ON CONVERGENCE AND GROWTH OF PARTIAL SUMS OF FOURIER SERIES

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1. Introduction

In the present paper we shall introduce a new method to estimate partial sums of Fourier series. This will give quite precise results and will in particular enable us to solve the long open problem concerning convergence a.e. for functions in L^2. We denote by $s_n(x)$ the nth partial sum of a function $f(x) \in L^1(-\pi, \pi)$ and have the following theorem.

THEOREM. (a) *If for some* $\delta > 0$

$$\int_{-\pi}^{\pi} |f(x)| (\log^+ |f(x)|)^{1+\delta} dx < \infty, \tag{1.1}$$

then

$$s_n(x) = o(\log \log n), \quad a.e.$$

(b) *If* $f(x) \in L^p$, $1 < p < 2$, *then*

$$s_n(x) = o(\log \log \log n), \quad a.e.$$

(c) *If* $f(x) \in L^2$, *then* $s_n(x)$ *converges a.e.*

Remarks. (a) This result should be compared with Kolmogorov's example of an a.e. divergent series in L^1. If we consider in detail the construction of Hardy–Rogosinski (see [1], pp. 306–308), we see that the following is actually true. Given $\varepsilon(n) \to 0$, $n \to \infty$, there is a function $f \in L^1$ such that

$$s_n(x) \neq O(\varepsilon(n) \log \log n), \quad a.e.$$

The best previous result in this case is $o(\log n)$.

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(b) The best previous result here is the Littlewood–Paley theorem (see [2], p. 166) $s_n(x)=o((\log n)^{1/p})$, a.e. It is rather obvious from the proof of (c) that we actually have convergence a.e. in this case also, and the proof of (b) will only be sketched.

(c) This result was conjectured by Lusin. The best earlier result is the Kolmogorov–Seliverstov–Plessner theorem $s_n(x)=o((\log n)^{\frac{1}{2}})$, a.e.

The proof is quite technical and it is convenient to give an outline of the idea behind the proof here.

We assume f real and extend f periodically. We then consider the modified Dirichlet formula

$$s_n^*(x)=\int_{-4\pi}^{4\pi}\frac{e^{-int}f(t)}{x-t}\,dt, \quad -\pi<x<\pi. \tag{1.2}$$

If ω is a subinterval of $(-4\pi, 4\pi)$, we let $E_\omega(f)$ denote the mean value of f over ω. We consider a suitable disjoint covering $\Omega=\{\omega_\nu\}$ of $(-4\pi,4\pi)$. If $x\in\omega_\nu=\omega^*(x)$, we write

$$s_n^*(x)=\int_{\omega^*(x)}\frac{e^{-int}f(t)}{x-t}\,dt+\sum_{\mu\neq\nu}\int_{\omega_\mu}\frac{E_{\omega_\mu}(e^{-int}f)}{x-t}\,dt + \sum_{\mu\neq\nu}\int_{\omega_\mu}\frac{e^{-int}f(t)-E_{\omega_\mu}(e^{-int}f)}{x-t}\,dt. \tag{1.3}$$

If x is "strictly" inside ω^*, the first term gives the main contribution. If ω^* has length $2\pi\cdot 2^{-s}$, s an integer, we modify in this term n to the closest integer of the form $h\cdot 2^s$. This gives only a small change in the value of the integral. After the modification and a change of variables $x=2^s\xi$, $t=2^s\tau$, we have an integral of the same form as $s_n^*(x)$ but localized to ω^*. We can now repeat the argument.

To prove that the second term is small, we choose Ω so that the mean values $E_{\omega_\mu}(e^{-int}f)$ are all small, specifically of the same magnitude as $\int_{-\pi}^{\pi}f\,e^{-int}\,dt$.

In the third term, finally, we use the fact that the numerator has a vanishing integral over each ω_μ. In this way we can change the first order singularity into a second order singularity, which is easier to handle (Lemma 5). The situation should be compared with the trivial formulas

$$\int_\delta^1\frac{dt}{t}=\log\frac{1}{\delta}; \qquad \delta\int_\delta^1\frac{dt}{t^2}<1.$$

For every combination of intervals $\omega^*(x)$ and integers n in formula (1.3) we get an exceptional set where the remainder terms are not small. In the proof of (a), which will first be given in Sections 2–5, we allow in a certain sense all combinations (n,ω^*).

The improvement that is needed to get (c) is a careful examination of which (n, ω^*)'s are necessary. The Parseval equation plays a fundamental role and to get the L^p-result, a sufficiently good substitute has to be found. In Section 12 we sketch how an interpolation argument gives (b). The author is indebted to L. Gårding, A. Garsia, L. Hörmander, J.-P. Kahane and I. Katznelson for many improvements in the presentation of the proof.

2. Some notations and lemmas

Let ω^*_{-1} be the interval $(-4\pi, 4\pi)$ and $\omega_{-10}, \omega_{00}, \omega_{10}, \omega_{20}$ be the intervals $(-4\pi, -2\pi), \ldots, (2\pi, 4\pi)$. We shall restrict x to $(-\pi, \pi)$ and ω_{00} and ω_{10} will be our basic intervals. We subdivide $\omega_{00} \cup \omega_{10}$ into $2 \cdot 2^\nu$ equal intervals of lengths $2\pi \cdot 2^{-\nu}$, $\nu = 1, 2, \ldots$. The resulting intervals are from left to right denoted $\omega_{j\nu}$, $j = -2^\nu + 1, \ldots, 2^\nu$.

We further define

$$\omega^*_{j\nu} = \omega_{j\nu} \cup \omega_{j+1,\nu}, \quad -2^\nu + 1 \leqslant j \leqslant 2^\nu - 1. \tag{2.1}$$

For the length of an interval ω we use the notation $|\omega|$.

Let f be a given real function with period 2π such that

$$\int_{-\pi}^{\pi} |f(x)| (\log^+ |f|)^{1+\delta} dx = 1. \tag{2.2}$$

$\delta > 0$ is fixed and Const. will indicate numbers only depending on δ. For each $\omega =$ some $\omega_{j\nu}$, $\nu \geqslant 0$, we define the Fourier transform

$$c_\alpha(\omega) = \frac{1}{|\omega|} \int_\omega f(x) \exp\{-2^\nu \alpha i x\}\, dx, \tag{2.3}$$

where α is any real number. Together with the Fourier coefficients $c_n(\omega)$, n an integer, we consider the non-negative numbers $C_n(\omega)$ defined by

$$C_n(\omega) = \sum_{\mu=-\infty}^{\infty} |c_{n+\mu/3}(\omega)| (1 + \mu^2)^{-1}. \tag{2.4}$$

We shall also set $p = (n, \omega)$ and use the notation $C(p)$.

It will be necessary to have some estimates of c_n and C_n, as $n \to \infty$, for functions of type (1.1). The following form of the Hausdorff–Young inequality is sufficient.

LEMMA 1. *If*

$$\frac{1}{|\omega|} \int_\omega |f(x)| (\log^+ |f|)^{1+\delta} dx \leqslant \lambda, \tag{2.5}$$

then there are numbers $a(\lambda)$ and $A(\lambda) > 0$ only depending on λ and δ such that

$$\sum_{-\infty}^{\infty} \exp\{-a\, C_n(\omega)^{-1/(1+\delta)}\} \leqslant A. \tag{2.6}$$

Proof. Except for the fact that no constants are mentioned, (2.6) is given in [2], p. 158, for $|c_n|$ instead of C_n. It then also follows for $c_{n\pm 1/3}$ if we consider $e^{\pm i\beta x} f(x)$, $\beta = \frac{1}{3} 2^\nu$. To obtain (2.6) from this result we observe that

$$C_n \leqslant \text{Const.} \sup_\mu \frac{1}{(1+|\mu|)^{\frac{1}{2}}} |c_{n+\mu/3}|.$$

This means that, $q = 1/(1+\delta)$,

$$\exp\{-b C_n^{-q}\} \leqslant \sum_\mu \exp\{-a|c_{n+\mu/3}|^{-q}(1+|\mu|)^{\frac{1}{2}q}\} \tag{2.7}$$

for some other constant $b>0$. Summing (2.7) with respect to n and observing that for $|c| \leqslant 2\lambda$

$$\sum_{\mu=-\infty}^{\infty} \exp\{-a|c|^{-q}(1+|\mu|)^{\frac{1}{2}q}\} \leqslant A_1(\lambda) \exp\{-a|c|^{-q}\},$$

we obtain (2.6).

With the intervals $\omega_{j\nu}^*$, defined by (2.1), we associate the analogous numbers

$$C_n^*(\omega_{j\nu}^*) = \max_{\omega'} C_n(\omega') \tag{2.8}$$

where ω' ranges over the four subintervals $\omega_{k,\nu+1}$ of $\omega_{j\nu}^*$. If $\omega_{j\nu}^* = \omega_{-1}^*$, C_n^* is simply $C_n(\omega_{00})$. We also use the notations $p^* = (n, \omega^*)$ and $C^*(p^*)$.

Finally, given an integer n and an interval $\omega = \omega_{j\nu}$, we expand n to the basis 2

$$n = \sum_{i \geqslant 0} \varepsilon_i 2^i, \quad \varepsilon_i = 0, 1,$$

and define

$$n[\omega] = 2^{-\nu} \sum_{i \geqslant \nu} \varepsilon_i 2^i. \tag{2.9}$$

If ω' is related to ω^* as in (2.8), we also write $n[\omega^*]$ for $n[\omega']$.

The reason for considering $C_n(\omega)$ together with $c_n(\omega)$ is seen from the following lemma.

LEMMA 2. *For any integer n and any $\omega = \omega_{j\nu}$ we have the inequality*

$$|c_{n\cdot 2^{-\nu}}(\omega)| \leqslant \text{Const.}\, C_{n[\omega]}(\omega).$$

It will be convenient for future reference to collect the method of proof of Lemma 2 in a special lemma.

LEMMA 3. *Let $\varphi(t) \in C^2(\omega)$, $|\omega| = 2\pi \cdot 2^{-\nu}$. Then we can represent $\varphi(t)$*

$$\varphi(t) = \sum \gamma_\mu \exp\{-i 2^\nu 3^{-1} \mu t\}, \quad t \in \omega, \tag{2.10}$$

where

$$(1+\mu^2)|\gamma_\mu| \leqslant \text{Const.} (\max_\omega |\varphi| + 2^{-2\nu} \max_\omega |\varphi''(t)|). \tag{2.11}$$

Proof of Lemma 3. By a change of variables, $t=2^{-\nu}\cdot\tau$, we see that we may assume $\omega=(0,2\pi)$. If we determine a polynomial with the same derivatives of order $\leqslant 2$ as $\varphi(t)$ at $t=0$ and vanishing with these derivatives at $t=-2\pi$, and do analogously for $(2\pi,4\pi)$, we realize thas we can extend φ to $(-2\pi,4\pi)$ and then periodically so that $\max|\varphi|+\max|\varphi''|$ increases at most by a fixed factor. By partial integrations we see that (2.11) holds for the expansion of this function.

Proof of Lemma 2. We apply Lemma 3 to the function $e^{-i\beta t}$, $\beta=n-n[\omega]\,2^{\nu}$, $\omega=\omega_{j\nu}$, so that $|\gamma_\mu|\leqslant \text{Const.}\,(1+\mu^2)^{-1}$. We find

$$c_{n2^{-\nu}}(\omega)=\frac{1}{|\omega|}\int_\omega e^{-int}f(t)\,dt=\frac{1}{|\omega|}\int_\omega e^{-i\beta t}\,e^{-in[\omega]2^{\nu}t}f(t)\,dt=\sum\gamma_\mu c_{n[\omega]+\mu/3}(\omega).$$

The estimate of γ_μ and the definition (2.4) prove the lemma.

The estimations of the remainder terms of (1.3) depend on the following two lemmas, the first of which is of well-known type.

LEMMA 4. *Let $E(t)$ be defined on an interval ω^* and assume $|E(t)|\leqslant c$. Let σ_x denote subintervals of ω^* containing x inside their middle halves. We define (as principal values) the maximal Hilbert transform*

$$H^*(x)=\sup_{\sigma_x}\left|\int_{\sigma_x}\frac{E(t)}{x-t}\,dt\right|.$$

Let T be the set $$T=\{x\,|\,H^*(x)>a,\ x\in\omega^*\}.$$

Then $$mT\leqslant \text{Const.}\exp\left(-\text{Const.}\frac{a}{c}\right)|\omega^*|.$$

Proof. By a change of scale we may assume $\omega^*=(0,1)$ and $c=1$. Let $u(z)$ be harmonic in $y>0$, with boundary values $=E(x)$ on $(0,1)$ and $=0$ otherwise. Let v be the conjugate of u, $v(\infty)=0$. It is easy to see that (see [1], p. 103)

$$H^*(x)\leqslant \text{Const.}\sup_{y>0}|v(x+iy)|+\text{Const.}$$

For $k<\pi/2$ we have (see [1], p. 254)

$$\int_{-1}^{2}e^{k|v(x+iy)|}\,dx<\text{Const.},$$

and applying the Hardy–Littlewood maximal theorem ([1], p. 155) to $\exp\{\pm\tfrac{1}{2}ki(u+iv)\}$, which is in L^2, Lemma 4 follows.

LEMMA 5. *Let* $\{\omega_k\}$ *be a disjoint covering of an interval* ω^* *and let* ω_k *have midpoint* t_k *and length* δ_k. *We define the function*

$$\Delta(x) = \sum_k \frac{\delta_k^2}{(x-t_k)^2+\delta_k^2}, \quad x \in \omega^*,$$

and the set

$$U = \{x \mid \Delta(x) > M, \ x \in \omega^*\}.$$

Then

$$mU \leqslant \text{Const.} \exp(-\text{Const.}\, M)\,|\omega^*|.$$

Proof. By a change of scale we may again assume $\omega^* = (0,1)$. Let $g(x) \geqslant 0$ be integrable and have its support in $(0,1)$ and let $g(x+iy)$, $y>0$, be the corresponding harmonic function in $y>0$. Then

$$\int_{-\infty}^{\infty} \Delta(x)\, g(x)\, dx = \pi \sum_k \delta_k\, g(t_k + i\delta_k) \leqslant \text{Const.} \int_0^1 \sup_{y>0} g(x+iy)\, dx,$$

since by Harnack's inequality $\delta_k g(t_k + i\delta_k) \leqslant 3 \int_{\omega_k} g(t+i\delta_k)\, dt$. By the theorem on maximal functions, [1], p. 155, this last integral is bounded if

$$\int_0^1 g \log^+ g\, dx \leqslant 2. \tag{2.12}$$

Let μ be the measure of the set where $\Delta(x) > M$, $x \in (0,1)$, and define

$$g(x) = \left(\mu \log \frac{1}{\mu}\right)^{-1}$$

on this set and 0 otherwise. This function satisfies (2.12) and hence

$$\frac{M}{\log \dfrac{1}{\mu}} \leqslant \text{Const.},$$

which proves Lemma 5.

3. Construction of the exceptional set

We start from a large number λ and an integer N. Depending on λ, we shall determine a number λ_1 and a set $E_N(\lambda, \lambda_1)$ such that outside E_N the partial sums $s_n(x)$, $n \leqslant 2^N$, satisfy $|s_n(x)| \leqslant \text{Const.}\, \lambda_1 \lambda \log N$. In this section we shall construct E_N and in later sections show that E_N has the desired properties. E_N will consist of four different parts S, T, U, V, which will be constructed in the steps (A)–(E) below.

(A) We first consider the set of all $\omega=\omega_{j\nu}\subset(-2\pi,2\pi)$ such that

$$\int_{\omega}|f(x)|\,(\log^+|f|)^{1+\delta}\,dx>\lambda\,|\omega|. \tag{3.1}$$

We then define S to be the union of all intervals

$$S:\ \omega_{j+k,\nu},\ k=0,\ \pm1,\ \pm2,\ \pm3,\quad \omega_{j\nu}\ \text{satisfies (3.1)}.$$

Selecting successively suitable intervals $\omega_{j\nu}$ for $\nu=0,1,2,\ldots$, it is easy to see that

$$mS\leqslant 7\lambda^{-1}\int_{-2\pi}^{2\pi}|f|\,(\log^+|f|)^{1+\delta}\,dx\leqslant\frac{14}{\lambda}. \tag{3.2}$$

We observe certain properties of the set S.

(Aα) S does not depend on N.

(Aβ) If (3.1) does not hold, then $(\lambda>e)$ $|c_n(\omega)|<2\lambda$ and

$$C_n(\omega)\leqslant 2\lambda\sum_{\mu=-\infty}^{\infty}\frac{1}{1+\mu^2}<10\lambda. \tag{3.3}$$

(Aγ) Given (j_0,ν_0), $-1\leqslant\nu_0$, recall the definition of $\omega^*=\omega^*_{j_0\nu_0}$ in (2.1) and of $C^*_n(\omega^*)$ in (2.8). If $\omega_{j_0,\nu_0}\not\subset S$, (3.1) holds also for the neighboring intervals with $\nu=\nu_0+1$ and it follows from (3.3) that

$$C^*_n(\omega^*)<10\lambda.$$

(B) Again, given $\omega^*=\omega^*_{j_0,\nu_0}$, $-1\leqslant\nu_0\leqslant N-1$, and an arbitrary, nonnegative integer n, we shall define a partition $\Omega_n(\omega^*)$ of ω^*. To simplify our notations we assume $\omega^*=(-4\pi,4\pi)$, but the general case consists simply in a change of scale.

An interval $\omega=\omega_{j\nu}$, $\nu_0+1=0\leqslant\nu\leqslant N-1$, belongs to $\Omega_n(\omega^*)$ if

(Bα) $C_{n[\omega]}(\omega)\leqslant 2C^*_n(\omega^*)$;

(Bβ) the condition (Bα) holds with $\omega_{j\nu}$ replaced by any $\omega_{k\mu}$, $\omega_{j\nu}\subset\omega_{k\mu}\subset\omega^*$, but not for a certain $\omega_{l,\nu+1}\subset\omega_{j\nu}$.

(Bβ′) If $\omega_{j\nu}\subset\omega_{k\mu}\subset\omega^*$, $k<j$, then (Bα) and (Bβ) do not hold for $\omega_{k\mu}$.

(Bγ) $\Omega_n(\omega^*)$ contains all ω_{jN} not included in the $\omega_{j\nu}$ defined by B(α, β, β′).

Loosely speaking, the definition means that we subdivide ω^* into as small intervals as possible so that (Bα) holds and $\nu\leqslant N$.

(C) As a preparation for the construction of the sets T and U, we shall make a careful definition of the interval $\omega^*(x)$ in (1.3) so that x is "strictly" inside $\omega^*(x)$.

Let x belong to the middle half of ω^* and consider the set of intervals $\tilde{\omega}^*$ which are obtained by taking every $\omega_{j\nu}\in\Omega_n(\omega^*)$ and adjoining $\omega_{j-1,\nu}$ or $\omega_{j+1,\nu}$. Among these intervals $\tilde{\omega}^*$ there are those which contain x in their middle half. We then define $\omega^*(x)$, corresponding to $\Omega_n(\omega^*)$ and the point x as such an interval $\tilde{\omega}^*$ for which $|\tilde{\omega}^*|$ is as large as possible. We observe that

$$|\omega^*(x)| \leq \tfrac{1}{2}|\omega^*|. \tag{3.4}$$

Furthermore, $\omega^*(x)$ has the following properties:

(Cα) x belongs to the middle half of $\omega^*(x)$.

(Cβ) $\omega^*(x)$ is a union of intervals $\omega_{j\nu} \in \Omega_n(\omega^*)$ since $|\tilde{\omega}^*|$ was assumed maximal.

(Cγ) If $\omega^*(x) = \omega_{j\nu} \cup \omega_{j\pm 1,\nu}$, $\omega = \omega_{j\nu} \in \Omega_n(\omega^*)$, it follows from (Bα), (Bβ) and (Cβ) that

$$C_{n[\omega]}(\omega_{j\nu}), \qquad C_{n[\omega]}(\omega_{j\pm 1,\nu}) \leq 2\, C_n^*(\omega^*).$$

(Cδ) The complement of $\omega^*(x)$ with respect to ω^* is by (Cβ) the union of certain intervals in $\Omega_n(\omega^*)$. For each such interval σ, the distance from x to σ exceeds half the length of σ.

We now define
$$H_n(x) = \int_{\omega^* - \omega^*(x)} \frac{E_n(t)}{x-t}\, dt,$$

where (note $\nu_0 = -1$)

$$E_n(t) = \frac{1}{|\omega|} \int_\omega f(x)\, e^{-inx}\, dx, \quad t \in \omega \in \Omega_n(\omega^*). \tag{3.5}$$

By the construction of $\Omega_n(\omega^*)$ and Lemma 2, $|E_n(t)| \leq \text{Const.}\, C_n^*(\omega^*)$. As in Lemma 4, we define the maximal transform $H_n^*(x)$ of $E_n(t)$ and the set T_n of points $x \in \omega^*$ such that

$$T_n:\ H_n^*(x) > \lambda_1\, C_n^*(\omega^*)^{\varrho} \log N,$$

where ϱ is a number, $0 < \varrho < \delta/(1+\delta)$, fixed from here on. By Lemma 4,

$$mT_n \leq \text{Const.} \exp\{-\text{Const.}\, \lambda_1\, C_n^*(\omega^*)^{-1+\varrho} \log N\}\, |\omega^*|.$$

We also observe that $|H_n(x)| \leq 2H_n^*(x)$.

(D) With the same definition of $\omega^*(x)$ we now set

$$R_n(x) = R(x) = \int_{\omega^* - \omega^*(x)} \frac{e^{-int} f(t) - E_n(t)}{x-t}\, dt, \tag{3.6}$$

where again we have normalized the situation to $\omega^* = (-4\pi, 4\pi)$. If ω_k denote the intervals in $\Omega_n(\omega^*)$, $\omega^* - \omega^*(x)$ is by (Cδ) the union of a certain subset (x) of the ω_k's.

Denote by δ_k the lengths and by t_k the midpoints of the ω_k's and define $\Delta(x)$ as in Lemma 5. We rewrite the formula (3.6), using the fact that the numerator has vanishing integral over each ω_k,

$$R(x)=\sum_{(x)}\int_{\omega_k}\frac{t-t_k}{(x-t)(x-t_k)}e^{-int}f(t)\,dt-\sum_{(x)}\int_{\omega_k}\frac{t-t_k}{(x-t)(x-t_k)}E_n(t)\,dt. \tag{3.7}$$

Using (Cδ) and $|E_n(t)|\leqslant$ Const. $C_n^*(\omega^*)$, we see that the last sum is dominated by Const. $C_n^*(\omega^*)\,\Delta(x)$.

We shall now prove that also the first sum of (3.7) has this bound and shall use Lemma 3. We write, if $t\in\omega=\omega_k$, $|\omega|=2\pi\cdot 2^{-\nu}$,

$$e^{-int}\frac{t-t_k}{(x-t)(x-t_k)}=e^{-i2^\nu n[\omega]t}\varphi(t).$$

By Lemma 3, $\varphi(t)$ can be written

$$\varphi(t)=\sum_\mu \gamma_\mu \exp\{-2^\nu 3^{-1} i\mu t\},\quad t\in\omega, \tag{3.8}$$

where, again observing (Cδ),

$$|\gamma_\mu|\leqslant\frac{\text{Const.}\ \delta_k}{(1+\mu^2)\,[(x-t_k)^2+\delta_k^2]}.$$

We multiply (3.8) by $f(t)$ and integrate over ω_k. If we observe (Bα), we obtain after summation over *all* k the desired bound so that

$$|R(x)|\leqslant \text{Const.}\ C_n^*(\omega^*)\,\Delta(x). \tag{3.9}$$

Let us now define $U_n(\omega^*,\lambda_1,N)$ as the set where

$$U_n\colon\ \Delta(x)>\lambda_1 C_n^*(\omega^*)^{\varrho-1}\log N.$$

It follows from (3.9) and Lemma 5 that

$$mU_n\leqslant \text{Const.}\ |\omega^*|\exp\{-\text{Const.}\ \lambda_1 C_n^*(\omega^*)^{-1+\varrho}\log N\}. \tag{3.10}$$

Observe that outside $T_n\cup U_n$, $|H_n(x)|$ and $|R_n(x)|$ are less than Const. $\lambda_1 C_n^*(\omega^*)^\varrho\log N$.

(E) Finally, let V, not depending on N, be the set, where

$$V\colon\ \sup_{\sigma_x}\left|\int_{\sigma_x}\frac{f(t)}{x-t}\,dt\right|>\lambda_1,$$

σ_x being defined as in Lemma 4. As is well known $mV \leqslant \eta(\lambda_1)$, where η does not depend on f and $\eta \to 0$, $\lambda_1 \to +\infty$ (see [1], p. 279).

These definitions made, the exceptional set $E_N(\lambda, \lambda_1)$ is defined by

$$E_N = (S \cup V) \cup \bigcup_{n,\,\omega^*} (T_n(\omega^*) \cup U_n(\omega^*)), \tag{3.11}$$

where ω^* runs over all intervals $\omega^*_{j\nu} \not\subset S$ for which $-1 \leqslant \nu \leqslant N-1$.

4. Estimate of the exceptional set

We shall here estimate the measure of E_N and must now determine the relation between the numbers λ and λ_1.

Let us recall that by (Aγ) the numbers $C_n^*(\omega^*)$ associated with (3.11) are $< 10\lambda$. Furthermore, by (2.8) $C_n^*(\omega^*) = C_n(\omega')$ for a certain subintervval ω' of ω^*. For a fixed ω', such a relation holds for at most two ω^*'s and $|\omega^*| = 4|\omega'|$. It therefore follows from (2.6) and the definition of S in (A) that

$$\sum_{(s)} \exp\{-a\, C_n^*(\omega^*)^{-q}\}\, 2\pi\, 2^{-s} \leqslant A(\lambda), \quad q = \frac{1}{1+\delta}, \tag{4.1}$$

where (s) indicates that the summation runs over all pairs (n, ω^*) used in the definition of E_N for which $|\omega^*| = 2\pi\, 2^{-s}$, $-2 \leqslant s \leqslant N-2$. Summing over s we get

$$\sum_{n,\,\omega^*} \exp\{-a\, C_n^*(\omega^*)^{-q}\}\, |\omega^*| \leqslant \text{Const.}\, A(\lambda)\, N. \tag{4.2}$$

In (4.2) we now only consider the set Q_0 of those pairs (n, ω^*) for which

$$Q_0\colon\ 1 \leqslant C_n^*(\omega^*) \quad (< 10\lambda).$$

It follows that

$$\sum_{Q_0} |\omega^*| \leqslant \text{Const.}\, A(\lambda)\, N. \tag{4.3}$$

Disregarding the set $(S \cup V)$, it follows from the estimates of mT_n and mU_n above that only at most the fraction

$$\text{Const.} \exp\{-\text{Const.}\, \lambda_1 (10\lambda)^{-1} \log N\}$$

of each interval ω^* that corresponds to Q_0 can belong to E_N. Hence choosing λ_1 sufficiently large as a function of λ, we can make this fraction $< \text{Const.}\, N^{-3}$. (4.3) then implies that the corresponding part of E_N has measure $< \text{Const.}\, A(\lambda)\, N^{-2}$.

Similarly, let Q_{i+1} be the set of (n, ω^*) for which

$$2^{-i-1} \leqslant C_n^*(\omega^*) < 2^{-i}, \quad i = 0, 1, 2, \ldots.$$

As above, $$\sum_{Q_i+1} |\omega^*| \leqslant \text{Const.}\, A(\lambda) \exp\{2a\, 2^{qi}\}\, N. \tag{4.4}$$

The corresponding fractions are at most

$$\text{Const.} \exp\{-\text{Const.}\, \lambda_1 2^{(1-\varrho)i} \log N\}.$$

We observe that $1-\varrho > q$. Hence it follows that if we take the above fraction of each term in (4.4) and sum over i, the result will for $\lambda_1 \geqslant \lambda_1(\lambda)$ be less than N^{-2}. This implies

$$mE_N \leqslant m(S \cup V) + \text{Const.}\, A(\lambda)\, N^{-2}.$$

Hence, since S and V are independent of N, we obtain

$$m\left(\bigcup_{N=N_0}^{\infty} E_N\right) \leqslant \varepsilon(\lambda, \lambda_1) + \text{Const.}\, A(\lambda)\, N_0^{-1}, \tag{4.5}$$

so that the measure of $E = \bigcup_{N_0}^{\infty} E_N$ is less than a prescribed ε if λ, λ_1 and N_0 are larger than certain bounds, which only depend on ε and δ.

5. Proof of Theorem (a)

As in the introduction, we assume f real with the integral (1.1) equal 1 and extend f periodically. We first compare the Dirichlet formula for the nth partial sum,

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)(x-t)}{\sin \dfrac{x-t}{2}} f(t)\, dt,$$

with $$t_n(x) = \frac{1}{2\pi i}\left(e^{inx} s_n^*(x) - e^{-inx} s_{-n}^*(x)\right),$$

where $s_n^*(x)$ is defined by (1.2). It is very easy to see that uniformly for $|x| < \pi$

$$|s_n(x)| \leqslant \text{Const.}\, |t_n(x)| + o(1), \quad n \to \infty. \tag{5.1}$$

We are going to prove that outside the set E of measure $< \varepsilon$ constructed in Section 4

$$|s_n^*(x)| \leqslant \text{Const.}\, \lambda_1 \lambda \log\log n, \quad n \geqslant 2^{N_0}. \tag{5.2}$$

By (5.1) this implies the same relation for $s_n(x)$. To obtain the $o(\log\log n)$ of the theorem, we simply consider $f(x) - T(x)$ which has a small integral (1.1), if the trigonometric polynomial T is suitable, and observe that the bounds on λ, λ_1 and N_0 only depend on δ and ε.

Let us now consider $0 < n < 2^N$, $N \geqslant N_0$, and a fixed x outside E_N. If we can prove $|s_n^*(x)| \leqslant \text{Const.}\, \lambda_1 \lambda \log N$, the proof is complete. In formula (1.3) we choose as Ω the covering $\Omega_n(-4\pi, 4\pi)$ defined by (Bα)–(Bγ) in Section 3. The interval $\omega^*(x)$ is defined in (C). Using the previous notations, (1.3) becomes

$$s_n^*(x) = \int_{\omega^*(x)} \frac{e^{-int} f(t)}{x-t}\, dt + H_n(x) + R_n(x). \tag{5.3}$$

Since $x \notin E_N$, the bounds $\text{Const.}\, \lambda_1 C_n^*(-4\pi, 4\pi)^\varrho \log N$ are valid for the two remainder terms.

In the integral (5.3) we are now going to replace n by the number $n_1 = n[\omega']\, 2^{\nu+1}$ if $\omega^*(x) = \omega_{j\nu}^*$ and $|\omega'| = 2\pi\, 2^{-\nu-1}$. Multiplying the integral in (5.3) by e^{imx}, $m = n - n_1$, and subtracting the integral with n replaced by n_1, we obtain the difference

$$\int_{\omega^*(x)} e^{-in_1 t} \frac{e^{im(x-t)} - 1}{x-t} f(t)\, dt. \tag{5.4}$$

If we observe that $C_{n[\omega]}(\omega) \leqslant 2 C_n^*(-4\pi, 4\pi)$ for the two intervals constituting $\omega^*(x)$, we see that we can estimate (5.4) by Lemma 3 applied to the function $u^{-1}(e^{imu} - 1)$. We find $\gamma_\mu = O(2^\nu \mu^{-2})$ since $|m| < 2^{\nu+1}$, and for (5.4) we obtain the estimate

$$\text{Const.}\, C_n^*(-4\pi, 4\pi). \tag{5.5}$$

This is smaller than the previous bound if $N \geqslant N_1(\lambda)$ and can be included in that term.

The relation (5.3) now becomes

$$s_n^*(x) = e^{i\theta_0} \int_{\omega^*} \frac{e^{-i2^{\nu+1} n[\omega'] t} f(t)}{x-t}\, dt + O(\lambda_1 \log N \cdot \alpha_0),$$

where $\omega^* = \omega^*(x) = \omega_{j\nu}^*$ for a certain $\omega_{j\nu}$ and where

$$\alpha_0 = C_n^*(-4\pi, 4\pi)^\varrho. \tag{5.6}$$

$O(\cdot)$ has the same dependence as the notion Const. Here, the integral over ω^* is of exactly the same type as the original integral with the fundamental difference that

$$\alpha_1 = C_{n[\omega']}^*(\omega^*)^\varrho > 2^\varrho \alpha_0$$

by the property (Bβ) of Ω_n and the definition (2.8) of C_m^*.

For the new interval ω^* we construct the covering $\Omega_{n[\omega']}(\omega^*)$, get a similar remainder term since x is so chosen that the estimates for the remainder terms are valid for *all* ω^* that come into account. We also get a new number $\alpha_2 \geqslant 2^\varrho \alpha_1$. The process cannot stop until we reach an interval $\omega^*(x) = I$ of length $2\pi \cdot 2^{-N+1}$, in which case $n[\omega'] = 0$ since $n < 2^N$. Hence

$$s_n^*(x) = e^{i\theta} \int_I \frac{f(t)}{x-t}\, dt + \sum_{i=0}^{r} O(\lambda_1 \log N \cdot \alpha_i),$$

where
$$\frac{\alpha_{i+1}}{\alpha_i} > 2^\varrho \qquad \text{and} \qquad \alpha_r < 10\lambda. \tag{5.7}$$

The first integral is bounded by Const. λ_1 since $x \notin V$ and since x belongs to the middle half of I. Since α_i grow exponentially by (5.7) and are bounded above, we obtain the desired bound $O(\lambda_1 \lambda \log N)$, and the proof is complete.

6. Summary of proof of (c)

The result for L^2 is proved by analyzing carefully the weak point in the preceding proof. This is the fact that we have allowed all combinations $p^* = (n, \omega^*)$ when we estimate the size of the exceptional set in (4.2). The factor N that is introduced in this way must then be compensated by the factor $\log N$ in the exponent. However, in the proof itself only certain special combinations $(n, \omega^*(x))$ will occur. Furthermore, in changing n to the closest n_1, we obtain the very small error $C^*(n; (-4\pi, 4\pi))$. Obviously, we can allow a larger gap between n and n_1, i.e., restrict the choice of the new combination $(n_1, \omega^*(x))$. The basis for the construction of those pairs $p^* = (n, \omega^*)$ that may be used during the proof is certain trigonometric polynomials $P_k(x, \omega)$. They will be constructed in the next section. It will also be convenient to modify the definition of the coverings Ω, and we must add new exceptional points. This is done in Sections 7, 8 and 9, and the proof of (c) is completed in Sections 10 and 11.

7. The polynomials $P_k(x, \omega)$

Denote by b_k the numbers 2^{-2^k}, $k = 0, 1, \ldots$. Let $f(x)$ be real and periodic and assume

$$\varepsilon^2 = \int_{-\pi}^{\pi} |f(x)|^2\, dx \leqslant 1. \tag{7.1}$$

Denote by $a_{\mu 0}$ those integral Fourier coefficients c_n, $n = \lambda_{\mu 0}$, of f over ω_{j0} which are of absolute value $\geqslant b_k$. We define

$$P_k(x; \omega_{j0}) = \sum_{\mu} a_{\mu 0}\, e^{i\lambda_{\mu 0} x}, \quad j = -1, 0, 1, 2.$$

On each of the two intervals ω_{l1} constituting ω_{j0}, we consider similarly those Fourier coefficients (now corresponding to even integers) of $f(x) - P_k(x; \omega_{j0})$ which are of modulus $\geqslant b_k$. They are denoted $a_{\mu 1}$, corresponding to $\lambda_{\mu 1}$, and will be called primitive for ω_{l1}. We define

$$P_k(x; \omega_{l1}) = P_k(x; \omega_{j0}) + \sum a_{\mu 1} e^{i\lambda_{\mu 1} x}.$$

It is clear how the construction proceeds, and we obtain the polynomials

$$P_k(x; \omega) = \sum_{(\omega)} a_\omega e^{i\lambda_\omega x}, \tag{7.2}$$

where the summation runs over a certain set (ω) of pairs $(a_\omega, \lambda_\omega)$. By the Parseval relation we have, summed over all ω, $|\omega| = 2\pi 2^{-\nu}$,

$$\sum_\omega \int_\omega |f(x) - P_k(x; \omega)|^2 \, dx + \sum_\omega \sum_{(\omega)} |a_\omega|^2 |\omega| = \int_{-4\pi}^{4\pi} |f(x)|^2 \, dx.$$

Hence if we define by an infinite sum where (P): a_ω primitive for ω, holds

$$A_k(x) = \sum_{x \in \omega} |a_\omega|^2,$$

it follows that
$$\sum_{(P)} |a_\omega|^2 |\omega| = \int_{-4\pi}^{4\pi} A_k(x) \, dx \leqslant 4\varepsilon^2 \leqslant 4. \tag{7.3}$$

Denote by X_k the set
$$X_k = \{x \mid A_k(x) > b_k^{-1}\}$$

so that $mX_k \leqslant 4b_k$. To every $\omega \subset X_k$ we associate its three right and left neighbors of equal length and define X_k^* as the union of all such intervals so that

$$mX_k^* \leqslant 7\, mX_k \leqslant 28\, b_k. \tag{7.4}$$

If $\omega \not\subset X_k$, then $P_k(x; \omega)$ has at most b_k^{-3} terms and

$$|P_k(x; \omega)| \leqslant \sum_{(\omega)} |a_\omega| \leqslant b_k^{-2}. \tag{7.5}$$

In analogy with Section 3 we shall also consider the set S of points x included in an interval ω such that

$$\int_\omega |f(x)|^2 \, dx > \varepsilon |\omega|. \tag{7.6}$$

Such ω's together with their six neighbors are denoted by S^*. Then

$$mS^* \leqslant 28\, \varepsilon. \tag{7.7}$$

Denote by M_k the set of $\omega \not\subset (X_k \cup S)$.

8. Allowed pairs p^*

We first consider the set F_k of pairs $p=(n,\omega)$, $\omega\in M_k$, for which $P_k(x;\omega)$ contains a primitive term $ae^{i\lambda x}$, $\lambda[\omega]=n$. By (7.3)

$$\sum_{F_k}|\omega|\leqslant 4b_k^{-2}. \tag{8.1}$$

We are now going to define in ($\tilde{\text{F}}$a) and ($\tilde{\text{F}}$b) below a larger set $\tilde{F}_k$ by associating with each $p\in F_k$ a number of elements $\tilde{p}$. Let P defined in (7.2) be the k-polynomial corresponding to ω.

($\tilde{\text{F}}$a) If $p=(n,\omega)\in F_k$, then $\tilde{F}_k$ contains every $\tilde{p}=(\tilde{n},\tilde{\omega})$ such that $\tilde{\omega}\in M_k$ and

$$\tilde{\omega}\subset\omega,\quad |\tilde{\omega}|\geqslant b_k^{10}|\omega|,\quad |\tilde{n}-\lambda_\omega[\tilde{\omega}]|\leqslant b_k^{-10}, \tag{8.2}$$

where λ_ω is an arbitrary exponent in $P(x;\omega)$.

The condition means that we include all neighbors within b_k^{-10} of all exponents in P not only on ω but also on sufficiently large subintervals $\tilde{\omega}$ of ω.

The number of possibilities for $\tilde{n}$ is $\leqslant \text{Const.}\, b_k^{-3}b_k^{-10}\log b_k^{-1}$.

($\tilde{\text{F}}$b) Again let $p=(n,\omega)\in F_k$ and consider $\tilde{p}=(\tilde{n},\tilde{\omega})$, $\tilde{\omega}\in M_k$, $\tilde{\omega}\subset\omega$, and the polynomial $P(x,\omega)$. Then, by definition, $\tilde{p}\in\tilde{F}_k$ if there are two different exponents λ_ω and λ'_ω in P such that

$$b_k^{10}\leqslant|\lambda_\omega-\lambda'_\omega|\,|\tilde{\omega}|\leqslant b_k^{-10}\quad\text{and}\quad|\tilde{n}-\lambda_\omega[\tilde{\omega}]|<b_k^{-10}.$$

To estimate the number of such pairs $\tilde{p}$, we first observe that P contains $\leqslant b_k^{-3}$ exponents so that the number of pairs $(\lambda_\omega,\lambda'_\omega)$ is $<b_k^{-6}$. For each fixed pair, the first inequality holds for $\leqslant\text{Const.}\log b_k^{-1}$ choices of lengths of $\tilde{\omega}$ and the second for $<2b_k^{-10}$ different $\tilde{n}$'s. This implies that, for *fixed* $p\in F_k$,

$$\sum_{(\tilde{\text{F}}\text{b})}|\tilde{\omega}|\leqslant\text{Const.}\,b_k^{-17}|\omega|,$$

and the same inequality clearly also holds for ($\tilde{\text{F}}$a).

If we also observe (8.1), we find

$$\sum_{(\tilde{F}_k)}|\tilde{\omega}|\leqslant\text{Const.}\,b_k^{-19}. \tag{8.3}$$

The above construction has the following consequence: *if* $p=(n,\omega)\notin\tilde{F}_k$, *then*

$$P(x,\omega)=Q_0(x,\omega)+Q_1(x,\omega), \tag{8.4}$$

where the polynomials Q_i *satisfy:*

$$Q_0(x) = \varrho\, e^{i\lambda x} + O(b_k^8), \quad x \in \omega, \tag{8.5}$$

where ϱ *is constant and* $\lambda = \text{some } \lambda_\omega$; *further* $|\varrho| \leqslant b_k^{-2}$ *by* (7.5);

$$Q_1(x) \text{ contains those } \lambda_\omega \text{ for which } |n - \lambda_\omega[\omega]| \geqslant b_k^{-10}. \tag{8.6}$$

Since $p \notin \tilde{F}_k$, the exponents in Q_0 satisfy by ($\tilde{\text{F}}$b) $|\lambda_\omega - \lambda'_\omega|\,|\tilde{\omega}| < b_k^{10}$, which gives (8.5). If Q_0 only contains one exponent, (8.5) is obvious.

With the primitive elements we also associate an exceptional set. If $p = (n, \omega)$ gives a primitive element, the set Y_k^* contains two intervals of lengths $2b_k^3|\omega|$, symmetric around the endpoints of ω. By (8.1)

$$mY_k^* \leqslant 16 b_k. \tag{8.7}$$

This set is of a purely technical nature and is introduced to secure the validity of Lemma 6 below.

With each $\tilde{p} = (\tilde{n}, \tilde{\omega})$ we associate the two intervals ω^*, $|\omega^*| = 4|\tilde{\omega}|$, which contain $\tilde{\omega}$. The set of such combinations $(n, \omega^*) = p^*$, $n = \tilde{n}$, for $\tilde{p} \in \tilde{F}_k$, is denoted F_k^* and the relation (8.3) becomes

$$\sum_{F_k^*} |\omega^*| \leqslant \text{Const.}\, b_k^{-19}. \tag{8.8}$$

Finally we shall need the following lemma.

LEMMA 6. *Let* $p^* = (n, \omega^*)$ *be given and assume* $p^* \notin F_k^*$. *Assume that for each* $\omega' \subset \omega^*$, $4|\omega'| = |\omega^*|$, $\omega' \not\subset X_k \cup Y_k^*$. *We further assume that for a certain choice* ω'_0 *of* ω' *the corresponding polynomial* P^0 *contains an exponent* λ^0 *such that* $|\lambda^0[\omega'_0] - n| < b_k^{-9}$. *Then the four polynomials* P *corresponding to different choices of* ω' *are all identical.*

Proof. Let ω' be another choice and assume that the corresponding polynomial does not contain λ^0. This exponent was primitive for a certain $\omega_1 \supset \omega'_0$. By the construction ($\tilde{\text{F}}$a), $|\omega_1| \geqslant b_k^{-10}|\omega'|$ because the opposite inequality would imply $(n, \omega'_0) \in \tilde{F}_k$, i.e., $p^* \in F_k^*$. Since $\omega_1 \not\supset \omega'$, it then follows that $\omega'_0 \subset Y_k^*$ against our assumption.

In the same way it then also follows from ($\tilde{\text{F}}$a) that none of the four polynomials $P(x, \omega')$ can contain exponentials that are primitive for intervals $\omega_2 \supset \omega'$, $|\omega_2| < b_k^{-10}|\omega'|$, and that every primitive interval ω_2 for one ω' must contain the three others.

9. The coverings $\Omega(p^*; l)$ and the exceptional set

Consider a pair $p^* = (n, \omega^*)$. If the following conditon $\Omega(l)$

$$\Omega(l):\ p^* \in F^*_{l+3},\ C^*(p^*) < b_{l-1},$$

holds, we construct the partition $\Omega(p^*; l)$ in analogy with Section 3 (B). The condition (Bα) is simply changed to

$$(\mathrm{B}^*\alpha) \qquad C(n[\omega]; \omega) < b_{l-1}.$$

N is here arbitrary but fixed, and the main point in the proof is to make the estimates independent of N. We may take f as a trigonometric polynomial of degree N.

As in (C) and (D) of Section 3, we form the functions $H^*(x)$ and $\Delta(x)$ corresponding to the partition and define the exceptional sets

$$T^*(p^*):\ H^*(x) > b_{l-1}^{\frac{1}{2}}, \qquad U^*(p^*):\ \Delta(x) > b_{l-1}^{-\frac{1}{2}},$$

and as there

$$m(T^*(p^*) \cup U^*(p^*)) \leqslant \text{Const.} \exp\{-\text{Const.}\, b_{l-1}^{-\frac{1}{2}}\}\,|\omega^*|. \tag{9.1}$$

We next observe that for $\omega^* \not\subset S^*$, $C^*(p^*) < b_L$ will automatically hold where $L = L(\varepsilon) \to \infty$, $\varepsilon \to 0$. We shall therefore only consider $l \geqslant L$ and define

$$T^* = \bigcup_{l=L}^{\infty} \bigcup_{\Omega(l)} T^*(p^*)$$

and similarly for U^*. Since $\Omega(l)$ defines a subset of F^*_{l+3}, we have by (9.1) and (8.8)

$$m(T^* \cup U^*) \leqslant \text{Const.} \sum_{l=L}^{\infty} b_l^{-19\cdot 16} \exp\{-\text{Const.}\, b_l^{-\frac{1}{2}}\}. \tag{9.2}$$

Similarly, we define $\qquad X^* = \bigcup_{L}^{\infty} X^*_k, \qquad Y^* = \bigcup_{L}^{\infty} Y^*_k,$

and recall the definition of V in (E) of Section 3, here with $\lambda_1 = \varepsilon^{\frac{1}{2}}$. We then define the exceptional set

$$E = S^* \cup T^* \cup U^* \cup X^* \cup Y^* \cup V.$$

The results (7.7), (8.7) and (9.2) show that

$$mE \leqslant \delta(\varepsilon) \to 0, \quad \varepsilon \to 0,$$

and there is no reference to N in the estimate of E. It now remains to prove that every partial sum of order $< N$ is small outside E.

10. Proof of Theorem (c). Three propositions

To get a better organization of the proof, we isolate certain parts of it in propositions 1–3 in this section and complete the proof in Section 11. Proposition 1 is purely technical and is needed because we have used integers relatively each ω in the definition of the polynomials P. We must then deduce estimates for fractional Fourier transforms. Proposition 2 gives an estimate of the change in $s_n^*(x)$ when we move n to another position n_1 for which we have an estimate of the remainder terms. Proposition 3 finally shows how that estimate for a pair $p_0^* = (n_0, \omega_0^*)$ can be obtained from the corresponding expression for $(\bar{n}, \bar{\omega}^*)$ where $\bar{\omega}^* \supset \omega_0^*$. This fact is the crucial one; it is here essential that $H^*(x)$ is a maximal Hilbert transform and that $\Delta(x)$ has positive terms.

In the sequel we shall consider x fixed outside E.

PROPOSITION 1. *Let $g(t) \in L^2(\omega^*)$ and let n be given. We assume*

$$\int_{\omega^*} |g(t)|^2\, dt \leq G^2 |\omega^*|$$

and for all $\omega' \subset \omega^*$, $|\omega^*| = 4|\omega'|$,

$$|c_m(\omega')| \leq \mu, \quad |m-n| < M.$$

Then

$$C_n^*(\omega^*; g) \leq \text{Const.}\left\{\mu \log M + \frac{G}{\sqrt{M}}\right\}.$$

Proof. Take a fixed ω' and normalize to $\omega' = (0, 2\pi)$ and $n = 0$. Let

$$e^{i\alpha t} \sim \sum_{-\infty}^{\infty} \alpha_\nu e^{i\nu t}, \quad 0 < t < 2\pi.$$

Then

$$|\alpha_\nu| < \frac{2}{|\alpha - \nu| + 1}$$

and

$$c_\alpha(\omega') = \sum c_\nu \bar{\alpha}_\nu.$$

If $|\alpha| \leq \frac{1}{2}M$, we have

$$|c_\alpha(\omega')| \leq \text{Const.}\left\{\mu \sum_{\nu=1}^{\frac{1}{2}M} \frac{1}{\nu} + \left\{\sum_{-\infty}^{\infty} |c_\nu|^2\right\}^{\frac{1}{2}} \left\{\sum_{\frac{1}{2}M}^{\infty} \frac{1}{\nu^2}\right\}^{\frac{1}{2}}\right\} \leq \text{Const.}\left\{\mu \log M + \frac{G}{\sqrt{M}}\right\} = q.$$

For $|\alpha| \geq \frac{1}{2}M$ we have $|c_\alpha| \leq 2G$. We find

$$C_n^* \leqslant q \sum_{-\infty}^{\infty} \frac{1}{1+\nu^2} + 4G \sum_{M}^{\infty} \frac{1}{1+\nu^2},$$

which proves the assertion.

For proposition 2, let us recall definition (1.2) of $s_n^*(x)$ for $\omega_{-1}^* = (-4\pi, 4\pi)$. The same integral over an arbitrary ω^* is denoted $s_n^*(x; \omega^*)$. We shall also assume ε sufficiently small.

PROPOSITION 2. *Assumptions. Let n_0 and $p_0^* = (n_0[\omega_0^*], \omega_0^*)$ be given and assume $\omega_0 \notin E$. Let l be defined by*

$$b_{l-1} > C^*(p_0^*) \geqslant b_l \quad (\textit{so that } l \geqslant L(\varepsilon)) \tag{10.1}$$

and suppose $p_0^* \notin F_{l+3}^*$. Let x belong to the middle half of ω_0^*.

Assertions. There exist $\omega_1^* \supseteq \omega_0^*$, x belonging to the middle half of ω_1^*, and n_1 such that

$$p_1^* = (n_1[\omega_1^*], \omega_1^*) \in F_{l+3}^* \tag{10.2}$$

and

$$|n_1[\omega_0^*] - n_0[\omega_0^*]| \leqslant \frac{1}{b_l}. \tag{10.3}$$

Furthermore, if we set $p_{10}^* = (n_1[\omega_0^*], \omega_0^*)$, then

$$\big| |s_{n_1}^*(x; \omega_0^*)| - |s_n(x; \omega_0^*)| \big| \leqslant \text{Const.}\, \{C^*(p_{10}^*) + b_l\} \tag{10.4}$$

for all n such that

$$|n_1[\omega_0^*] - n[\omega_0^*]| < b_l^{-3/2} \tag{10.5}$$

and in particular for $n = n_0$.

Proof. Let ω_0' be the subinterval of ω_0^* for which $C_{n_0[\omega_0']}(\omega_0') = C^*(p_0^*)$ and let ω' be an arbitrary interval $\omega' \subset \omega^*$, $|\omega^*| = 4|\omega'|$. Let P_0 and P be the corresponding $(l+3)$-polynomials. The definition of $(l+3)$-polynomials implies for every such ω'

$$|c_m(\omega', f - P)| < b_{l+3}, \quad m \text{ an integer.}$$

Since $|P| \leqslant b_{l+3}^{-2}$, we can use Proposition 1 with

$$G = 3b_{l+3}^{-2}, \quad \mu = b_{l+3}, \quad M = b_{l+3}^{-10}, \quad n \text{ arbitrary.}$$

We obtain

$$C_n(\omega'; f - P) \leqslant b_{l+2} \quad (l \geqslant L(\varepsilon),\ \varepsilon < \varepsilon_0) \tag{10.6}$$

for all n.

In particular for $\omega' = \omega_0'$ and $n = n_0[\omega_0^*]$, (10.6) yields

$$C_n(\omega_0'; P_0) \geqslant C^*(p_0^*; f) - b_{l+2} \geqslant b_l - b_{l+2}.$$

Since $p_0^* \notin F_{l+3}^*$ by our assumption, we can use (8.4–6) with $k = l+3$ and find

$$\text{Const.}\,\frac{|\varrho|}{|\lambda[\omega_0'] - n_0[\omega_0']|} \geq b_l. \tag{10.7}$$

We now choose $n_1 = \lambda$; $\lambda[\omega_0']$ is used in (10.6) and we obtain

$$|\varrho| \leq \text{Const.}\,(C^*(p_{10}^*) + b_{l+2}) < 1 \quad (\varepsilon < \varepsilon_1). \tag{10.8}$$

We finally insert this improved estimate of ϱ in (10.7) and have verified (10.3). The relation (10.2) also holds for $n_1 = \lambda$ since λ is an exponent in P. ω_1' is the corresponding primitive interval or its left neighbor so that $\omega_1^* \supseteq \omega_0^*$ and contains x in its middle half. We also observe that by Lemma 6 the polynomials P corresponding to the four choices of ω' coincide and we have a unique P on ω_0^*.

To prove (10.4), we write

$$s_n^*(\cdot\,; f) = s_n^*(\cdot\,; f-P) + s_n^*(\cdot\,; P).$$

Since (10.6) holds for $f-P$, it follows from (5.4) and (5.5) used less than $b_l^{-3/2}$ times that

$$|e^{inx}s_n^*(\cdot\,; f-P) - e^{in_1x}s_{n_1}^*(\cdot\,; f-P)| \leq \text{Const.}\,b_l^{-3/2}b_{l+2},$$

if n satisfies (10.5). Since $P = Q_0 + Q_1$ satisfies (8.5) and (8.6) and since P has at most b_{l+3}^{-3} terms an easy calculation shows that

$$|e^{inx}s_n^*(\cdot\,; P) - e^{in_1x}s_{n_1}^*(\cdot\,; P)| \leq \text{Const.}\,\{|\varrho| + b_l\}$$

for these values of n. The basic fact for this calculation is the relation

$$\int_{\omega_0^*} \frac{e^{-i\lambda t}}{x-t}\,dt = \pi i\,\text{sign}\,(\lambda)\,e^{-i\lambda x} + O((\lambda|\omega_0^*|)^{-1}).$$

The proof of proposition 2 is now complete.

PROPOSITION 3. *Assumptions as in Proposition 2.*

Assertions. There exist $\bar\omega$, x belonging to the middle half of $\bar\omega^*$, an integer m, $m \leq l$, and an integer $\bar n$ such that

$$|\bar n[\omega_0^*] - n_0[\omega_0^*]| \leq \text{Const.}\,b_l^{-1}, \tag{10.9}$$

$$\bar p^* = (\bar n[\bar\omega^*], \bar\omega^*) \in F_{m+3}^*, \tag{10.10}$$

$$C^*(\bar p^*) < b_{m-1}. \tag{10.11}$$

The relations (10.10) *and* (10.11) *imply that $\Omega(\bar p^*; m)$ is defined. For this partition it holds that*

$\bar{\omega}^*(x) \subset \omega_0^*$ *strictly and* $\omega_0^* - \bar{\omega}^*(x)$ *is a union of intervals belonging to* $\Omega(\bar{p}^*; m)$. (10.12)

Furthermore, if p_{10}^* *is given by Proposition 2, then*

$$C^*(p_{10}^*) < b_{m-1}. \tag{10.13}$$

Proof. Denote by Σ the set of triplets $(n; \omega^*; k)$ where n and k are integers and ω^* an interval $\omega_{j\nu}^*$ such that

$$\Sigma: \begin{cases} (1) & \omega^* \supseteq \omega_0^* \text{ and } x \in \text{middle half of } \omega^*; \\ (2) & k \leqslant l \text{ and } C^*(p_{10}^*), \text{ defined in Prop. 2, } < b_{k-1}; \\ (3) & \text{if } n_1 \text{ is defined in Prop. 2, then } |n_1[\omega_0^*] - n[\omega_0^*]| \leqslant \sum_{j=k}^{l} b_j^{-1}; \\ (4) & (n[\omega^*], \omega^*) \in F_{k+3}^*. \end{cases}$$

We first show that Σ is not empty. If $C^*(p_{10}^*) < b_{l-1}$, then $(n_1; \omega_1^*; l) \in \Sigma$. If $C^*(p_{10}^*) \geqslant b_{l-1}$, we define k by $b_k \leqslant C^*(p_{10}^*) < b_{k-1}$. If $p_{10}^* \in F_{k+3}^*$, then $(n_1; \omega_0^*; k) \in \Sigma$. If $p_{10}^* \notin F_{k+3}^*$, we use p_{10}^* as p_0^* in Proposition 2 to construct a new pair (n_2, ω_2^*). By (10.3)

$$|n_2[\omega_0^*] - n_1[\omega_0^*]| \leqslant b_k^{-1}$$

so that Σ (3) holds. It is now clear that $(n_2; \omega_2^*; k) \in \Sigma$.

Define $(\bar{n}; \bar{\omega}^*; m)$ of Proposition 3 as an element of Σ for which k is minimal. Then (10.9) holds by (3) in the definition of Σ since (10.3) holds and

$$\sum_{j=1}^{l} b_j^{-1} \leqslant \text{Const.}\, b_l^{-1}.$$

(10.10) follows from (4) and (10.13) is included in (2). It remains to prove (10.11) and (10.12).

Assume first $C^*(\bar{p}^*) \geqslant b_{m-1}$ and define k by $b_k \leqslant C^*(\bar{p}^*) < b_{k-1}$, $k < m$. Since m is minimal, $(\bar{n}; \bar{\omega}^*; k) \notin \Sigma$, i.e. $\bar{p}^* \notin F_{k+3}^*$. We then use $\bar{p}^*$ as p_0^* in Proposition 2 and obtain $\bar{n}_1$, $\bar{\omega}_1^*$ by that construction. Since

$$\begin{aligned} |\bar{n}_1[\omega_0^*] - n_1[\omega_0^*]| &\leqslant |\bar{n}_1[\omega_0^*] - \bar{n}[\omega_0^*]| + |\bar{n}[\omega_0^*] - n_1[\omega_0^*]| \\ &\leqslant |\bar{n}_1[\bar{\omega}^*] - \bar{n}[\bar{\omega}^*]| + \sum_{m}^{l} b_j^{-1} \\ &\leqslant \sum_{k}^{l} b_j^{-1}, \end{aligned}$$

$(\bar{n}_1; \bar{\omega}_1^*; k) \in \Sigma$, which again contradicts the minimality of m. We have thus proved (10.11).

We now know that we can use the construction $\Omega(m)$ on $\bar{p}^*$ and get $\bar{\omega}^*(x)=\bar{\omega}_1^*$ and a corresponding $\bar{p}_1^*$. Then $C^*(\bar{p}_1^*)\geqslant b_{m-1}$ and if $\bar{\omega}_1^*\supseteq\omega_0^*$, we could use the argument above to obtain a contradiction. Hence $\bar{\omega}_1^*\subset\omega_0^*$ strictly, and it is then easy to see that $\omega_0^*-\bar{\omega}^*(x)$ has the stated property.

11. Proof of Theorem (c), completed

Let us again consider formula (5.3) for $s_n^*(x)$, x fixed not in E. We have used the covering $\Omega((n,\omega_{-1}^*);k)$, $b_k\leqslant C_n^*(\omega_{-1}^*)<b_{k-1}$. Since all $c_n(\omega_{00})$, $|c_n|\geqslant b_{k+3}$, are coefficients in $P_{k+3}(x;\omega_{00})$, it follows from Proposition 1 and ($\tilde{\text{F}}$a) in Section 8 that $(n;\omega_{-1}^*)\in F_{k+3}^*$. The remainders in (5.3) therefore are $O(b_{k-1}^{\frac{1}{2}})$.

We write $\omega^*(x)=\omega_0^*$ and $p_0^*=(n[\omega_0^*],\omega_0^*)$. If $b_{l-1}>C^*(p_0^*)\geqslant b_l$, then $l<k$. If $p_0^*\in F_{l+3}^*$, we can make the construction $\Omega(p_0^*:l)$ and the remainders will be $O(b_{l-1}^{\frac{1}{2}})$. If $p_0^*\notin F_{l+3}^*$, we construct p_1^* and n_1 according to Proposition 2 and $\bar{n}$, $\bar{\omega}$ and m according to Proposition 3. We write, using $\Omega(\bar{p}^*;m)$ only on ω_0^*,

$$\int_{\omega_0^*}\frac{e^{-i\bar{n}t}f(t)}{x-t}\,dt=\int_{\bar{\omega}^*(x)}\frac{e^{-i\bar{n}t}f(t)}{x-t}\,dt+\bar{H}_{\bar{n}}(x)+\bar{R}_{\bar{n}}(x).$$

By the estimate (10.9) for $\bar{n}$ and (10.4) and (10.13), the left side integral differs in modulus from the corresponding integral in (5.3) by $O(b_{m-1})$. Since x belongs to the middle half of $\bar{\omega}^*$, ω_0^* and $\bar{\omega}^*(x)$, it follows that

$$|\bar{H}_{\bar{n}}(x)|\leqslant 2H_{\bar{n}}^*(x)<2b_{m-1}^{\frac{1}{2}}$$

since $x\notin T^*$. A similar inequality holds for $\bar{R}_{\bar{n}}(x)$ since $\omega_0^*-\bar{\omega}^*(x)$ satisfies (10.12) so that Const. $b_{m-1}\Delta(x)$ is a majorant also of $\bar{R}_{\bar{n}}(x)$. Finally we observe that unless $|\bar{\omega}^*(x)|=2\pi 2^{-N+1}$, which case is easy, since f is of degree N,

$$C^*(\bar{n}[\bar{\omega}^*(x)],\bar{\omega}^*(x))\geqslant b_{m-1}.$$

We can therefore repeat the argument and obtain, as in the proof of Theorem (a),

$$s_n^*(x)=\sum_{j=L(\varepsilon)}^{\infty}(O(b_j^{\frac{1}{2}})+O(b_j))+\sqrt{\varepsilon}\to 0,\quad \varepsilon\to 0,$$

for all $x\notin E$, $mE\to 0$, $\varepsilon\to 0$. Since no estimates depend on N, Theorem (c) is proved.

12. Theorem (b)

As mentioned before, it seems clear that $s_n(x)$ converges a.e. also if $f\in L^p$, $p>1$. We shall therefore only outline the proof of (b).

Let N be fixed and define in this case the exceptional sets as in (c) by the inequalities, $n \leqslant 2^N$,

$$T^*: \quad H^*(x) > b_{l-1}^{\frac{1}{2}};$$

respectively, $$U^*: \quad \Delta(x) > b_{l-1}^{-\frac{1}{2}}.$$

As in the proof of (a) we allow the construction $\Omega(p^*; l)$, $b_l \leqslant C^*(p^*) < b_{l-1}$ for *all* pairs p^* for which l is such that $b_{l-1} < (\log N)^{-k}$, where k is a suitable constant depending on p ($f \in L^p$). As in the proof of (a), it follows from the Hausdorff–Young inequality that the corresponding exceptional set has measure $O(N^{-2})$.

If $b_{l-1} \geqslant (\log N)^{-k}$, we have to select certain pairs $p = (n, \omega)$ for which we allow the construction. For ω_{00} we include all n such that $|c_n| \geqslant b_l$. The set $\{p\} = \Phi_l$, being defined for $|\omega| \geqslant 2\pi 2^{-\nu}$, we include $p' = (n', \omega')\, |\omega'| = 2\pi 2^{-\nu-1}$ if

$$|c(p'; f)| \geqslant b_l \quad \text{and} \quad \big||\omega| n' - |\omega'| n\big| > b_l^{-K} |\omega| \tag{12.1}$$

holds for every $(n, \omega) \in \Phi_l$ with $\omega \supset \omega'$; K is a suitable constant.

We now consider functions $\varphi(x)$, $|x| < 4\pi$, of the form

$$\varphi(x) = e^{i\theta} \cdot e^{\frac{2\pi n}{|\omega|} ix}, \quad x \in \omega, \ (n, \omega) \in \Phi_l,$$

where for fixed φ the different ω's do not intersect and are maximal with respect to this property. From the separation (12.1) it follows that this system is "almost" orthogonal. From this we obtain L^p estimates by a standard interpolation. It is then easy to define a set X^* for the set Φ_l as we did for F_l in (b).

By the above construction, we have for each $(n[\omega^*], \omega^*) = p^*$ with $C^*(p^*) \geqslant b_l$ a $\bar{p}^* \in \Phi_l$, as in Proposition 3, such that

$$|\bar{n}[\omega^*] - n[\omega^*]| < \text{Const.}\, b_l^{-K} = b.$$

Since $x \notin S^*$, $|s_{\bar{n}} - s_n| = O(\log 1/b)$ and summing this for $b_l \geqslant (\log N)^{-k}$, we obtain the desired estimate.

References

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[2]. Ibid., Vol. II.

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