STRUCTURE IN SIMPLEXES

BY

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1. Introduction

The Choquet theory of simplexes has provided an elegant approach to direct integral decompositions in several areas of analysis. In the various contexts, one identifies "irreducible" elements with the extreme points of a simplex. The decomposition of a general element into irreducibles then corresponds to the unique barycentric representation of a point in the simplex as a probability measure "on" the extreme points. One has, for example, that the invariant probability measures for a locally compact transformation group on a compact space form a simplex, and the extreme points are then just the ergodic measures [11] (see [23, § 10]). Similarly the normalized traces on a C^*-algebra form a simplex, and the extreme points are the factor traces [26, p. 116]. It would seem likely that a classification of simplexes will provide more information in these applications than just an existence proof for the decompositions.

In this paper we shall introduce a "structure" topology on the extreme points $E(K)$ of a simplex K. A closed set in this topology is just the extreme points of a closed face of K. That $E(K)$ with the structure topology is analogous to Jacobson's structure space of a ring (see [12, Ch. 9]) is best seen by considering the "affine space" $\mathcal{A}(K)$ of K. $\mathcal{A}(K)$ consists of the continuous affine functions on K. It is an ordered vector space with a distinguished "order unit", the constant function 1. Owing to studies of Kadison and Lindenstrauss (see § 2), the spaces that arise in this fashion have been completely characterized. We prove that the closed faces in K are in one-to-one correspondence with the closed "ideals" in $\mathcal{A}(K)$, and that $E(K)$ with the structure topology may be identified with the maximal ideal space of $\mathcal{A}(K)$, with the "hull-kernel" topology.

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As ideals in $\mathcal{A}(K)$ need not have an order unit, it has been necessary to enlarge the category of ordered spaces under consideration. In § 2 a "simplex space" is defined to be an ordered vector space A with complete norm and closed positive cone, for which A^* is an L-space in the sense of Kakutani [17; 18] (see below). A representation theorem for such spaces is proved, and those of the form $\mathcal{A}(K)$, K a simplex, are characterized as those with an "order identity".

In § 3 the notion of ideal is introduced, and it is shown that the closed ideals are just the annihilators of weakly* closed faces of the positive cone in A^*. We prove that proper closed ideals and the resulting quotients are simplex spaces.

The structure topology is introduced in § 4. The structure spaces of ideals and quotients are seen to correspond to open and closed sets in the expected manner. The simplex spaces A with order identity have a compact structure topology, and the topology is Hausdorff if and only if A is a Kakutani M-space.

§ 5 is devoted to a brief discussion of open problems, and an example of a simplex K such that the structure topology on $E(K)$ is not of second category.

We are indebted to E. Alfsen for introducing us to the theory of simplexes, and making available to us the manuscript of [1]. We also wish to thank D. A. Edwards and J. Semadeni for explaining to us the role of Lindenstrauss's work in the theory of simplexes (see § 2). We are indebted to R. Phelps for the use of a preliminary version of [23]. The reader will find in the latter an excellent exposition of the theory of simplexes, together with a comprehensive bibliography (see also [6]). An introduction to L- and M-spaces may be found in [19] (see also [7, Ch. VI]).

Throughout this paper, vector spaces will be assumed to have non-zero elements, and cones are assumed proper, i.e., not containing a non-zero element and its negative. In normed spaces, the subscript α on a subset indicates the corresponding intersection with the closed ball of radius α.

2. The affine space of a simplex

Let K be a compact convex subset of a locally convex space. A real function a on K is *affine* if

$$a(\alpha p+(1-\alpha)q)=\alpha a(p)+(1-\alpha)a(q)$$

for all p, q in K and $0\leqslant\alpha\leqslant 1$. The *affine space* $\mathcal{A}(K)$ of K is the vector space of all continuous affine functions on K, together with the ordering defined by the cone $\mathcal{A}(K)^+$ of nonnegative continuous affine functions. The function e defined by $e(p)=1$ for all p in K is an Archimedean order unit for $\mathcal{A}(K)$, i.e., for each a in $\mathcal{A}(K)$, there is a scalar α with $a\leqslant\alpha e$, and if $a\leqslant\alpha e$ for all $\alpha>0$, then $a\leqslant 0$.

Following Kadison [13; 14], we call a partially ordered vector space A together with a distinguished Archimedean order unit e a *function system.* Kadison has proved a representation theorem for function systems, which we shall outline. For each a in a function system A, define

$$L(a) = \sup\{\alpha\colon \alpha e \leqslant a\}, \tag{2.1}$$

$$M(a) = \inf\{\alpha\colon a \leqslant \alpha e\}, \tag{2.2}$$

$$\|a\| = \max\{|L(a)|,\ |M(a)|\}. \tag{2.3}$$

We refer to $\|\ \|$, which is a norm on A, as the *order norm.* It coincides with the usual supremum norm if $A = \mathcal{A}(K)$, and in that case it is complete. Letting $P(A)$ be the positive linear functions on A, $P(A)$ is contained in A^*, the bounded linear functions. We give A^* the weak* topology, and define the *state space* $S(A)$ to be $P(A) \cap H(A)$ with the relative topology, where $H(A)$ is the set of f in A^* with $f(e)=1$. If $A = \mathcal{A}(K)$, the map $p \to \hat{p}$, where $\hat{p}(a) = a(p)$ for p in K, $a \in A$, is an affine homeomorphism of K onto $S(A)$. Thus rather than study compact convex sets K, we may restrict our attention to the state spaces of (norm) complete function systems A. In [13], Kadison showed that

$$L(a) = \inf\{p(a)\colon p \in S(A)\}, \tag{2.4}$$

$$M(a) = \sup\{p(a)\colon p \in S(A)\}, \tag{2.5}$$

$$\|a\| = \sup\{|p(a)|\colon p \in S(A)\}. \tag{2.6}$$

Define $\hat{a}(p) = p(a)$ for a in A, p in $S(A)$. As $a \geqslant 0$ if and only if $L(a) \geqslant 0$, it follows that $a \to \hat{a}$ is an isometric order isomorphism of A into $\mathcal{A}(S(A))$. It has been known that this map is onto. A proof readily follows from [16, Lemma 4.3].

If C is a cone in a vector space E, a *base* of C is an intersection of C with a hyperplane H, where H is a hyperplane in E not containing 0 that meets all of the generators of C, i.e., for all x in $C - \{0\}$, there is an $\alpha > 0$ with $\alpha x \in H$. C is a *lattice cone* if it defines a lattice ordering on $C - C$. It suffices to show each pair x, y in C has a minimum in C (see [23, § 9]). We say that a convex set K in a vector space is a *simplex* if it is affinely isomorphic to the base of some lattice cone. It will then follow that any cone with a base affinely isomorphic to K is a lattice cone.

An ordered vector space A satisfies the *Riesz decomposition property* if a, b_1, $b_2 \in A^+$ and $a \leqslant b_1 + b_2$ imply the existence of a_1, $a_2 \in A^+$ with $a = a_1 + a_2$ and $a_i \leqslant b_i$. Equivalently, if a_i, $b_j \in A^+$ with $i = 1, \ldots, m$; $j = 1, \ldots, n$ and $\sum a_i = \sum b_j$, then there exist $c_{ij} \in A^+$ with $a_i = \sum_j c_{ij}$, $b_j = \sum_i c_{ij}$ (see [4, Ch. II]). Another equivalent condition is that if a_i, $b_j \in A$ and $a_i \leqslant b_j$, then there is an element c with $a_i \leqslant c \leqslant b_j$ for all i, j [21, Lemma 6.2]. Vector lattices satisfy the Riesz decomposition property (see [4, Ch. II]), but the converse is false (see Theorem 4.8).

We recall that a vector lattice V together with a complete norm is an *L-space* if $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$, and for $x, y \geqslant 0$, $\|x+y\| = \|x\| + \|y\|$. V is an *M-space* if rather than the second equality, one has for $x, y \geqslant 0$, $\|x \vee y\| = \max\{\|x\|, \|y\|\}$.

The following theorem is due to J. Lindenstrauss [21] (see [25, Theorem 5]). An elegant proof by D. A. Edwards was presented in [8]. Earlier partial results occur in [20] (I am indebted to Z. Semadeni for this reference) and [24].

THEOREM 2.1. *Let A be a complete function system. Then the following conditions on A are equivalent:*

(1) *$S(A)$ is a simplex,*

(2) *A has the Riesz decomposition property,*

(3) *A^* with the ordering defined by $P(A)$ and the uniform norm is an L-space.*

In § 3 we shall consider "ideals" in function systems satisfying the conditions of Theorem 2.1. As these subspaces will in general not have order units, they will not be function systems. Noting that the primary function of the distinguished order unit is to define the order norm, we introduce a more inclusive category of ordered spaces with norm.

An ordered vector space A together with a complete norm is a *simplex space* if A^+ is closed, and the Banach space A^* together with the order defined by the cone $P(A)$ of *bounded* positive linear functions is an L-space. The first condition is equivalent to the assumption that if $p(a) \geqslant 0$ for all p in $P(A)$, then $a \geqslant 0$ (see [19, § 23.2]). Any M-space is a simplex space.

If A is a simplex space, we define the *state space* $S(A)$ to be $\{p \in P(A): \|p\| = 1\}$. As this is generally not compact with the weak* topology, it is useful to consider instead the set $P_1(A) = P(A) \cap A_1^*$ with the weak* topology.

THEOREM 2.2. *If A is a simplex space, then $P_1(A)$ is a simplex, and A may be identified with the ordered Banach space of continuous affine functions on $P_1(A)$ vanishing at 0. In particular, A satisfies the Riesz decomposition property.*

Proof. Define $\hat{a}(p) = p(a)$ for $a \in A$ and $p \in P_1(A)$. $a \to \hat{a}$ is trivially an order isomorphism. We have

$$\|a\| = \sup\{|f(a)| : f \in A_1^*\}, \qquad \|\hat{a}\| = \sup\{|p(a)| : p \in P_1(A)\},$$

hence $\|\hat{a}\| \leqslant \|a\|$. If $f \in A_1^*$, then as A^* is an L-space, $f = f^+ - f^-$, where $f^+, f^- \in P(A)$, and

$$\|f\| = \|f^+\| + \|f^-\|.$$

Assuming $f^+, f^- \neq 0$, let $p = f^+/\|f^+\|$, $q = f^-/\|f^-\|$. Then $p, q \in P_1(A)$, and

$$|f(a)| \leqslant \|f^+\| \, |p(a)| + \|f^-\| \, |q(a)| \leqslant (\|f^+\| + \|f^-\|)\|\hat{a}\| \leqslant \|\hat{a}\|.$$

If f^+ or $f^- = 0$, the resulting inequality is trivial, and we conclude $a \to \hat{a}$ is an isometry. It follows from [16, Lemma 4.3] that all continuous affine functions on $P_1(A)$ vanishing at 0 are of the form $\hat{a}$.

As A^* is an L-space, $\| \ \|$ is positive and positive linear on $P(A)$. As $A^* = P(A) - P(A)$, it extends to a strictly positive linear function on A^*. That $P_1(A)$ is a simplex follows from the following (see [23, Prop. 11.3]):

LEMMA 2.3. *Suppose that P is a cone in a vector space E, φ is a strictly positive linear function on E, and $P_1 = \{x \in P: \varphi(x) \leqslant 1\}$. If P is lattice ordered, P_1 is a simplex.*

Proof. Let $E' = E \times R$, R the reals, be ordered by the cone

$$P' = \{(x, \alpha): x \geqslant 0, \alpha \geqslant 0\}.$$

P' is a lattice cone as $\qquad (x, \alpha) \wedge (y, \beta) = (x \wedge y, \min(\alpha, \beta)).$

The function $\psi(x, \alpha) = \varphi(x) + \alpha$ is linear, and as φ is strictly positive, the hyperplane

$$G = \{(x, \alpha): \psi(x, \alpha) = 1\}$$

meets all generators of P'. It is readily verified that the map

$$\theta(x) = (x, 1 - \varphi(x))$$

is an affine isomorphism of P_1 onto the base $P' \cap G$.

Making the identification of Theorem 2.2, we will write $a(p) = p(a)$ for $a \in A$, $p \in P_1(A)$.

If K is a convex set in a vector space, a *face* (or "extremal subset") of K is a convex subset Q such that if $\alpha x + (1-\alpha) y \in Q$, with $x, y \in K$ and $0 < \alpha < 1$, then $x, y \in Q$. A face of a face is again a face, and the set of *extreme points* $E(K)$ consists of just the one-point faces.

The following refinement of the Riesz decomposition property is central to our investigation. We first encountered the 2^{-n} thechnique for forcing convergence in Edwards' proof of Theorem 2.1 [8].

THEOREM 2.4. *Suppose that K is a compact simplex in a locally convex vector space, and that Q is a closed face in K. If a_1, a_2, b are in $\mathcal{A}(K)$ with $a_i \leqslant b$ and $a_1 | Q \leqslant a_2 | Q$, then there exists an element c in $\mathcal{A}(K)$ for which $a_i \leqslant c \leqslant b$, and $c | Q = a_2 | Q$.*

Proof. Let $a_1 \vee a_2$ be the maximum of the functions a_i, and $(a_1 \vee a_2)^-$ be the lower envelope of the continuous affine functions on K majorizing $a_1 \vee a_2$. We assert that $(a_1 \vee a_2)^- | Q = a_2 | Q$.

Given $q \in Q$, let μ be the maximal probability measure on K with resultant $r(\mu)=q$. The support S_μ of μ must be contained in Q. For if $p \in S_\mu$, there is a net of positive continuous functions f_γ with $f_\gamma d\mu$ probability measures converging weakly* to the point mass δ_p. The resultants $q_\gamma = r(f_\gamma d\mu)$ lie in Q as $0 \leqslant q_\gamma \leqslant \|f_\gamma\| q$, and converge to $r(\delta_p)=p$, hence $p \in Q$. As μ is maximal, the Borel set

$$B(a_1 \vee a_2) = \{p \in P_1(A): (a_1 \vee a_2)^-(p) = a_1 \vee a_2(p)\}$$

has complement of measure zero. Thus for μ almost all p, $p \in Q$, $(a_1 \vee a_2)^-(p) = a_2(p)$, and

$$\int (a_1 \vee a_2)^-(p)\, d\mu(p) = \int a_2(p)\, d\mu(p) = a_2(q).$$

$(a_1 \vee a_2)^-$ is upper semi-continuous, and affine as K is a simplex (see [6, Theorem 11]), hence

$$\int (a_1 \vee a_2)^-(p)\, d\mu = (a_1 \vee a_2)^-(q)$$

(see [6, Lemma 10]), and $(a_1 \vee a_2)^- | Q = a_2 | Q$.

It follows from the Riesz decomposition property that the functions c_γ in $\mathcal{A}(K)$ with $a_1, a_2 \leqslant c_\gamma \leqslant b$ form a decreasing net. As they converge point-wise on Q to the continuous function $(a_1 \vee a_2)^- | Q = a_2 | Q$, the convergence is uniform on Q (see [22, § 16 A]). Thus we may select c_1 in $\mathcal{A}(K)$ with $a_1, a_2 \leqslant c_1 \leqslant b$ and $c_1 | Q \leqslant a_2 | Q + 2^{-1}$. Suppose that we have defined c_n, and that it satisfies $a_1, a_2 \leqslant c_n \leqslant b$, $c_n | Q \leqslant a_2 | Q + 2^{-n}$. Then

$$a_1, a_2, c_n - 2^{-n} \leqslant b, c_n$$

and as $(c_n - 2^{-n}) | Q \leqslant a_2 | Q$, our previous argument implies

$$(a_1 \vee a_2 \vee (c_n - 2^{-n}))^- | Q = a_2 | Q.$$

Examining the functions c_γ in $\mathcal{A}(K)$ with

$$a_1, a_2, c_n - 2^{-n} \leqslant c_\gamma \leqslant b, c_n,$$

uniformity of convergence on Q provides us with c_{n+1} satisfying

$$a_1, a_2 \leqslant c_{n+1} \leqslant b,$$

$$c_{n+1} | Q \leqslant a_2 | Q + 2^{-(n+1)},$$

and

$$\|c_n - c_{n+1}\| \leqslant 2^{-n}.$$

The functions c_n converge uniformly to a continuous affine function c. c is the desired element of $\mathcal{A}(K)$.

COROLLARY 2.5. *If A is a simplex space, $A = A^+ - A^+$.*

Proof. As $P(A)$ is a cone, 0 is an extreme point in $P(A)$, hence $\{0\}$ is a face in $P_1(A)$. If $a \in A$, we have $0, a \leqslant \|a\| 1$, and $a \mid \{0\} = 0 \mid \{0\}$. From Theorem 2.2, $P_1(A)$ is a simplex, hence there is a continuous affine function c on $P_1(A)$ with $0, a \leqslant c$, and $c(0) = 0$. We have $c \in A$ (Theorem 2.2), and $a = c - (c - a)$ with $c, c - a \geqslant 0$.

Using an argument of Kaplansky (see [7, pp. 98–99]):

COROLLARY 2.6. *Any positive function on a simplex space is bounded.*

Proof. If p is positive but not bounded, choose a sequence $a_n \in A$ with $\|a_n\| \leqslant 1$ and $p(a_n) \geqslant 4^n$. As $0, a_n \leqslant 1$ on $P_1(A)$, there is a continuous affine function c_n on $P_1(A)$ with $0, a_n \leqslant c_n \leqslant 1$ and $c_n(0) = 0$, i.e., $\|c_n\| \leqslant 1$ and $c_n \in A^+$. We have $\sum 2^{-n} c_n$ converges to an element $c \in A$. But $c \geqslant 2^{-n} c_n$ implies $p(c) \geqslant 2^n$ for all n, a contradiction.

COROLLARY 2.7. *If A is a simplex space and $f \in A^*$, then for $a \in A^+$,*

$$f^+(a) = \sup \{f(b) : 0 \leqslant b \leqslant a\}.$$

Proof. As $f^+, f^- \geqslant 0$, we have $0 \leqslant b \leqslant a$ implies

$$f(b) \leqslant f^+(b) + f^-(b) \leqslant f^+(a) + f^-(a).$$

Thus we may define $g(a)$ to be the indicated supremum. From Theorem 2.2, A satisfies the Riesz decomposition property. It follows that g is positive-linear on A^+ (see [7, p. 98]). g has a unique extension to a positive, linear function g_1 on A. From Corollary 2.6, $g_1 \in A^*$. It is clear that $0, f \leqslant g_1 \leqslant f^+$, hence $g_1 = f^+$.

We say that an element e in a simplex space A is an *order identity* if $p(e) = 1$ for all $p \in S(A)$. If e' is another such element, $p(e) = p(e')$ for all $p \in P(A)$, hence as $A^* = P(A) - P(A)$, $e = e'$. The spaces described in Theorem 2.1 are just the simplex spaces with an order identity. More precisely:

PROPOSITION 2.8. *If A is a complete function system satisfying the conditions of Theorem 2.1, then with the order norm, A is a simplex space, and the distinguished order unit is an order identity. Conversely, if A is a simplex space with an order identity e, then e is an Archimedean order unit, and the corresponding order norm coincides with the given norm.*

Proof. If A is a complete function system, it follows from (2.4) that if $p(a) \geqslant 0$ for all $p \in P(A)$, then $L(a) \geqslant 0$, i.e., $a \geqslant 0$. Thus we have the first assertion. In the second situation, if $a \in A$, then for all $p \in P(A)$,

$$p(a) \leqslant \|p\| \, \|a\| = p(\|a\| e).$$

It follows that $a \leqslant \|a\|e$. If $a \leqslant \alpha e$ for all $\alpha > 0$, $p(a) \leqslant \alpha p(e)$, hence $p(a) \leqslant 0$ for all $p \in P(A)$, and $a \leqslant 0$. If $\| \ \|_e$ is the order norm defined by e, we have from (2.6),

$$\|a\|_e = \sup\{|p(a)|: p \in P'(A),\, p(e) = 1\},$$

where $P'(A)$ consists of all positive functions on A. From Corollary 2.6, $P'(A) = P(A)$, hence

$$\|a\|_e = \sup\{|p(a)|: p \in P_1(A)\} = \|a\|,$$

the second equality following from Theorem 2.2.

If A is a simplex space, then $S(A)$ is a face in $P_1(A)$, as if $\alpha p + (1-\alpha) q \in S(A)$ with $p, q \in P_1(A)$, $0 < \alpha < 1$, then $\alpha\|p\| + (1-\alpha)\|q\| = 1$, and $\|p\| = \|q\| = 1$.

PROPOSITION 2.9. *If A is a simplex space, then A has an order identity if and only if $S(A)$ is closed in $P_1(A)$.*

Proof. If e is an order identity for A, then $S(A) = P_1(A) \cap H(A)$, where

$$H(A) = \{f \in A^*: f(e) = 1\}$$

is weak* closed. Conversely, suppose that $S(A)$ is closed. The Hahn-Banach Theorem and a simple compactness argument provide us with an element $a \in A$ such that $a | S(A) \geqslant 1$. From Corollary 2.5, we may choose $b \in A^+$ with $a \leqslant b$. We have $0 \leqslant 1$, b and $b | S(A) \geqslant 1 | S(A)$, hence applying Theorem 2.4, there exists a continuous affine function e with $0 \leqslant e \leqslant 1$, b, and $e | S(A) = 1 | S(A)$. As $b(0) = 0$, $e \in A$, and e is an order identity.

3. Ideal theory

In order to make further applications of Theorem 2.4, we must describe the faces of $P_1(A)$ for a simplex space A. In the broader context of Lemma 2.3, let $H = \{x: \varphi(x) = 1\}$, and $S = H \cap P$. The following facts are known and readily verified. A subset Q of the cone P is a face in P if and only if it is itself a cone, and $0 \leqslant y \leqslant x$ with $x \in Q$ implies $y \in Q$. The map $Q \to Q \cap S$ is a one-to-one correspondence between the faces $Q \neq \{0\}$ of P and the faces of S. The map $Q \to Q \cap P_1$ is a one-to-one correspondence between the faces of P and those in P_1 containing 0. The other faces in P_1 are just the faces of S.

As an illustration of the arguments used to prove these facts, we show that any face Q_1 of P_1 with $Q_1 \nsubseteq S$ has the form $Q \cap P_1$, Q a face in P. Say that $x \in Q_1 - S$, and $x \neq 0$. Then $0 < \varphi(x) < 1$, and

$$x = \varphi(x)\frac{x}{\varphi(x)} + (1 - \varphi(x)) \cdot 0,$$

i.e., as Q_1 is a face, $0, x/\varphi(x) \in Q_1$. In any event, $0 \in Q_1$, and if $0 \neq x \in Q_1$, then $x/\varphi(x) \in Q_1$. If $0 \leq y \leq x$ and $x \in Q_1$, then $y \in Q_1$. For assuming $y \neq 0$ and $u = x - y \neq 0$,

$$\frac{x}{\varphi(x)} = \alpha \frac{y}{\varphi(y)} + \beta \frac{u}{\varphi(u)},$$

where $\alpha = \varphi(y)/\varphi(x)$, $\beta = \varphi(u)/\varphi(x)$, and $\alpha + \beta = 1$. As $x/\varphi(x) \in Q_1$, we have $y/\varphi(y) \in Q_1$. As Q_1 is convex,

$$y = \varphi(y) \frac{y}{\varphi(y)} + (1 - \varphi(y))\, 0 \in Q_1.$$

Let Q be the non-negative scalar multiples of elements in Q_1. If $x \in Q$ and $x \neq 0$, then $x/\varphi(x) \in Q_1$, as if $x = \alpha x_0$, $x_0 \in Q_1$, then $x/\varphi(x) = x_0/\varphi(x_0) \in Q_1$. Given $x, y \in Q$ with $x, y \neq 0$, $x/\varphi(x), y/\varphi(y) \in Q_1$, hence

$$\frac{x+y}{\varphi(x+y)} = \frac{\varphi(x)}{\varphi(x+y)} \frac{x}{\varphi(x)} + \frac{\varphi(y)}{\varphi(x+y)} \frac{y}{\varphi(y)} \in Q_1$$

and $x + y \in Q$. If $0 \leq y \leq x$ with $x \in Q$, then assuming $x \neq 0$, $0 \leq y/\varphi(x) \leq x/\varphi(x) \in Q_1$, hence $y/\varphi(x) \in Q_1$ and $y \in Q$. It follows that Q is a face of P. Trivially, $Q_1 \subseteq Q \cap P_1$. If $x \in Q \cap P_1$, $x/\varphi(x) \in Q_1$, and $\varphi(x) \leq 1$. Thus $0 \leq x \leq x/\varphi(x)$, $x \in Q_1$, and $Q_1 = Q \cap P_1$.

If A is a simplex space with order identity, then the map $Q \to Q \cap S(A)$ is a one-to-one correspondence between the closed faces $Q \neq \{0\}$ of $P(A)$ and the closed faces of $S(A)$. For if Q is a face in $P(A)$ with $Q \cap S(A)$ closed, then the latter is compact, and

$$Q \cap A^*_\alpha = \{\beta p : p \in Q \cap S(A), 0 \leq \beta \leq \alpha\}$$

is closed for all α. It follows that Q is closed (see [5, Ch. IV, § 2, Theorem 5]). Similarly in any simplex space, $Q \to Q \cap P_1(A)$ is a one-to-one correspondence between the closed faces of $P(A)$ and the closed faces of $P_1(A)$ containing 0.

Say that V is an ordered vector space. Following Kadison [13], we say that a linear subspace J of A is an *order ideal* if $J^+ = J \cap A^+$ is a face of A^+. J is an *ideal* if in addition it is positively generated, i.e., $J = J^+ - J^+$.

If A is a simplex space, and J and Q are subsets of A and $P_1(A)$, respectively, define

$$J_\perp = \{p \in P_1(A) : p \mid J = 0\},$$

$$Q_\perp = \{a \in A : a \mid Q = 0\}.$$

THEOREM 3.1. *Let A be a simplex space. If J is an ideal in A, $J_\perp$ is a closed face in $P_1(A)$ containing 0, and $J_{\perp\perp}$ is the closure of J. If Q is a closed face in $P_1(A)$ containing 0, $Q_\perp$ is an ideal in A, and $Q_{\perp\perp} = Q$.*

Proof. We have $J_\perp = J^0 \cap P_1(A)$, where J^0 is the polar, i.e., annihilator of J. $J^0 \cap P(A)$ is a face in $P(A)$ as if $0 \leqslant q \leqslant p \in J^0$, then $p|J^+ = 0$ implies $q|J^+ = 0$, hence as $J = J^+ - J^+$, $q|J = 0$. Thus $J_\perp$ is a closed face in $P_1(A)$ containing 0.

$Q_\perp$ is clearly a closed linear subspace of A. If $0 \leqslant y \leqslant x$ and $x \in Q_\perp$, then $y \in Q_\perp$, hence $Q_\perp \cap A^+$ is a face in A^+ and $Q_\perp$ is a closed order ideal. $Q_\perp$ is positively generated, for suppose that $a \in Q_\perp$. Then $0, a \leqslant \|a\|$ and $a|Q = 0$. Applying Theorem 2.4, there exists a continuous affine function c on $P_1(A)$ with $0, a \leqslant c$ and $c|Q = 0$. As $c(0) = 0$, $c \in A$ (Theorem 2.2). We have $a = c - (c - a)$, where $c, c - a \in (Q_\perp)^+$.

Let $\bar{J}$ be the closure of J. Trivially, $\bar{J} \subseteq J_{\perp\perp}$. If $a_0 \notin \bar{J}$, the Hahn-Banach Theorem enables us to select $f \in A^*$ with $f|J = 0$ and $f(a_0) \neq 0$. If $a \in J^+$, $0 \leqslant b \leqslant a$ implies $b \in J^+$ and $f(b) = 0$. From Corollary 2.7 it follows that $f^+(a) = 0$, $f^+|J^+ = 0$, and as $J = J^+ - J^+$, $f^+|J = 0$. Since $f^- = (-f)^+$, this argument shows that $f^-|J = 0$. As $f = f^+ - f^-$, either $f^+(a_0) \neq 0$ or $f^-(a_0) \neq 0$. Thus we obtain $p \in J_\perp$ with $p(a_0) \neq 0$, i.e., $a_0 \notin J_{\perp\perp}$.

$Q \subseteq Q_{\perp\perp}$ is again trivial. Say that $p_0 \notin Q_{\perp\perp}$. As Q is closed, convex, and contains 0, Q^{00} coincides with Q (see [5, Ch. IV, § 1, Prop. 3]). Thus there is an element $a \in A$ with $a(p_0) > 1$, $a(q) \leqslant 1$ for all $q \in Q$. From Theorem 2.4, there is a continuous affine function c on $P_1(K)$ such that $0, a - 1 \leqslant c$ and $c|Q = 0$. We have $c \in A$, $c \in Q_\perp$, but $c(p_0) \geqslant (a-1)(p_0) > 0$, i.e., $p_0 \notin Q_{\perp\perp}$.

COROLLARY 3.2. *$J \to J_\perp$ defines a one-to-one order inverting correspondence between the closed ideals in A and the closed faces containing 0 in $P_1(A)$.*

Theorem 3.1 is false for non-commutative C^*-algebras, as the null space of a pure (i.e., extremal) state that is not a character is not positively generated. For arbitrary function systems it is not clear that there are *any* non-zero positive elements in the null-space of a pure state. Note that such an element will exist if and only if the state is "supported". For further information on the null-spaces of faces, see [3; 10; 15; 9].

If J is a proper subspace of an ordered vector space V, we give J the *relative* ordering, i.e., that defined by the cone $J \cap V^+$, and V/J the *quotient* ordering, that defined by the image of V^+ under the canonical linear map of V onto V/J. In general the quotient ordering will not be "proper" as the image of V^+ need not be a cone. If J is a closed subspace of a Banach space V, we give J and V/J the relative and quotient norms. The following is well-known.

LEMMA 3.3. *If J is a proper ideal in an ordered vector space V, then J and V/J are ordered vector spaces. If J is closed and V is an L-space, then J and V/J are L-spaces.*

Proof. Suppose V is an ordered vector space. If θ is the quotient map, we must show

$\theta(V^+)$ is a cone. If $x_1+J=x_2+J$ with $x_1\geqslant 0$, $x_2\leqslant 0$, choose $r\in J$ with $x_1+r=x_2$. Then as J is an ideal, $r=r_1-r_2$, $r_i\in J^+$. We have

$$x_1+(-x_2)+r_1=r_2,$$

hence $0\leqslant x_1\leqslant r_2$. As J is an order ideal, $x_1\in J$, and $x_1+J=x_2+J=J$.

Say that V is an L-space, J a closed ideal. To prove J is an L-space, it suffices to show that J is closed under the lattice operations of V. If $r\in J$, say that $r=r_1-r_2$, $r_i\in J^+$. Then as $0, r\leqslant r_1$, we have $0\leqslant r^+\leqslant r_1$ and $r^+\in J$. If $r, s\in J$, then $r\vee s=r+(s-r)^+\in J$.

If $x\in V$, we claim that $x^++J=\max(x+J, J)$. As $x^+\geqslant x, 0$, we have $x^++J\geqslant x+J, J$. Suppose that $y+J\geqslant x+J, J$. Choose $r, s\in J$ with $y+r\geqslant x$ and $y+s\geqslant 0$. Then $r^+, s^+\in J$ and

$$y+r^++s^+\geqslant x, 0,$$

$$y+r^++s^+\geqslant x^+,$$

and $y+J\geqslant x^++J$. It readily follows that V/J is a vector lattice.

We have for any $x, y\in V$,

$$\|x+y+J\|\leqslant\|x+J\|+\|y+J\|.$$

If $x, y\in A^+$ and $\varepsilon>0$, choose $r\in J$ with

$$\|x+y+r\|\leqslant\|x+y+J\|+\varepsilon.$$

As $r^+\in J$, we have $(x+r^+)+J=x+J$, hence changing notation we may assume $r^+=0$. We have

$$\|(x+y-r^-)^+\|\leqslant\|x+y-r^-\|. \tag{3.1}$$

Letting

$$s=(x+y)\wedge r^-=x+y-(x+y-r^-)^+,$$

we have $s\in J$ and from (3.1)

$$\|x+y-s\|\leqslant\|x+y-r^-\|.$$

As $0\leqslant s\leqslant x+y$, we have from the Riesz decomposition property that $s=t+u$ where $0\leqslant t\leqslant x$ and $0\leqslant u\leqslant y$. As $0\leqslant t, u\leqslant s$, and $x-t\geqslant 0$, $y-u\geqslant 0$, it follows that $t, u\in J$ and

$$\|x+J\|+\|y+J\|\leqslant\|x-t\|+\|y-u\|=\|x+y-s\|\leqslant\|x+y+J\|+\varepsilon.$$

Thus we have $x+J, y+J\geqslant 0$ implies

$$\|x+y+J\|=\|x+J\|+\|y+J\|. \tag{3.2}$$

In particular,

$$\|x+J\|\leqslant\|x^++J\|+\|x^-+J\|=\|\,|x|+J\|.$$

Conversely, given $\varepsilon>0$, choose x with $\|x\|\leqslant\|x+J\|+\varepsilon$.

Then

$$\|\,|x|+J\|\leqslant\|\,|x|\,\|=\|x\|\leqslant\|x+J\|+\varepsilon.$$

It follows that $\|\,|x+J|\,\|=\|x+J\|$.

Finally, if $0\leqslant x+J\leqslant y+J$, then $0\leqslant(y-x)+J$, and from (3.2),

$$\|x+J\|\leqslant\|x+J\|+\|(y-x)+J\|=\|y+J\|.$$

THEOREM 3.4. *If J is a proper closed ideal in a simplex space A, then J and A/J are simplex spaces.*

Proof. The restriction map $\varrho: A^* \to J^*$ is onto, norm-decreasing, and order-preserving. If J^0 is the polar of J, the induced map ϱ_1 of A^*/J^0 onto J^* is an isometry (see [7, p. 26]). To show that ϱ_1 is an order-isomorphism, it suffices to prove that each bounded positive linear function p on J extends to a positive function on A.

From the Hahn-Banach Theorem there exists a function $f \in A^*$ extending p. If $r \in J^+$, $0 \leqslant s \leqslant r$ implies $s \in J$, hence $f(s) \leqslant f(r)$, and from Corollary 2.7, $f^+(r) = f(r)$. As $J = J^+ - J^+$, $f^+ | J = p$. f^+ is the desired extension.

J^0 is a norm-closed subspace of the L-space A^*. If $0 \leqslant q \leqslant p \in J^0$, then $q | J^+ = 0$ implies $q | J = 0$, hence J^0 is an order ideal. If $f \in J^0$, $f | J^+ = 0$ implies $f^+ | J^+ = 0$ hence $f^+ \in J^0$. Similarly $f^- \in J^0$, hence J^0 is an ideal. From Lemma 3.3, A^*/J^0 is an L-space, hence we conclude J^* is an L-space, and J is a simplex space.

From Lemma 3.3, the quotient ordering on A/J is proper. The map ψ of J^0 into $(A/J)^*$ defined by $\psi(f)(a+J) = f(a)$ is an order preserving isometry onto (see [7, p. 25]). It is an order isomorphism, as if $p \in P(A/J)$ and h is the canonical map of A onto A/J, then $p \circ h \in (J^0)^+$ and $\psi(p \circ h) = p$. From Lemma 3.3, J^0 is an L-space. All that remains is to show that if $p(a+J) \geqslant 0$ for all $p \in P(A/J)$, then $a + J \geqslant 0$. The hypothesis implies that $q(a) \geqslant 0$ for all $q \in J_\perp$. From Theorem 2.4 there exists a continuous affine function c on $P_1(A)$ with $0, a \leqslant c$ and $c | J_\perp = a | J_\perp$. As $c(0) = a(0) = 0$, $c \in A$ (Theorem 2.2), and $c - a \in J_{\perp\perp} = J$ (Theorem 3.1). Thus $a + J = c + J \geqslant 0$.

PROPOSITION 3.5. *The map $J \to J^+$ is a one-to-one correspondence between the ideals (closed ideals) in A, and the faces (closed faces) in A^+.*

Proof. If F is a face in A^+, let $J = F - F$. Then trivially $J \cap A^+ \supseteq F$. If $a, b \in F$, and $a - b \in A^+$, then $0 \leqslant a - b \leqslant a$ implies $a - b \in F$, hence $J \cap A^+ = F$. The assertion regarding ideals and faces is thus clear. To prove the remainder of the theorem, it suffices to show that $F_{\perp\perp}$ is a closed ideal in A with $F_{\perp\perp} \cap A^+ = \overline{F}$.

We have $F_\perp = Q \cap P_1(A)$, where Q is the positive annihilator of F. Q is a closed face in $P(A)$, as if $p | F = 0$, and $0 \leqslant q \leqslant p$, then $q | F = 0$. Thus $F_\perp$ is a closed face of $P_1(A)$ containing 0, and $F_{\perp\perp}$ is an ideal in A (Theorem 3.1). Trivially, $\overline{F} \subseteq F_{\perp\perp} \cap A^+$. Suppose that $a \notin \overline{F}$. As $\overline{F} = F^{00}$, there exists a function $f \in A^*$ with $f | F \leqslant 1$ and $f(a) > 1$. As F is a cone, $f | F \leqslant 0$. If $0 \leqslant s \leqslant r \in F$, then $s \in F$ and $f(s) \leqslant 0$. From Corollary 2.7 $f^+ | F = 0$, i.e., $f^+ \in F_\perp$. As $f^+(a) \geqslant f(a) > 1$, $a \notin F_{\perp\perp}$.

If F_γ is a family of faces in A^+, it is clear that $\bigcap F_\gamma$ is again such a face. It follows that if J_γ is a collection of ideals, then $(\bigcap J_\gamma)^+$ is a face in A^+, i.e., $\bigcap J_\gamma$ is an order ideal.

We have been unable to determine if $\bigcap J_\gamma$ is necessarily an ideal, i.e., whether or not $\bigcap J_\gamma$ coincides with $(\bigcap J_\gamma)^+ - (\bigcap J_\gamma)^+$. It follows from Proposition 3.5 that the latter is an ideal, which is closed if the J_γ are closed. We shall denote this ideal by $\wedge J_\gamma$. If $\bigcap J_\gamma$ is an ideal, then $\bigcap J_\gamma = \wedge J_\gamma$.

We note that the intersection J of finitely many ideals $J_1, \ldots, J_n$ is an ideal. For if $x \in J$, we may choose $y_i \in J_i$ with $0, x \leqslant y_i$. From the Riesz decomposition property, there is an element $u \in A$ with $0, x \leqslant u \leqslant y_i$. Our assertion follows as $x = u - (u - x)$ where $u, u - x \in J^+$.

If F_γ are faces in A^+, then the collection $\sum F_\gamma$ of finite sums is a face in A^+. It is trivial that $\sum F_\gamma$ is a cone. If $0 \leqslant y \leqslant \sum x_i$ with $x_i \in F_{\gamma(i)}$, then choose y_i with $y = \sum y_i$ and $0 \leqslant y_i \leqslant x_i$. We have $y_i \in F_{\gamma(i)}$, hence $y \in \sum F_\gamma$. If J_γ are ideals in A, then $(\sum J_\gamma)^+ = \sum J_\gamma^+$. For if $\sum x_i \geqslant 0$ with $x_i \in J_{\gamma(i)}$ choose $y_i, z_i \in J_{\gamma(i)}^+$ with $x_i = y_i - z_i$. Then

$$0 \leqslant \sum y_i - \sum z_i \leqslant \sum y_i \in \sum J_\gamma^+,$$

implies that $\sum y_i - \sum z_i \in \sum J_\gamma^+$. The converse inclusion is trivial. Thus $\sum J_\gamma$ is an order ideal, and as it is positively generated, it is an ideal.

Alfsen has shown that the closure of a face in $P_1(A)$ need not be a face [1, Theorem 1]. This fact is relevant to the intersection problem. Given ideals J_γ, it is readily proved that the convex span Q of the faces $J_\gamma^\perp$ is a face in $P_1(A)$ containing 0. If the weak* closure of Q were a face, one could use Theorem 2.4 to show that $\bigcap J_\gamma$ is an ideal.

4. The structure space

Let A be a simplex space. An ideal M in A is *maximal* if it is a proper subset of A, and coincides with any proper ideal containing it. As the closure of an ideal is again an ideal (Theorem 3.1), if M is maximal either $\bar{M} = A$ or M is closed. If A has an order identity e, then M must be closed. It suffices to show that $\|e + M\| = 1$, as then $\|e + \bar{M}\| = 1$. If there were an element $u \in M$ with $\|e - u\| = 1 - \varepsilon$, $\varepsilon > 0$, we would have from Proposition 2.8, and (2.3)

$$1 - L(u) = M(e - u) \leqslant 1 - \varepsilon,$$

hence $u \geqslant L(u)e$ where $L(u) \geqslant \varepsilon$. It would follow that $e \in M$, and as e is an order unit, $M = A$, a contradiction.

Suppose that M is a closed maximal ideal in a simplex space A. From Corollary 3.2, $M^\perp$ is minimal among the closed faces of $P_1(A)$ containing but not equal to $\{0\}$. As $M^\perp$ is compact and convex, it must have an extreme point $p_M \neq 0$. $M^\perp$ is a face in $P_1(A)$, hence p_M is an extreme point of $S(A)$. The ray passing through p_M is a closed face in $P(A)$, and $\{\alpha p_M: 0 \leqslant \alpha \leqslant 1\}$ is a closed face in $P_1(A)$ containing 0. It must coincide with $M^\perp$ as the latter is minimal, hence $M = \{p_M\}^\perp$. This determines a one-to-one correspondence between the set of maximal closed ideals $\max A$, and the extreme states $ES(A) = EP_1(A) - \{0\}$. In the proof of Theorem 3.4 we showed that $P_1(A/M)$ may be identified with $M^\perp$. In particular, $a + M \geqslant 0$ if and only if $p_M(a) \geqslant 0$, and $\|a + M\| = |p_M(a)|$. Thus p_M determines an order isomorphism and isometry of A/M onto the reals, the latter given its natural ordering and norm.

Let $\max A$ be the set of closed maximal ideals in A. If J is a subset of A, the *hull* of J is the set

$$h(J) = \{M \in \max A : J \subseteq M\}.$$

If S is a subset of $\max A$, the *kernel* of S is the closed ideal

$$k(S) = \wedge\{M : M \in S \cup \{A\}\}.$$

Note that if $S = \emptyset$, $k(S) = A$.

THEOREM 4.1. *If J is an ideal in a simplex space A, then $kh(J)$ is the closure $\bar{J}$ of J.*

Proof. From Theorem 3.1 and the Krein-Milman theorem, we have $\bar{J} = J^{\perp\perp} = [E(J^\perp)]^\perp$. As $J^\perp$ is a face in $P_1(A)$, $E(J^\perp) = J^\perp \cap EP_1(A)$. Thus

$$\bar{J} = \bigcap \{p \in EP_1(A) : p \mid J = 0\}^\perp = \bigcap \{M \in \max A \cup \{A\} : M \supseteq J\} = kh(J).$$

It follows that the subsets of A invariant under the "operation" kh are just the closed ideals. As $hkh(J) = h(J)$ for any subset J of A, the hulls are those subsets of $\max A$ invariant under hk, and each is the hull of a closed ideal.

THEOREM 4.2. *If A is a simplex space, the hulls are the closed sets of a topology on* $\max A$.

Proof. Due to Proposition 3.5, we may regard $\max A$ as the closed maximal faces N of A^+, and a hull as the closed maximal faces containing a given closed face. If F_γ are closed faces, it follows from the discussion after Proposition 3.5 that $\sum F_\gamma$ is a face. The closure F_0 of $\sum F_\gamma$ is again a face (Proposition 3.5) and it is clear that $\bigcap h(F_\gamma) = h(F_0)$. Thus an intersection of hulls is a hull.

If F_1 and F_2 are closed faces, $F_1 \cap F_2$ is a closed face. Trivially $h(F_1) \cup h(F_2) \subseteq h(F_1 \cap F_2)$. Conversely, suppose that N is a closed maximal face in A^+ with $F_1 \cap F_2 \subseteq N$, but that $F_i \nsubseteq N$, $i = 1, 2$. Then as $F_i + N$ are faces, $F_1 + N = F_2 + N = A^+$. Choose $u \in A^+$ with $p_M(u) = 1$ where $M = N - N$, and $r_i \in F_i$, $s_i \in N$ with

$$u = r_1 + s_1 = r_2 + s_2.$$

As a consequence of the Riesz decomposition property there exist $t_{ij} \in A^+$ with

$$r_1 = t_{11} + t_{12}, \qquad r_2 = t_{11} + t_{21},$$
$$s_1 = t_{21} + t_{22}, \qquad s_2 = t_{12} + t_{22}.$$

As $t_{11} \leqslant r_i \in F_i$, $t_{11} \in F_1 \cap F_2 \subseteq N$, and as $t_{12} \leqslant s_2$, $t_{12} \in N$. Thus $r_1 \in N$, $u = r_1 + s_1 \in N$, and $p_M(u) = 0$, a contradiction. We must either have $F_1 \subseteq N$ or $F_2 \subseteq N$, i.e., $h(F_1 \cap F_2) \subseteq h(F_1) \cup h(F_2)$. The union of two hulls is a hull, and noting that $\max A = h(\{0\})$ and $\emptyset = h(A)$, we are done.

We call the above topology on $\max A$ the *structure topology*, and $\max A$ with this topology the *structure space*. This topology is weaker than the *weak* topology*, obtained from the identification of $\max A$ and $ES(A)$.

LEMMA 4.3 (*see Lemma 3.3*). *Suppose that J is a proper closed ideal in an L-space V. The canonical homomorphism θ of V onto V/J restricts to an injection of $E(V_1^+) - J$ into $E((V/J)_1^+)$.*

Proof. As the norm restricted to V^+ extends to a strictly positive linear function on V, the discussion at the beginning of the previous section is applicable. In particular, a point $p \neq 0$ in V_1^+ is extremal if and only if $\|p\| = 1$, and $0 \leqslant q \leqslant p$ implies $q = \alpha p$ for some scalar α.

If $p \in E(V_1^+) - J$, we assert that $\|p + J\| = 1$. If $r \in J$, then $0 \leqslant p \wedge |r| \leqslant |r|$ and $|r| \in J$, hence $p \wedge |r| \in J$. As p is extreme, $p \wedge |r| = \alpha p$ for some scalar α. If $\alpha \neq 0$, we have $p \in J$, a contradiction. Thus $p \wedge |r| = 0$, and

$$\|p + r\| = \|p\| + \|r\| \geqslant 1.$$

It follows from the proof of Lemma 3.3 that θ is a lattice homomorphism. Thus if $p \in E(V_1^+) - J$, $q \geqslant 0$, and $\theta(q) \leqslant \theta(p)$, then letting $q \wedge p = \alpha p$,

$$\theta(q) = \theta(q) \wedge \theta(p) = \theta(q \wedge p) = \alpha\theta(p),$$

i.e., $\theta(p) \in E((V/J)_1^+)$. If in addition, $q \in E(V_1^+) - J$ with $q \neq p$, then $q \wedge p = 0$, hence $\theta(q) \wedge \theta(p) = 0$, and $\theta(q) \neq \theta(p)$.

THEOREM 4.4. *If J is a proper closed ideal in a simplex space A, then there are natural homeomorphisms of* $\max A/J$ *onto $h(J)$, and* $\max J$ *onto* $\max A - h(J)$ *for the structure topologies.*

Proof. Let θ be the canonical linear map of A onto A/J, and say that I is an ideal in A/J. $\theta^{-1}(I)$ is an order ideal, as if $0 \leqslant b \leqslant a \in \theta^{-1}(I)$, then $0 \leqslant \theta(b) \leqslant \theta(a)$, $\theta(b) \in I$, and $b \in \theta^{-1}(I)$. If $a \in \theta^{-1}(I)$, there is an element $\theta(b) \in I^+$ with $\theta(a) \leqslant \theta(b)$. We may assume that $b \geqslant 0$. Then $a \leqslant b + u$, for some element $u \in J$. Choosing $v \in J^+$ with $u \leqslant v$, we have $a \leqslant b + v$, where $b + v \in \theta^{-1}(I)^+$. Thus $\theta^{-1}(I)$ is positively generated. The map $I \to \theta^{-1}(I)$ is clearly a one-to-one inclusion preserving map of the ideals in A/J onto the ideals in A containing J. It follows that the map $\varphi(M) = \theta^{-1}(M)$ is a one-to-one map of $\max A/J$ onto $h(J)$. If I is a closed ideal in A/J, then $\varphi(h(I)) = h(\theta^{-1}(I))$. If H is a closed ideal in A with $h(H) \subseteq h(J)$, then $H \supseteq J$, and $\varphi(h(\theta(H))) = h(H)$. Thus φ is a homeomorphism.

We saw in the proof of Theorem 3.4 that the restriction map $\varrho: A^* \to J^*$ induced an isometric order isomorphism of A^*/J^0 onto J^*. From Lemma 4.3 it follows that ϱ is an injection of $EP_1(A) - J^0$ into $EP_1(J)$. ϱ maps $P_1(A)$ onto $P_1(J)$ as any positive function on J extends to a positive function on A of the same norm (again, see the proof of Theorem 3.4). Letting $\varrho_1 = \varrho | P_1(A)$, if F is a closed face in $P_1(J)$, then $\varrho_1^{-1}(F)$ is a non-empty closed face in $P_1(A)$. In particular, if $q \in EP_1(J) - \{0\}$, then $\varrho_1^{-1}(\{q\})$ is a closed non-empty face in $P_1(A)$ and must contain an extreme point $p \notin J^0$. We have $p \in EP_1(A)$ and $\varrho(p) = q$, i.e., ϱ maps $EP_1(A) - J^0$ onto $EP_1(J) - \{0\}$.

For each $M \in \max A - h(J)$ we have $p_M \in EP_1(A) - J^0$, $\varrho(p_M) \in EP_1(J) - \{0\}$, and

$$M \cap J = \{\varrho(p_M)\}^{\perp} \in \max J.$$

It follows from the above discussion that the map $\eta(M) = M \cap J$ is a one-to-one map of $\max A - h(J)$ onto $\max J$. If I is a closed ideal in A, $I \cap J$ is an ideal in A (see § 3), and thus in J. Denoting a hull taken in $\max J$ by h_J, it is clear that

$$\eta(h(I) \cap [\max A - h(J)]) \subseteq h_J(I \cap J).$$

If $M \in \max A - h(J)$ and $M \supseteq I \cap J$, then $M \supseteq I$ (see the proof of Theorem 4.2) hence we have the converse inclusion. Thus η is a homeomorphism, as an ideal in J is an ideal in A.

As one might expect from C^*-algebra theory, the elements of A "vanish at infinity" on $\max A$, i.e., letting $a(M) = p_M(a)$ for $M \in \max A$,

PROPOSITION 4.5. *If A is a simplex space, $a \in A$, and $\alpha > 0$, then the set*

$$C = \{M \in \max A: |a(M)| \geq \alpha\}$$

is compact.

Proof. Suppose that F_γ is a decreasing net of closed sets in $\max A$ with $C \cap F_\gamma \neq \emptyset$. Let $J_\gamma = k(F_\gamma)$, $J_0 = \bigcup J_\gamma$, and $J_1 = \bar{J}_0$. Then J_0 is an ideal, hence J_1 is an ideal (Theorem 3.1). As $F_\gamma \neq \emptyset$, we have $J_\gamma \neq A$, and identifying $P_1(A/J_\gamma)$ with $J_\gamma^\perp$,

$$\|a + J_\gamma\| = \sup\{|p(a)|: p \in J_\gamma^\perp\} = \sup\{|p(a)|: p \in E(J_\gamma^\perp)\} = \sup\{|a(M)|: M \in F_\gamma\} \geq \alpha.$$

We used the fact that the sets of points on $J_\gamma^\perp$ at which a assumes its maximal and minimal values, respectively, are closed faces in $J_\gamma^\perp$ and thus contain extreme points. It follows that $\|a + J_1\| \geq \alpha$, hence $J_1 \neq A$, and again, there is an $M \in h(J_1)$ with $|a(M)| = \|a + J_1\| \geq \alpha$. As $h(J_1) = \bigcap h(J_\gamma)$, we have $\bigcap (F_\gamma \cap C) \neq \emptyset$.

COROLLARY 4.6. *If A has an order identity, then* $\max A$ *is compact.*

If one had that $M \to |a(M)|$ was lower semi-continuous, i.e., $\{M: |a(M)| \leq \alpha\}$ was closed for all α, one could prove that $\max A$ is always locally compact. Unfortunately, in contrast with C^*-algebras, these functions are generally not semi-continuous (see Theorem 4.8). We do have:

PROPOSITION 4.7. *If A is a simplex space, and $a \in A^+$, then the set $\{M: a(M)=0\}$ is closed.*

Proof. As $\{a\}_\perp$ is obviously a closed face in $P_1(A)$ containing 0, $J=\{a\}_{\perp\perp}$ is an ideal. We have

$$\{p \in EP_1(A): p(a) = 0\} = E(J_\perp),$$

hence

$$\{M \in \max A: a(M) = 0\} = h(J).$$

If X is a compact Hausdorff space, the ordered Banach space $C(X)$ of continuous functions on X is an M-space, and thus a simplex space, and the constant function e is an order identity. Such spaces are readily characterized:

THEOREM 4.8. *If A is a simplex space with order identity, then the following are equivalent:*

(1) $\max A$ *is Hausdorff.*

(2) A *is a lattice.*

(3) A *is an M-space.*

(4) $ES(A)$ *is closed in* $S(A)$.

(5) *There is a natural isometric order isomorphism of A onto $C(ES(A))$.*

(6) *The functions $M \to |a(M)|$ with a in A are upper (lower) semi-continuous.*

Proof. If $\max A$ is Hausdorff, then for each $a \in A^+$, and $\alpha > 0$, $\{M: a(M) \geqslant \alpha\}$ is compact (Proposition 4.5), hence closed. Trivially $\{M: a(M) \geqslant 0\}$ is closed, and a is upper semi-continuous. If $a \in A$ is arbitrary, $a + \|a\|e$ is positive, hence $a + \|a\|e$ and a are upper semi-continuous. Taking negatives, we conclude that a is continuous. This would also have followed if $|a|$ were known to be lower semi-continuous. As the functions $M \to a(M)$ define the weak* topology on $\max A$, and that topology contains the hull-kernel topology, the two topologies coincide. $ES(A)$ is thus weak* compact (Corollary 4.6), and closed in $S(A)$.

The implication (4) $\Rightarrow$ (5) is due to Bauer [2]. In any C^*-algebra $\mathfrak{A}$, the closed faces of $\mathfrak{A}^+$ are just the positive parts of the closed left ideals [9, Theorem 2.4] hence letting $\mathfrak{A} = C(ES(A))$, of the closed two-sided ideals. As the latter are positively generated, the algebraic and order notions of closed ideal coincide. The closed maximal ideals in the order sense are the closed ideals of co-dimension one (see above). As $\mathfrak{A}$ has an identity, the latter coincide with the maximal ideals in the algebraic sense. Thus we are considering the usual structure space of a commutative Banach algebra, and (5) $\Rightarrow$ (1) is a well-known result (see [22, p. 57]).

(5) $\Rightarrow$ (3) $\Rightarrow$ (2) are trivial. (2) $\Rightarrow$ (4) follows readily from [19, § 24.2].

5. Problems and an example

It should be possible to give a definition of simplex space that is analogous to that for M-spaces. Presumably it may be defined as an ordered Banach space satisfying the Riesz decomposition property and certain other, as yet undetermined, conditions. Is the structure space of a simplex space always locally compact? How may Theorem 4.8 be generalized to simplex spaces without order identity? If A is a separable simplex space, when do the hull-kernel and weak* topologies generate the same Borel structure on max A? This question is of importance in the applications to C^*-algebras. It would suffice to show the first structure is countably separated.

In attempting to find analogies with the theory of C^*-algebras, one is confronted with numerous problems. Can one develop the notion of the spectrum of an element? Is there a class of simplex spaces with almost Hausdorff structure spaces, analogous to GCR algebras?

In order to obtain a more detailed theory, it may be necessary to restrict the spaces under consideration. Counter to the situation for C^*-algebras, the structure space of a simplex space need not be second category. To see this, consider Alfsen's construction of a simplex in the proof of [1, Theorem 1] (see also [21, p. 78]). Letting $\alpha_n = 2^{-n}$ for $1 \leqslant n < \infty$, and $\alpha_\infty = -1$, one obtains a simplex K for which $E(K)$ is countable, and the only closed faces of K are K itself, and the faces spanned by finitely many points in $E(K)$. It follows that if $A = \mathcal{A}(K)$, $\max A$ has only countably many points, and the only hulls are the finite sets and $\max A$ (the "Zariski topology"). Each point comprises a closed set without interior, and $\max A$ is a countable union of such sets.

We note that Alfsen's original example has a non-Hausdorff, but almost Hausdorff structure space.

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