi \) be the map \( \psi^{-1} : \mathcal{B}(\mathcal{A}) \rightarrow \mathcal{A} \). Then \( \psi \) is normal and has an extension to a normal \( \mathcal{C}^* \)-isomorphism of \( \mathcal{A} \) onto \( \mathcal{A} + \mathcal{A} \). By [4, Corollary to Theorem 7] (or by [10, Theorem 3.3]) \( \psi \) is the sum of a normal \(*\)-isomorphism \( \psi_1 \), and a normal \(*\)-anti-isomorphism \( \psi_2 \) of \( \mathcal{B}(\mathcal{A}) \) into \( \mathcal{B}(\mathcal{A}) \). Since \( \mathcal{A} = \psi(\mathcal{B}(\mathcal{A}) \mathcal{A}) \) the proof is complete.

6. Non-reversible JW-algebras

It turns out that a JW-algebra can be decomposed along its center into three parts, one part being the self-adjoint part of a von Neumann algebra, one part more like the JW-algebras given by (1), (3), and (4) in Theorem 5.2, and a third part, which is practically a global form of a spin factor.

**Lemma 6.1.** Let \( \mathcal{A} \) be a reversible JW-algebra. Then there exist central projections \( E \) and \( F \) in \( \mathcal{A} \) with \( E + F = I \) such that \( E \mathcal{A} \) is the self-adjoint part of a von Neumann algebra, and \( \mathcal{A}(\mathcal{F}) \cap \mathcal{A}(\mathcal{F}) = \{0\} \).

**Proof.** Let \( \mathcal{K} = \mathcal{A}(\mathcal{K}) \cap \mathcal{M}(\mathcal{K}) \). Then \( \mathcal{K} \) is an ideal in \( \mathcal{A} \) [10, Remark 2.2], hence its weak closure \( \mathcal{K}^- \) is an ideal in \( \mathcal{A}^- \). Thus there exists a central projection \( E \) in \( \mathcal{A}^- \) such that \( \mathcal{K}^- = E(\mathcal{A}) - [2, p. 45] \), and \( E \in \mathcal{K}^- \). Now \( \mathcal{K} \) is reversible, hence \( \mathcal{K}_{SA} \subset \mathcal{K} \), and \( (\mathcal{K}^-)_{SA} = (\mathcal{S}^-)_{SA} \subset \mathcal{K} \). Thus \( E \in \mathcal{K} \), and \( \mathcal{K}^- \subset \mathcal{K} \). Clearly \( \mathcal{K}^- \) is a von Neumann algebra. Let \( F = I - E \). Then \( F \) is central in \( \mathcal{A} \), and

\[
\mathcal{A}(\mathcal{F}) \cap \mathcal{A}(\mathcal{F}) = F(\mathcal{A}(\mathcal{F}) \cap \mathcal{A}(\mathcal{F})) = F \mathcal{A} \subset FE(\mathcal{A}) = \{0\}.
\]

The proof is complete.
Lemma 6.2. Let $\mathfrak{A}$ be a JC-algebra with identity $I$. Let $\mathfrak{J}$ denote the set of operators $A \in \mathfrak{A}$ such that $BAC + C^*AB^* \in \mathfrak{A}$ for all $B, C \in \mathfrak{A}(\mathfrak{A})$. Then $\mathfrak{J}$ is a uniformly closed Jordan ideal in $\mathfrak{A}$. Moreover, $\mathfrak{J}$ is a reversible JC-algebra.

Proof. Let $A, B \in \mathfrak{J}$, $S, T \in \mathfrak{A}(\mathfrak{A})$. Then

$$S(A + B) T + T^* (A + B) S^* = (SAT + T^* AS^*) + (SBT + T^* BS^*) \in \mathfrak{A},$$

so $\mathfrak{J}$ is linear. Let $A \in \mathfrak{J}$, $B \in \mathfrak{A}$, $S, T \in \mathfrak{A}(\mathfrak{A})$. Then

$$S(AB + BA) T + T^* (AB + BA) S^* = (SA(BT) + (BT)^* AS^*) + ((SB) A T + T^* A(SB)^*) \in \mathfrak{A},$$

so $\mathfrak{J}$ is a Jordan ideal in $\mathfrak{A}$. Since multiplication is uniformly continuous $\mathfrak{J}$ is uniformly closed. Let $A_1 \in \mathfrak{J}$, $A_2, \ldots, A_n \in \mathfrak{A}$. Let $A = \prod_{i=1}^n A_i$. Then $A_1 A + A^* A_1 \in \mathfrak{A}$ by definition of $\mathfrak{J}$. We show $A_1 A + A^* A_1 \in \mathfrak{J}$, hence $\mathfrak{J}$ is in particular reversible (with $A_2, \ldots, A_n \in \mathfrak{J}$). Let $B, C \in \mathfrak{A}(\mathfrak{A})$. Then

$$B(A_1 A + A^* A_1) C + C^* (A_1 A + A^* A_1) B^* = (B A_1 (AC) + (AC)^* A_1 B^*) + ((BA^*) A_1 C + C^* A_1 (BA^*)^*) \in \mathfrak{A}.$$

The proof is complete.

Definition 6.3. Let $\mathfrak{A}$ be a JC-algebra. We say $\mathfrak{A}$ is totally non reversible if the ideal $\mathfrak{J}$ in Lemma 6.2 is zero.

Theorem 6.4. Let $\mathfrak{A}$ be a JW-algebra. Then there exist three central projections $E, F, G$ in $\mathfrak{A}$ with $E + F + G = I$ such that

1. $E \mathfrak{A}$ is the self-adjoint part of a von Neumann algebra.
2. $F \mathfrak{A}$ is reversible and $\mathfrak{A}(F \mathfrak{A}) \cap \mathfrak{A}(F \mathfrak{A}) = \{0\}$.
3. $G \mathfrak{A}$ is totally non reversible.

Proof. Let $\mathfrak{J}$ be the ideal found in Lemma 6.2. $\mathfrak{J}$ is weakly closed. In fact, if $A_a \in \mathfrak{J}$, $A_a \rightarrow A$ weakly, then for all $S, T \in \mathfrak{A}(\mathfrak{A})$, $S A_a T + T^* A_a S^* \rightarrow SAT + T^* AS^*$ weakly. Since $\mathfrak{A}$ is weakly closed $SAT + T^* AS^* \in \mathfrak{A}$, $A \in \mathfrak{J}$. Let $H$ be the central projection in $\mathfrak{A}$ such that $H \mathfrak{A} = \mathfrak{J}$ (see [11]). Then $H \mathfrak{A}$ is reversible, and the existence of $E$ and $F$ follows from Lemma 6.1. Let $G = I - H$. We must show $G \mathfrak{A}$ is totally non reversible. Let $A \in G \mathfrak{A}$. If for all $B, C$ in $\mathfrak{A}(G \mathfrak{A}) = G \mathfrak{A}(\mathfrak{A})$, $BAC + C^* AB^* \in G \mathfrak{A}$, then, since $B = GS$, $C = GT$, $S, T \in \mathfrak{A}(\mathfrak{A})$, $BAC + C^* AB^* = G(SAT + T^* AS^*) \in G \mathfrak{A}$. But $A = GA$. Thus $SAT + T^* AS^* \in G \mathfrak{A} \subset \mathfrak{A}$ for all $S, T$ in $\mathfrak{A}(\mathfrak{A})$. But then $A \in \mathfrak{J} = H \mathfrak{A}$, $A = 0$. Thus $G \mathfrak{A}$ is totally non reversible. The proof is complete.

Corollary 6.5. A JW-factor is either reversible or totally non reversible.
Theorem 6.6. A totally non reversible JW-algebra is of type $I_\infty$.

Proof. From [11, Theorem 5] there exists a central projection $E$ in $\mathfrak{A}$—the JW-algebra in question—such that $\mathfrak{A}E$ is of type $I$ and $\mathfrak{A}(I-E)$ has no type $I$ portion. If $\mathfrak{A}(I-E)+0$ the "halving lemma" [11, Theorem 17] yields the existence of at least four orthogonal equivalent projections in $\mathfrak{A}(I-E)$ with sum $I$, hence $\mathfrak{A}(I-E)$ is reversible by Lemma 3.1, contrary to assumption. Thus $\mathfrak{A}$ is of type $I$. By [11, Theorems 15 and 16] there exists an orthogonal family $\{P_n\}$ of central projections in $\mathfrak{A}$ such that $P_n=0$ or $\mathfrak{A}P_n$ is of type $I_n$ for all cardinals $n$, and $\sum_n P_n=I$. However, if $n>3$ and $P_n+0$ then $\mathfrak{A}P_n$ is reversible by Lemma 3.1, contrary to assumption. If $P_1+0$ then $\mathfrak{A}P_1$ is abelian hence reversible. Thus $\mathfrak{A}$ is of type $I_\infty$, the proof is complete.

7. JW-factors of type $I_2$

Following [11] we define a spin system to be a set $\mathfrak{B}$ of symmetries $\pm I$ such that $TS+ST=0$ for $S, T \in \mathfrak{B}, S \neq T$. If $\mathfrak{B}$ is a spin system let $\mathfrak{S}$ denote the weak closure of the real linear space spanned by $\mathfrak{B}$. If a JW-factor can be written in the form $RI \otimes \mathfrak{A}$ with $\mathfrak{A}$ as above, it is said to be a spin factor.

Theorem 7.1. Let $\mathfrak{A}$ be a JW-factor. Then the following are equivalent.

1. $\mathfrak{A}$ is of type $I_2$.

2. $\mathfrak{A}$ is a spin factor.

If dim $\mathfrak{A}$ as a vector space over $\mathbb{R}$ is greater than $10(1)$ then the above conditions are equivalent to

3. $\mathfrak{A}$ is totally non reversible.

Proof. (3) $\Rightarrow$ (1). This follows from Theorem 6.6.

(1) $\Rightarrow$ (3). Assume dim $\mathfrak{A}>10$ and that (3) does not hold. Then $\mathfrak{A}$ is reversible (Corollary 6.5). Let $E_1$ and $E_2$ be non zero abelian projections in $\mathfrak{A}$ with $E_1+E_2=I$. Then dim $\mathfrak{A}=1+1+\dim \mathfrak{S}_{12}$, as a vector space over $\mathbb{R}$. Since $\mathfrak{A}$ is reversible it follows from Corollary 2.2 that $E_j(\mathfrak{A})E_j$ is isomorphic to $M_2$, $C$, or $C \otimes C$ hence $E_j(\mathfrak{A})E_j$ can be imbedded in $M_4(j-1,2)$. Hence $\mathfrak{S}_{12}$ can be imbedded in $M_6$, and dim $\mathfrak{A} \leq 1+1+\dim M_6=2+8=10$, contrary to assumption.

(2) $\Rightarrow$ (1). Let $\mathfrak{A}$ be a spin factor. Then $\mathfrak{A}=RI \otimes \mathfrak{S}$, $\mathfrak{S}$ as above. By [11, Corollary 29] every non zero operator in $\mathfrak{S}$ is a positive multiple of a symmetry. Thus every operator in $\mathfrak{A}$ is of the form $T=axI+\beta S$, $S$ a symmetry in $\mathfrak{S}$, $a, \beta \in \mathbb{R}$. Since $S=E-F$ with $E$ and $F$ projections in $\mathfrak{S}$ such that $E+F=I$, $T$ has at most two spectral projections. Thus $\mathfrak{A}$ is of type $I_\infty$.

(1) In fact it suffices to assume dim $\mathfrak{A}>6$, see e.g. [5].
(1) \implies (2). Let \( \mathcal{A} \) be of type \( I_\alpha \) and \( E \) and \( F \) orthogonal abelian projections in \( \mathcal{A} \) such that \( E + F = I \). Then every operator in \( \mathcal{A} \) is of the form \( A = \alpha E + \beta F + EAF + FAE \), \( \alpha, \beta \in \mathbb{R} \), and where \( FAE \) (resp. \( EAF \)) is a scalar multiple of a partial isometry of \( E \) onto \( F \) (resp. \( F \) onto \( E \)) (Lemma 3.2). Let \( x \) and \( y \) be vectors of norm \( 2^{-1} \) in \( E \) and \( F \) respectively. Let \( \mathcal{A}_E \) be the state \( \omega_\alpha + \omega_\beta \) on \( \mathcal{A} \). In view of Lemma 3.2 it is easy to show \( \mathcal{A}_E \) is a faithful trace of \( \mathcal{A} \) in the sense of [11]. Define an inner product on \( \mathcal{A} \) by \( \langle A, B \rangle = Tr(\langle A B + B A \rangle) \). Let \( \| \|_2 \) denote the corresponding norm on \( \mathcal{A} \). Then \( \mathcal{A} \) is a real pre-Hilbert space. We show \( \mathcal{A} \) is closed. In fact, it is straightforward to show \( \| \| \leq 2^{1/2} \| \|_2 \). If \( A_n \) is a Cauchy sequence in \( \mathcal{A} \) with respect to \( \| \|_2 \) then \( \| A_n - A_m \| \leq 2^{1/2} \| A_n - A_m \|_2 \to 0 \). Hence there exists \( A \in \mathcal{A} \) such that \( A_n \to A \) uniformly. Since \( Tr \) is uniformly continuous \( \| A_n - A \|_2 \to 0 \), \( \mathcal{A} \) is a real Hilbert space. Denote it by \( \mathcal{H} \). Now \( I \) and \( E - F \) are orthogonal unit vectors in \( \mathcal{H} \). Extend them to an orthonormal base \( (S_\lambda) \) for \( \mathcal{H} \). If \( A \in \mathcal{A} \) is orthogonal to \( I \) and \( E - F \) then \( EAE = FAF = 0 \). Thus, if \( S_\lambda \) is in the base and \( S_\lambda + I \) and \( E - F \), then \( S_\lambda - V_\lambda + V_\lambda^* \) with \( V_\lambda \) a partial isometry of \( F \) onto \( E \). Let \( S_\lambda \) and \( S_\mu \) be distinct elements in the base different from \( I \) and \( E - F \). Then

\[
S_\lambda S_\mu + S_\mu S_\lambda = (V_\lambda + V_\lambda^*) (V_\mu + V_\mu^*) + (V_\mu + V_\mu^*) (V_\lambda + V_\lambda^*)
= (V_\lambda V_\mu^* + V_\mu V_\lambda^*) + (V_\mu V_\lambda^* + V_\lambda V_\mu^*) V_\lambda
= \lambda E + \lambda F = \lambda I
\]

by Lemma 3.2. Since \( S_\lambda \) and \( S_\mu \) are orthogonal, \( 0 = Tr(S_\lambda S_\mu + S_\mu S_\lambda) = \lambda \). Thus \( S_\lambda S_\mu + S_\mu S_\lambda = 0 \). Let \( \mathcal{B} \) be the set of \( S_\lambda \) distinct from \( I \). Then \( \mathcal{B} \) is a spin system. If \( \mathcal{A} \) denotes the weakly closed linear space generated by \( \mathcal{B} \) then \( \mathcal{A} = \mathbb{R} I \oplus \mathcal{B} \), \( \mathcal{A} \) is a spin factor. The proof is complete.

8. Reversible JW-algebras

It would be easy by Theorem 5.2 to show that the von Neumann algebra generated by a JW-factor of type \( I_\alpha, n \geq 3 \), is itself of type \( I \). It is possible, however, to give a global version of this fact. For this some facts on central carriers will be needed. If \( \mathcal{A} \) is a JW-algebra or a von Neumann algebra the central carrier of a projection \( E \) in \( \mathcal{A} \) with respect to \( \mathcal{A} \) is the least central projection in \( \mathcal{A} \) greater than or equal to \( E \). It will be denoted by \( C_\mathcal{A}(E) \).

**Lemma 8.1.** Let \( \mathcal{A} \) be a JW-algebra and \( E \) a projection in \( \mathcal{A} \). Then

\[
C_\mathcal{A}(\mathcal{A}) - [\mathcal{A} E] = [\mathcal{A}' E] = C_\mathcal{A}(\mathcal{A}'').
\]
Proof. By [2, Corollaire 1, p. 7] $C_4(\mathfrak{H}) = [\mathfrak{H}^* \mathfrak{E}]$. Clearly $[\mathfrak{H} \mathfrak{E}] \subseteq [\mathfrak{H}^* \mathfrak{E}]$. Now $[\mathfrak{H} \mathfrak{E}] \in \mathfrak{H}$.

In fact, if $x \in \mathfrak{E}$, $A, B \in \mathfrak{H}$ then

$$BAx = (BAE + AEB)x - EABx \in [\mathfrak{H} \mathfrak{E}] \vee [\mathfrak{H} \mathfrak{E}].$$

Thus $B$ leaves $[\mathfrak{H} \mathfrak{E}]$ invariant, $[\mathfrak{H} \mathfrak{E}] \in \mathfrak{H}$. Moreover, $[\mathfrak{H} \mathfrak{E}] \in \mathfrak{H}$. In fact, if $A \in \mathfrak{H}$, and $r(B)$ denotes the range projection of an operator $B$, then $r(AE) = r(AE(AE)^*) = r(AEA) \in \mathfrak{H}$, by spectral theory and the fact that $\mathfrak{A}$ is weakly closed. Thus $[\mathfrak{H} \mathfrak{E}] = V_{AE} r(AE) \in \mathfrak{H}$, as asserted. Thus $[\mathfrak{H} \mathfrak{E}]$ belongs to the center of $\mathfrak{H}$, which in turn is contained in the center of $\mathfrak{H}^*$. Since $C_4(\mathfrak{H}) = [\mathfrak{H}^* \mathfrak{H}] \supseteq [\mathfrak{H}] \supseteq \mathfrak{E}$, $[\mathfrak{H} \mathfrak{E}] = C_4(\mathfrak{H})$. Since clearly $C_4(\mathfrak{H}^*) \supset C_4(\mathfrak{H})$ the proof is complete.

THEOREM 8.2. If $\mathfrak{H}$ is a reversible JW-algebra of type I then $\mathfrak{H}^*$ is a von Neumann algebra of type I.$^1$

Proof. There exists an abelian projection $E$ in $\mathfrak{H}$ with $C_4(\mathfrak{H}) = I$. Let $\varphi$ be an irreducible representation of $E(\mathfrak{H}) E$. Since $(\mathfrak{H})$ equals the uniform closure of $\mathfrak{H}(\mathfrak{H}) + \mathfrak{H}(\mathfrak{H})$, $\varphi$ is an irreducible representation of $E(\mathfrak{H}) E$. Since $(E(\mathfrak{H}) E)_{sa} = E(\mathfrak{H}) E$ is abelian, $\varphi(E(\mathfrak{H}) E)$ is isomorphic to either $R$, $C$, or $Q$, by Corollary 2.3. Thus $\varphi(E(\mathfrak{H}) E)$ is isomorphic to either $C$ or $M_2$, hence $E(\mathfrak{H}) E$ is a CCR-algebra (see [8]). By [9, Theorem 6] $E(\mathfrak{H})^* E = (E(\mathfrak{H}) E)^*$ is a von Neumann algebra of type $I$, hence $E(\mathfrak{H})^* E$ is of type $I$. Let $F$ be an abelian projection in $E(\mathfrak{H})^* E$ with $C_4(E(\mathfrak{H})^* E) = F$ [2, Théorème 1, p. 123]. Then $F$ is abelian in $\mathfrak{H}^*$ since $E(\mathfrak{H}) F = F(E(\mathfrak{H}) E)$. Let $P$ be a central projection in $\mathfrak{H}^*$ such that $P > E$. Then $PE > F$. But $PE$ belongs to the center of $E(\mathfrak{H})^* E$, hence $PE = E$, and $P > E$. But by Lemma 8.1 $C_4(\mathfrak{H}) - C_4(\mathfrak{H}) = I$. Thus $P = I$, $C_4(\mathfrak{H}) = I$, $\mathfrak{H}^*$ is of type $I$ [2, Théorème 1, p. 123]. The proof is complete.

References


$^1$ In the paper referred to in footnote $^1$, p. 165, we shall show the converse of this theorem.

Received March 3, 1965